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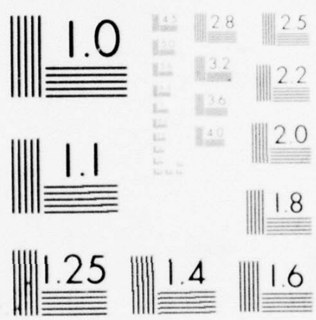

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STRUCTURAL DESIGN BY SINGULAR IMBEDDING

by

Wiwat Sangtian
Charles H. Goodspeed
Steven J. Fenves

A Technical Report of a Research Program
Sponsored by
THE OFFICE OF NAVAL RESEARCH
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DEPARTMENT OF CIVIL ENGINEERING
CARNEGIE INSTITUTE OF TECHNOLOGY
Carnegie-Mellon University

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CHAPTER 1. INTRODUCTION

1.1. Objective

Numerous structural analysis programs are available today to assist in the structural design process. However, the complete structural design involves not only analysis, which determines the structural response on the basis of given element properties, but also the selection of element properties so that imposed constraints (e.g. stresses and displacements) are met. Today, two general methods are available for designing structures to meet specified constraints: optimization methods, in which the analysis and the satisfaction of constraints are performed simultaneously, but at great expense; and iterative schemes, where successive analyses and modifications performed through a man-computer interaction until the constraints are specified.

The objective of this study is to develop a combined method for designing selected components of a structure subject to a set of performance criteria simultaneously with the analysis process.

1.2. Background

The method presented here is based on the work of A.M. Murray-Lasso in computer-aided circuit design [1,2]. In the method originally presented, the actual elements of the network are augmented by fictitious elements which impose constraints on the network variables (voltages and currents). These elements are represented by singular admittance matrices, hence the term "singular embedding". The constraints expressed by the singular elements produce additional equations appended to the network equations.

In this method, selected elements may be designed by treating their element voltage drops and currents as separate variables. After the system equations, including the additional constraints, are solved for the unknowns, the computed voltage drops and currents can be used to determine the necessary element resistances, i.e. design the elements.

1.3 Scope of Report

Chapter 2 presents the basic approach to design by singular embedding, by essentially interpreting the method in structural engineering terminology.

Chapter 3 consists of three parts. First, the basic method is significantly extended by introducing a large number of constraints occurring in structural design, involving limits, ratios, and ranges on the design and response parameters. The system equations including these constraints are also formulated. The second part discusses methods of solution for the combined analysis-constraint equations, and the post-processing to obtain the design parameters. The chapter concludes with a number of examples and a brief discussion of the computational errors introduced.

The report concludes with a brief summary of findings and conclusions arising from the study in chapter 4.

The new approach using singular imbedding, presented herein, can help the designer to design structures to meet performance characteristics without having to perform the complete analysis to check the requirements and without having to select initial sizes for all members.

CHAPTER 2. APPROACH

2.1. Introduction

To understand the concepts of the approach to design presented herein, the requirements on a structure to be designed are first examined. For example, in a floor system the midspan deflection is normally held to a specified ratio of the span length; in a frame the sidesway is limited to specified drift limits; in members the stresses must not exceed allowable values for the material used. Common to all these problems is that the network variables, such as forces and displacements, or their ratios, are constrained. The design problem is to determine the element properties in accordance with the constraints. If the problem is specified with sufficient variables, one or more solutions will exist consistent with the imposed constraints. Conversely, if the problem is specified with insufficient variables inconsistent equations develop and there is no solution.

Network analysis methods for structural analysis [3,4,5] require that the member properties for all members of the structure are known. Therefore, the network method can be used only for analyzing structures whose properties are known, to determine if the results (i.e. joint displacements, member distortions or member forces) meet the specified requirements placed on the design. Design of statically indeterminate structures to meet a set of specified requirements is normally an iterative procedure, in which the initial element properties are selected based on approximate analysis methods [6]. Once the members are selected, the structure is analyzed and the results compared with the specified requirements. If the results of the analysis match the specified requirements, the design is considered satisfactory. If not, the elements are redesigned and the structure reanalyzed on the bases of new member properties. A complete analysis is needed to determine if the structure performs in the specified manner whenever a redesign is made. The new approach using singular imbedding, presented herein, can help the designer to design structures to meet performance characteristics without having to perform the complete analysis to check the requirements and without having to select initial sizes for all members.

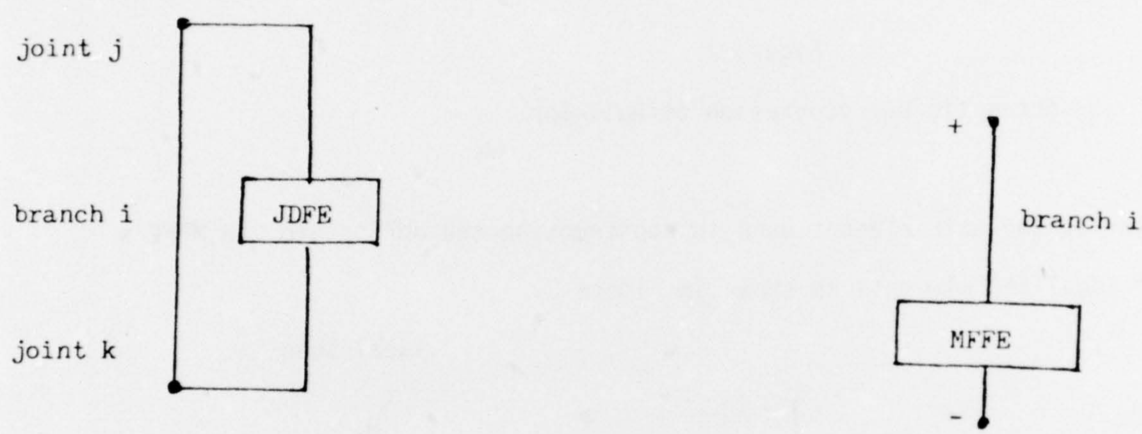
2.2. Singular Elements

A new element is introduced to represent constraints. Since adding this element to the network will cause the structural stiffness matrix to become singular, this constraining element is called a singular element. Although the structural variables (i.e., displacements and forces) are vector quantities, constraints imposed on structural design problems are generally specified individually (i.e. as single displacement or force components). Therefore, a singular element is a scalar element.

Two types of singular elements are utilized. These two singular elements are Joint Displacement Forcing Elements (JDFE) and Member Force Forcing Elements (MFFE), which are used for constraining joint displacements and member forces, respectively.

The JDFE is defined as a singular element which constrains a displacement component at a specified joint to a specified value, while no force is allowed to occur in the element. Since displacements are across variables in the structural problem, to constrain the joint displacement of joint j the JDFE is applied parallel with branch i, which is connected between joints k and j as shown in Figure 1a. One end of the JDFE is connected to joint j whose displacement is constrained, and the other end is connected to a fixed joint, k. Since no forces pass through the JDFE, the continuity law and equilibrium law are affected only to the extent that displacement at joint j relative to the fixed joint k is now constrained to be equal to the specified value.

The MFFE is defined as a singular element which constrains a member force component to a specified value while allowing no displacements to occur in the element. Since forces are through variables in the structural problem, the MFFE is applied in series with branch i, at the negative end of the branch, to constrain a member force in branch i, as shown in Figure 1b. The continuity law and equilibrium law are affected only to the extent that the member force in branch i is now constrained to be equal to the specified value because the MFFE allows no displacements to exist.



a) Displacement at joint j constrained by Joint Displacement Forcing Element

b) Member force in branch i constrained by Member Force Forcing Element

Figure 1,

Singular Element

The JDFE's and the MFFE's can be thought of in terms of idealized elements. The necessary elements are nullators 1, ideal branches, and ideal loads [1]. A nullator is defined as an element which allows only zero displacement and force to occur. A nullator is used to show the relationship between nodes. A nullator can be represented schematically as shown in Figure 2.

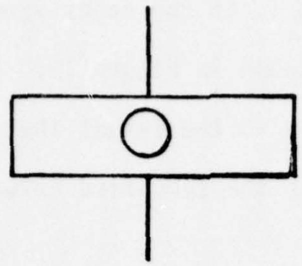


Figure 2

Schematic Representation of Nullator

The nullator is the main element used in constructing the JDFE's and the MFFE's in terms of idealized elements as shown in Figure 3.

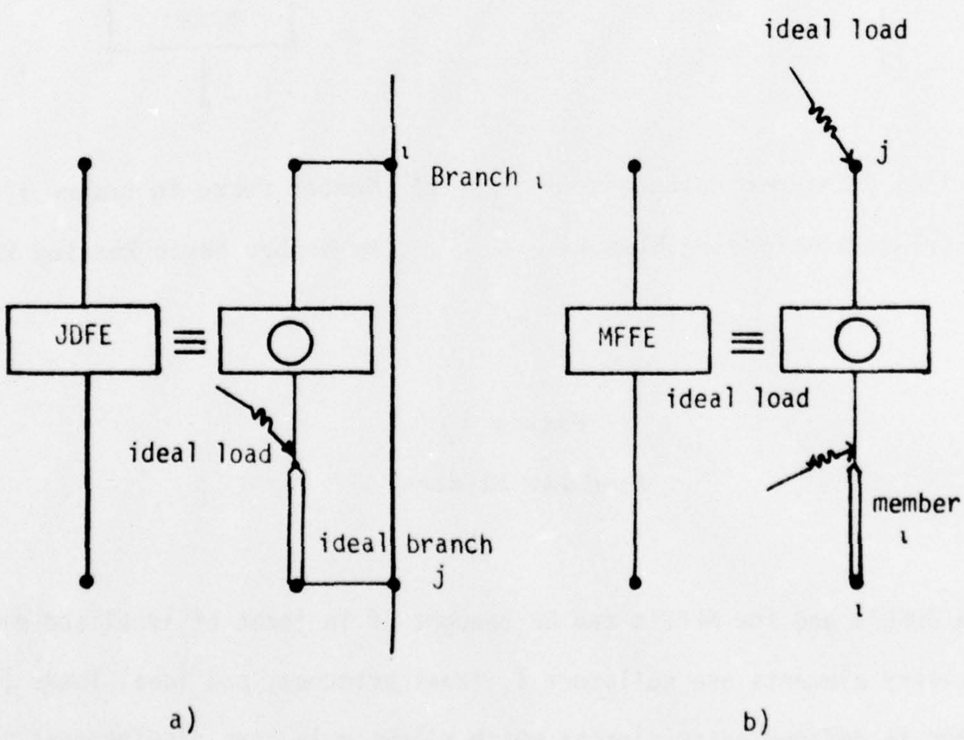


Figure 3

Equivalent Representation for

JDFE and MFFE Using Nullator

In Figure 3a, an equivalent representation for the JDFE using a nullator is shown. The element is composed of a nullator and an ideal branch connected in series. The ideal branch is a scalar element whose properties are: (1) it has stiffness in the component direction of interest (i.e. axial, shear, or bending) equal to unity, and (2) its direction is always negatively incident on a common node to which the nullator and ideal branch are connected. The nullator is connected to the joint whose displacement is constrained. The ideal load is applied at the node on which the ideal branch is negatively incident. The value of this ideal load is equal to the value of the constrained joint displacement. The sign of the ideal load is the same as the sign of the constrained displacement. If the JDFE is used for constraining a displacement at a joint in the x-direction, the ideal load will be an axial load. It will be a shear load for a displacement in the y-direction and a moment for a rotational displacement.

In Figure 3b, the equivalent representation for the MFFE using a nullator is shown. Only a nullator and ideal loads are needed to represent a MFFE. The nullator is inserted in series between the negative end of the member in which a member force is constrained and the node on which the member is negatively incident. The ideal loads are applied at the nodes on which the nullator is incident. The sign of the ideal loads depends on the sign of the constrained force. If the sign of the constrained force is positive, a positive ideal load is applied at the node connecting the nullator and the member and a negative ideal load is applied at the other node of the nullator. These will be reversed if the sign of the constrained force is negative.

The singular elements, JDFE and MFFE, are added to the network graph representing the structure. Since the structural stiffness matrix is composed of the individual stiffness matrices, the insertion of the singular elements into the structure will affect the structural stiffness matrix and the nodal equations. The nodal equations for the structure will be written by first nullators from the structure. The nodal equations with nullators removed can be written as

$$[K_t] \left\{ u' \right\} = \left\{ P' \right\} \quad (1)$$

where

K_t is the structural stiffness matrix

u' is a vector of joint displacements in global coordinates.

P' is a vector of applied joint loads in global coordinates.

The effect of the nullators on the nodal equations can be handled in two ways. First, the nodal equations (Eq. (1)) are augmented by the set of constraints:

$$[A_n] [u'] = [\gamma] \quad (2)$$

where

A_n is a matrix expressing the constraints for nullators

γ is the vector of constraint value.

This method will be expanded to handle a variety of constraints in Chapter 3. The second method adjusts the stiffness matrix in accordance with the constraints. We can illustrate the method by connecting a nullator between two nodes say i and j. Since the nullator allows no displacement to occur u'_i and u'_j are constrained to be equal. Writing the displacements for nodes i and j as $u'_i = u'_j = u'_{ij}$ the nodal equations can be written as

$$\begin{bmatrix} k_{t11} & k_{t12} & \cdots\cdots & (k_{t1i} + k_{tij}) & \ldots k_{t1N} \\ k_{t21} & k_{t22} & \cdots\cdots & (k_{t2i} + k_{t2j}) & \ldots k_{t2N} \\ \vdots & & & & \\ k_{tN1} & k_{tN2} & \cdots\cdots & (k_{tNi} + k_{tNj}) & \ldots k_{tNN} \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{ij} \\ \vdots \\ u' \end{Bmatrix} = \left\{ P' \right\}$$

Therefore, the addition of m nullators causes the addition of m columns in K_t and reduces the dimension of the vector u' by m. The reduced set of equations for a planar frame (i.e. three degrees of freedom per node) can be denoted as

$$[K_t]_{3n \times (3n-m)} \left\{ u' \right\}_{(3n-m)\times 1} = \left\{ P' \right\}_{3n\times 1} \quad (3)$$

where

K_t is the structural stiffness matrix excluding nullators

n is the number of free joints

m is the number of nullators

P' is the vector of applied joint loads.

By summing columns the number of variables becomes smaller than the number of equations; thus the nodal equations are no longer independent after adding singular elements to the network. This is compensated for by introducing member forces in variable members as unknowns and explicitly adding them into the nodal equations.

2.3. Variable Members

An important step in design is the determination of the necessary member properties. This aspect can also be included in the approach explicitly. The design problems applicable are those in which the properties of some members (cross-sectional area and moment of inertia for members of planar structured) are known, while for some other members these properties must be determined so that the structure will perform in a specified manner. The members whose properties are known will be referred to as defined members and the members with unknown properties will be referred to as variable members. The variable members will be included in the analysis by first explicitly adding their applied member forces (fixed-end forces) into the nodal equations. This is done by determining the joint loads induced by the applied member forces using the relation $A^tP = P'$, and then adding these derived joint loads with signs reversed to the nodal equations. After the variable members are selected, the nodal equations can be written as

$$[K_d]\left\{u'\right\} = \left\{P'\right\} - [A_v^t]\left\{P_v\right\} \tag{4}$$

where

K_d is the structural stiffness matrix of the structure containing only the defined members.

A_v is the branch-node incidence matrix associated with the variable members.

P_v are the applied member forces of the variable members, in local coordinates.

The vectors u' and P_v are both variables and may be combined by matrix partitioning as

$$[A_v^t \,\vdots\, K_d]\left\{\frac{P_v}{u'}\right\} = \left\{P'\right\} \tag{5}$$

By introducing member forces in variable members as unknowns and explicitly adding them to the nodal equations, the number of variables is increased. Thus the degrees of freedom removed by imposing singular elements into the network, as represented in Equation (3), are replaced by the unknowns introduced by the variable members. These additional unknowns are shown by rewriting Eq. (5) as

$$[A_v^t \,\vdots\, K_d]_{3n\times\{(3n-m)+3nvm\}}\left\{\frac{P_v}{u'}\right\}_{\{(3n-m)+3nvm\}\times 1} = \left\{P'\right\}_{3n\times 1} \tag{6}$$

where nvm is the number of variable members. The solution of Eq. (6), yields both the member forces P_v and the joint displacements u'. Member forces in the variable members are obtained directly from the solution, whereas member distortions in the variable members are calculated by the relation $u_v = A_v u'$, where u_v is the vector of variable member distortions.

2.4. Summary

In review, the following steps are performed in the singular imbedding approach:

1. The singular elements are imbedded and variable members designated in the network graph representing the structure;
2. The structural stiffness matrix is written for the specified members with the nullators removed;
3. The loads representing the variable members are added to the nodal equations;
4. The nodal equations are appended with the set of constraint equations.

In the next section, these procedures will be generalized in order to deal with different types of constraints.

CHAPTER 3. GENERAL PROCEDURE

3.1. Introduction

If a wide range of constraints is to be considered, a corresponding range of singular elements has to be available. Therefore, a more general procedure than the one presented so far must be developed. A general procedure is to describe any constraint as an equation in terms of the unknown variables in the augmented nodal equations namely the joint displacement and member forces. As a result, inequality constraints as well as equality constraints can be applied.

3.2. Types of Constraints

Structural variables used as constraints can be either behavior variables such as member forces, stresses, and displacements, or design variables such as cross sectional area and moment of inertia. A constraint can be used for restricting the structural variables to a desired value (an equality constraint) or designating a desired range (an inequality constraint). The general constraint equation is a scalar row, enabling structural variables to be specified along specified degrees of freedom. For example, only the displacement in the x-direction at a joint may be constrained, or all displacements at a joint can be constrained. Different types of constraints can be used together in the same problem, pertaining to both defined members and variable members. The following is the development of general constraint equations for structural design problems.

a) Member Distortions - Let the value of the constraint be u^* for a member distortion component; it can be related to the unknown joint displacements, u', by the continuity law, $u^* = Au'$. For each equality constraint the equation is:

$$(a_g)_m u' = u^*_0 \tag{7}$$

where

g is the member in which the member distortion is constrained;

m is the component of the constrained displacement; and

a_g is the g^{th} row of the branch-node incidence matrix.

For an inequality constraint, the following inequalities apply:

$$(a_g)_m \acute{u} \geq u_1^\star \quad \text{(lower limit)} \tag{8a}$$

$$(a_g)_m \acute{u} \leq u_2^\star \quad \text{(upper limit)} \tag{8b}$$

b) Joint Displacements - Let the value of the constraint be $\acute{u}$ * for a joint displacement component. For each equality constraint the equation is:

$$\acute{u}_{i,m} = \acute{u}_0^\star \tag{9}$$

where

i is the joint at which the displacement is constrained; and

m is the component of the constrained displacement.

For an inequality constraint the inequalities are:

$$\acute{u}_{i,m} \quad \acute{u}_1{}^\star \tag{10a}$$

$$\acute{u}_{i,m} \quad \acute{u}_2{}^\star \tag{10b}$$

c) Member Forces - Let the value of the constraint be P* for a member force component. If the member considered is a variable member, the following equality equation applies:

$$P_{g,m} = P_0^\star \tag{11}$$

where

g is the member in which member force is constrained; and

m is the component of the constrained force.

For each inequality constraint, the equation is:

$$\left| \begin{array}{l} P_{g,m} \geq P_1^* \\ \\ P_{g,m} \leq P_2^* \end{array} \right| \quad \begin{array}{r} (12a) \\ \\ (12b) \end{array}$$

If the member considered is a defined member, the following equality equation applies for each equality constraint:

$$\left(K_{gg}\right)\left(a_g\right)_m \quad \acute{u} = P_0^* \qquad (13)$$

where

k_{gg} is the member stiffness submatrix written for member g, in member coordinates.

For each inequality constraint, the relation is:

$$\left| \begin{array}{l} \left(k_{gg}\right)\left(a_g\right)_m \quad \acute{u} \geq P_1^* \\ \left(k_{gg}\right)\left(a_g\right)_m \quad \acute{u} \leq P_2^* \end{array} \right| \quad \begin{array}{r} (14a) \\ (14b) \end{array}$$

d) Member Properties - Member properties which may be constrained for a planar frame cross-sectional area, moment of inertia, and modulus of elasticity. Only cross-sectional area and moment of inertia will be used as constraints. For a variable member for which the cross-sectional area is constrained to be equal to A_0^*, the equality equation is:

$$P_{g,x} - \left(\left[\left(k^*_{gg}\right) \right]^{A_o^*} \left(a_g\right) \right)_x \quad \acute{u} = 0 \qquad (15)$$

where

g represents the variable member;

x represents the axial component;

$P_{g,x}$ is the unknown axial force in member g; and

$\left[\left(k^*_{gg}\right)\right]^{A_0^*}$ is calculated using A_0^*

For a variable member for which the cross-sectional area is constrained to the range between A_1^* and A_2^* $(A_1^* \quad A_2^*)$, the inequalities are:

$$P_{g,x} - \left(\left[\left(k^*_{gg}\right) \quad A_1^*\right]\left(a_g\right)\right)_x \quad \acute{u} \leq 0 \qquad (16a)$$

$$P_{g,x} - \left[\left(k^*_{gg}\right) \quad A_2^*\right]\left(a_g\right)\Big)_x \quad \acute{u} \leq 0 \qquad (16b)$$

where

$\left[\left(k^*_{gg}\right) \quad A^*_1\right]$ is calculated using A_1^*; and

$\left[\left(k^*_{gg}\right) \quad A^*_2\right]$ is calculated using A_2^*.

For a variable member for which the moment of inertia is constrained to I_0^*, the equality equations are:

$$\left| \begin{array}{l} P_{g,y} - \left(\left[\left(k^*_{gg}\right) \quad I_0^*\right]\left(a_g\right)\right)_y \quad \acute{u} = 0 \qquad (17a) \\ M_g - \left(\left(k^*_{gg}\right) \quad I_0^* \quad a_g \quad \theta \quad \acute{u} = 0 \qquad (17b) \end{array} \right|$$

where

y represents the transverse component;

θ represents the rotational component;

$P_{g,y}$ is the unknown shear force in member g;

M_g is the unknown bending moment in member g; and

$\left[k^*_{gg} \quad I_0^*\right]$ is calculated using I_0^*.

For a variable member for which the moment of inertia is bounded by I_1^* and I_2^* $(I_1^* \quad I_2^*)$, the inequalities are

$$\left| \begin{array}{l} P_{g,y} - \left(\left(k^*_{gg}\right) \quad I_1^*\left(a_g\right)\right)_y \quad \acute{u} \geq 0 \qquad (18a) \\ M_g - \left(\left(k^*_{gg}\right) \quad I_1^*\left(a_g\right)\right)_\theta \quad \acute{u} \geq 0 \qquad (18b) \end{array} \right|$$

$$\left| P_{g,y} - \left(\left(k^*_{gg}\right)^{I^*_2} \left(a_g\right)\right)_y \quad \acute{u} \leq 0 \right| \qquad (19a)$$

$$\left| M_g - \left(\left(k^*_{gg}\right)^{I^*_2} \left(a_g\right)\right)_\theta \quad \acute{u} \leq 0 \right| \qquad (19b)$$

where

$\left[\left(k^*_{gg}\right)^{I^*_1}\right]$ is calculated using I^*_1; and

$\left[\left(k^*_{gg}\right)^{I^*_2}\right]$ is calculated using I^*_2.

Since the modulus of elasticity has to be known in determining variable member properties, a modulus of elasticity constraint will be considered in the determination of the variable member properties.

e) Stresses - The procedure for constraining stress in a variable member can be done in two ways. In the first method, the member forces in variable members can be calculated by solving the nodal equations. (Eg. (5)). These member forces are then used to find the section properties which will give the allowable or constrained stresses. In the second, the constrained stresses are included in the nodal equations directly. This can be done by calculating the member forces induced by the constrained stresses. As a result the member forces will be in terms of the unknown section properties; therefore, the section properties of variable members are the unknown variables in the nodal equations. By solving the nodal equations the section properties can be determined.

Stresses induced in defined members can also be constrained. Since the member properties are defined, the member forces induced by the constrained stresses can be found. Thus a stress constraint is transformed to a force and the procedure for constraining member forces can be used.

f) Displacement Ratio - This constraint will force one displacement to have a specified relation with another displacement. It constrains the ratio of two or more displacements without constraining their specific values. For each equality constraint involving member displacement ratios the equality equation is

$$(a_g)_m \acute{u} - \mu (a_p)_q \acute{u} = 0 \qquad (20)$$

where

μ is the ratio value;

m and q are the components of the constrained distortions; and

a_g and a_p are the g^{th} and p^{th} row of the branch-node incidence matrix, respectively.

For each inequality constraint, the equation is:

$$\left| \begin{array}{l} (a_g)_m \acute{u} - \mu (a_p)_q \acute{u} \;\; 0 \\ (a_g)_m \acute{u} - \mu (a_p)_q \acute{u} \;\; 0 \end{array} \right| . \qquad \begin{array}{r} (21a) \\ (21b) \end{array}$$

For joint displacement ratios, the equality for each constraint is:

$$\acute{u}_{i,m} - \mu \acute{u}_{j,g} = 0 \qquad (22)$$

where

i and j are the joint at which displacements are constrained; and

m and q are the components of the constrained displacements.

For each inequality constraint, the equation is:

$$\left| \begin{array}{l} \acute{u}_{i,m} - \mu \acute{u}_{j,q} \;\; 0 \\ \acute{u}_{i,m} - \mu \acute{u}_{j,q} \;\; 0 \end{array} \right| \qquad \begin{array}{r} (23a) \\ (23b) \end{array}$$

g) Force Ratio - The concept of a force ratio constraint is similar to the displacement ratio constraint: constrains the ratio of two or more forces, but not their specific values. For variable members, the equality for each constraint is:

$$P_{g,m} - \mu P_{p,q} = 0 \qquad (24)$$

where

g and p are the variable members in which forces are constrained;

m and q are the conponents of the constrained forces; and

μ is the ratio value.

For each inequality constraint, the equation is:

$$P_{g,m} - \mu P_{p,q} \geq 0 \qquad (25a)$$

or

$$P_{g,m} - \mu P_{p,q} \leq 0 \qquad (25b)$$

For defined members, the equation for each equality constraint is:

$$\left(\left(k_{gg}\right)\left(a_g\right)\right)_m \acute{u} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} = 0 \qquad (26)$$

For each inequality constraint, the equation is:

$$\left(\left(k_{gg}\right)\left(a_g\right)\right)_m \acute{u} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} \geq 0 \qquad (27a)$$

or

$$\left(\left(k_{gg}\right)\left(a_g\right)\right)_m \acute{u} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} \leq 0 \qquad (27b)$$

Member forces in a variable member and a defined member can be constrained together. For each equality constraint, the equation is:

$$P_{g,m} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} = 0 \qquad (28)$$

For each inequality constraint, the equation is

$$P_{g,m} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} \geq 0 \qquad (29a)$$

or

$$P_{g,m} - \mu\left(\left(k_{pp}\right)\left(a_p\right)\right)_q \acute{u} \leq 0 \qquad (29b)$$

The constraint equations for the various types of constraints are summarized in Table 1 and Table 2.

Table 1

Constraint Equations for Variable Members

Type of Structural Variable	Type of Constraint	
	Equality	Inequality
member distortion	$(a_g)_m u' = u_0^*$	$(a_g)_m u' \geq u_1^*$ $(a_g)_m u' \leq u_2^*$
joint displacement	$u'_{i,m} = u_0'^*$	$u'_{i,m} \geq u_1'^*$ $u'_{i,m} \leq u_2'^*$
member force	$P_{g,m} = P_0^*$	$P_{g,m} \geq P_1^*$ $P_{g,m} \leq P_2^*$
member property **1. sectional area**	$P_{g,x} - \left(\left[k^*_{gg}\right]^{A^*_o}\left[a_g\right]\right)_x u' = 0$	$P_{g,x} - \left(\left[k^*_{gg}\right]^{A^*_1}\left[a_g\right]\right)_x u' \geq 0$ $P_{g,x} - \left(\left[k^*_{gg}\right]^{A^*_2}\left[a_g\right]\right)_x u' \leq 0$
2. moment of inertia	$P_{g,y} - \left(\left[k^*_{gg}\right]^{I^*_o}\left[a_g\right]\right)_y u' = 0$ $M_g - \left(\left[k^*_{gg}\right]^{I^*_o}\left[a_g\right]\right)_\theta u' = 0$	$P_{g,y} - \left(\left[k^*_{gg}\right]^{I^*_1}\left[a_g\right]\right)_y u' \geq 0$ $M_g - \left(\left[k^*_{gg}\right]^{I^*_1}\left[a_g\right]\right)_\theta u' \leq 0$ $P_{g,y} - \left(\left[k^*_{gg}\right]^{I^*_2}\left[a_g\right]\right)_y u' \geq 0$ $m_g - \left(\left[k^*_{gg}\right]^{I^*_2}\left[a_g\right]\right)_\theta u' \leq 0$

Table 1 (continued)

Constraint Equations for Variable Members

Type of Structural Variable	Type of Constraint	
	Equality	Inequality
stress	Same as member force	
displacement ratio 1. member distortion	$(a_g)_m u' - \mu(a_p)_q u' = 0$	$(a_g)_m u' - \mu(a_p)_q u' \geq 0$ $(a_g)_m u' - \mu(a_p)_q u' \leq 0$
2. joint displacement	$u'_{i,m} - \mu u'_{j,q} = 0$	$u'_{i,m} - \mu u'_{j,q} \geq 0$ $u'_{i,m} - \mu u'_{j,q} \leq 0$
force ratio	$P_{g,m} - \mu P_{p,q} = 0$	$P_{g,m} - \mu P_{p,q} \geq 0$ $P_{g,m} - \mu P_{p,q} \leq 0$

Table 2

Constraint Equations for Defined Members

Type of Structural Variable	Type of Constraint	
	Equality	Inequality
member distortion	Same as variable member	
joint displacement	Same as variable member	
member force	$([k_{gg}][a_g])_m u' = P_0^*$	$([k_{gg}][a_g])_m u' \geq P_1^*$ $([k'_{gg}][a_g])_m u' \leq P_2^*$
stress	Same as member force	
displacement ratio	Same as variable member	
force ratio	$([k_{gg}][a_g])_m u' - \mu([k_{pp}][a_p])_q u' = 0$	$([k_{gg}][a_g])_m u' - \mu([k_{pp}][a_p])_q u' \geq 0$ $([k_{gg}][a_g])_m u' - \mu([k_{pp}][a_p])_q u' \leq 0$

3.3. General Form of Nodal Equations

In the previous section, constraints were presented for both equality and inequality constraints. By adding these equalities and/or inequalities to the nodal equations, the specified constraints are explicitly included in the analysis. The general procedure of the singular imbedding approach can now be described as follows:

1. Generate the nodal equations for that portion of the structure which does not contain variable members as

$$\left[A_o^t K_o A_o\right]\{u'\} = \{P'\} \tag{30}$$

where A_o and K_o are written only for the defined members.

2. Add member forces in variable members to the nodal equations by describing them as the unknown applied member forces, P_v. By adding these applied member forces to the nodal equations, the variable members are included in the analysis explicitly.

The nodal equations become:

$$\left[A_o^t K_o A_o\right]\{u'\} = \{P'\} - \left[A_v^t\right]\{P_v\} \tag{31a}$$

adding $A_v^t\, P_v$ to both sides, the equations are:

$$\left[A_v^t \;\vdots\; A_o^t K_o A_o\right]\left\{\frac{P_v}{u}\right\} = \{P'\} \tag{31b}$$

3. Add the constraint equations and/or inequalities to Eq. (31b), yielding:

$$\begin{bmatrix} A_v^t & A_o^t K_o A_o \\ \alpha & \beta \end{bmatrix}\begin{Bmatrix} P_v \\ u \end{Bmatrix} \begin{matrix} = \\ \gtrless \end{matrix} \begin{Bmatrix} P' \\ \gamma \end{Bmatrix} \tag{32}$$

where the added terms

$$\left[\alpha \;\vdots\; \beta\right]\left\{\frac{P_v}{u'}\right\} \gtrless \{\gamma\}$$

represents the constraint equations and/or inequalities;

α,β are partitioned matrices representing coefficients of the unknown variables P_v and $\acute{u}$, respectively.

γ is the vector of constant values of the constraint equalities and/or inequalities.

Equation (39) is the final design equation in which the variable members are designed according to the specified constraints. By solving this equation, the force-displacement relations of the variable members are determined. From these relations, variable member properties are calculated through the member stiffness equations for each member.

3.4. Methods of Solution

Equation (32) is a system of equality and inequality equations. If all equations are equalities, the Gauss-Jordan method (7) can be used for solving Eq. (32). If some of the constraints are inequalities, the linear programming technique is used for the solution. For the case of all equalities, the linear programming technique can still be applied.

3.4.1. Gauss-Jordan Method

Writing equation (32) specifying the dimensions of the matrices yields

$$\begin{bmatrix} A_v^t & A_o^t K_o A_o \\ \hline \alpha & \beta \end{bmatrix}_{(ndf(n+nc))x(ndf(n+ndf)xnvm)} \left\{ \begin{array}{c} P_v \\ \hline \acute{u} \end{array} \right\}_{(ndfx(n+ndf)xnvm)x1} \tag{33}$$

$$= \left\{ \begin{array}{c} P \\ \hline \gamma \end{array} \right\}_{(ndfx(n+nc))x1}$$

rewritten as

$$\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}_{nexnv} \{X\}_{nvx1} = \{W\}_{nex1} \tag{34}$$

where

Q_1 and Q_2 are partitioned matrices representing the matrix

$$\begin{bmatrix} A_v^t & A_o^t K_o A_o \\ \hline \alpha & \beta \end{bmatrix}$$

$\{X\}$ represents the unknown variable vector $\dfrac{P_v}{u'}$, and

$\{W\}$ is the constant vector representing the vector $\dfrac{P}{Y}$

n is the number of free joints

ndf is the number of degrees of freedom

nc is the number of constraints.

nvm is the number of variable members.

ne is the number of equations which is equal to (ndfx(n+nc))

nv is the number of unknown variables which is equal to (ndfx(n+ndf)xnvm)

If the number of the unknown variables nv is equal to the number of the equations ne, a unique solution for P_v and u' will be determined. If the number of the unknown variables nv is larger than the number of the remaining nv-ne variables. However, if the number of the equations is larger than the number of unknown variables (over-constrained problem), Eq. (34) will be unsolvable.

After applying the Gauss-Jordan method to Eq. (34) the result can be represented as

$$\begin{bmatrix} I & \vdots & \theta \end{bmatrix} \begin{Bmatrix} X_1 \\ \hline X_2 \end{Bmatrix} = \{\omega\} \qquad (35)$$

where

$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ is a vector of member forces in variable members and joint displacements; and

I is the unit (identity) matrix

θ and ω are the resulting submatrices after transformation.

The solution, X_1, can be written in terms of X_2 as:

$$\{X_1\} = \{\omega\} - [\theta]\{X_2\} \tag{36}$$

If the number of unknown variables and equations is equal, the vector X_2 will be empty. Otherwise, the number of elements in vector X_2 is equal to the difference between the number of unknown variables and the number of equations, as stated earlier. Elements in vector X_2 can be either member forces in variable members or joint displacements or a combination thereof. The member stiffness matrix is used to determine the magnitudes.

3.4.2. Linear Programming Technique

The second method for solving Eq. (32) is the linear programming technique (8,9). This technique can be applied to the case when any combination of equality and inequality constraints is given. The starting point of this method is also Eq. (32) except that the condition of the constraint equations now can be either equal to, greater than, less than, or any combination thereof. Hence Eq. (32) is written as:

$$\begin{pmatrix} A_v^t & A_o^t K_o A_o \\ \hline \alpha & \beta \end{pmatrix}_{(3n+nc)x(3n+3nvm)} \begin{Bmatrix} P_v \\ \hline u' \end{Bmatrix}_{(3n+3nvm)x1} \gtreqless \begin{Bmatrix} P' \\ \hline \gamma \end{Bmatrix}_{(3n+nc)x1} \tag{37}$$

or, in the form of Eq. (34) as:

$$\begin{pmatrix} Q_1 & Q_2 \end{pmatrix}_{nexnv} \{X\}_{nvx1} \gtreqless \{W\}_{nex1} \tag{38}$$

Linear programming requires that the vector denoted $\{W\}$ have only positive entries. This requirement can be achieved by multiplying all rows in Eq. (38) which have a netative entry in the $\{W\}$ vector by (-1) yielding

$$\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}\{X\} \gtreqless \{Z\} \tag{39}$$

where

$\left(Y_1 \mid Y_2 \right), \{Z\}$ are obtained from $\left(Q_1 \mid Q_2 \right)$ and $\{W\}$, respectively, by multiplying the required rows by -1.

Linear programming also requires that the variables in Eq. (39) be positive. The variables P_v and u' representing negative values are written as

$$P_v = P_v'' - P_v''' \tag{40a}$$

$$u' = u'' - u''' \tag{40b}$$

where P_v'', P_v''', u'', and $u''' \geq 0$.

Substituting the values of P_v and u' expressed in Eq. (40a) and (40b) respectively, into Eq. (39) yields

$$\left(Y_1' \mid Y_2' \right) \{X\} \geq \{Z\} \tag{41}$$

Since only a feasible solution to the problem is desired, only phase I of the simplex method is used. By applying phase I of the simplex method Eq. (41) is reduced to a canonical form identical to Eq. (35), namely:

$$\left(I \mid \theta \right) \begin{pmatrix} X_1 \\ \hline X_2 \end{pmatrix} = \{\omega\} \tag{42}$$

rewritten as in Eq. (43), the solution is:

$$\{X_1\} = \{\omega\} - (\theta) \{X_2\} \tag{43}$$

where

X_1 represents the basic variables, and

X_2 represents the nonbasic variables.

The nonbasic variables are those variables to which values can be assigned arbitrarily. The basic variables can then be calculated.

3.5. Determination of Variable Member Properties

Properties of variable members are determined by using the member stiffness equation. The equation in matrix form is:

$$P_g = (k_{gg})u_g \tag{44}$$

Expanded for a plane frame for a particular member the equation is

$$\begin{Bmatrix} P_x \\ P_y \\ M \end{Bmatrix} = \begin{pmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{pmatrix} \begin{Bmatrix} u_x \\ u_y \\ \theta \end{Bmatrix} \tag{45}$$

By solving Eq. (34) or Eq. (41), whichever is applicable, variable member forces and joint displacements are obtained. Variable member distortions can then be determined from the joint displacements. If the modulus of elasticity is specified, the cross-sectional area and moment of inertia can be determined by substituting member forces and member distortions for the variable members into Eq. (45). Some variable member forces and/or joint displacements may be expressed in terms of other variable member forces and/or joint displacements; that is; the vector X_2 in Eq. (36) or Eq. (43) may not be empty. Therefore, the number of unknown variables in Eq. (45) for any one member may be larger than the number of equations. If the number of elements in vector X_2 pertaining to the member in question is one, Eq. (45) can be solved directly because there will be three unknowns namely the cross-sectional area, the moment of inertia, and an element in vector X_2. Otherwise, the number of unknowns exceeding three has to be specified before the member stiffness equations can be solved for the member properties. A designer may arbitrarily select variables for which to designate values. If variable member forces and/or joint displacements are selected, these variables can be specified either before or after substituting member forces and member distortions into the member stiffness equations. If cross-sectional area and/or moment of inertia are selected, their values will be specified after the member stiffness equations are written.

In the case when the modulus of elasticity has not been chosen, a trial procedure can be used to determine the variable member properties. The techniques applied for the case of known modulus of elasticity are still applicable except that cross-sectional area and moment of inertia will be determined by trial and error. By substituting the modulus of elasticity into Eq. (45), the cross-sectional area and moment of inertia associated with that modulus of elasticitly are found. This computational procedure will be performed for different values of the modulus of elasticity. As a result, different values of cross-sectional area and moment of inertia are determined. A designer can then select the type of material and variable member properties desired from the set of feasible options.

3.6. Design Problems

Two example problems are used to illustrate the procedures and to demonstrate the capability of the singular imbedding approach. The first design problem consists of a set of specified equality constraints. Since constraints are equalities, the Gauss-Jordan method is applied for solving the equations. The second problem consists of inequality constraints. The linear programming technique is used for solving for the second example as inequality constraints are present. Since the linear programming technique can be used for the case of either all equality constraints, inequality constraints, or the combination of both, a computer program was written for this technique.

Example 1: For the structure shown in Figure 4, design the beam (member 2) such that the displacement at joint 2 in the x-direction is 1.43 inches and the bending moment in the beam at joint 4 is 4.5 kips - inches.

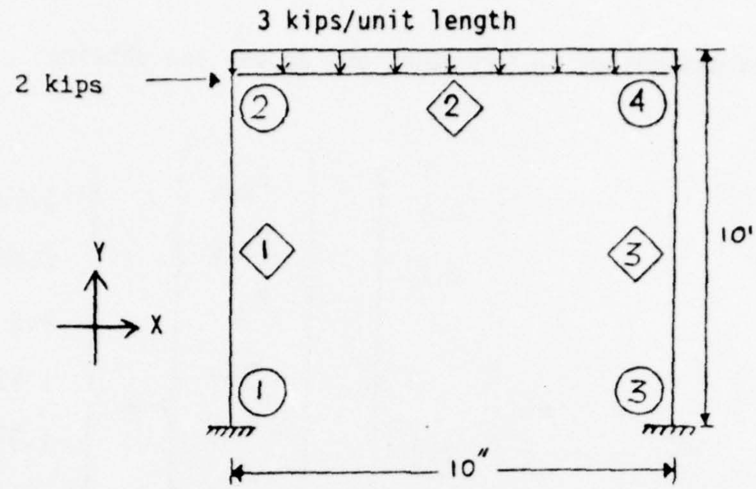


Figure 4 Structure, Example 1

Design information:

Column properties are: $A_1=A_3=10.\ in.^2$, $I_1=I_3=10.\ in.^4$

Modulus of elasticity, $E = 10\ kips/in^2$

Member 2 is a variable member.

The constraint equations are

$$\begin{Bmatrix} u'_{2,x} = 1.43 \\ M_2 = 4.5 \end{Bmatrix}$$

Expanding equation (40) yields

$$\left[\begin{array}{ccc|cccccc} -1 & 0 & 0 & 1.2 & 0 & 6 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & -10 & -1 & 6 & 0 & 40 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 40 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} P_{2,x} \\ P_{2,y} \\ M_2 \\ u'_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,x} \\ u'_{4,y} \\ u'_{4,\theta} \end{Bmatrix} = \begin{Bmatrix} 2. \\ -15. \\ -25. \\ 0 \\ -15. \\ 25. \\ 1.43 \\ 4.5 \end{Bmatrix}^*$$

Applying the Gauss-Jordan method to the equations above, one obtains

$$
\left[\begin{array}{cccccccc:c}
1 & & & & & & & & 0.3 \\
 & 1 & & & & & & & 0.2 \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & 1 & & & & 0.02 \\
 & & & & & 1 & & & 0.05 \\
 & & & & & & 1 & & -0.02 \\
 & & & & & & & 1 & 0.15
\end{array}\right]
\left\{\begin{array}{c}
P_{2,x} \\ P_{2,y} \\ M_2 \\ u'_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,y} \\ u'_{4,\theta} \\ \hdashline u'_{4,x}
\end{array}\right\}
=
\left\{\begin{array}{c}
-3.075 \\ 1.047333 \\ 4.5 \\ 1.43 \\ -1.395267 \\ -0.465167 \\ -1.604733 \\ 0.5125
\end{array}\right\}
$$

Multiplying and solving the equations yields

$$
\begin{bmatrix}
P_{2,x} \\ P_{2,y} \\ M_2 \\ u'_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,y} \\ u'_{4,\theta}
\end{bmatrix}
\begin{bmatrix}
-3.075-0.3\ u'_{4,x} \\ 1.047333-0.2\ u'_{4,x} \\ 4.5 \\ 1.43 \\ -1.395267-0.02\ u'_{4,x} \\ -0.465167-0.05\ u'_{4,x} \\ -1.604733+0.02\ u'_{4,x} \\ 0.5125-0.15\ u'_{4,x}
\end{bmatrix}
$$

Displacements in member 2 are found using $u = A u'$; hence, after substitution

$$u_{2,x} = -1.43 + u'_{4,x}$$

$$u_{2,y} = 4.442204 + 0.54 u'_{4,x}$$

$$u_{2,\theta} = 0.977667 - 0.1 u'_{4,x}$$

By substituting the values for the forces, distortions, length and modulus of elasticity for member 2 into equation (52), the following is obtained

$$\begin{Bmatrix} -3.075 - 0.3u'_{4,x} \\ 1.047333 - 0.2u'_{4,x} \\ 4.5 \end{Bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0.12I & -0.6I \\ 0 & -0.6I & 4I \end{bmatrix} \begin{Bmatrix} -1.43 + u'_{4,x} \\ 4.442204 + 0.54u'_{4,x} \\ 0.977667 - 0.1u'_{4,x} \end{Bmatrix}$$

Expanded and rearranged

$$-3.075 - 0.3u'_{4,x} = -1.43A + u'_{4,x}.A \qquad \text{(i)}$$

$$1.047333 - 0.2u'_{4,x} = -0.053536I + 0.1248\, u'_{4,x}.I \qquad \text{(ii)}$$

$$4.5 = 1.245346I - 0.724\, u'_{4,x}.I \qquad \text{(iii)}$$

There are three equations and three unknowns; therefore, these equations are solvable. From (iii):

$$I = \frac{4.5}{1.245346 - 0.724\, u'_{4,x}} \qquad \text{(iv)}$$

Substituting both values of $u'_{4,x}$ into equation (iv) yields

$$u'_{4,x} = 1.095666 \qquad I = 9.95$$

$$u'_{4,x} = 9.739541 \qquad I = -0.775$$

Only the positive value of I is acceptable. Therefore, the values of $u'_{4,x} = 1.095666$ and $I = 9.95$ will be used to determine the value of A. Substituting the value of $u'_{4,x}$ into equation (i) yields: A = 10.18

Therefore, the final design for member 2, is A - 10.18 $in.^2$, and I = 9.95 $in.^4$

Example 2: Design the beam (member 2) such that the axial distortion of the beam is greater than or equal to -02 in.

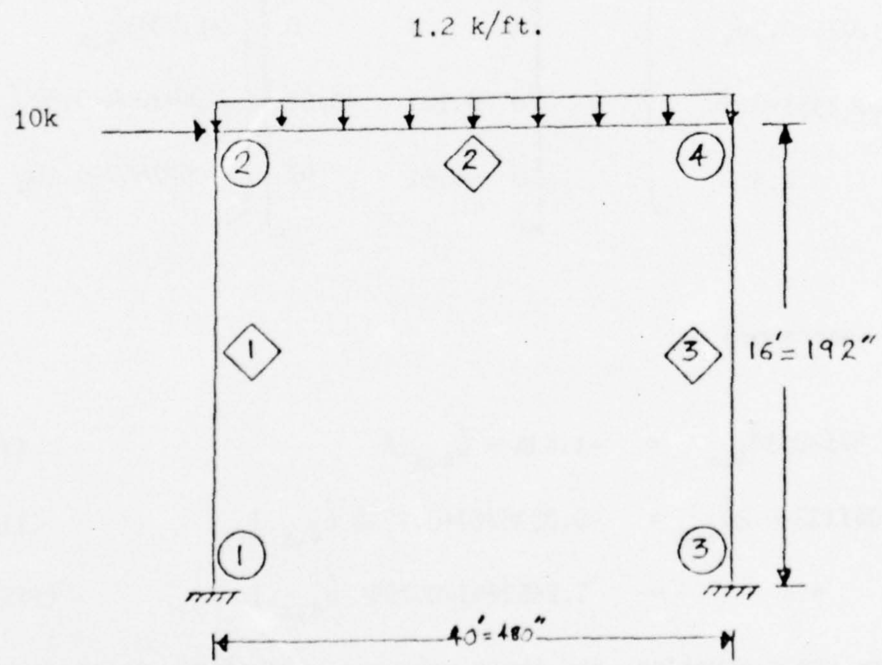


Figure 5 Structure, Example 2

Design Information:

Column properties: $A_1=A_3=22.7$ $in.^2$, $I_1=I_3=457.$ $in.^4$

Modulus of elasticity, E = 29,000 ksi.

Member 2 is a variable member.

The constraint equation is:

$$-u'_{2,x} + u'_{4,x} \geq -0.02$$

Equation (37) is written as

$$\left[\begin{array}{ccc:cccccc} -1 & 0 & 0 & 22.469 & 0 & 2157.065 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 3428.645 & 0 & 0 & 0 & 0 \\ 0 & -480 & -1 & 2157.065 & 0 & 276104.125 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 22.469 & 0 & 2157.065 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3428.645 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2157.065 & 0 & 276104.125 \\ \hdashline 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\times \begin{Bmatrix} P_{2,x} \\ P_{2,y} \\ M_2 \\ u'_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,x} \\ u'_{4,y} \\ u'_{4,\theta} \end{Bmatrix} \begin{matrix} = \\ \\ \geq \end{matrix} \begin{Bmatrix} 10. \\ -24. \\ -1920. \\ 0. \\ -24. \\ 1920. \\ -0.02 \end{Bmatrix}$$

Multiplying the rows which have a negative entry by -1, writing each variable in terms of the difference of two positive quantities and solving the equations by using phase I of the simplex method yields:

$$
\begin{bmatrix}
1 & & & & & & & \vdots & -0.533 & 0.00417 \\
 & 1 & & & & & & \vdots & 0.178 & -0.00139 \\
 & & 1 & & & & & \vdots & -0.000155 & 0.0000012 \\
 & & & 1 & & & & \vdots & -0.00232 & 0.0000145 \\
 & & & & 1 & & & \vdots & 0.178 & -0.00139 \\
 & & & & & 1 & & \vdots & 0.000155 & -0.0000012 \\
 & & & & & & 1 & \vdots & -0.00139 & 0.0000145
\end{bmatrix}
\begin{Bmatrix}
P_{2,y} \\ P_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,x} \\ u'_{4,y} \\ u'_{4,\theta} \\ \cdots \\ P_{2,x} \\ M_2
\end{Bmatrix}
=
\begin{Bmatrix}
10.6367 \\ -2.6503 \\ -0.003897 \\ 0.03224 \\ -2.6703 \\ -0.010102 \\ 0.02782
\end{Bmatrix}
$$

Here the basic variables are $P_{2,y}$, $u'_{2,x}$, $u'_{2,y}$, $u'_{2,\theta}$, $u'_{4,x}$, $u'_{4,y}$, and $u'_{4,\theta}$. The nonbasic variables are $P_{2,}$ and M_2. Multiplying and solving the equation yields

$$
\begin{bmatrix}
P_{2,y} \\ u'_{2,x} \\ u'_{2,y} \\ u'_{2,\theta} \\ u'_{4,x} \\ u'_{4,y} \\ u'_{4,\theta} \\ u_{2,x} \\ u_{2,y} \\ u_{2,\theta}
\end{bmatrix}
\begin{bmatrix}
10.6367 + 0.533\, P_{2,x} - 0.00417\, M_2 \\
-2.6503 - 0.178\, P_{2,x} + 0.00139\, M_2 \\
-0.003897 + 0.000155\, P_{2,x} - 0.0000012\, M_2 \\
0.03224 + 0.00232\, P_{2,x} - 0.0000145\, M_2 \\
-2.6703 - 0.178\, P_{2,x} + 0.00139\, M_2 \\
-0.010102 - 0.000155\, P_{2,x} + 0.0000012\, M_2 \\
0.02782 + 0.00139\, P_{2,x} - 0.0000145\, M_2 \\
-0.02 \\
-15.481405 - 1.11391\, P_{2,x} + 0.00696\, M_2 \\
-0.00442 - 0.00093\, P_{2,x}
\end{bmatrix}
$$

By substituting the values for the forces, distortions, length, and modulus of elasticity for member 2 into equation (45), the following is obtained:

$$\begin{Bmatrix} P_{2,x} \\ 10.6367 + 0.533\,P_{2,x} - 0.00417\,M_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} 60.4167A & & \\ 0 & 0.00315I & -0.7552I \\ 0 & -0.7552I & 241.67I \end{bmatrix}$$

$$\times \begin{Bmatrix} -0.02 \\ -15.481405 - 1.11391\,P_{2,x} + 0.00696\,M_2 \\ -0.00442 - 0.00093\,P_{2,x} \end{Bmatrix}$$

Expanding and rearranging:

$$P_{2,x} = -1.208334A \qquad \text{(i)}$$

$$0.533\,P_{2,x} - 0.00417\,M_2 + 0.04531 + 0.00281\,P_{2,x}.I - 0.00002\,M_2.I = -10.6367 \qquad \text{(ii)}$$

$$M_2 - 10.623381I - 0.61647\,P_{2,x}.I + 0.00526\,M_2.I = 0. \qquad \text{(iii)}$$

There are four unknowns and three equations; therefore, one of the unknowns must be specified. Let I = 1340. $in.^4$ By substituting the value of I into Eq. (ii) and (iii), respectively, the following equations are obtained

from (ii) $4.2984\,P_{2,x} - 0.03097\,M_2 = -71.5129$ (iv)

(iii) $-826.0698\,P_{2,x} + 8.0484\,M_2 = 14235.3292$ (v)

Solving (W) and (V) simultaneously yields:

$P_{2,x} = -14.946555$

$M_2 =$

Substituting the value of $P_{2,x}$ into Eq. (i), yields A = 12.37. Therefore, for member 2, use A = 12.37 $in.^2$, I = 1340 $in.^4$

3.7. Computational Error

In order to study the accuracy of the singular imbedding approach, the results obtained by this approach were compared with STRESS [11] results. Initially, structures were analyzed using the STRESS program. After constraints and variable members were selected, the structures were designed to meet the specified constraints by using the singular imbedding approach. The analysis results obtained from STRESS were used as the imposed constraints on the designs. Thus, the difference between the two sets of results was a direct indication of the accuracy of the singular embedding approach.

Singular Imbedding			Stress
Variable Constraint	Member 2 Joint 2	Member 2 Joint 4	Member 2
P_x	-16.8521	-16.8509	-16.8518
P_y	1.5081	1.5091	1.5078
M	33.3941	33.6624	33.5499
U_x	-0.02114	-0.0201	-0.0210
U_y	3.5103	3.5055	3.5103
θ	0.0112	.0112	0.0112
A	13.19	13.86	13.3
I*	584.35	586.12	583.19
I**	577.56	574.98	

Table

Comparison of Results for Single-Bay, Single-Story Structure

$A_1 = A_3 = 22.7$ in^2

$I_1 = I_3 = 457$ in 4

E = 29,000 ksi

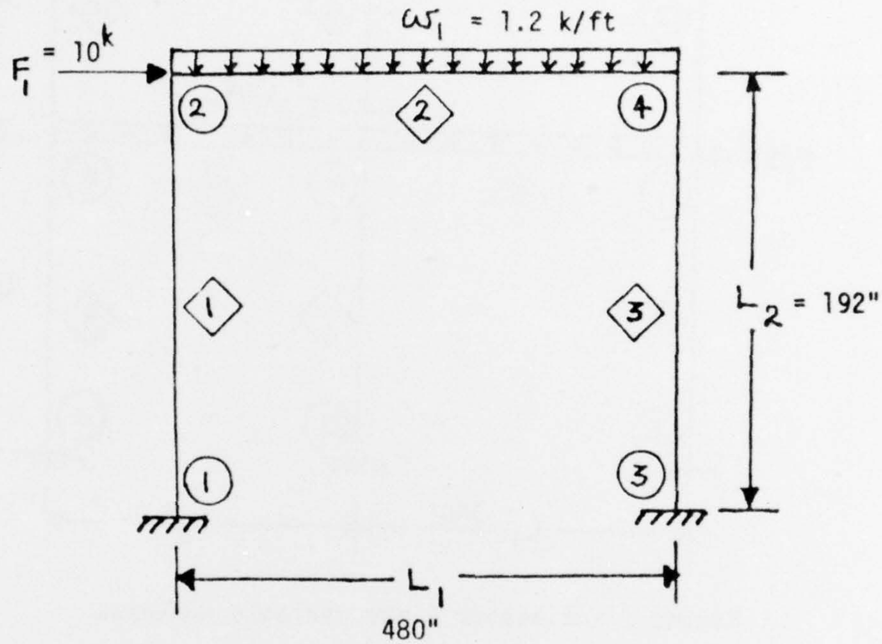
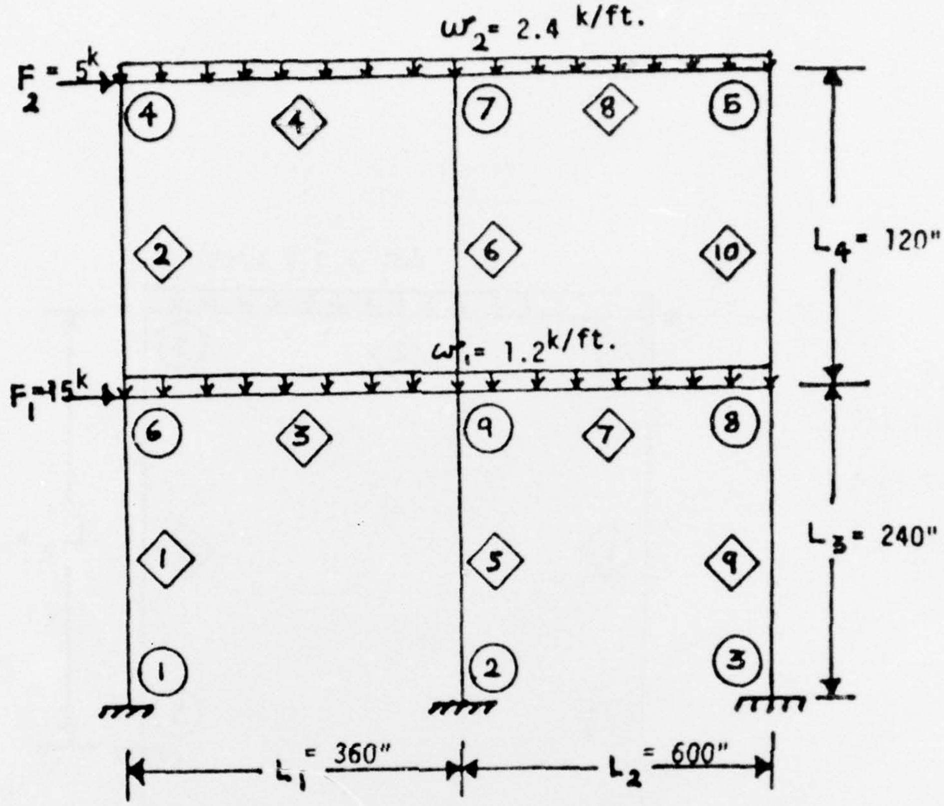


Figure 6 Single Bay, Single Story Structure



Member 1 and member 2 are variable members.

member 3: $A = 11.8\ in.^2$, $I = 612\ in.^4$

member 4: $A = 18.0\ in.^2$, $I = 1540\ in.^4$

member 5: $A = 16.2\ in.^2$, $I = 1340\ in.^4$

member 6: $A = 16.2\ in.^2$, $I = 1340\ in.^4$

member 7: $A = 20.0\ in.^2$, $I = 1820\ in.^4$

member 8: $A = 27.7\ in.^2$, $I = 3270\ in.^4$

member 9: $A = 27.7\ in.^2$, $I = 3270\ in.^4$

member 10: $A = 27.7\ in.^2$, $I = 3270\ in.^4$

$E = 29{,}000$ ksi.

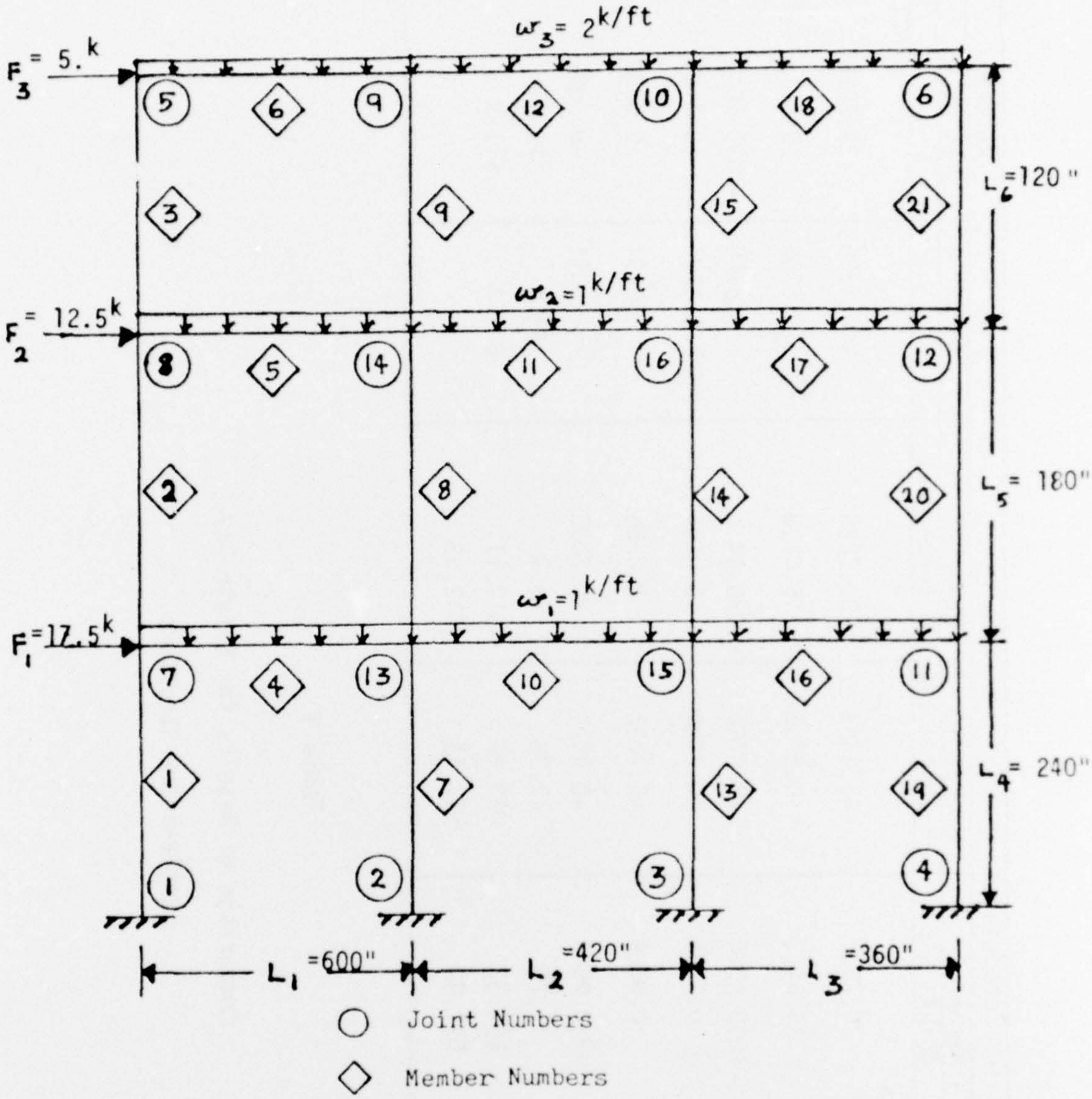
Figure 7 Two-Story, Two-Bay Structure

	Singular Imbedding				Stress	
Variable Constraints	Member 1 Joints 4&6	Member 2 Joints 4&6	Member 1 M forces 1&2	Member 2 M forces 1&2	Member 1	Member 2
P_x	-46.4870	-29.7231	-46.4863	-29.7228	-46.4836	-29.7228
P_y	-0.6701	15.6120	-0.6708	15.6113	-0.6708	15.6113
M	2.6628	-1030.4900	2.8748	-1030.40	2.8748	-1030.4041
U_x	-0.0326	-0.0104	-0.0326	-0.0104	-0.0326	-0.0104
U_y	-0.1697	0.0886	-0.1697	0.0886	-.1697	0.0886
θ	-0.0011	-0.0006	-0.0011	-0.0006	-.0011	-0.0006
A	11.8	11.78	11.78	11.8	11.8	11.8
I	610.63	611.33	612.17	612.17	611.3	611.3
I	556.87	612.80	612.13	612.13		

Table 3

Comparison of Results for Single-Bay,

Single-Story Structure



a) Framed Structure

Figure 8

Framed Structure and Network Graph of

Three Story, Three Bay Structure

Member 1, member 2, and member 3 are variable members.

Member properties of member 4: A = 24.8 $in.^2$, I = 2830 $in.^4$

member 5: A = 27.7 $in.^2$, I = 3270 $in.^4$

member 6: A = 41.6 $in.^2$, I = 7460 $in.^4$

member 7: A = 20.0 $in.^2$, I = 1480 $in.^4$

member 8: A = 16.2 $in.^2$, I = 1340 $in.^4$

member 9: A = 20.0 $in.^2$, I = 1820 $in.^4$

member 10: A = 16.2 $in.^2$, I = 1140 $in.^4$

member 11: A = 16.2 $in.^2$, I = 1340 $in.^4$

member 12: A = 27.7 $in.^2$, I = 3270 $in.^4$

member 13: A = 20.0 $in.^2$, I = 1480 $in.^4$

member 14: A = 16.2 $in.^2$, I = 1340 $in.^4$

member 15: A = 16.2 $in.^2$, I = 1340 $in.^4$

member 16: A = 14.4 $in.^2$, I = 971 $in.^4$

member 17: A = 14.4 $in.^2$, I = 971 $in.^4$

member 18: A = 22.4 $in.^2$, I = 2100 $in.^4$

member 19: A = 20.0 $in.^2$, I = 1480 $in.^4$

member 20: A = 20.0 $in.^2$, I = 1480 $in.^4$

member 21: A = 20.0 $in.^2$, I = 1480 $in.^4$

E = 29.000 ksi.

Figure 8 (cont.)

CHAPTER 4. SUMMARY AND CONCLUSIONS

4.1. Summary

A mathematical tool called singular element was introduced for constraining a structure to perform in a specified manner. The singular element is composed of a nullator, an ideal branch, and an ideal load. Properties of the nullator and the ideal branch were presented. Two types of the singular elements, the Joint Displacement Forcing Element and the Member Force Forcing Element for constraining joint displacements and member forces, respectively were introduced. It was shown that by adding the singular elements to the network the number of unknown variables in the nodal equation is reduced. Additional degrees of freedom were obtained by introducing member forces in variable members as unknown applied member forces and explicitly adding them into the nodal equations. The variable member properties were then directly determined such that the specified constraints placed on the problem were satisfied.

Types of constraints which can be handled by the approach were presented. They are member distortions, joint displacements, member forces, stresses, member properties, force ratios, and displacement ratios. The constraints can be either equalities or inequalities. Thus the designer does not have to designate specific constraints on the problem; instead upper and/or lower bounds of acceptable performance may be used as constraints.

There is one limitation inherent in the method presented. If all members connected to a joint are selected as variable members, displacements at that joint have to be prescribed as constraints; otherwise, the values of the joint displacements will be calculated as zero, and it will be impossible to find the section properties of the variable members connected to that joint.

Two methods, the Gauss-Jordan method and the linear programming technique, were introduced for solving the nodal equations. The Gauss-Jordan method is used when all constraints are equalities. The linear programming technique can be applied when the constraints are all equalities, but must be used when some of the constraints are inequalities. After solving for the system variables, the stiffness equation for a variable member was solved to determine the variable member properties. Since system variables may not be independent, member forces and member distortions of the variable members may be expressed in terms of some system variables.

4.2. Conclusions

The singular imbedding approach presented has some advantages over a conventional design method in designing a structure to meet specified requirements, especially in the computational aspects. Instead of having to iterate on analysis and design until the requirements are met, a single computational procedure is used when the singular imbedding approach is used. The approach also provides flexibility in designing variable members to meet the constraints imposed on the structure. Hence, the designer will save time and will have a choice in selecting properties for the variable members.

The approach has been shown to be efficient for designing a structure to meet a set of specified requirements. The class of problems which is best suited to this approach is the following: given a structure with prespecified topology and geometry, with a significant fraction of members prespecified, and a relatively small number of constraints, design the remaining variable members to yield the desired performance. This class covers a sufficiently wide range of problems to make the method attractive. As an example, in a tall building, lateral displacement or drift limitations may be imposed, and many members may already be prespecified, either because of gravity loading requirements or other considerations, such as repeated use of a single structural shape to reduce costs.

	Singular Imbedding						Stress		
Variable Constraint	Member 1 J.D. 5,7,8	Member 2 J.D. 5,7,8	Member 3 J.D. 5,7,8	Member 1 M.F. 1,2,3	Member 2 M.F. 1,2,3	Member 3 M.F. 1,2,3	Member 1	Member 2	Member 3
P_x	-90.4307	-69.6143	-46.2620	-90.4297	-69.8158	-46.2738	-90.4297	-69.8158	-46.2738
P_y	-1.8419	6.6668	37.3024	-1.8439	6.6629	37.3007	-1.8439	6.6629	37.3007
M	-56.7332	-332.1060	2862.9200	-57.1303	-337.9130	-2864.5800	-57.1303	-332.9127	-2864.5811
U_x	-0.0576	-0.0332	-0.0068	-0.0580	-0.0347	-0.0058	-0.0576	-0.0332	-0.0068
U_y	-0.4149	-0.3092	0.0141	-0.4126	0.3149	0.0114	-0.4149	0.3092	0.0141
θ	-.0027	-0.0020	-0.0012	-0.0027	0.0020	-0.0012	-0.0027	0.0020	-0.0012
A	13.0	13.0	28.29	12.89	12.49	33.21	13.0	13.0	28.3
I	842.11	843.52	2099.49	871.94	813.52	2158.08	843.03	843.03	2099.39
I	837.45	841.00	2098.14	741.24	792.93	2142.61			

Table 4

Comparison of Results for Single-Bay,

Single-Story Structure

Appendix 1. Nomenclature

A	=	Branch-node incidence matrix, also cross-sectional area of a member
A_n	=	Matrix expressing the constraints for k nullators
A_o	=	Branch-node incidence matrix for the graph of defined structural members
A_v	=	Branch-node incidence matrix for the graph of variable structural members
E	=	Modulus of elasticity
H	=	Translation matrix
I	=	Moment of inertia of a member, also unit matrix
K_d	=	Structural stiffness matrix of the structure which contains only defined members and singular elements
K_o	=	Diagonal matrix of member stiffness matrices of structural members
K_{sm}	=	Structural stiffness matrix of structural members
K_t	=	Structural stiffness matrix of the structure with singular elements
L	=	Length of a member
n	=	Number of free joints
m	=	Number of nullators
nc	=	Number of constraints
ne	=	Number of equations in the nodal equations
nm	=	Number of structural members
nv	=	Number of unknown variables in the nodal equations
nvm	=	Number of variable members
P	=	Vector of member forces
P_v	=	Vector of member forces in variable members
P'	=	Vector of joint loads
T	=	Rotation matrix

u = Vector of member distortions

u' = Vector of joint displacements

γ = Constraint magnitude

Superscripts:

t = Matrix transposition

$*$ = Constrained values

APPENDIX II - REFERENCES

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