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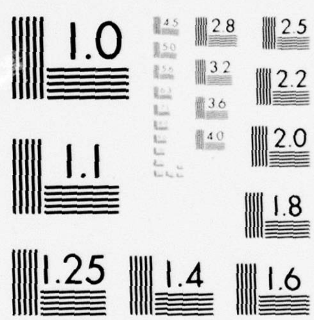

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TECHNICAL REPORT ARMID-TR-77001

INCORPORATING AN INTERNAL MASS-SPRING SYSTEM INTO THE TRAJECTORY EQUATIONS FOR A SIX-DEGREE-OF-FREEDOM MISSILE

EDWARD B. LACHER

APRIL 1977

U.S. ARMY ARRADCOM

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
MANAGEMENT INFORMATION SYSTEMS DIRECTORATE
DOVER, NEW JERSEY

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This report describes the analysis necessary to extend the existing six-degree-of-freedom missile trajectory equations to include a missile containing an internal mass-spring system. The analysis is divided into two main parts, dealing with the internal mass-spring system alone and with the combined projectile and mass system. This work may be considered an extension of the basic trajectory analysis documented in Picatinny Arsenal Technical Report 3391, June 1966. The major result consists of

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an additional differential equation describing the motion of the internal mass-spring system. to be added to the basic trajectory equations.

This analysis was applied to extending a six-degree-of-freedom missile trajectory computer program at Picatinny Arsenal, and a brief discussion of this is included.

The citation in this report of the names of commercial firms or commercially available products or services does not constitute official endorsement or approval of such commercial firms, products, or services by the US Government.

ACKNOWLEDGMENTS

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The author wishes to thank Eugene Friedman for his help in developing the model used as a basis of this analysis.

TABLE OF CONTENTS

INTRODUCTION

This report describes the analysis necessary to extend the six-degree-of-freedom missile trajectory equations to include a missile containing an internal mass-spring system. It also describes how this analysis is applied to implementing corresponding modifications to the most commonly used six-degree-of-freedom missile trajectory computer program at Picatinny Arsenal, to calculate the trajectories of such projectiles with internal mass systems.

The basic analysis, described after the discussion of the model, falls naturally into two main parts. The first is a treatment of the internal mass-spring system alone and consists of the derivation of the additional equation describing the motion of this internal system only. This is the major step required to incorporate a treatment of the internal movable mass into the existing trajectory equations. The second part deals with the combined projectile and mass system whose motion is governed by the existing basic trajectory equations as previously developed in Reference 1. Since the structure of those equations is unchanged, they are not repeated here. However, now that the total missile is not rigid, a number of modifications to equations related to the combined system must be made. These are related to the variation of the center of gravity and involve mostly aerodynamic considerations.

The application of the analysis developed here to extend the capability of a trajectory computer program to handle a modified missile with an added movable internal mass has been completed. Since that computer program was based on the analysis in Reference 1, the present report describing analysis for extending the program can be considered an extension of that document, and much of the notation is borrowed from it. Therefore, it is recommended that Reference 1 be consulted in conjunction with the present report.

Although this work is geared to developing analysis to be applied to a particular trajectory computer program, most of the topics in this report consider matters relevant to any six-degree-of-freedom missile trajectory computation. This includes the final section in which a brief summary is presented of the computer program changes required.

DISCUSSION

Model

The total model used in this study is divided into a number of separate parts. The physical description is presented first, followed by the assumptions made limiting the scope of the study. Next, the coordinate systems required for the analysis are described, and finally the overall analytical approach is summarized. Together these explain the basic physical and analytical preliminaries and provide an expanded introduction to the two-part analysis of the projectile and internal mass system which follows.

Physical System

The system under consideration in this analysis is an axially symmetric projectile containing a mass which is constrained to move along the longitudinal projectile axis only and is connected to the projectile by a spring. The basic physical arrangement and internal configuration of the projectile is shown schematically in Figure 1. The internal mass is connected to the projectile by a spring (with constant k) from the point Q, fixed in the projectile, to the point P, located at the center of gravity of the mass. All distances shown in the figure are measured from the point Q, positive along the direction labelled $+X_H$. In particular, d represents the displacement of the internal mass and d_{eq} the equilibrium position of the spring, both measured to the center of gravity of the mass. The center of gravity labeled "total" includes both the projectile body and the internal mass, while that of the "body" excludes the internal mass.

For purposes of clarity it is essential to differentiate between the missile body and the total or combined system. When referring to the basic projectile body alone, excluding the internal movable mass, but including any other components (such as fuzes) rigidly attached to the body, the term "missile body" will be used. However, when discussing the aerodynamics, the more customary term "missile skin" will generally be used, but it should be taken to mean the same as "missile body". Both these terms always indicate that the internal mass-spring system is not included. When referring to the "combined system," "overall system," or "total system," however, both the body and the mass-spring system are always included. References to the "mass-spring," "mass alone," internal mass," or "internal" system indicate the internal mass-spring system excluding

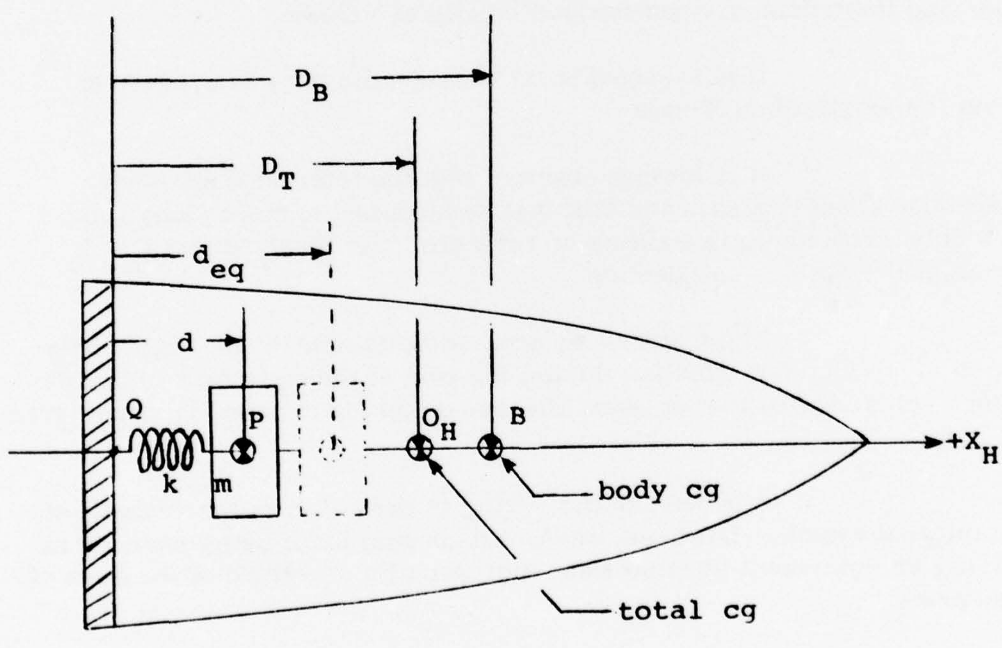

Fig 1 Internal configuration of projectile

the missile body or skin. The terms "missile" and "projectile" are used interchangeably and no significance should be attached to the use of one in place of the other.

Assumptions

For purposes of this study, it was necessary to focus only on certain effects of the addition of the mass-spring system on the projectile motion, rather than to solve the problem in complete generality. This resulted in some simplification of the task by making possible a number of assumptions. Some of these assumptions were also necessitated by the desire to implement the calculations by modifying an existing trajectory program with its own limitations. These limitations either were considered acceptable for the current study or the model was modified. The assumptions and limitations are summarized briefly as follows:

It is assumed that the projectile body is symmetric about the longitudinal X-axis.

It is further assumed that the internal mass is also symmetric about this axis and that it is constrained to move along this axis only, with negligible tilting or rotation. The result is that $I_Z = I_Y$ for all of the systems in question.

The trajectories are limited to non-thrust flights only which involve no fuel burning during the part of the trajectory to be analyzed. This implies that no mass changes or jet effects need be considered.

The mass of the spring is neglected in the analysis of the internal system; however, one could account for it using methods involving an equivalent internal mass increased by one-third of the mass of the spring.

The present analysis is applied to short trajectories so that only the "flat earth" model described in Reference 1 is needed.

Coordinate Systems

In order to find the equations of motion for the physical system in question, it is necessary to establish and inertial coordinate system, I, whose origin may be assumed to be at the center of the earth. A fixed plane system, H (for Horizontal plane), with origin at the center of gravity of the total projectile-mass combination will be utilized (as in Ref 1). The H-system X-axis lies along the missile axis, the Y-axis is constrained to lie in a horizontal plane, and the Z-axis is chosen so as to form a right handed (rectangular) coordinate system. Because the only allowable motion of the mass m is along the projectile axis, the only non-ignorable coordinate in the H-system will be the X-axis.

Approach

In this physical system consisting of two distinct parts (the projectile body and the internal mass), there are a number of possible approaches to describing the motions. Since the object of this study is to analyze both the motion of the internal mass-spring and the motion of the total projectile system, it is logical to choose these as the two systems on which to concentrate. The first part of the analysis treats the internal mass-spring subsystem alone and the second treats the total system consisting of the projectile body and internal mass combined.

In order to investigate the motion of a physical system, it is necessary to include the governing equations of motion and the forces and moments acting on the system. These are discussed separately for each of the two systems. Because of the differeing nature of the systems and the applicability of much of the rigid-body trajectory analysis, the emphasis is different in each case.

In analyzing the motion of the internal mass-spring system, one first applies Newton's Second Law to the internal mass alone, taking into account all forces acting on it. The differential equation derived through this process is the desired new equation of motion of the internal system. The forces considered acting on the internal mass are the spring force, gravity, and a frictional term. No consideration of aerodynamic forces (or winds) is necessary because the mass is inside the missile body. No moments are considered because for the present analysis the internal mass is allowed only one degree of freedom inside the projectile, without rotation. These considerations are treated in the section dealing with the internal mass-spring system.

To analyze the motion of the total system, one must add the new differential equation from the internal system to the standard equations described in Reference 1. This forms a larger set of differential equations which, when solved simultaneously (numerically), describe the motions of both the internal and combined systems. No new derivations of those standard equations of motion are required for this combined system. Aerodynamic and gravitational forces and moments are included, and no jet effects need be considered because of the assumption of no thrust. The basic methodology (and computer program) was used previously to calculate the trajectories of rigid missiles only. The present, compound, non-rigid system whose center of gravity is no longer fixed in the system, requires a number of modifications. These involve aerodynamic forces and moments and moments of inertia, and are considered in the discussion of the combined system.

It should be noted that it would entail more work to consider the projectile body alone, without the internal mass (instead of the total system) as the second system of interest. If this were done, the existing trajectory computer program would then need to be applied to the projectile body alone, as if it were being flown separately, and its effects on the internal mass would then be calculated. This alternate procedure would be awkward and would not make the fullest use of the existing analysis and computer program.

Internal Mass-Spring System

The basic equation to be applied to the mass-spring system is simply Newton's Second Law, $\vec{F} = m\vec{a}$. In view of the assumed constraints on the mass m, the only direction in which motion of this mass can occur is along the X-axis of the H-system. Using the double subscript XH to indicate components of vectors along this direction, the only resulting non-trivial equation is

$$F_{XH} = ma_{XH} \tag{1}$$

This equation must be applied in an inertial frame only, so that the term on the right must refer to the proper component of the acceleration of the mass with respect to the inertial coordinate system I. To make Equation 1 more explicit then, it can be re-written as

$$F_{XH} = m \left. \frac{d_I^2 \vec{r}_m}{dt^2} \right|_{XH} \tag{2}$$

A few words of explanation are in order to clarify the notation used for the vector derivative appearing on the right hand side of Equation 2. By a (time) derivative of a vector relative to a coordinate system is meant that one first expresses the vector in components of that system and then differentiates with respect to time. The two systems used here are the inertial (I) coordinate system necessary to apply Newton's Second Law, and the H coordinate system along whose X-axis the mass moves. The two systems are in motion with respect to each other, so that derivatives of a vector are in general different relative to the two systems. The vector derivative in Equation 2 is relative to the I-system, but only its XH direction component is required.

The method followed expresses both sides of Equation 2 in terms of quantities in the H-system; that is, in terms of the displacement d (along the X-axis in this system) and its derivatives. The reason for this approach is that one can most easily express forces in the H-system and that one desires an equation in this system in order to use the previous analysis and computer program. The result will be a single second-order ordinary differential equation for d, which can then be added to the standard equations of motion for the projectile and mass system.

Derivative Term

To facilitate the conversion of the right hand side of Equation 1 to the H-system, one can make use of Figure 2 in which the origin of the H-system is located at the overall center of gravity of the total projectile. Thus,

$$\vec{r}_m = \vec{r} + \vec{R} \tag{3}$$

expresses the position vector $\vec{r}_m$ of the center of gravity (P) of the internal mass with respect to the I-system, in terms of the position vector $\vec{r}$ of P in the H-system and the vector $\vec{R}$ indicating the position of the H-system origin O_H. All of these vectors vary with time as the projectile moves.

Differentiating Equation 3 relative to the I-system,

$$\frac{d_I \vec{r}_m}{dt} = \frac{d_I(\vec{r} + \vec{R})}{dt} = \frac{d_I \vec{r}}{dt} + \frac{d_I \vec{R}}{dt} \tag{4}$$

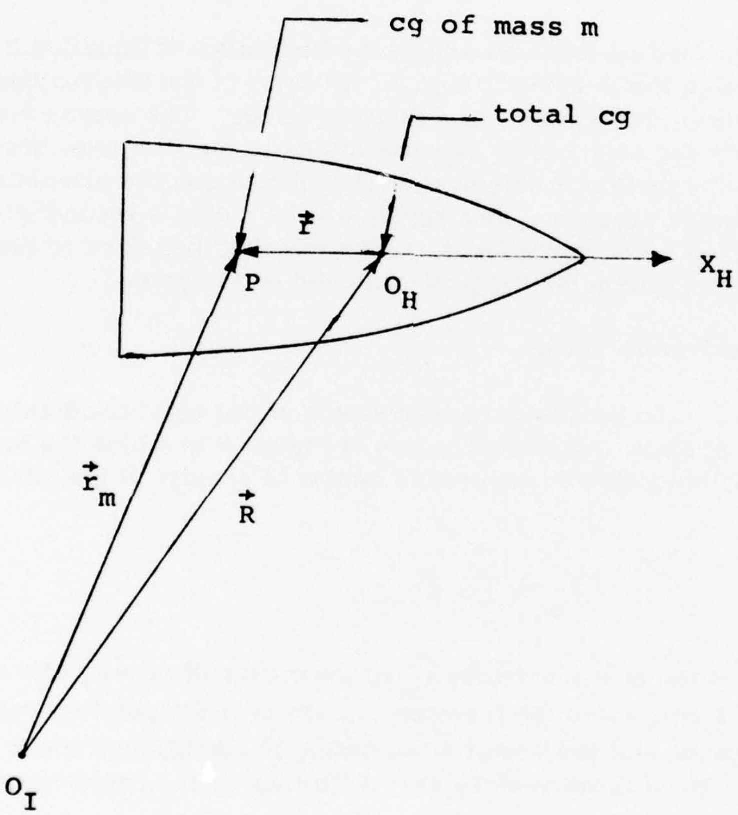

Fig 2 Overview of projectile position

The standard formula for expressing the time derivative of a vector $\vec{A}$ relative to a coordinate system I, in components of another system H in motion with respect to the first and having angular velocity $\vec{\omega}$ with respect to it, is given by

$$\frac{d_I \vec{A}}{dt} = \frac{d_H \vec{A}}{dt} + \vec{\omega} \times \vec{A} \tag{5}$$

This equation can be found in many mechanics texts (among them Ref 2) and also in Reference 1.

Applying Equation 5 to the first term on the right in Equation 4 yields

$$\frac{d_I \vec{r}_m}{dt} = \frac{d_H \vec{r}}{dt} + \vec{\omega} \times \vec{r} + \frac{d_I \vec{R}}{dt} \tag{6}$$

Differentiating and then applying Equation 5 to the term in square brackets below,

$$\frac{d_I^2 \vec{r}_m}{dt^2} = \frac{d_I}{dt}\left(\frac{d_H \vec{r}}{dt} + \vec{\omega} \times \vec{r} + \frac{d_I \vec{R}}{dt}\right) = \frac{d_I}{dt}\left(\frac{d_H \vec{r}}{dt} + \vec{\omega} \times \vec{r}\right) + \frac{d_I^2 \vec{R}}{dt^2}$$

$$= \frac{d_H}{dt}\left(\frac{d_H \vec{r}}{dt} + \vec{\omega} \times \vec{r}\right) + \vec{\omega} \times \left(\frac{d_H \vec{r}}{dt} + \vec{\omega} \times \vec{r}\right) + \frac{d_I^2 \vec{R}}{dt}$$

This results in

$$\frac{d_I^2 \vec{r}_m}{dt^2} = \frac{d_H^2 \vec{r}}{dt^2} + 2\vec{\omega} \times \frac{d_H \vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r} + \frac{d_I^2 \vec{R}}{dt} \tag{7}$$

Cross-Product Terms

To simplify the cross-product terms in Equation 7, a few observations are necessary. Since the motion of the internal mass is along the H-system X-axis only, that is, parallel to the unit vector $\vec{i}_H$, one is concerned with the component of Equation 7 in this direction only. Using Figures 1 and 2 it can be seen that

$$\vec{r} = (d - D_T)\,\vec{i}_H \qquad (8A)$$

the vector $\vec{r}$ having no $\vec{j}_H$ or $\vec{k}_H$ components. Similarly, when differentiated relative to the H-system

$$\frac{d_H \vec{r}}{dt} = (\dot{d} - \dot{D}_T)\,\vec{i}_H \qquad (8B)$$

$$\frac{d_H^2 \vec{r}}{dt} = (\ddot{d} - \ddot{D}_T)\,\vec{i}_H \qquad (8C)$$

It is evident from Equations 8A and 8B that both $\vec{\omega} \times \frac{d_H \vec{r}}{dt}$ and $\frac{d\vec{\omega}}{dt} \times \vec{r}$ have no components in the $\vec{i}_H$ direction and therefore can be dropped.

Representing the angular velocity $\vec{\omega} = \omega_{XH}\,\vec{i}_H + \omega_{YH}\,\vec{i}_H + \omega_{ZH}\,\vec{k}_H$, it is clear that

$$\vec{\omega} \times \vec{r} + (d - D_T)\,(\omega_{ZH}\,\vec{j}_H - \omega_{YH}\,\vec{k}_H) \qquad (9)$$

then,

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\,(d - D_T)\,(\omega_{YH}^2 + \omega_{ZH}^2)\,\vec{i}_H + \ldots$$

where only the $\vec{i}_H$ component is of interest.

Substituting the results of these observations into Equation 7 and multiplying by m yields an equation without cross products.

$$m \left. \frac{d_I^2 \vec{r}_m}{dt^2} \right|_{XH} = m(\ddot{d} - \ddot{D}_T) - m(d - D_T)(\omega_{YH}^2 + \omega_{ZH}^2) + m \left. \frac{d_I^2 \vec{R}}{dt^2} \right|_{XH} \tag{10}$$

D_T Terms

The aim is to obtain a single equation for the single unknown d in terms of quantities known or computed from the basic trajectory equations. The variable D_T and its derivatives, describing the position of the (movable) overall center of gravity relative to the projectile fixed-point Q, are not independent of d and must be eliminated.

The origin of the H-system, O_H, lies at the center of gravity of the overall system. Thus the sum of the first moments of the total mass about this point is zero. That is,

$$m_B (D_B - D_t) + m(d - D_t) = O \tag{11}$$

Solving for D_T,

$$D_T = \frac{m_B D_B + md}{m_B + m} \tag{12A}$$

Differentiating twice,

$$\dot{D}_T = \frac{m\dot{d}}{m_B + m} \tag{12B}$$

$$\ddot{D}_T = \frac{m\ddot{d}}{m_B + m} \tag{12C}$$

Substituting these into Equation 10 yields

$$m \left. \frac{d_I^2 \vec{r}_m}{dt^2} \right|_{XH} = \frac{m m_B}{m_B + m} \ddot{d} + (D_B - d) \frac{m m_B}{m_B + m} \left(\omega_{YH}^2 + \omega_{ZH}^2 \right)$$

$$+ m \left. \frac{d_I^2 \vec{R}}{dt^2} \right|_{XH} \qquad (13)$$

Derivative of $\vec{R}$

To evaluate the derivative in the last term on the right in Equation 13, it is convenient to call the velocity vector of the projectile center of gravity with respect to the I system $\vec{V} = \frac{d_I \vec{R}}{dt}$. Then using Equation 5

$$\frac{d_I^2 \vec{R}}{dt^2} = \frac{d_I}{dt} \left(\frac{d_I \vec{R}}{dt} \right) = \frac{d_I \vec{V}}{dt} = \frac{d_H \vec{V}}{dt} + \vec{\omega} \times \vec{V} \qquad (14)$$

Using the notation $\vec{V} = V_{XH} \vec{i}_H + V_{YH} \vec{j}_H + V_{ZH} \vec{k}_H$ then

$$\dot{V}_{XH} = \left. \frac{d_H \vec{V}}{dt} \right|_{XH} \quad \text{and}$$

$$\vec{\omega} \times \vec{V} = (\omega_{YH} V_{ZH} - V_{YH} \omega_{ZH}) \vec{i}_H + \ldots \qquad (15)$$

Combining these

$$\left. \frac{d_I^2 \vec{R}}{dt^2} \right|_{XH} = \dot{V}_{XH} + (\omega_{YH} V_{ZH} - V_{YH} \omega_{ZH}) \qquad (16)$$

Substituting Equation 16 into Equation 13 and using

$G_B = \frac{m_B}{m_B + m}$, the right hand side of Equation 2 becomes finally

$$m \left. \frac{d_I^2 \vec{r}_m}{dt^2} \right|_{XH} = m\, G_B\, \ddot{d} + (D_B - d)\, m\, G_B\, (\omega_{YH}^2 + \omega_{ZH}^2) + m\, \dot{V}_{XH}$$

$$+ m\, (\omega_{YH} V_{ZH} - V_{YH}\, \omega_{ZH}) \qquad (17)$$

Forces

Returning to the left hand side of Equation 2, it remains necessary to find the forces on the internal mass in the direction of the H-system X-axis only. There are three forces which will be considered: the spring force, gravity, and a frictional or damping term. No aerodynamic forces need be considered for this system because the mass lies entirely inside the projectile.

The spring force is $-k(d-d_{eq})$, and the gravity term can be expressed as the component of the gravitational force along the $\vec{i}_H$ direction, which is denoted by mg_{XH}.

The frictional term is used for the purpose of obtaining a damping effect on the motion of the mass, and it is actually a viscous damping form $-b\dot{d}$. Here, b represents the damping coefficient where the force is directed against the motion of the internal mass and is proportional to its velocity. Although this is not the customary means of representing static or dynamic friction, for the purposes of this analysis one can easily investigate some effects of friction in this way. In the actual application of this analysis, little use was made of this frictional term.

Substituting these forces into Equation 2 using Equation 17 yields the final result

$$-k\,(d - d_{eq}) - b\dot{d} + mg_{XH} = m\, G_B \ddot{d} + m\, G_B\, (D_B - d)\, (\omega_{YH}^2 + \omega_{ZH}^2)$$

$$+ m\, \dot{V}_{XH} + m\, (\omega_{YH} V_{ZH} - V_{YH}\, \omega_{ZH})$$

(18)

In the actual trajectory computer program, the above equation is solved for the second derivative $\ddot{d}$, and this is reduced to a system of two first-order equations expressing the derivatives of d and $\dot{d}$.

Combined Projectile and Mass System

The overall approach to describing the motion of the combined projectile body and internal mass system has been divided into two parts. The first involved the internal mass-spring system and the second involves the total system. In each case the general equations of motion are applied to obtain the specific equations desired. Only the external forces and/or torques acting on the corresponding system are considered. This procedure yields the required set of equations describing the motions of both the internal mass and the entire projectile.

Since the basic structure of the acceleration terms (on the right hand side of $\vec{F} = m\vec{a}$) is the same as for the standard trajectory analysis, no new motion equation derivations are necessary. The resulting equations of motion for the combined system are simply the new equation (Eq 18) plus the original trajectory equations.

In the preceding sections, the forces on the internal mass alone (along the direction of its motion) have been considered. No torques were necessary for these calculations, because only longitudinal motion was permitted with no rotation. It remains to consider here the external forces and moments acting on the total missile system. Since the movable mass is a part of the combined system, its effect on the combined system is internal only. The only external forces in the present problem are the same ones already treated in the analysis of ordinary missile trajectories (Ref 1). However, some consideration as to how the motion of the internal mass affects the trajectory of the overall system will show why a number of modifications to these forces and moments computed for the equations of motion are now necessary.

Using the present approach, one does not consider the forces and torques applied to the projectile body by the internal mass, because they were accounted for when the forces exerted on the internal mass were considered. In view of this, it might not be evident how the present model accounts for the internal mass affecting the motion of the combined system. The answer is that as the mass moves inside the projectile, it transmits effects of its motion to the overall system by constantly changing the center of gravity and moments of the combined system. Because the moment arms of the aerodynamic forces are all computed about this center and because the gravitational force is applied at this point, changing these quantities at once changes all the torques on the combined system from what they would have been if applied to the projectile body alone. In addition, since the moments of inertia are constantly used in the rotational equations for the total system, changing them will at once also change all these equations. Thus when the combined set of equations is solved simultaneously by the computer program, the internal motion is described by Equation 18 and the resulting changes in the center of gravity and moments have the proper effect on the total projectile.

All of this can be summed up by noting that the relative position of the total center of gravity in the missile and the relative velocity of this center of gravity are both used extensively in the trajectory program. Since they are now both variable, their repeated computation must be added to the standard analysis. The following sections consider in more detail the modifications relating to the forces and moments resulting from the movement of the total center of gravity in the projectile, in order to insure that this mechanism proceeds correctly in the trajectory computation.

Aerodynamic Forces

The actual aerodynamic forces acting on a projectile depend (among other variables) on the shape of the projectile body, the angle of attack, and the velocity of the projectile. The expressions for the various aerodynamic forces are the same here as in the standard trajectory analysis and they will not be repeated. But the velocity they depend on is the velocity of the projectile body or skin with respect to the air, $V_{skin/air}$. In the present situation, the total missile is not rigid due to the movement of the internal mass, so that the projectile skin and the total center of gravity are in motion with respect to each other. Therefore, the velocity of the center of gravity with respect to the air, $V_{cg/air}$, which was formerly used in the aerodynamic calculations, is no longer equal to $V_{skin/air}$ which should more properly be used, so that it becomes necessary to compute $V_{skin/air}$ separately for the combined analysis. However, since this difference in velocity exists only in the component along th H-system X-axis, one need modify this component only. One can write the scalar equation, assuming XH-direction components only,

$$V_{skin/air} = V_{skin/cg} + V_{cg/air} \tag{19}$$

Since one of the basic functions of the trajectory program is to compute the motion of the overall center of gravity, $V_{cg/air}$ can be calculated using the complete system of equations (including Eq 18). The first term on the right, $V_{skin/cg}$, is calculated using Equation 12B, noting first from the figures that the point Q is fixed with respect to the projectile skin and must move with it. Since the distance D_T is measured from Q to the overall center of gravity, measured positive along the $+ X_H$-axis, the derivative $\dot{D}_T$ as given by Equation 12B is just the component of the velocity of the center of gravity with respect to the projectile skin. Thus $V_{skin/cg}$ is just $-\dot{D}_T$, and this is used in Equation 19 to insure that the proper velocity is used in computing aerodynamic forces.

Generally, these forces are computed by multiplying the expressions involving velocity, projectile diameter, and air density by proportional factors called aerodynamic coefficients. These latter quantities are supplied to the program in tables dependent upon the angle of attack and Mach number. Since the Mach number used for this purpose should be $V_{skin/air}$ (divided by the speed of sound) rather that $V_{cg/air}$, the above adjustment in velocity given by Equation 19 is again necessary to assure that proper values are found for these coefficients themselves.

Aerodynamic Moments

The same remarks and use of Equation 19 apply to the computation of aerodynamic moments, which similarly depend on the velocity of the skin with respect to the air and involve aerodynamic coefficients which do also. In addition, some of the moment terms arise from the fact that the aerodynamic forces act through a point in the projectile referred to as the center of pressure. Like the aerodynamic coefficients, these centers of pressure are also found from tables, themselves functions of Mach number. So again, the remarks above relating to the use of Equation 19 to insure the proper values are relevant. Finally, since these moments are to be calculated about the overall center of gravity, which, in the present case, is moving in relation to the projectile skin, the moment arms must always be computed as the distance between the center of pressure and the current position of the overall center of gravity. This position is computed using Equation 12A first; then the moment arms can be calculated correctly.

Moments of Inertia

During the trajectory calculation, the moments of inertia of the total system used in the torque equations must repeatedly be updated as a consequence of the motion of the internal mass. Since this motion is along the XH-axis only, the moment of inertia of the total system about this axis, denoted by $I_X(T)$, will not change, but the moment of inertia about the YH-axis $I_Y(T)$ will change. Referring to the fixed moments of inertia of the internal mass and body alone about axes parallel to the YH-axis but passing through their own centers of gravity as $I_Y(m)$ and $I_Y(B)$, respectively, their corresponding moments of inertia about the YH-axis (passing through the total center of gravity) are given by

$$I_{YH}(m) = I_Y(m) + m(D_T - d)^2 \tag{20A}$$

$$I_{YH}(B) = I_Y(B) + m_B(D_T - D_B)^2 \tag{20B}$$

Since the origin of the H-system is always located at the overall center of gravity, the quantity $I_Y(T)$, representing the moment of inertia of the total projectile about a Y-axis parallel to the YH-axis and through its own center of gravity, is precisely $I_{YH}(T)$. Also, the total projectile is merely the sum of the internal mass and the projectile body so that the same is true for the corresponding moments of inertia. Together these imply that

$$I_Y(T) = I_{YH}(T) = I_{YH}(m) + I_{YH}(B) \tag{21}$$

Substituting Equations 20A and 20B into Equation 21 yields the necessary result

$$I_Y(T) = I_Y(m) + I_Y(B) + m(D_T - d)^2 + m_B(D_T - D_B)^2 \tag{22}$$

which is used to compute the moment of inertia of the total system using the current values of D_T and d. It should be remembered that because of the assumptions of symmetry, $I_Z(\ ) = I_Y(\ )$ for m, B, and T, so that Equation 22 also applies with Z substituted for Y.

Although the moments are updated as necessary in the program using Equation 22, all terms involving time derivatives of moments of inertia have been neglected in this study (see footnote, page 9, Ref 1), as have any effects of rate of change of velocity on aerodynamic forces. No change in the moment I_X occurs, so that $\dot{I}_X = 0$. The moment I_Y (and also I_Z) varies because of the motion of the internal mass, so that $\dot{I}_Y$ is actually not zero. It has been assumed here that the effect of the derivative term $\dot{I}_Y$ itself is small compared to the more basic effect of the variation of I_Y at each step in the computation, which is accounted for by the updating using Equation 22.

The present analysis has been limited to stages of trajectories without fuel burning, so that no mass changes occur. Therefore no jet reaction is considered (see footnote, page 7, Ref 1) and no changes in moments due to mass changes occur.

Application to the Trajectory Computer Program

From the standpoint of the particular trajectory computer program which was modified, the main changes consisted of adding variables representing the displacement d and its derivative $\dot{d}$ into the program and defining $\ddot{d}$ by Equation 18. Connected with this was the necessity of shifting the total center of gravity (using Eq 12A) and moments of inertia (using Eq 22) and of recomputing the velocity of the skin with respect to the air for the aerodynamics (using Eq 12) in the innermost loop of the differential equation solving code.

Other changes involved adding new inputs describing the internal system, converting to proper program internal units, initializing variables and minor changes to the output to provide information on the internal system. The alternate set of equations referring to the vertical (V) coordinate system (Ref 1) had previously been removed so that consideration of this system was not necessary.

In addition, for the present study an initial "release time" was implemented in the trajectory program so that one could specify when in the course of the trajectory the spring would be released. Until this time is reached in the flight, the internal system is constrained in its initial position, as specified by the initial condition on d. When this time is reached, the spring is in effect released and an initial condition on its velocity d can be applied.

All of the necessary additions and modifications based on this analysis have been implemented. This extended an existing six-degree-of-freedom missile trajectory program at Picatinny Arsenal to accommodate a missile containing an internal mass-spring system as described here. When the original equations plus the added equation of motion for the internal mass (Eq 18) are solved simultaneously (with the modifying equations described in the discussion of the combined system), the trajectory of the total projectile can be computed. This provides both the trajectory of the combined system and a solution for d and $\dot{d}$ describing the position and velocity of the mass within the projectile.

REFERENCES

1. Barnett, Bruce D., "Trajectory Equations for a Six-Degree-of-Freedom Missile Using a Fixed-Plane Coordinate System," Technical Report 3391, Picatinny Arsenal, Dover, NJ, June 1966

2. Symon, Keith R., Mechanics, Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1953

SYMBOLS

D_T	Directed distance from Q to total cg
D_B	Directed distance from Q to body or skin cg
d	Directed distance from Q to cg of the internal mass
d_{eq}	Value of d at spring equilibrium position
k	Spring constant
$\vec{r}$	Position vector of internal mass in H-system
$\vec{r}_m$	Position vector of internal mass in I-system
$\vec{R}$	Vector from origin of I-system to origin of H-system
m	Mass of internal movable object
m_B	Mass of missile body or Skin alone
G_B	Fraction of total mass attributable to missile body or skin
$\vec{F}$	Force vector
$\vec{a}$	Acceleration vector
$\vec{\omega}$	Angular velocity vector of H-system relative to I-system
$\vec{V}$	Velocity vector
b	Damping coefficient
g_{XH}	Gravitational acceleration (i_H component only)
$\vec{i}_H$, $\vec{j}_H$, $\vec{k}_H$	Unit vectors in directions of H-system, X, Y and Z-axes
$V_{skin/air}$	Velocity of missile skin with respect to air (i_H component only)

$V_{cg/air}$	Velocity of total missile cg with respect to air (i_H component only)
$V_{skin/cg}$	Velocity of missile skin with respect to total cg (i_H component only)
$I_{YH}(m)$	Moment of inertia of internal mass about YH-axis
$I_{YH}(B)$	Moment of inertia of missile body alone about YH-axis
$I_{YH}(T)$	Moment of inertia of total missile about YH-axis
$I_Y(m)$	Moment of inertia of internal mass about Y-axis through its own center of gravity
$I_Y(B)$	Moment of inertia of missile body alone about Y-axis through its own center of gravity
$I_Y(T)$	Moment of inertia of total missile about Y-axis through its own center of gravity

NOTATION

I	Refers to inertial coordinate system which may be considered to have its origin at the center of the earth.
H	Refers to horizontal fixed-plane coordinate system, whose origin is at the center of gravity of the total missile.
$\dot{d}$, $\ddot{d}$	Dots above scaler quantities indicate time derivatives
V_{XH}	Component of vector $\vec{V}$ in direction of $\vec{i}_H$ (H-system X-axis) This component may also be indicated by $\Big\|_{XH}$
$\frac{d_I(\ )}{dt}$	Denotes time derivative of vector realtive to I coordinate system
$\times$	Denotes vector cross product

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