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On Singular Singularly-Perturbed Initial Value Problems

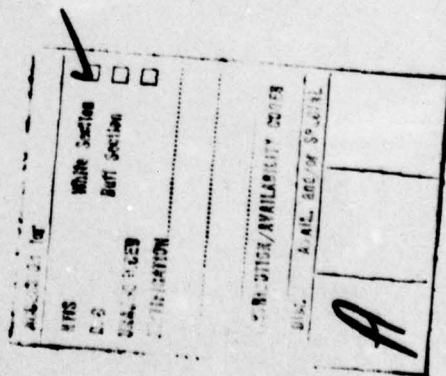
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DEDICATED TO ARTHUR AND EVA ERDELYI ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY

Abstract

Consider the vector initial value problem $\epsilon \dot{y} = f(t, y, \epsilon)$, $y(0) = y^0(\epsilon)$ with $f(t, y, 0) = F(t)y + G(t)$ for a singular matrix $F(t)$ of constant rank with stable eigenvalues and zero eigenvalues having simple elementary divisors. This paper shows how to determine the unique limiting solution when the reduced problem $FY_0 + G = 0$ is solvable and how to obtain a uniform asymptotic expansion for the solution on finite t intervals.



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1. Introduction

We wish to consider the vector initial value problem

$$(1) \quad \epsilon \dot{y} = f(t, y, \epsilon), \quad y(0) = y^0(\epsilon)$$

in the nearly linear situation that

$$(2) \quad f(t, y, 0) \equiv F(t)y + G(t).$$

When $F(t)$ is stable on some interval $0 \leq t \leq T$, it is a classical result of Tihonov [17] that the limiting solution of the initial value problem (1) will converge to the solution $Y_0 = -F^{-1}G$ of the reduced equation

$$(3) \quad 0 = F(t)Y_0 + G(t)$$

for $0 < t \leq T$ as the small positive parameter ϵ tends to zero. (The result continues to be valid for all $t > 0$ if Y_0 decays exponentially to zero as $t \rightarrow \infty$ (cf. Hoppensteadt [7]).) The solution generally converges nonuniformly at $t = 0$ since we cannot expect that $y^0(0) = Y_0(0)$. Assuming infinite differentiability of f in t and y and asymptotic series expansions in ϵ for both f and y^0 , we can indeed show that the asymptotic solution $y(t, \epsilon)$ of (1) is of the form

$$(4) \quad y(t, \epsilon) = Y(t, \epsilon) + \Pi(\tau, \epsilon)$$

throughout $0 \leq t \leq T$ where Y and Π have asymptotic expansions in ϵ and the terms of Π all tend to zero as the stretched (or boundary layer)

variable

$$(5) \quad \tau = t/\epsilon$$

tends to infinity. Thus, the limiting asymptotic solution for $t > 0$ is provided by the outer solution $Y(t, \epsilon)$, and the nonuniform convergence of the solution from $y^0(\epsilon)$ to $Y(0, \epsilon)$ near $t = 0$ is accomplished through the boundary layer correction $\Pi(\tau, \epsilon)$.

If $F(t)$ has unstable eigenvalues, we cannot expect the initial value problem to have bounded solutions as $\epsilon \rightarrow 0$ unless we restrict the initial values to a lower dimensional manifold (cf. Levin [11]). It is still reasonable, however, to allow arbitrary initial values if the matrix $F(t)$ is singular. (The usual expansion procedure will then break down.) For an M -vector y , let us assume

(H1) the $M \times M$ matrix $F(t)$ has a constant rank k , $0 \leq k < M$, for all t in $0 < t \leq T$, its nonzero eigenvalues all have negative real parts there, and its null space is spanned by $M - k$ linearly independent eigenvectors.

Under (H1), we shall find that the asymptotic solution of (1) - (2) remains in the form (4) whenever the limiting equation (3) is consistent, i.e. $G \in \mathcal{R}(F)$, the range of F . Because F is singular, (3) no longer has a unique solution. Its solution is determined up to an arbitrary element of $N(F)$, the null space of F , so additional analysis is required to determine the unique limiting solution for $t > 0$. This is an instance of a "singular singular-perturbation problem" where the reduced problem (obtained when

$\epsilon = 0$) has an infinity of solutions in regions of uniform convergence.

More generally, one could consider singular problems (1) where $f_y(t, y, 0) = 0$ for some t values along some solutions y of the reduced equation $f(t, y, 0) = 0$. Difficult turning point problems would then be faced (cf. Wasow [20]). A somewhat more restricted class of nonlinear problems is considered in O'Malley and Flaherty [15]. Their analysis seeks to eventually develop numerical algorithms for such problems. Among other possible applications are high gain feedback control systems (cf. O'Malley [13]), singular control problems (O'Malley [14]), and certain Cauchy problems in Banach spaces (Gordon [6]). We note that the structure of the asymptotic solutions will generally differ considerably if the elementary divisors of F corresponding to zero eigenvalues are nonlinear. An example is provided by

$$f(t, y, \epsilon) = F(t, \epsilon)y \quad \text{for} \quad F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & -1 & 0 \end{pmatrix}. \quad \text{Further, if } F \text{ remains non-}$$

singular, but has purely imaginary eigenvalues, rapidly oscillating solutions would result (cf. Hoppensteadt and Miranker [9]).

2. Preliminary Linear Algebra

Under hypothesis (H1), we are guaranteed that the matrix F can be put into its row-reduced form by an orthogonal matrix $E(t)$, i.e.

$$(6) \quad EF = \begin{bmatrix} U \\ 0 \end{bmatrix}$$

where $U(t)$ is a $k \times M$ matrix of rank k . Indeed, E is guaranteed by the singular value decomposition of F , i.e. $F = E'DH$ for orthogonal matrices E and H and diagonal D (cf., e.g., Stewart [16]). E can be obtained numerically by performing a sequence of Householder transformations

(cf. Golub [4]). Moreover, the differentiability of E will follow that of the coefficients under the constancy of rank assumption (cf. Golub and Pereyra [5]).

The use of such orthogonal matrices to reduce F to echelon form seems more convenient than the diagonalization procedures which have been more common in the singular perturbation literature. Among other differences, it avoids the use of eigenvectors of F . The success of traditional procedures on closely related problems is demonstrated in Fife [3], Butuzov and Vasil'eva [1], and Vasil'eva [19]. Alternative analogous methods which don't involve orthogonal matrices may be necessary for nonlinear problems (cf. O'Malley [14]).

Writing

$$(7) \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

where E_1 is $k \times M$, we have the orthogonality condition

$$(8) \quad E_2 F = 0.$$

Furthermore, the orthogonality of E implies that

$$(9) \quad E_1 E_2' = 0, \quad E_1 E_1' = I_k, \quad E_2 E_2' = I_{M-k}, \quad \text{and} \quad E_1' E_1 + E_2' E_2 = I_M.$$

These last relationships imply that

$$(10) \quad P = E_1' E_1 = P^2 \quad \text{and} \quad Q = E_2' E_2 = Q^2$$

are complementary projections. Indeed, a direct sum decomposition of M -space results with

$$(11) \quad R(Q) = N(F') \quad \text{and} \quad R(P) = R(F).$$

This follows since $F'Q = (E_2F)'E_2 = 0$ shows that Q projects into $N(F')$ and the dimensions of $R(Q)$ and $N(F')$ are both $M - k$. Finally, (7) and

$$(8) \quad \text{show that} \quad EFE' = \begin{pmatrix} E_1FE'_1 & E_1FE'_2 \\ 0 & 0 \end{pmatrix} \quad \text{where the } k \times k \text{ matrix}$$

$$(12) \quad S \equiv E_1FE'_1 \quad \text{is stable}$$

since it shares the k stable eigenvalues of F . Indeed, the purpose of (H1) is to provide an orthogonal matrix E such that EF is row-reduced with EFE' having k stable eigenvalues.

3. The Transformation Approach

Making the one-one transformation

$$(13) \quad z = E(t)y,$$

z will satisfy

$$(14) \quad \epsilon z' = EFE'z + EG + \epsilon \left[\dot{E}E'z + \frac{1}{\epsilon} E(f(t, E'z, \epsilon) - f(t, E'z, 0)) \right].$$

Setting

$$(15) \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} E_1y \\ E_2y \end{pmatrix},$$

then, provides a transformed singular perturbation problem

$$(16) \quad \begin{cases} \varepsilon z_1' = Sz_1 + E_1 F E_2' z_2 + E_1 G + \varepsilon g_1(z_1, z_2, t, \varepsilon) \\ \varepsilon z_2' = E_2 G + \varepsilon g_2(z_1, z_2, t, \varepsilon) \end{cases}$$

where $g_2 = \dot{E}_2 E' z + \frac{1}{\varepsilon} E_2 (f(t, E' z, \varepsilon) - f(t, E' z, 0))$. The asymptotic behavior of solutions to (16) differs considerably in the two cases $E_2 G = 0$ and $E_2 G \neq 0$. Since $E_2 G = 0$ if and only if $QG = 0$, Q projecting onto $N(F')$, we're guaranteed that $E_2 G = 0$ if and only if the reduced problem (3) is consistent. Taking

$$(17) \quad E_2 G \equiv 0 \text{ throughout } [0, T],$$

then, reduces (16) to a "nonsingular singular-perturbation problem." For it, classical singular perturbation theory (cf., e.g., Hoppensteadt [8] or O'Malley [12]) implies that (16) - (17) has a limiting solution for $t > 0$

which is the unique solution $\begin{pmatrix} z_{10} \\ z_{20} \end{pmatrix}$ of the reduced problem

$$(18) \quad \begin{cases} 0 = Sz_{10} + E_1 F E_2' z_{20} + E_1 G \\ \dot{z}_{20} = g_2(z_{10}, z_{20}, t, 0), \quad z_{20}(0) = E_2(0) y^0(0), \end{cases}$$

presuming that the corresponding nonlinear initial value problem

$$(19) \quad \dot{z}_{20} = g_2(-S^{-1} E_1 (F E_2' z_{20} + G), z_{20}, t, 0), \quad z_{20}(0) = E_2(0) y^0(0)$$

has a solution for $0 \leq t \leq T$. Furthermore, that theory also shows that the nonuniform convergence of z near $t = 0$ is determined as the decaying solution of the boundary layer problem

$$(20) \quad \begin{cases} \frac{dT_{10}}{d\tau} = S(0)T_{10}, & T_{10}(0) = E_1(0)y^0(0) - Z_1(0) \\ \frac{dT_{20}}{d\tau} = 0. \end{cases}$$

Thus, $T_{10}(\tau) = e^{-S(0)\tau} T_{10}(0)$ and $T_{20}(0) = 0$ for $\tau \geq 0$. The bifurcation character (cf., e.g., Cesari [2]) of these limiting problems makes them differ from the classical ones. Here, the reduced problem (18) consists of an algebraic problem for a k -vector and an initial value problem for an $M - k$ vector, while the boundary layer problem (20) involves a k dimensional initial value problem and an $M - k$ dimensional algebraic problem.

4. A Direct Solution

Since the boundary layer correction $\Pi(\tau, \epsilon)$ will be asymptotically negligible for each $t > 0$, the outer solution

$$(21) \quad Y(t, \epsilon) \sim \sum_{j=0}^{\infty} Y_j(t) \epsilon^j$$

must satisfy the original nearly linear system (1) as a power series in ϵ . Rewriting the equation as

$$(22) \quad F(t)Y = -G(t) - [f(t, Y, \epsilon) - f(t, Y, 0)] + \epsilon \dot{Y}$$

and substituting in the expansion (21) implies that the terms of (21) must

successively satisfy the algebraic equations

$$(23) \quad F(t)Y_j = \xi_{j-1}(t)$$

where

$$(24) \quad \begin{cases} \xi_{-1} = -G, & \xi_0 = \dot{Y}_0 - f_\epsilon(t, Y_0, 0), \\ \text{and} \\ \xi_k = \dot{Y}_k - f_{y\epsilon}(t, Y_0, 0)Y_k + \eta_{k-1}(t), & k \geq 1 \end{cases}$$

for a known function η_{k-1} of $Y_0, Y_1, \dots, Y_{k-1}$. Assuming consistency of (3), $G \in R(F)$ so, by (8), $QG = 0$. Consistency of (23), by the Fredholm alternative, requires $Q\xi_{j-1} = 0$, $j \geq 1$, so we obtain the singular differential equations

$$(25) \quad \begin{cases} Q\dot{Y}_0 = Qf_\epsilon(t, Y_0, 0) \\ \text{and} \\ Q\dot{Y}_k = Qf_{y\epsilon}(t, Y_0, 0)Y_k - Q\eta_{k-1}, & k \geq 1. \end{cases}$$

Taking (23) and (25) together nearly allows us to obtain the outer solution Y termwise. Specifically, multiplying (23) by E_1 implies that $E_1 F(E_1' E_1 + Q)Y_j = E_1 \xi_{j-1}$, $j \geq 0$, so the invertibility of $S = E_1 F E_1'$ and the definition of P imply that

$$(26) \quad P Y_j = -A F(Q Y_j) + A \xi_{j-1}$$

where $A = E_1' S^{-1} E_1$. Thus $P Y_j$ and thereby Y_j are completely determined as

linear functions of QY_j . Putting (25) and (26) together, then, implies the differential equations

$$(27) \quad \begin{cases} (QY_0)' = \dot{Q}(B(QY_0) - AG) + Qf_\epsilon(t, B(QY_0) - AG, 0) \\ \text{and} \\ (QY_k)' = [\dot{Q} + Qf_{y\epsilon}(t, Y_0, 0)][B(QY_k) + A\xi_{k-1}] - Q\eta_{k-1}, \quad k \geq 1 \end{cases}$$

since (26) implies that

$$(28) \quad Y_j = B(QY_j) + A\xi_{j-1}$$

for the projection $B = I - AF$. Note that the representation (4) implies the yet unspecified initial values

$$(29) \quad Q(0)Y_k(0) = Q(0)y_k^0 - Q(0)\pi_k(0), \quad k \geq 0$$

needed to completely determine the outer solution $Y(t, \epsilon)$.

If we anticipate that $Q(0)\pi_0(0) = 0$ (paralleling the result that $T_{20}(0) = 0$ in (20)), it will follow that the limiting outer solution is given by

$$(30) \quad Y_0 = B(QY_0) - AG$$

where QY_0 , the projection of the limiting solution onto $N(F')$, satisfies the nonlinear initial value problem

$$(31) \quad \begin{cases} (QY_0)' = \dot{Q}(B(QY_0) - AG) + Qf_\epsilon(t, B(QY_0) - AG, 0) \\ Q(0)Y_0(0) = Q(0)y^0(0). \end{cases}$$

We shall assume that the solution to (31) exists throughout $0 \leq t \leq T$ (in analogy to our existence assumption for (19)). We note that (31) shows that the limiting solution of (1) - (2) is determined by a dynamical system in an $M - k$ dimensional space. Further, we observe that unique solutions QY_k and Y_k for $k > 0$ follow from the linear differential equations of (27) and the algebraic relation (28).

By the form of (4), the boundary layer correction Π must satisfy the system

$$(32) \quad \frac{d\Pi}{d\tau} = \epsilon \frac{dy}{dt} - \epsilon \frac{dY}{dt} = f(\epsilon\tau, Y(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon) - f(\epsilon\tau, Y(\epsilon\tau, \epsilon), \epsilon)$$

and the terms of its expansion

$$(33) \quad \Pi(\tau, \epsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau) \epsilon^j$$

should tend to zero as τ tends to infinity. In the quasilinear situation when (2) holds, equating coefficients successively in (32) implies the linear system

$$(34) \quad \frac{d\Pi_j}{d\tau} = F(0)\Pi_j + \zeta_{j-1}(\tau)$$

where ζ_{j-1} is a linear combination of the preceding coefficients Π_l , $l < j$, with polynomial coefficients in τ , and $\zeta_{-1} \equiv 0$. Since $QF = 0$, it follows

that $\frac{d}{d\tau} (Q(0)\Pi_j(\tau)) = Q(0)\zeta_{j-1}(\tau)$. Thus, the decay requirement implies that

$$(35) \quad Q(0)\Pi_j(\tau) = -Q(0) \int_{\tau}^{\infty} \zeta_{j-1}(s)ds.$$

We note, in particular, that (35) allows the termwise determination of the initial values (29) needed to fix the outer solution. It remains to successively find the $P(0)\Pi_j(\tau)$'s. Multiplying (34) by $E_1(0)$ implies that $\frac{d}{d\tau} (E_1(0)\Pi_j) = S(0)(E_1(0)\Pi_j) + E_1(0)F(0)(Q(0)\Pi_j) + E_1(0)\zeta_{j-1}$. Using (35) and $I_M = E_1'(0)E_1(0) + Q(0)$ implies the decaying solutions

$$(36) \quad \begin{aligned} \Pi_j(\tau) = & E_1'(0)e^{S(0)\tau}E_1(0)\Pi_j(0) - Q(0) \int_{\tau}^{\infty} \zeta_{j-1}(s)ds \\ & + E_1'(0) \int_0^{\tau} e^{S(0)(\tau-r)}E_1(0)[\zeta_{j-1}(r) - F(0)Q(0) \int_r^{\infty} \zeta_{j-1}(s)ds]dr. \end{aligned}$$

Thus, the boundary layer terms are determined successively as exponentially decaying vectors up to the initial value $E_1(0)\Pi(0,\epsilon)$, just as the outer expansion coefficients are determined up to $Q(0)Y(0,\epsilon)$. Since the limiting outer solution is known from (30) - (31) and higher order terms follow successively from (27) - (29), the unique higher order boundary layer correction terms follow from (36) with

$$(37) \quad E_1(0)\Pi_j(0) = E_1(0)(y_j^0 - Y_j(0)), \quad j \geq 0.$$

For $j = 0$, we'll have

$$(38) \quad \Pi_0(\tau) = E_1'(0)e^{S(0)\tau}E_1(0)(y^0(0) - B(0)Q(0)y^0(0) - A(0)G(0)).$$

Since $QE_1' = 0$, it follows that Π_0 lies within the k dimensional

$R(F) = R(P)$. The function of this zero order boundary layer correction term, then, is to "instantly" transfer from the prescribed initial vector $y^0(0)$ to $Y_0(0^+)$ where the reduced equation (3) is satisfied. Finally, we observe that the expansion (4) which we've formally generated will, of course, agree with that obtained by our earlier transformation procedure and is justified by it.

Summarizing, we have

Theorem

Under hypothesis (H1); (H2), that (3) is consistent; and (H3), that the solution (QY_0) of (34) exists for $0 \leq t \leq T$, the unique solution of the initial value problem (1) - (2) is asymptotically of the form (4) uniformly in $0 \leq t \leq T$ as $\epsilon \rightarrow 0$.

5. The Inconsistent Problem

If $E_2 G \neq 0$, the transformed system (16) cannot be easily solved as (16) - (17) was. Then, the limiting system (3) has no solutions and we cannot expect to have an outer solution of the form (21).

By variation of parameters, we know that solution of (1) - (2) must satisfy the integral equation

$$(39) \quad y(t, \epsilon) = V(t, \epsilon) y^0(\epsilon) + \frac{1}{\epsilon} \int_0^t V(t, \epsilon) V^{-1}(s, \epsilon) \{G(t) + f(t, y, \epsilon) - f(t, y, 0)\} ds$$

where V is the fundamental matrix solution of the linear system

$$(40) \quad \dot{V} = \frac{1}{\epsilon} F(t) V, \quad V(0, \epsilon) = I_M.$$

Such fundamental matrices can be constructed following the work of Turrittin

and earlier authors (cf. Turrittin [18]). Under natural assumptions, we can show that $\|Y(t, \epsilon)Y^{-1}(s, \epsilon)\| \leq K$ for $0 \leq s, t \leq T$ for some constant K provided no eigenvalues of F are unstable. When F is stable, we instead have a decaying bound $Ke^{-\kappa(t-s)/\epsilon}$ and this allows the corresponding outer solution for (1) - (2) to be bounded. When F has eigenvalues with zero real parts, other estimates are appropriate (cf. Hoppensteadt and Miranker [9]).

For

$$(41) \quad f(t, y, \epsilon) = F(t, \epsilon)y + G(t, \epsilon)$$

linear in y , (39) makes it natural to seek the solution to (1) in the form

$$(42) \quad y(t, \epsilon) = \frac{1}{\epsilon} Z(t, \epsilon) + \Pi(\tau, \epsilon)$$

when hypothesis (H1) holds for F_0 with $(F(t, \epsilon), G(t, \epsilon)) \sim \sum_{j=0}^{\infty} (F_j(t), G_j(t))\epsilon^j$.

Here the outer solution $\frac{1}{\epsilon} Z(t, \epsilon)$ and the boundary layer correction Π have power series expansions in ϵ and $\Pi \rightarrow 0$ as $\tau \rightarrow \infty$. Now the limiting solution $\frac{1}{\epsilon} Z_0(t)$ for $t > 0$ satisfies the homogeneous system $F_0(t)Z_0 = 0$. Applying the Fredholm alternative to the equation for $F_0 Z_1$ requires that $Q(\dot{Z}_0 - F_1 Z_0 - G_0) = 0$ where Q is the projection onto $N(F'_0)$. Using our earlier notation, then, we have the unbounded limiting solution $\frac{1}{\epsilon} Z_0(t)$ with

$$(43) \quad Z_0 = B(QZ_0)$$

where QZ_0 satisfies the linear system

$$(44) \quad (QZ_0)' = (\dot{Q} + QF_1)B(QZ_0) + QG_0, \quad Q(0)Z_0(0) = 0.$$

The boundary layer correction is calculated as before. When $QG_0 \equiv 0$, $Z_0 = 0$ and the solution reduces to the form (4) obtained for the consistent problem.

If f is nonlinear in y , the expansion (42) cannot generally be used. (A term like y^k would provide a term $\frac{1}{\epsilon^k} Z_0^k$ in the outer region.) For polynomial dependence on y , an appropriate scheme might be devised for the inconsistent problem though we shall not do so (cf. Kersten [10], however, for a related discussion concerning special boundary value problems).

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