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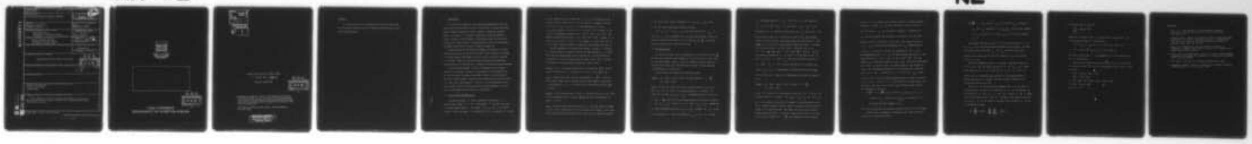

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YALE UNIVERSITY
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Linear Time Coloring of Planar Graphs

R. J. Lipton[*] and R. E. Miller[†]

Research Report #92

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Abstract

It is shown how to color a planar with 5-colors in linear time. The "batching" method used may have additional applications to other graph theory problems.

I. *Introduction*

The coloring of graphs has had longstanding mathematical interest. Indeed, much of this interest stems from the wealth of applications that rely on finding colorings of graphs. However, finding the chromatic number of a graph, i.e., coloring a graph with the minimal number of colors, is known to be NP-complete in the sense of Karp and Cook [1,3]. This has led to much interest in algorithms that either find approximate colorings or only work for special classes of graphs [2].

We describe here an algorithm for finding a 5-coloring of a planar graph and prove that it runs in linear time. Since finding a 3-coloring of a planar graph is NP-complete [6], this result can only be improved to 4-coloring. While our current method uses some of the well-known reducibilities for 5-coloring [5], it differs in several essential features.

As in the known methods, to find a 5-coloring we do a recursive reduction of the graph. However, at each step of the recursion we show how one can reduce the graph by removing a "batch" of vertices rather than just a single vertex. This is why our algorithm runs in linear time, i.e., $O(n)$ time, rather than in n^2 time [4]. It is interesting to note that ensuring a batch of sufficient size is itself dependent on the fact that a planar graph can be 7-colored in linear time.

II. *Basic Results and Terminology*

An undirected graph $G = (V,E)$ consists of a finite set V of *vertices* and a subset E of $V \times V$ called *edges*. If $(u,v) \in E$ then there is an edge between vertex u and vertex v, and u and v are called *adjacent*. Since the graph G is undirected $(u,v) \in E$ implies that $(v,u) \in E$, but for simplicity we just assume that E is a set of unordered pairs. A vertex is said to have *degree d* if it has exactly d adjacent vertices. A set $V_k = \{v_1, v_2, \ldots, v_k\} \subseteq V$ is called *independent* if for any v_i, v_j $V_k, \epsilon (v_i, v_j) \notin E$. A *coloring* of G is a map $C: V \to N$, where N is the set of nonnegative integers and if $(u,v) \in E$ then $C(u) \neq C(v)$. The number of colors *used* to color G is $|C(v): v \in V| = C(G)$. A graph G is said to be *k-colored* if $C(G) \leq k$. For a graph $G = (V,E)$ with $|V| = n$ we let $T_k(n)$ denote the number of steps (or time) needed to k-color G.

We are interested in coloring planar graphs. A graph G is *planar* if it can be drawn in a plane so that no two edges intersect. It is well known that any planar graph can be 5-colored [3]. We will describe an algorithm to 5-color any planar graph for which $T_5(n) = O(n)$. Previous algorithms for 5-coloring planar graphs had $T_5(n) \geq \Omega(n^2)$.

Lemma 1: Let G be a planar graph with n vertices and $D_6(G)$ be the number of vertices in G of degree 5, 4, 5, or 6. Then $D_6(G) > \frac{n}{4}$.

Proof: Assume $D_6(G) \leq \frac{n}{4}$. Then the total degree $\geq 7 \times \frac{3n}{4} + 3 \times \frac{n}{4} = 6n$. By the Euler formula [3] it is known that the total degree of any planar n vertex graph is $\leq 6n - 12$. Thus we have a contradiction and the lemma is proved. □

Lemma 2: Given a planar graph G with n vertices, one can in $O(n)$ time find an independent set of at least $\frac{n}{28}$ vertices, with vertices each of degree d, $3 \leq d \leq 6$.

Proof: By Lemma 1 there must be vertices $v_1, \ldots, v_m$, $m > \frac{n}{4}$, that have degree d with $3 \leq d \leq 6$. Clearly such a set of vertices can be found in $O(n)$ time simply by inspecting each vertex in turn. Now we use a "greedy" algorithm to, in linear time, find an independent set $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ with $k \geq \frac{m}{7}$. The greedy algorithm proceeds as follows:

Let $v_{i_1} = v_1$. If $v_{i_1}, \ldots v_{i_j}$ have been selected, let $v_{i_{j+1}}$ be the first v_r that is adjacent to no vertex already selected. Since the degree of each of these vertices is 6 or less and $\frac{m}{7} > \frac{n}{28}$, it follows that the selection can be done in $0(n)$ time. $\square$

III. *The Main Results*

So far we have shown that an independent set of vertices of size $> \frac{n}{28}$ in which each vertex has small degree $(d \leq 6)$ can be found in $0(n)$ time for any planar graph. Clearly, vertices of degree 2 need not enter the discussion since they can be deleted from the graph, and from coloring considerations in an obvious way.

We now turn to the question of coloring the graph.

Lemma 3: (1) $T_7(n) \leq T_7(\lambda n) + 0(n)$ for some constant $0 < \lambda < \frac{27}{28}$.

(2) $T_7(n) = 0(n)$.

Proof: Clearly (2) follows from (1) by a simple induction, so we only prove (1). Let $G = (V,E)$ be a planar graph with n vertices. By Lemma 2 we can find $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ an independent set of vertices of G with $k > \frac{n}{28}$. Now let H be the graph induced by the set of vertices $V - \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$. That is, H is G with $\{v_{i_1}, \ldots, v_{i_k}\}$ removed and all edges touching any vertex in $\{v_{i_1}, \ldots, v_{i_k}\}$ removed. Let V_{i_j} be the set of vertices of G that are adjacent to v_{i_j}, $j=1,2,\ldots,k$. We call

V_{i_j} the neighborhood set of v_{i_j}. Since for all i_j, the degree of v_{i_j} is ≤ 6, we have $|V_{i_j}| \le 6$. Also, since $\{v_{i_1},\ldots,v_{i_k}\}$ is an independent set H contains all vertices in the V_{i_j} sets, j=1,2,...,k. Now we wish to prove (1). Clearly (1) is true for all planar graphs having 7 or fewer vertices, providing a basis for induction on n. Now assume (1) is true for all planar graphs having fewer than n vertices. Then, we can 7-color H in time $T_7(\lambda n)$. Note here that for H $0 < \lambda < (1 - \frac{1}{28}) = \frac{27}{28}$ since H has at most $\frac{27}{28}$n vertices. Then we can extend the coloring of H to 7-color G in the obvious way: For each v_{i_j}, j=1,2,...,k color v_{i_j} any color not used by vertices of V_{i_j}. Since $|V_{i_j}| \le 6$ this is always possible in a 7-coloring, and since $\{v_{i_1},\ldots,v_{i_k}\}$ is independent no interaction between colorings occur over the set. Therefore (1) immediately follows and the lemma is proved. □

Theorem: (1) $T_5(n) \le T_5(\lambda n) + O(n)$ for some $0 < \lambda < \frac{27}{28}$.

(2) $T_5(n) = O(n)$.

Proof: As in Lemma 3 it is sufficient to prove (1). Let G = (V,E) be a planar graph of n vertices. Let $\{v_{i_1},\ldots,v_{i_k}\}$, V_{i_j} for j=1,...,k, and H be defined as in the proof of Lemma 3. We now proceed to prove (1) in a manner similar to that of Lemma 3. Clearly (1) is true for all graphs of 5 or fewer vertices. Now assume (1) is true for all planar graphs having fewer than n vertices. Then we can 5-color H in time $T_5(\lambda n)$, where as before $0 < \lambda < \frac{27}{28}$. The extension of the 5-coloring from H to G is, however, more complex than the 7-coloring extension in Lemma 3. For any v_{i_j} for which fewer than 5 colors are used to color the nodes of V_{i_j} the extension is immediate. Checking each v_{i_j} for this condition and extending the coloring in this way, when possible, clearly can be done in time $O(k)$. This leaves a subset of $\{v_{i_1},\ldots,v_{i_k}\}$ for which each neighborhood set V_{i_j} required exactly 5 colors. Let this set of vertices be designated by $\{x_i,\ldots,x_m\}$, with neighborhood sets $V_i,\ldots,V_m$. We have 5-colored the graph H', the graph induced by $V - \{x_i,\ldots,x_m\}$ in time $T_5(\lambda n) + O(k)$. All that remains is to extend the coloring to $\{x_1,\ldots,x_m\}$. Since the neighborhood set of each x_i uses 5-colors, the extension must do some changing of colors. The interchange techniques for 5-coloring [3] are called into play. Let x and y be vertices of H'. We say that $x \underset{\alpha\beta}{\equiv} y$ where $\alpha,\beta\in\{0,1,2,3,4\}$ (the colors used) provided there is a path of vertices $x = z_1,\ldots,z_\ell = y$ from x to y each of which is colored either α or β. Obviously, since this is a coloring α and β alternate along the path. The following is a key fact:

> (*) $\forall\ 1 \le i \le m\ \exists\ r,s,\alpha,\beta$ such that $y_{i_r} \in V_i$, $y_{i_s} \in V_i$, y_{i_r} is colored α, y_{i_s} is colored β, $y_{i_r} \underset{\alpha\beta}{\not\equiv} y_{i_s}$, and neither α nor β is used by any other neighbor of x_i.

Fact (*) is proved like the reduction results in Ore [3] for 5-coloring.

We next claim, by renaming if necessary, since there are only 10 choices for α,β that (**) holds:

(**) $\forall\ 1 \le i \le m_0$ such that y_{i_0} is colored 0, y_{i_1} is colored 1, y_{i_0} and y_{i_1}, neighbors of x_i, $y_{i_0} \neq_{01} y_{i_1}$ and no other neighbor of x_i is colored 0 or 1; where $m_0 \ge \epsilon_0 m$ for some constant $\epsilon_0 > 0$.

Now consider just the vertices of H' that are colored 0 or 1. Let $C_1,\ldots,C_t$ be the connected components formed by these vertices. Now (**) essentially states first that y_{i_1} and y_{i_2} are never in the same component, and second that we can find a "batch" of m_0 such vertices from $\{x_i,\ldots,x_m\}$.

Next form a bipartite graph B as follows. The input vertices of B are $\{x_1,\ldots,x_m\}$; the output vertices of B are $\{C_1,\ldots,C_t\}$. There is an edge from x_j to C_k if and only if y_{i_0} or y_{i_1} is in C_k. Clearly this is a bipartite graph and each x_j has degree exactly 2. We now claim that B is planar. This can be seen by using the contraction operations in Ore [3].

Now let B' be the planar graph obtained from B by, for each x_j, replacing a path C_i,x_j,C_k by an edge from C_i to C_k and deleting x_j. The vertices of B' are $C_1,\ldots,C_t$ with $t \le n$. Clearly B' has m edges. Now in linear time we can 7-color B'. If deg(v) = the degree of vertex v, and $V_1,V_2,\ldots,V_7$ is the partition induced on the vertices of B' by the 7-coloring we see that:

$$2m = \sum_{i=1}^{t} \deg(C_i) = \sum_{i=1}^{7} \sum_{C_j \in V_i} \deg(C_j).$$

So that for some V_i we have:

$$\sum_{C_j \in V_i} \deg(C_j) \geq \frac{2m}{7} .$$

Assume for convenience that V_1 is this block. Now consider $C_j \in V_1$ in the bipartite graph B. They satisfy:

(1) no y_{i_0} or y_{i_1} is in a C_j and a C_k with $j \neq k$.

(2) at least $\frac{2m}{7}$ y_{i_0}'s or y_{i_1}'s are in some C_j.

The first follows since V_1 is independent in B', the second by the way V_1 was selected.

Now we can interchange the colors in each $C_j \in V_1$. By (1) and (2) this causes at least $\frac{2m}{7}$ x_i's to be surrounded by only 4 colors. Thus we obtain

$T_5(n) \leq T_5(\lambda n) + O(k) + \Gamma(m)$

$\Gamma(m) \leq \Gamma(\delta m) + O(m)$

so $\Gamma(m) = O(m)$ and $T_5(n) \leq T_5(\lambda n) + O(k) + O(m)$

or $T_5(n) \leq T_5(\lambda n) + O(n)$

since $m \leq k \leq n$. $\square$

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