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AN INEQUALITY FOR SUMS OF DYADS AND TENSORS.(U)
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An inequality for sums of dyads and tensors.*

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Given finite rank transformation R on Hilbert space where $\sum_{i=1}^{N}(u_i \times v_i) = R$, then it is shown that $2\text{rank}(R) \leq r(U) + r(V) \leq \text{rank}(R)+N$, where $r(U) = \dim(\text{span}(u_1,u_2,\ldots,u_N)$, $r(V) = \dim(\text{span}(v_1,v_2,\ldots,v_N))$.

Applications to sums of decomposable Kronecker products are given.

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An inequality for sums of dyads and tensors*

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ABSTRACT: Given a finite rank transformation R on Hilbert space with dyadic sum decomposition

$$\sum(u_i \times v_i) = R,$$

then it is shown that

$$2\cdot \mathrm{rank}(R) \leq r(U) + r(V) \leq \mathrm{rank}(R) + N,$$

where $r(U) = \dim(\mathrm{span}(u_1, u_2, \ldots u_N))$ and
$r(V) = \dim(\mathrm{span}(v_1, v_2, \ldots v_N))$.

Applications to sums of decomposable Kronecker products and to summs of dyads are presented.

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Introduction. In previous works, relations between dyadic and Kronecker products of vectors (definitions follow) are explored, cf. [2], [3]. In fact, consider the general situation where finite rank linear transformation R on infinite-dimensional Hilbert space, H, is the sum of dyadic products. If the number of terms of this sum is known, then these dyadic terms can be fairly well characterized [3, Thm. 3.2]. In this paper, we consider dyadic sum decompositions for R where N, the number of terms, is not known *a priori*, and present a sharp inequality which ties together

(i) the rank of R,

(ii) the ranks (dimension of the spans) of the dyad component vectors, and

(iii) N, the number of distinct dyads which sum to R.

This inequality proves useful for establishing necessary conditions for certain special questions, e.g., when do N dyads sum to a single Kronecker product, or when do N dyads sum to (another) dyad? These questions, in turn, relate to the complexity question in the computation of matrix products, cf., [4], [1].

2. Definitions and Preliminaries. L(H,K) denotes all bounded linear transformations from Hilbert space H to Hilbert space K. Among the elements of L(H,K) are the *dyads* (rank one transformations) $(x \times y)$ defined for each $y \in H$, $x \in K$ by requiring that for all $z \in H$, $(x \times y): z \to \langle z,y \rangle x$, where $\langle\ ,\ \rangle$ is the inner product on H. We proceed to give the Kronecker or tensor product

$A \otimes B^t$: First, for $A \in L(H,K)$, A^*, the *adjoint* of A, is that element of $L(K,H)$ given by $\langle Ay,x\rangle = \langle y,Ax\rangle$ for all $y \in H$, $x \in K$. As an example, $(x \times y)^* = (y \times x)$ for all dyads. $\overline{H}$ denotes the Hilbert space of linear functionals on H. That is, for $x \in H$, $\overline{x} \in \overline{H}$ is defined by $\overline{x}: y \to \langle y,x\rangle$ for all $y \in H$. This leads to the definition of $A^t \in L(\overline{K},\overline{H})$ where $A \in L(H,K)$. In fact, for all $x \in H$, $\overline{y} \in \overline{K}$, we define $A^t(\overline{y})(x) = \overline{y}(A(x))$. Finally, for any $A \in L(H_1,K_1)$, $B \in L(H_2,K_2)$ we deine the *Kronecker* (or *tensor*) *product* $A \otimes B^t$ by $A \otimes B^t: C \to ACB$ for all $C \in L(K_2,H_1)$.

We will use *rk(R)* to denote the *rank of a transformation R*, i.e., $rk(R)$ is the dimension of the range of R. Also, if $U = \{x_1,x_2,\ldots,x_N\} \subset H$, then we will use *r(U)* to denote the *rank of the set U*, i.e., $r(U)$ is the dimension of *span* $\langle U\rangle$, the linear span of the set U.

Before arriving at our inequality, we will be using the following characterization of dyadic sums:

Theorem 2.1 ([9, Th. 3.2]). Given finite-rank linear transformation $R \in L(H,K)$ and the set $U = \{u_1,u_2,\ldots,u_n,\ldots,u_N\} \subset K$ where the range of R is a subspace of span $\langle U\rangle$. Assume (by re-ordering if necessary) that the first $n \le N$ elements of U form a basis for span $\langle U\rangle$ (i.e., $n = r(U)$, the rank of U). Accordingly the $N-n \ge 0$ remaining vectors $u_{n+1},u_{n+2},\ldots,u_N$ define $N-n$ scalars $\{\alpha_i^{(j)}: i = 1,2,\ldots,n,\ j = n+1, n+2,\ldots,N\}$ by the equations

$$u_j = \sum_{i=1}^{n} \alpha_i^{(j)} u_i\ ,\ j = n+1, n+2,\ldots,N.$$

Then for N-n <u>arbitrary</u> vectors $\{v_{n+1}, v_{n+2}, \ldots, v_N\} \subset H$ we have the representation

$$\sum_{i=1}^{N} (u_i \times v_i) = R \tag{2.1}$$

if and only if each "earlier" v_i is given by

$$v_i = R^*(\hat{u}_i) - \sum_{j=n+1}^{N} \bar{\alpha}_i^{(j)} v_j, \quad i = 1,2,\ldots,n=r(U), \tag{2.2}$$

where $\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n\} \in \text{span}\,\langle U\rangle$ is the unique biorthonormal complement to $\{u_1, u_2, \ldots, u_n\}$ in span $\langle U\rangle$ (i.e., $\langle \hat{u}_i, u_j\rangle = \delta_{ij}$, the Kronecker delta). The summation in (2.2) is taken to be zero in case $n = N$.

3. <u>The Inequality</u>

<u>Theorem 3.1</u>. Given finite-rank linear transformation $R \in L(H,K)$ and sets of vectors $U = \{u_1, u_2, \ldots, u_N\} \subset K$, $V = \{v_1, v_2, \ldots, v_N\} \subset H$ such that

$$\sum_{i=1}^{N} (u_i \times v_i) = R\ . \tag{3.1}$$

Then

$$2\cdot \mathrm{rk}(R) \le r(U) + r(V) \le \mathrm{rk}(R) + N\ , \tag{3.2}$$

where $rk(R)$ = dimension (range of R), and

$$r(U) = \text{dimension (span } \langle U \rangle)$$
$$r(V) = \text{dimension (span } \langle V \rangle).$$

<u>Proof</u>: By re-ordering the terms of sum (3.1) if necessary, we will assume that the first $n = r(U)$ elements, $u_1, u_2, \ldots, u_n$ of U, form a basis for span $\langle U \rangle$. Thus, the ordered set V lends itself to characterization (2.2). In fact,

$$r(V) = \text{rank}(\text{span}\langle v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_N \rangle) \;, \qquad (3.3)$$

where $v_i = R^*(\hat{u}_i) - \sum_{j=n+1}^{N} \bar{\alpha}_i^{(j)} v_j$, $i = 1,2,\ldots n$ (from (2.2)).

Equivalently,

$$r(V) = \text{rank}(\text{span}\langle R^*(\hat{u}_1), R^*(\hat{u}_2), \ldots, R^*(\hat{u}_n), v_{n+1}, \ldots, v_N \rangle) \quad (3.4)$$

The equivalence of (3.3) and (3.4) follows by observing that each of the N vectors in (3.4) belongs to the linear span of the N vectors in (3.3), and <u>vice</u> <u>versa</u>. From (3.4) we now obtain

$$\begin{aligned} r(V) &\le \text{rank}(\text{span}\langle R^*(\hat{u}_1), \ldots, R^*(\hat{u}_n)\rangle) + \text{rank}(\text{span}\langle v_{n+1}, \ldots, v_N \rangle) \\ &\le rk(R^*) + N - n \qquad (3.5) \\ &= rk(R) + N - r(U) \;, \end{aligned}$$

which gives us the right-hand side of inequality (3.2). Obtaining the left-hand side of (3.2) is immediate, since from (3.1) we deduce that span $\langle U \rangle \supset$ range R, while span $\langle V \rangle \supset$ range R* (recall $(u_i \times v_i)^* = (v_i \times u_i)$). Thus, $r(U) \geq rk(R)$ and $r(V) \geq rk(R^*) = rk(R)$ implying

$$2 \cdot rk(R) \leq r(U) + r(V). \tag{3.6}$$

Finally, (3.5) with (3.6) establishes (3.2) and the proof is done. ■

Is the inequality sharp? The left side of (3.2) yields equality whenever the entire N-element sets U and V are linearly independent (i.e., when $n = N = rk(R)$). In following the proof of the right-hand inequality for (3.2), we observe the two inequalities in (3.5). The first inequality yields equality if and only if

$$\text{span}\langle R^*(\hat{u}_1), R^*(\hat{u}_2), \ldots, R^*(\hat{u}_n)\rangle \cap \text{span}\langle v_{n+1}, v_{n+2}, \ldots, v_N\rangle = \{0\}.$$

That is, by choosing each of the N-n arbitrary vectors $v_{n+1}, \ldots, v_N$ in H outside the range of R*. The second inequality of (3.5) becomes equality if and only if the N-n element set $\{v_{n+1}, v_{n+2}, \ldots, v_N\}$ is linearly independent.

4. Final Remarks. In [3, Th. 4.2, 4.3], it is shown that

$$\sum(u_i \times v_i) = R \text{ if and only if } \sum(u_i \otimes v_i) = R' \qquad (4.1)$$

where the passage from R to R' is a well-defined linear relationship. This provides a dual form to (3.2) with tensor products replacing the dyads of (3.1) and this R' replacing R. As an easy special case, let us use (3.2) and dyad-tensor duality to justify the following statements for non-zero u_i, v_i, x_i, $y_i \in H$, $i = 1, 2, 3$.

<u>Proposition</u>. Suppose

$$(u_1 \times v_1) + (u_2 \times v_2) = (u_3 \times v_3) \text{ , and} \qquad (4.2a)$$

$$(x_1 \otimes u_1) + (x_2 \otimes y_2) = (x_3 \otimes y_3) \text{ .} \qquad (4.2b)$$

Then all the u_i's or else all the v_i's are non-zero scalar multiples of each other. Similarly, all the x_i's or else all the y_i's are scalar multiples of each other.

<u>Proof</u>. The proof of this assertion will <u>not</u> appeal to the definitions of the dyad $(u_i \times v_i)$ or of the tensor $(x_i \otimes y_i)$, since inequality (3.2) applies. In fact, write (4.2a) as

$$(u_1 \times v_1) + (u_2 \times v_2) - (u_3 \times v_3) = 0 \text{ (i.e., } N = 3, R = 0) \qquad (4.2a')$$

from which we obtain via (3.2) that

$$2 \cdot 0 \leq r(\{u_1, u_2, u_3\}) + r(\{v_1, v_2, v_3\}) \leq 0 + 3. \qquad (4.3)$$

Since we have assumed no u_i or v_i is zero, the ranks $r(u), r(v) \geq 1$. At the same time, the upper bound of 3 given by (4.3) assures us that both $r(u) = 2$ and $r(v) = 2$ can not happen, i.e., at least one of the terms $r(u)$, $r(v)$ in (4.3) equals one, or all the u_i's or all the v_i's are scalar multiples of each other. By our duality result, (4.1), (4.2a') is equivalent to

$$(u_1 \otimes v_1) + (u_2 \otimes v_2) - (u_3 \otimes v_3) = 0 ,$$

and the same conclusion obtains, i.e., in (4.2b), either $r(\{x_1,x_2,x_3\})$ or $r(\{y_1,y_2,y_3\})$ equals one, or all the x_i's or all the y_i's are scalar multiples of each other if (4.2b) is given.

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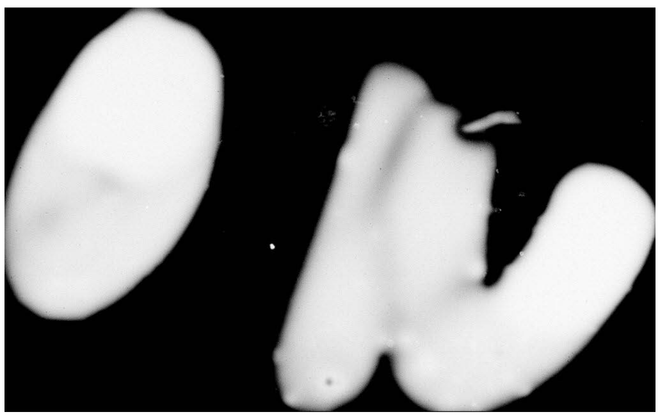

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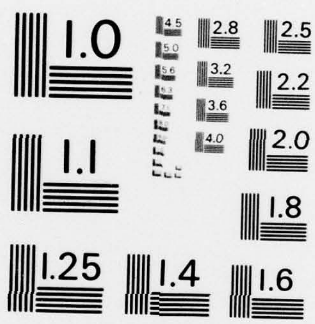

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SUPPLEMENTARY

INFORMATION

Errata

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