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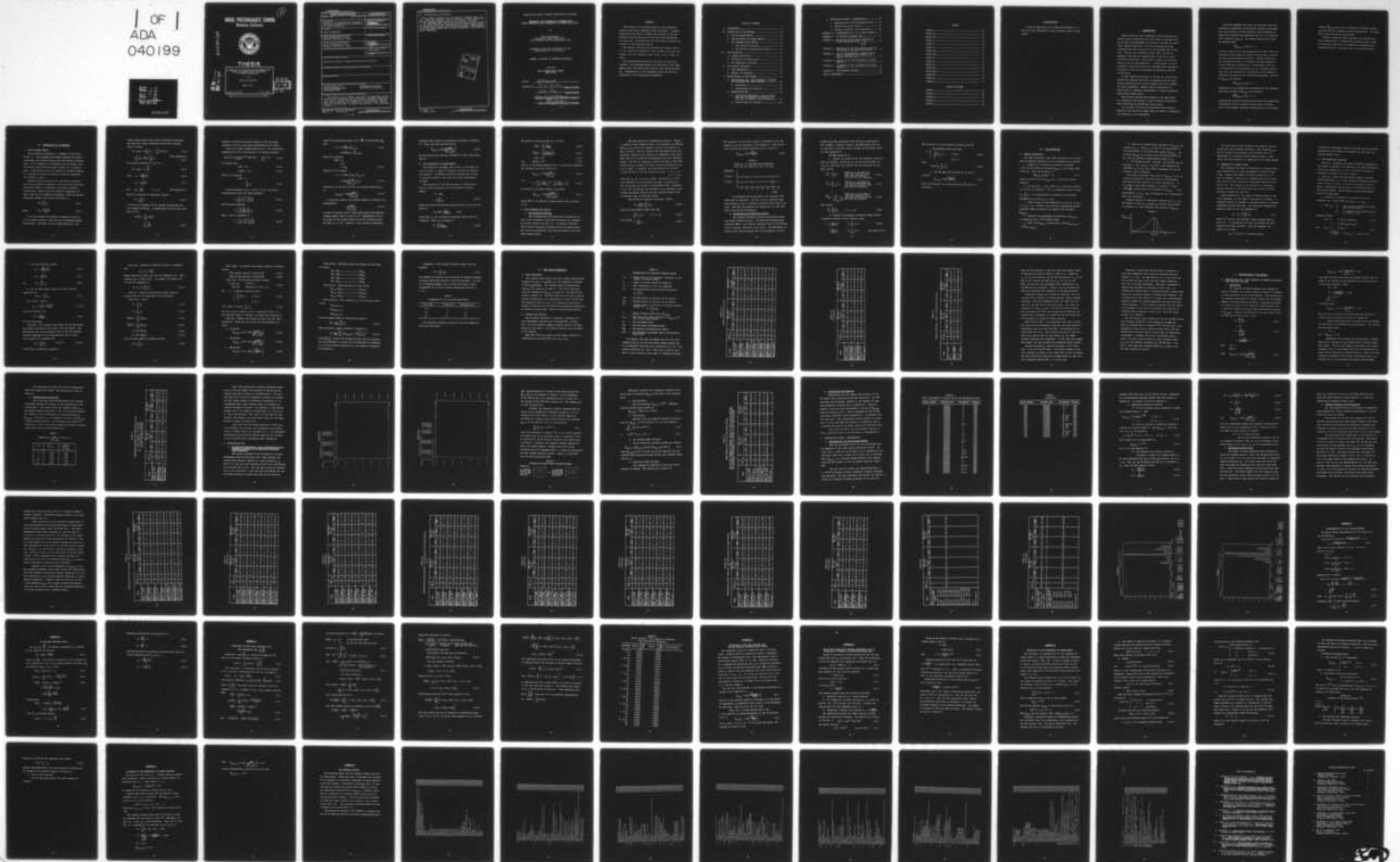
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THESIS

COMPARISON OF LOG-GAMMA AND LIEBERMAN-ROSS
LOWER CONFIDENCE LIMIT PROCEDURES
ON SYSTEM RELIABILITY

by

Teddy Ray Maynard

March 1977

Thesis Advisor:

W. M. Woods

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COMPARISON OF LOG-GAMMA AND LIEBERMAN-ROSS
LOWER CONFIDENCE LIMIT PROCEDURES ON SYSTEM RELIABILITY

by

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Lieutenant, United States Navy
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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

This thesis is an accuracy study of three empirical series system lower confidence limit procedures. Computer simulations were used to compare the accuracy of the procedures using the same set of data with 500 replications for each case. Modifications were then made to improve the accuracy of the Log-Gamma method.

The systems simulated had reliabilities ranging from .6 to .95. They were composed of four, fifteen, or forty components, and had component sample sizes of ten, twenty, or fifty.

The Lieberman-Ross method was accurate but had high variance. The Log-Gamma method was conservative with small sample sizes, but became more accurate with increased testing. Randomization of the Log-Gamma method was the most successful of the modifications attempted.

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I. INTRODUCTION

There exist many lower confidence limit procedures for series system reliability which are based on component data. All of them are approximate in one way or another for realistic testing situations. Two such procedures are the Lieberman-Ross (LR) procedure and the Log-Gamma (LG) procedure. The LR lower confidence limit (LCL) is based on operating time data and assumes time to failure is exponentially distributed. The LG LCL is based on attributes data and thus is non-parametric. A modification of the LG procedure, called the Continuous Log-Gamma (CLG) procedure, is based on time data and assumes an exponential failure distribution.

In this thesis the accuracy of the LR, LG, and CLG procedures are compared when data is generated from the exponential distribution for serial systems with four, fifteen, and forty components. Sample sizes are generated in a manner which is somewhat representative of data accumulated from missile flight tests.

The accuracy analysis was extended to the case where all components were assumed to have a Weibull distribution with increasing and decreasing failure rates.

In all cases, the LR and LG procedures were slightly modified from their more common forms in order to accommodate zero failures on all components.

Since the component test data was generated using the exponential assumption, the LR method enjoyed an advantage over the LG method in that continuous LCL procedures are exact while discrete LCL procedures are not. All discrete 100(1- α)% LCL's, $\hat{R}_{S,L(\alpha)}$, for system reliability R_S have the property that

$$P[\hat{R}_{S,L(\alpha)} \leq R_S] \geq 1 - \alpha$$

For some values of R_S , this probability is 100%. Thus discrete LCL's will usually be lower in value than a continuous LCL using the same set of data. Discrete LCL's have the advantage, however, of remaining the same regardless of the underlying distribution. Consequently, they are not susceptible to inaccuracies due to a mistaken assumption about the probability distribution of the operating component time data as are continuous procedures. That is, if $\hat{R}_{S,L(\alpha)}$ is discrete

$$P[\hat{R}_{S,L(\alpha)} \leq R_S] \geq 1 - \alpha$$

regardless of the probability distribution of the component time data, whereas if $\hat{R}_{S,L(\alpha)}^*$ is continuous

$$P[\hat{R}_{S,L(\alpha)}^* \leq R_S] = 1 - \alpha$$

provided the required assumption made about the probability distribution of the component failure data is correct. Even if the assumed continuous distribution is not correct,

however, $\hat{R}_{S,L}^*(\alpha)$ may still be reasonably accurate, depending upon the true component failure distribution. Of course these distributions are never known.

Finally, continuous LCL's required more refined data than discrete LCL's. Discrete LCL procedures can be used if continuous time data is available but continuous LCL procedures cannot be employed with only attributes data.

II. DESCRIPTION OF THE METHODS

A. THE LOG-GAMMA METHOD

The following development is a summary of that given in Ref. 1. The Log-Gamma (LG) method requires only attributes data, and assumes nothing about the failure distribution. It is designed for independent series systems, does not require equal component sample sizes, and is approximate. A modification of the LG method for systems composed of a mixture of series and parallel-connected components was explored by Ref. 2.

Suppose a system consists of K components in series, and that N_i copies of component i are put on test and operated until failure or completion of mission, whichever occurs first. Denote F_i as the number of components of type i that did not complete the mission. Then the maximum likelihood estimate for system reliability is

$$\hat{R}_s = \prod_{i=1}^K \hat{R}_i \quad (2.1)$$

where
$$\hat{R}_i = \frac{N_i - F_i}{N_i} \quad (2.2)$$

In the LG procedure the method of moments is used to fit the random variable $-\ln \hat{R}_s$ with the two-parameter gamma distribution. The gamma is then transformed into a chi

square distribution, about which probability statements are made and a lower confidence limit (LCL) obtained.

That is, define

$$S = -\ln R_S = -\ln \prod_{i=1}^K R_i = -\sum_{i=1}^K \ln(1-Q_i) \quad (2.3)$$

$$\approx \sum_{i=1}^K (Q_i + \frac{1}{2}Q_i^2) \equiv \sum_{i=1}^K T_i \quad \text{(See Appendix A)} \quad (2.4)$$

An unbiased estimator \hat{T}_i for T_i is

$$\hat{T}_i = A_i \hat{Q}_i + B_i \frac{\hat{Q}_i^2}{2} \quad (2.5)$$

where $A_i = \frac{2N_i - 3}{2N_i - 2}$ (2.6)

$$B_i = \frac{N_i}{N_i - 1} \quad (2.7)$$

and $\hat{Q}_i = \frac{F_i}{N_i} \quad i = 1, K$ (See Appendix B)

That \hat{T}_i is unbiased is important, because

$$\hat{S} = \sum_{i=1}^K \hat{T}_i \quad (2.8)$$

is used as an estimator for S , thereby accumulating any bias present in the \hat{T}_i . An approximate value for the variance of \hat{S} is

$$\begin{aligned} \text{Var}(\hat{S}) &= \sum_{i=1}^K \text{Var}(\hat{T}_i) \\ &\approx \sum_{i=1}^K \frac{\hat{T}_i}{N_i} \end{aligned} \quad (2.9)$$

Appendix C derives the actual variance of \hat{T}_i and shows equation (2.9) is an excellent approximation for $\text{Var}(\hat{S})$.

Next, fit \hat{S} with a gamma distribution. The probability distribution of \hat{S} is then given by the density function

$$f_{\hat{S}}(x; r, \theta) = \frac{1}{\Gamma(r) \theta^r} x^{r-1} \text{Exp}(-x/\theta) \quad \begin{array}{l} x \geq 0, r > 0 \\ \theta > 0 \end{array} \quad (2.10)$$

It follows that

$$E(\hat{S}) = r\theta \quad (2.11)$$

$$\text{Var}(\hat{S}) = r\theta^2 \quad (2.12)$$

Since \hat{S} is unbiased,

$$\begin{aligned} E(\hat{S}) &= S \\ &= \frac{K}{\sum_{i=1}^K T_i} \end{aligned} \quad (2.13)$$

Solving equations (2.9), (2.11), (2.12), and (2.13) simultaneously gives the shape parameter

$$r = \left(\frac{K}{\sum_{i=1}^K T_i} \right)^2 / \frac{K}{\sum_{i=1}^K \frac{T_i}{N_i}} \quad (2.14)$$

and the scale parameter

$$\theta = \left(\frac{K}{\sum_{i=1}^K \frac{T_i}{N_i}} \right) / \frac{K}{\sum_{i=1}^K T_i} \quad (2.15)$$

Thus, r can be estimated by

$$\hat{r} = \left(\frac{K}{\sum_{i=1}^K \hat{T}_i} \right)^2 / \frac{K}{\sum_{i=1}^K \frac{\hat{T}_i}{N_i}} \quad (2.16)$$

Since \hat{S} is distributed gamma (r, θ) , $\frac{2\hat{S}}{\theta}$ is distributed χ^2_{2r} .

Then

$$\begin{aligned} 1 - \alpha &= P\left(\frac{2\hat{S}}{\theta} \geq \chi^2_{1-\alpha, 2r}\right) \\ &= P(2r\hat{S} \geq r\theta \cdot \chi^2_{1-\alpha, 2r}) \end{aligned} \quad (2.17)$$

Since \hat{S} is unbiased

$$\begin{aligned} E(\hat{S}) &= S \\ &= r\theta \\ &= -\ln R_S \end{aligned} \quad (2.18)$$

Equation (2.17) becomes

$$\begin{aligned} 1 - \alpha &= P(2r\hat{S} \geq -\ln R_S \cdot \chi^2_{1-\alpha, 2r}) \\ &= P\left(\text{Exp}\left\{\frac{-2r\hat{S}}{\chi^2_{1-\alpha, 2r}}\right\} \leq R_S\right) \end{aligned} \quad (2.19)$$

Therefore, the LG 100(1- α)% LCL for system reliability R_S is

$$\hat{R}_{S,L(\alpha)} = \text{Exp}\left(\frac{-2r\hat{S}}{\chi^2_{1-\alpha, 2r}}\right) \quad (2.20)$$

To preclude usage of non-integer degrees of freedom, the approximation

$$\frac{[2\hat{r}]}{\chi^2_{1-\alpha, [2\hat{r}]}} \quad (2.21)$$

is used in equation (2.20), where $[2\hat{r}]$ denotes the smallest integer greater than or equal to $2\hat{r}$. Approximation (2.21) was shown to have little effect on the LG procedure accuracy [Ref. 2, p. 13]. However, it may produce monotonicity

problems under certain circumstances (discussed in Chapter V). Thus, the 100(1- α)% LCL becomes

$$\hat{R}_{S,L(\alpha)} = \text{Exp}\left(\frac{-[2\hat{r}] \hat{S}}{\chi^2_{1-\alpha}, [2\hat{r}]}\right) \quad (2.22)$$

and was the LG LCL initially simulated by this study (Chapter IV).

B. THE CONTINUOUS LOG-GAMMA METHOD

The Continuous Log-Gamma (CLG) method is developed in detail by Ref. 3, pages C-1 through C-20 (as the "Classical" method). It differs from the LG method only in that it requires component time data and assumes exponential failures.

The derivation of the CLG procedure is analogous to that of the LG procedure, and is outlined as follows.

Define

$$\begin{aligned} Z &\equiv -\ln R_s \\ &= -\ln \prod_{i=1}^K R_i \end{aligned} \quad (2.23)$$

Using time data and assuming exponentiality, R_i can be estimated by

$$\hat{R}_i = \text{Exp}\left(-\frac{F_i}{T_{\text{sum}_i}}\right) \quad i=1, K$$

where T_{sum_i} is the accumulated operating time of the i th component. Then Z is estimated by

$$\hat{Z} = \sum_{i=1}^K \frac{F_i}{T_{\text{sum}_i}} \quad (2.24)$$

The method of moments gamma fit to \hat{Z} uses

$$E(\hat{Z}) = \sum_{i=1}^K \frac{F_i}{T_{sum_i}} \quad (2.25)$$

$$\widehat{Var}(\hat{Z}) = \frac{K}{\sum_{i=1}^K \left(\frac{F_i}{T_{sum_i}} \right) / N_i} \quad (2.26)$$

$$E(\hat{Z}) = L\lambda \quad (2.27)$$

and $Var(\hat{Z}) = L\lambda^2 \quad (2.28)$

where L is the shape parameter and λ is the scale parameter.

The resultant CLG $100(1-\alpha)\%$ LCL is

$$\hat{R}_{S,L}(\alpha) = \text{Exp} \left(\frac{-2\hat{L}\hat{Z}}{\chi^2_{1-\alpha, 2\hat{L}}} \right) \quad (2.29)$$

with

$$\hat{L} = \left(\sum_{i=1}^K \frac{F_i}{T_{sum_i}} \right)^2 / \sum_{i=1}^K \left(\frac{F_i}{T_{sum_i}} / N_i \right) \quad (2.30)$$

To preclude non-integer degrees of freedom

$$\hat{R}_{S,L}(\alpha) = \text{Exp} \left(\frac{-[2\hat{L}]\hat{Z}}{\chi^2_{1-\alpha, [2\hat{L}]}} \right) \quad (2.31)$$

where $[2\hat{L}]$ is the smallest integer greater than or equal to $2\hat{L}$.

C. THE LIEBERMAN-ROSS METHOD

1. The Original Procedure

The Lieberman-Ross (LR) method was designed as an exact lower confidence limit (LCL) procedure for independent series systems (see Ref. 4). It requires component times between successive failures from an assumed exponential failure distribution, and does not require equal component sample sizes.

The test plan may be described as follows. Suppose N_i copies of each component type i are available for testing. Place a copy of the i th component on test, and replace it when it fails. Continue testing with replacement until all available copies of one component are exhausted (this procedure need not be conducted simultaneously for all component types). For the i th component, denote the time to the first failure as T_{i1} , between the first and second failure as T_{i2} , and so forth. In general, the time between the j th and $(j-1)$ th failure on the i th component is T_{ij} , $i = 1, \dots, K$
 $j = 1, \dots, N_i$

where the T_{ij} are in mission units. Reference 4, p. 840, also describes the LR procedure for Type II censoring at the r th failure of R items on simultaneous test. Reference 3, p. C-32, illustrates the procedure using component times between successive failures and the observed success time of the last copy of each type tested.

The LR LCL is computed as follows. Define

$$U \equiv \min_{1 \leq i \leq K} \left(\sum_{j=1}^{N_i} T_{ij} \right) \quad (2.32)$$

Let I_i be the greatest index such that

$$\sum_{j=1}^{I_i} T_{ij} \leq U \quad i = 1, \dots, K \quad (2.33)$$

Also define

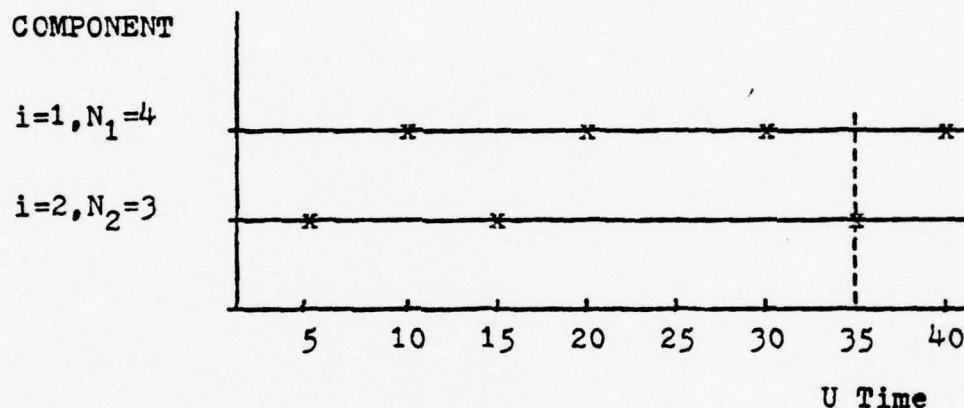
$$I \equiv \sum_{i=1}^K I_i \quad (2.34)$$

The variable I is the total number of failures in all components up to and including U (see Figure 1). Then the LR $100(1-\alpha)\%$ LCL for system reliability R_s is given by

$$\hat{R}_{S,L(\alpha)} = \text{Exp}\left(\frac{-\chi^2_{\alpha, 2I}}{2U}\right) \quad (2.35)$$

Figure 1

Depiction of Lieberman-Ross Parameters
 $U=35$ and $I=6$, when $K=2$, $N_1=4$, and $N_2=3$



To implement the LR procedure, the component data order must be specified. If only a set of component data were available, then a different procedure would have to be used. Note that the opposite chi square tail is used from that of the LG and CLG methods.

2. The Modified Lieberman-Ross Method

The original procedure utilizes accumulated component times between failures. The Modified Lieberman-Ross procedure (MLR), uses instead component times to one mission unit or failure, whichever occurs first. The MLR method is thereby more readily applied than the LR method to actual

system test data, especially when testing is destructive (for example, a missile firing). The MLR method also can be used when no failures occur, whereas the LR method could not (see Appendix D).

The MLR procedure is

a. Suppose N_i copies of the i th component are available for testing. Define T_{ij} as for the LR procedure, except all times past one mission unit are truncated to one. Next, define

$$T_{i1}^* = \sum_{j=1}^{i_1} T_{ij} \quad \text{where } i_1 \text{ is the index of the first component time less than one mission unit for the } i\text{th component} \quad (2.36)$$

$$T_{i2}^* = \sum_{j=i_1+1}^{i_2} T_{ij} \quad \text{where } i_2 \text{ is the index of the second component time less than one mission unit for the } i\text{th component} \quad (2.37)$$

$$\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ T_{iF_i}^* = \sum_{j=i_{(F_i-1)+1}}^{i_{F_i}} T_{ij} \end{array} \quad \begin{array}{l} \text{where } i_{F_i} \text{ is the index of} \\ \text{the last component time} \\ \text{less than one mission unit} \\ \text{for the } i\text{th component} \end{array} \quad (2.38)$$

Then obtain

$$T_i^* = \sum_{j=1}^{F_i} T_{ij}^* \quad i = 1, \dots, K \quad (2.39)$$

b. Compute the minimum accumulated times between successive failures in any component type.

$$U = \begin{cases} \min_{1 \leq i \leq K} T_i^* & \text{if } \sum_{i=1}^K F_i > 0 \\ \min_{1 \leq i \leq K} N_i & \text{if } \sum_{i=1}^K F_i = 0 \end{cases} \quad \begin{array}{l} (2.40) \\ \text{(see Appendix D)} \end{array}$$

The variable I_i is now defined in terms of the T_i^* .

$$I_i \equiv \begin{cases} \text{the greatest index such that} \\ \sum_{j=1}^{I_i} T_{ij}^* \leq U & \text{if } F_i > 0 \\ 0 & \text{if } F_i = 0 \end{cases} \quad (2.41)$$

Then

$$I \equiv \max \left\{ 1, \sum_{i=1}^K I_i \right\} \quad (2.42)$$

c. Now the MLR $100(1-\alpha)\%$ LCL for R_S can be computed

$$\hat{R}_{S,L}(\alpha) = \text{Exp} \left\{ \frac{-\chi_{\alpha, 2I}^2}{2U} \right\} \quad (2.43)$$

Like the LR method, the component data order must be specified.

III. THE SIMULATION

A. GENERAL METHODOLOGY

Any lower confidence limit (LCL) procedure can be evaluated by computer simulation of its distribution as follows:

1. Suppose it is desired to evaluate a proposed $100(1-\alpha)\%$ LCL procedure, denoted $\hat{R}_{S,L(\alpha)}$, for system reliability R_S . Then the assertion is

$$P[\hat{R}_{S,L(\alpha)} \leq R_S] \geq 1 - \alpha \quad (3.1)$$

Equality should hold if $\hat{R}_{S,L(\alpha)}$ is a continuous random variable.

2. $R_S = f(R_1, R_2, \dots, R_K)$, where R_i is the true reliability of the i th component. For an independent series system,

$R_S = \prod_{i=1}^K R_i$. If exponentiality is assumed, an equivalent statement is $R_S = f(\lambda_1, \lambda_2, \dots, \lambda_K)$.

3. Assign values to the parameters α , K , and R_i (or λ_i), $i = 1, 2, \dots, K$. Simulate life testing by generating random component times-to-failure, and compute the resultant

$\hat{R}_{S,L(\alpha)}$.

4. Generate the approximate distribution of $\hat{R}_{S,L(\alpha)}$ by repeating step 3 five hundred times.

5. Order the $\hat{R}_{S,L(\alpha)}$ realizations to get $\hat{R}_{S,L(\alpha)(1)}$, \dots , $\dots, \hat{R}_{S,L(\alpha)(500)}$.

6. Find the $(1-\alpha)$ 500th order statistic of $\hat{R}_{S,L(\alpha)}$ and denote it $A_{1-\alpha}$. Thus $A_{1-\alpha}$ is the $(1-\alpha)$ th percentile of the distribution of the LCL random variable $\hat{R}_{S,L(\alpha)}$. If in fact, $\hat{R}_{S,L(\alpha)}$ is an exact 100(1- α)% LCL procedure for R_s , then $A_{1-\alpha}$ should be approximately equal to R_s .

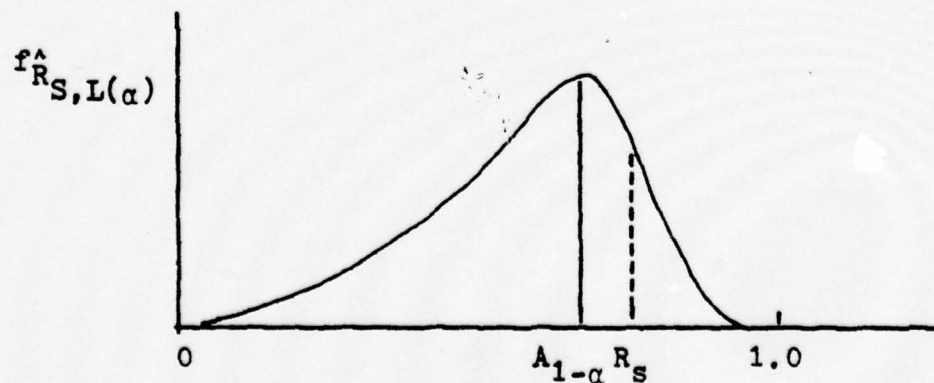
7. By repeating steps 3 through 6 for various sets of $K, N_1, N_2, \dots, N_K, R_1, R_2, \dots, R_K$ and α , and comparing the resultant $A_{1-\alpha}$ to R_s , the overall performance of the LCL procedure can be evaluated.

The actual confidence level given by $\hat{R}_{S,L(\alpha)}$ can be obtained by finding the order statistic of the generated distribution which matches (or closest matches) R_s . If the index of this order statistic is denoted i^* , then $(100\%) \frac{i^*}{500} = \text{Actual Level of Confidence of } \hat{R}_{S,L(\alpha)}$ (3.2)

Equivalently, if $A_{1-\alpha} < R_s$, the procedure is a conservative one, and vice-versa.

Figure 2 depicts a conservative outcome with $A_{1-\alpha} < R_s$. The degree to which $A_{1-\alpha}$ equals R_s reflects the validity of equation (3.1) and therefore the accuracy of $\hat{R}_{S,L(\alpha)}$.

Figure 2



Not only should an LCL procedure be accurate, but also should have small standard deviation. Even though $\hat{R}_{S,L}(\alpha)$ may be an exact LCL procedure, its practical value is diminished if it produces widely dispersed LCL's. This study used both criteria for comparison of the three methods.

B. GENERATION OF SAMPLE SIZES

The component test data are generated in a manner somewhat representative of data accumulated in missile flight tests. Suppose some number of series systems are to be used for testing. Failure of a component may fail the system, but still allow testing on components upstream of the failed one. For example, if the system is a five-stage missile, failure of the third stage may or may not allow testing on stages four and five.

Input parameter N_1 , the number of mission tests on the first component, is the number of systems to be tested. A component failure is assumed to preclude testing on upstream components one-half the time. The simulation computes the number of tests on the i th component as

$$N_i = N_1 - \left[\frac{i-1}{2} \sum_{j=1}^{i-1} F_j \right] \quad (3.3)$$

where " $[]$ " denotes "the smallest integer greater than or equal to". To illustrate, let $N_1=20$ and assume the first component had three failures. Then the component two sample size becomes

$$N_2 = 20 - \left[\frac{1}{2}(3) \right] = 18 \text{ mission trials}$$

It should be understood, however, that the three procedures themselves do not require this scheme, but are valid for varying sample sizes.

C. THE SIMULATION ALGORITHM

Step one: Initialize the number of series components K , the number of systems to be tested N_1 , the true reliability of the i th component R_i , $i=1,2,\dots,K$, and the number of replications desired ("Do Loop" index).

Step two: Using the probability integral transform (PIT) method [Ref. 5, p. 168], generate exponential failure times for the N_1 copies of the first component as shown below. Exponential failure data is a requirement for the CLG and LR methods. The LG method, being non-parametric, can use the same data.

a. Generate N_1 uniform (0,1) random numbers for component one. Denote these $U_{11}, U_{12}, \dots, U_{1N_1}$

$$b. \text{ Let } F_{1j} = \begin{cases} 0 & \text{if a success occurred for component one on trial } j \\ 1 & \text{if component one failed on trial } j, j=1,2,\dots,N_1 \end{cases}$$

Appendix E shows

$$F_{1j} = \begin{cases} 0 & \text{if } 0 \leq U_{1j} < \text{Exp}(-\lambda_1) \\ 1 & \text{if } \text{Exp}(-\lambda_1) \leq U_{1j} \leq 1 \end{cases} \quad (3.4)$$

$$\text{and } T_{1j} = \begin{cases} 1 & \text{if } F_{1j} = 0 \\ \frac{-\ln(U_{1j})}{\lambda_1} & \text{if } F_{1j} = 1 \quad j=1,2,\dots,N_1 \end{cases} \quad (3.5)$$

c. For the LG method, compute

$$A_1 = \frac{(2N_1 - 3)}{(2N_1 - 2)} \quad (3.6)$$

$$B_1 = \frac{N_1}{N_1 - 1} \quad (3.7)$$

and
$$F_1 = \sum_{j=1}^{N_1} F_{1j} \quad (3.8)$$

d. For the CLG method, compute the test time for component one.

$$T_{\text{sum}_1} = \sum_{j=1}^{N_1} T_{1j} \quad (3.9)$$

Step three: Compute

$$\hat{T}_1 = A_1 \left(\frac{F_1}{N_1} \right) + \frac{B_1}{2} \left(\frac{F_1}{N_1} \right)^2 \quad (3.10)$$

for the LG method, and

$$\hat{Z}_1 = \frac{F_1}{T_{\text{sum}_1}} \quad (3.11)$$

for the CLG method.

Step four: The component test times for the MLR method are defined differently from that of the CLG method. The MLR method uses only accumulated time to the last failure for each component; if no failures occur this is zero.

Thus, compute for component one

$$T_1^* = \sum_{j=1}^{N_1} T_{1j}^* \quad \text{if } F_1 > 0 \quad (3.12)$$

where T_{ij}^* is as defined on page 20.

Step five: Compute the number of trials on component two,

$$N_2 = N_1 - \left[\frac{1}{2} F_1 \right]$$

Repeat steps two, three, and four for component two. Then compute $N_3 = N_1 - \left[\frac{1}{2} (F_1 + F_2) \right]$. In general, the sample size for the i th component is

$$N_i = N_1 - \left[\frac{1}{2} \sum_{j=1}^{i-1} F_j \right] \quad (3.13)$$

Step six: Continue alternating through steps two through five for the remainder of the components.

Step seven: Obtain

$$\hat{S} = \sum_{i=1}^K \hat{T}_i \quad (3.14)$$

$$\hat{Z} = \sum_{i=1}^K \hat{Z}_i \quad (3.15)$$

$$\widehat{\text{Var}}(\hat{S}) = \sum_{i=1}^K (\hat{T}_i / N_i) \quad (3.16)$$

$$\widehat{\text{Var}}(\hat{Z}) = \sum_{i=1}^K (\hat{Z}_i / N_i) \quad (3.17)$$

$$\hat{r} = \hat{S}^2 / \widehat{\text{Var}}(\hat{S}) \quad (3.18)$$

$$\hat{L} = \hat{Z}^2 / \widehat{\text{Var}}(\hat{Z}) \quad (3.19)$$

and the total number of replica failures

$$FT \equiv \sum_{i=1}^K F_i \quad (3.20)$$

Step eight: To preclude non-integer degrees of freedom, denote

$$[2\hat{r}] = \max \{2, \text{smallest integer} \geq 2\hat{r}\} \quad (3.21)$$

$$[2\hat{L}] = \max \{2, \text{smallest integer} \geq 2\hat{L}\} \quad (3.22)$$

Step nine: For the MLR procedure, compute

$$U = \begin{cases} \min N_i & \text{if } FT = 0 \\ \min T_i^* & \text{otherwise, } i=1,2,\dots,K \end{cases} \quad (3.23)$$

and $I_i = \begin{cases} \text{greatest index such that} \\ \sum_{j=1}^{I_i} T_{ij}^* \leq U & \text{if } F_i > 0 \\ 0 & \text{if } F_i = 0 \end{cases} \quad (3.24)$

then obtain $I = \max \left\{ 1, \sum_{i=1}^K I_i \right\} \quad (3.25)$

The zero failure portion of the U computation sets U to the observed number of "system" successes (see Appendix D).

Step ten: Compute the replica one LCL's for the three procedures. Define RD, RT, and RL for discrimination as below

a. LG method

$$\hat{R}_{S,L}(\alpha) \equiv RD = \text{Exp} \left\{ \frac{-[2\hat{r}] \hat{S}}{\chi_{1-\alpha, [2\hat{r}]}} \right\} \quad (3.26)$$

b. CLG method

$$\hat{R}_{S,L}(\alpha) \equiv RT = \text{Exp} \left\{ \frac{-[2\hat{L}] \hat{Z}}{\chi_{1-\alpha, [2\hat{L}]}} \right\} \quad (3.27)$$

c. MLR method

$$\hat{R}_{S,L}(\alpha) \equiv RL = \text{Exp} \left\{ \frac{-\chi_{\alpha, 2I}^2}{2U} \right\} \quad (3.28)$$

Step eleven: Replicate steps two through ten 500 times to obtain

RD₁, RD₂,, RD₅₀₀
 RT₁, RT₂,, RT₅₀₀
 RL₁, RL₂,, RL₅₀₀
 FT₁, FT₂,, FT₅₀₀

Step twelve: Order the LCL's in step eleven.

RD₍₁₎, RD₍₂₎,, RD₍₅₀₀₎
 RT₍₁₎, RT₍₂₎,, RT₍₅₀₀₎
 RL₍₁₎, RL₍₂₎,, RL₍₅₀₀₎

Step thirteen: Next, compute A_{1-α} for the three procedures.

RD_{(500(1-α))}
 RT_{(500(1-α))}
 RL_{(500(1-α))}

and the average number of failures per replica

$$\hat{T} = \frac{1}{500} \sum_{j=1}^{500} FT_j \quad (3.29)$$

Each procedure's sample variance is computed by

$$s^2 = \frac{1}{499} \sum_{j=1}^{500} \left[(\hat{R}_{S,L(\alpha)})_j - \overline{(\hat{R}_{S,L(\alpha)})} \right]^2 \quad (3.30)$$

where $\overline{(\hat{R}_{S,L(\alpha)})}$ denotes the average LCL over the 500 replicas. Now each procedure's accuracy can be determined by comparing A_{1-α} to true system reliability R_s, and relative dispersion by comparing s.

Reference 1, the original LG method paper, used the quantity

$$TT = \sum_{i=1}^K N_i Q_i \quad (3.31)$$

as a measure of accuracy and the amount of testing necessary to assure sufficient accuracy of the LG procedure. As long as no component sample size is below ten, Table I shows recommended TT for the LG method (Reproduced from Ref. 1, p. 19).

TABLE I
Recommended TT for the Log-Gamma Method

100(1- α)%	Minimum TT	Reasonable TT
80	3	4
90	8	10
95	15	20

The simulation computes equation (3.29) for comparison with the above Table.

IV. THE INITIAL SIMULATION

A. CASE DESCRIPTION

The initial cases analyzed had true system reliabilities ranging from near .70 upwards and were composed of fifteen or forty components. The average number of failures per replica, $\hat{T}\hat{T}$, ranged up to near 15. The first simulation used the formulas for $\hat{R}_{S,L}(\alpha)$ on page 28. The results are listed in Table III. Two cases were also ran with a Weibull failure distribution, vice exponential, to test the CLG and MLR procedures' sensitivity to the exponential assumption. Appendix E develops the PIT method for generating exponential and Weibull failure data. Table IV lists Weibull results.

B. RESULTS AND ANALYSIS

The LG method conformed to Reference 1 assertions in that performance improved with increasing $\hat{T}\hat{T}$, although Ref. 1 did not generate component sample sizes in the same manner as this study. This should, however, have no effect on the results.

Small $\hat{T}\hat{T}$, such as in cases one and three, resulted in conservative LG and CLG LCL's (ie., $A_{1-\alpha} < R_s$).

TABLE II

NOMENCLATURE FOR SIMULATION RESULTS TABLES

N_1	:	Sample size of 1st component; analogous to the number of systems tested
K	:	Number of series connected components
R_i	:	True reliability of the i th component
R_s	:	True independent series system reliability $R_s = \prod_{i=1}^K R_i$
min	:	Minimum number of failures in one replica
max	:	Maximum number of failures in one replica
\hat{TT}	:	Average number of failures per replica; approximation for $TT = \sum_{i=1}^K N_i Q_i$
s	:	Sample standard deviation of $\hat{R}_{S,L(\alpha)}$
$A_{1-\alpha}$:	The $500(1-\alpha)$ th order statistic of $\hat{R}_{S,L(\alpha)}$ for each of the three methods
LG	:	The Log-Gamma method
CLG	:	The Continuous Log-Gamma method
MLR	:	The Modified Lieberman-Ross method
RLG	:	The Randomized Log-Gamma method (modification three only)

For example, the case one sample size for the first component was $N_1 = 20$, and the series system modeled had $K = 15$ components each with true reliability $R_i = .99$. True system reliability $R_s = .860$. There were as few as zero ($\text{min} = 0$) and as many as eight ($\text{max} = 8$) component failures

TABLE III
Initial Simulation Results

CASE	\hat{TT}	R_s	80% ICL's $A_{1-\alpha/s}$			90% ICL's $A_{1-\alpha/s}$		
			LG	CIG	MLR	LG	CIG	MLR
			Case 1: $N_1=20$ $K=15, R_1=.99$ $i=1, 15$ $\min=0, \max=8$	2.9	.860	.799/.074	.802/.086	.851/.255
Case 2: $N_1=50$ $K=15, R_1=.99$ $i=1, 15$ $\min=1, \max=17$	7.2	.860	.851/.052	.850/.054	.861/.253	.829/.050	.837/.054	.855/.264
Case 3: $N_1=20$ $K=15, R_1=.99$ $i=1, 14; R_{15}=.995$ $\min=0, \max=8$	2.8	.864	.799/.073	.803/.085	.854/.254	.680/.078	.770/.088	.856/.263
Case 4: $N_1=50$ $K=15, R_1=.99$ $i=1, 14; R_{15}=.995$ $\min=1, \max=16$	7.0	.864	.851/.051	.850/.053	.866/.251	.838/.049	.837/.054	.857/.262
Case 5: $N_1=20$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ $\min=0, \max=11$	4.3	.792	.749/.096	.749/.104	.776/.224	.680/.093	.722/.104	.773/.230
Case 6: $N_1=50$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ $\min=3, \max=20$	10.5	.792	.787/.058	.784/.059	.794/.225	.775/.058	.773/.059	.796/.235

TABLE III
(continued)

CASE	\hat{T}_T	R_g	80% ICL's $A_{1-\alpha/s}$			90% ICL's $A_{1-\alpha/s}$		
			LG	CLG	MLR	LG	CLG	MLR
Case 7: $N_1=20$ $K=15, R_1=.85$ $R_1=.99; i=2, 15$ $\min=0, \max=13$	5.5	.738	.703/.089	.698/.095	.718/.245	.677/.081	.676/.090	.708/.240
Case 8: $N_1=50$ $K=15, R_1=.85$ $R_1=.99; i=2, 15$ $\min=5, \max=29$	13.7	.738	.730/.061	.728/.062	.732/.235	.733/.061	.730/.062	.746/.237
Case 10: $N_1=50$ $K=40, R_1=.995$ $i=1, 40$ $\min=1, \max=18$	9.4	.818	.810/.056	.808/.058	.810/.227	.795/.056	.794/.058	.806/.278

TABLE IV
Initial Results: Weibull Sensitivity Analysis

CASE	\hat{T}	R_s	80% ICL's $A_{1-\alpha/s}$			90% ICL's $A_{1-\alpha/s}$		
			LG	CLG	MLR	LG	CLG	MLR
Case 1 with $\beta = .5$ min=0, max=8	2.9	.860	.799/.074	.801/.087	.851/.266	.680/.078	.767/.090	.851/.271
Case 1 with $\beta = 3.0$ min=0, max=8	2.9	.860	.799/.074	.803/.085	.851/.233	.680/.078	.772/.088	.856/.249

over the 500 replicas of case one, with the average number of failures per replica being 2.9 ($\widehat{TT} = 2.9$). Comparing $A_{1-\alpha}$ to R_g for accuracy, the LG 80% procedure had $A_{1-\alpha} = .799$ and $s = .074$, whereas the MLR had $A_{1-\alpha} = .851$ and $s = .255$. Thus, in case one both procedures were conservative, but the MLR was more accurate. However, the LG procedure had less than one-third the dispersion of the MLR procedure.

Increased testing, as in cases two and four, not only improved their accuracy but showed smaller sample standard deviation. The same comparison holds for case five with case six, and case seven with case eight. Also note the closeness of both $A_{1-\alpha}$ and s for the LG and CLG methods. This basically says that even though more assumptions are required and information is needed for the CLG method than the LG method, it yields no better results.

Case three of Ref. 1 is essentially case five of Table III; case five is essentially case six, and case eleven is essentially case two (note that Ref. 1 miscomputed TT in cases three and five). The accuracy achieved by case five was not as high as that of case three, but the sample standard deviation was comparable. In the other two comparable cases, both the accuracy and dispersion match closely.

The MLR procedure is the most accurate of the three, both in low and high \widehat{TT} cases. The sample standard deviation, however, is three to five times that of the LG method. The most disturbing thing about the MLR method is that even with increased testing (\widehat{TT}), s is still high.

Reference 6 shows that the LR method is expected to have high dispersion above about ten observed failures (Figure 1, p. 22). The MLR method, even in the cases with ten or more failures, still has much higher dispersion than the LG and CLG procedures. Reference 6 attempted to improve the LR method by reduction of data loss. By ordering the failure times in increasing order, maximum data usage and chi square degrees of freedom were attained. The resultant LCL's were found to be biased in an unpredictable direction (based on a single parameter) and were even more widely dispersed. It was therefore proposed that all possible orderings of the data be considered, and research is currently being conducted in that area. Thus this study pursued it no further.

The two Weibull cases showed that the CLG and MLR methods were slightly sensitive to the exponential assumption. Since $T \sim \text{exponential}(\lambda)$ implies $T^{1/\beta} \sim \text{Weibull}(\beta, \lambda)$ (see Appendix E), then for $\beta=.5$ and $T < 1$ mission unit, the generated Weibull failure time is sooner than its exponential counterpart. Likewise, for $\beta=3.0$, the Weibull time is later. In fact, case one with $\beta=.5$ gave slightly lower $A_{1-\alpha}$ for both methods (excepting the 80% MLR LCL). With $\beta=3.0$, both cases gave slightly higher $A_{1-\alpha}$ (again, with the same exception as above).

V. MODIFICATIONS OF THE METHODS

A. MODIFICATION ONE: EXACT DEGREES OF FREEDOM AND ADJUSTMENT FOR ZERO FAILURES

1. Description

The original LG method employed only integer degrees of freedom; the initial results of Chapter IV adhered to this convention for all three methods. The accessibility of computers today makes chi square values with non-integer degrees of freedom readily available. Rounding up degrees of freedom by usage of the "smallest integer greater than or equal to" function can also cause monotonicity problems in $\hat{R}_{S,L}(\alpha)$. Consider LG 80% LCL's in the case where $N_1=20$, and only one failure occurs. Then

$$\hat{S} = \frac{\sum_{i=1}^{K=15} \hat{T}_i}{15} \quad (5.1)$$

$$= .05$$

$$\hat{r} = \frac{\frac{\hat{S}^2}{15}}{\sum_{i=1}^{15} \frac{1}{N_i}} \quad (5.2)$$

$$= \frac{(.05)^2}{(.05/20)} = 1$$

then $2\hat{r} = 2$

$$[2\hat{r}] = 2$$

thus, $\hat{R}_{S,L}(\alpha) = \text{Exp} \left\{ \frac{-[2\hat{r}]\hat{S}}{\chi_{1-\alpha, [2\hat{r}]}} \right\} \quad (5.3)$

$$\hat{R}_{S,L(.2)} = \text{Exp} \left\{ \frac{-2(.05)}{\chi^2_{.8,2}} \right\} = .799$$

Now, take the exact same case and outcome, except that instead of one failure in one component, two failures occur in the same component. Then,

$$\hat{S} = .103$$

$$\hat{r} = \frac{(.103)^2}{(.103/20)} = 2.053$$

$$2\hat{r} = 4.105$$

$$[2\hat{r}] = 5$$

It follows that,

$$\hat{R}_{S,L(.2)} = \text{Exp} \left\{ \frac{-5(.103)}{\chi^2_{.8,5}} \right\} = .803$$

The above LCL is higher even though one more failure occurred, due to rounding up degrees of freedom $2\hat{r}$.

Another improvement felt necessary was an adjustment for the event of no failures for the LG method. The initial procedure here gives

$$\hat{S} = 0$$

and $\hat{R}_{S,L(\alpha)} = 1.00$

Therefore, the following was incorporated: Suppose there are no failures in one replica with N_1 initial component trials. Then set the number of failures on the first component as $F_1 = \frac{1}{2}$ and compute $2\hat{r}$ as if one failure had occurred (this effectively constrains $2\hat{r} \geq 2$). This ad hoc and simplistic adjustment will produce zero-failure LCL's as a function of N_1 , and satisfies monotonicity requirements.

The simulation was again run with the adjustments above for certain test cases. The results are listed in Table VI.

2. Observations and Analysis

Due to the fact that the MLR method never assumes non-integer degrees of freedom, it was unaffected by this modification. Both cases tested had slightly lower $A_{1-\alpha}$ and sample standard deviation, s for the LG and CLG methods. The smaller s can be attributed to less discreteness in $\hat{R}_{S,L}(\alpha)$. The lower $A_{1-\alpha}$ is caused by the fact that $\chi^2_{1-\alpha,2n}$ increases faster than $2n$. By rounding up the degrees of freedom $2n$ to $[2n]$, the initial simulation produced slightly higher LCL's than did modification one.

TABLE V

Behavior of $\frac{2n}{\chi^2_{1-\alpha,2n}}$ for $\alpha = .2$

$2n$	$\chi^2_{.8,2n}$	$\frac{2n}{\chi^2_{.8,2n}}$
2	.446	4.48
3	1.01	2.97
5	2.34	2.14
10	6.18	1.62
30	23.4	1.28

TABLE VI
SIMULATION RESULTS WITH MODIFICATION ONE
a) Exact Degrees of Freedom
b) A Zero-Failure Adjustment
for the Log-Gamma Method

CASE	\hat{T}	R_B	80% ICL's $A_{1-\alpha}/s$			90% ICL's $A_{1-\alpha}/s$		
			LG	CLG	MIR	LG	CLG	MIR
Case 1: $N_1=20$ $K=15, R_1=.99$ $i=1, 15$ $\min=0, \max=8$	2.9	.860	.783/.064	.779/.077	.851/.255	.680/.050	.679/.080	.853/.263
Case 5: $N_1=20$ $K=15, R_1=.995$ $i=1, 14; R_1=.85$ $\min=0, \max=8$	4.3	.792	.740/.088	.739/.098	.776/.224	.680/.074	.679/.093	.773/.230

While this modification slightly decreased dispersion, it did not improve the accuracy of the LG and CLG methods (and had no effect on the MLR method). The only real gain was to solve an infrequent monotonicity problem. If only integer degrees of freedom are available, it is recommended that a more refined scheme of rounding non-integer outcomes be devised. For instance, in the earlier example, $2\hat{r} = 4.105$ should be rounded down to 4 vice up to 5. The LG LCL becomes .779, which is lower than the one-failure .799 outcome, but still higher than the .738 three failure outcome (see Table VII).

Being that the CLG method performs no better than the LG method, but requires more assumptions and information, no further effort was expended on it. All subsequent modifications are directed towards improving the accuracy of the LG method while maintaining small dispersion.

B. MODIFICATION TWO

1. An Accuracy Improvement, and an Adjustment for the Degrees of Freedom in the 90% Lower Confidence Limit Procedure

The primary weakness of the LG method is its poor performance when few failures occur. More testing, and thereby more failures, results in a more accurate $A_{1-\alpha}$. Table VII lists and plots possible LG LCL's for zero through ten failures when $N_1 = 20$. Note the gaps caused by discreteness up to one failure. Also note that the 90% LCL for two and three failures is higher than that for one failure.

TABLE VII

LOG GAMMA 80 AND 90% LL(S)10-10 FAILURES

NI= 20. FAILURES	ONLY ONE FAILURE PER COMPONENT WHAT	80%LLC	90%LLC
0	2.0300	0.9114	0.7231
1	2.0300	0.7753	0.6476
2	3.0674	0.7482	0.6571
3	3.5189	0.6426	0.6165
4	6.5849	0.6076	0.5781
5	11.0677	0.6064	0.5381
6	13.9965	0.5665	0.5011
7	15.9252	0.5266	0.4637
8	17.5045	0.4899	0.4256
9	19.0510	0.4534	0.3956
10			

RCF (LL(")) AND 90% LL("*) VS. FAILURES

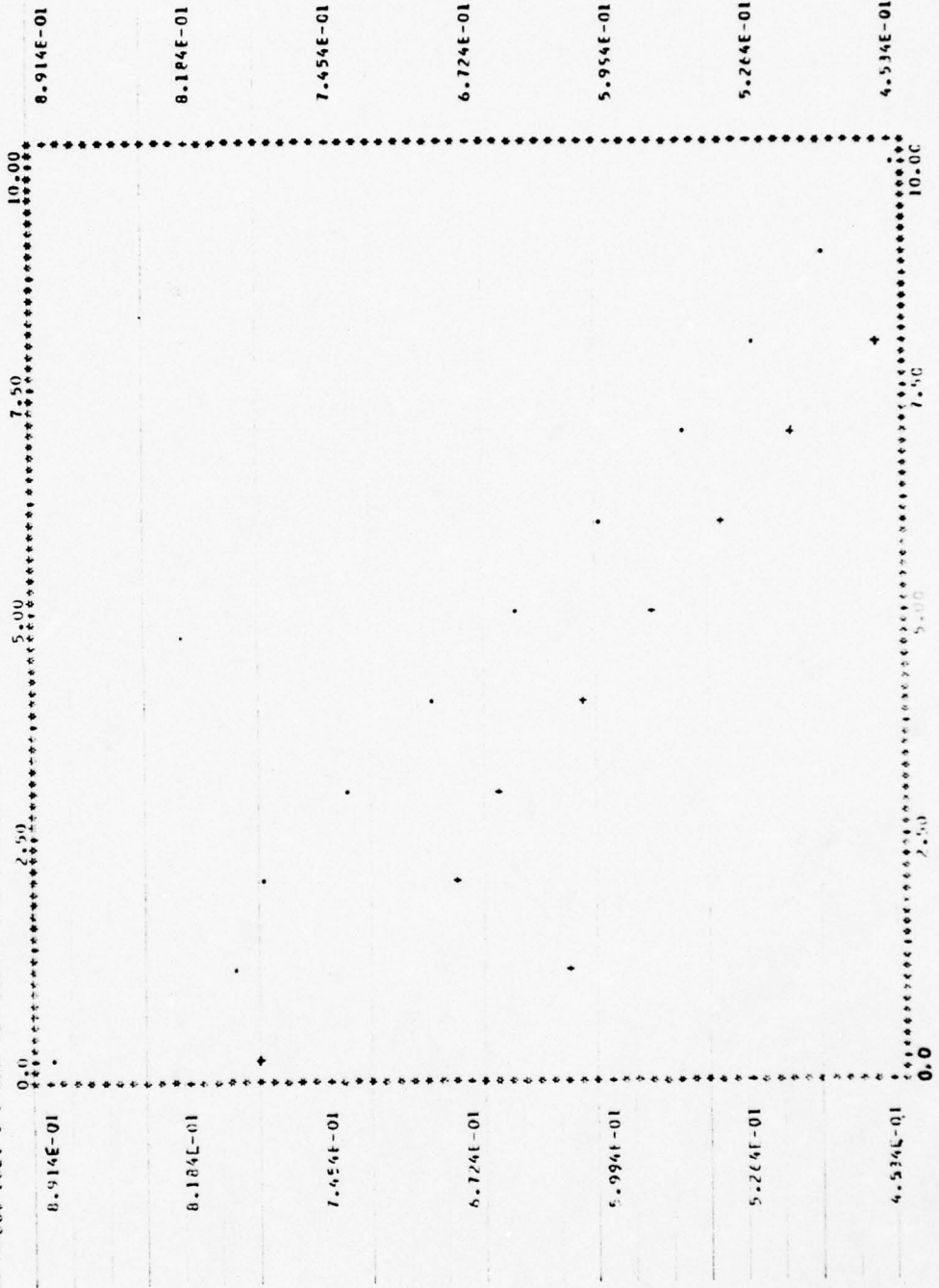
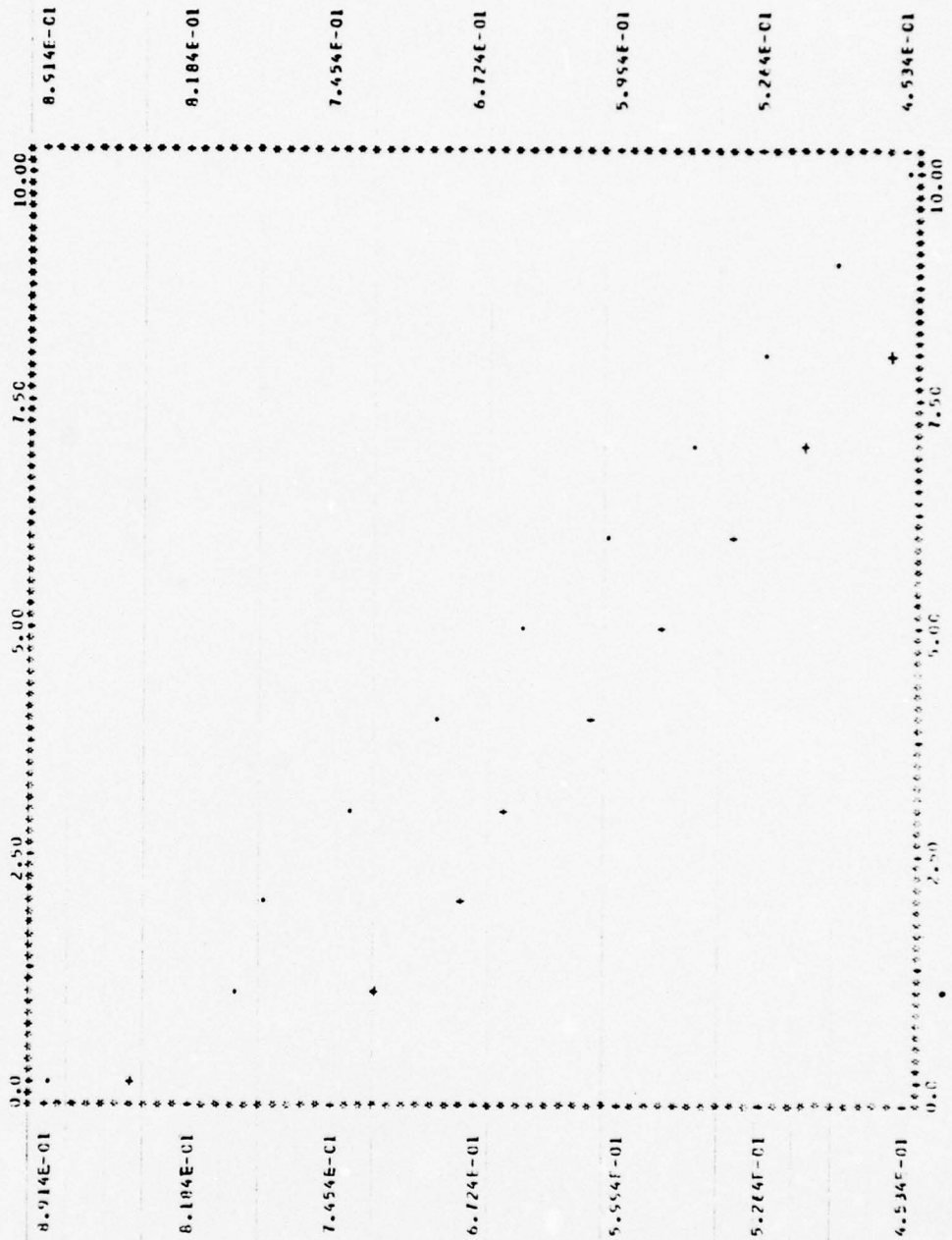


TABLE VIII
LOG GAMMA 90 AND 90Z LCL510-10 FAILURES

NO. OF FAILURES	ONLY ONE FAILURE PER COMPONENT	70Z LCL	70Z MAT-90Z LCL
0	2.0300	0.8450	2.5000
1	2.0300	0.7793	2.5000
2	5.9474	0.6756	2.9374
3	7.9489	0.6221	7.9869
4	9.9495	0.5781	9.9865
5	11.9503	0.5381	11.9877
6	13.9512	0.5011	13.9903
7	15.9522	0.4637	15.9252
8	17.9525	0.4296	17.9025
9	19.9510	0.3956	19.8510
10			

LOG LCL(%) AND 90Z LCL(%) VS. FAILURES



This non-monotonicity is caused by the sharp drop-off in $\chi^2_{1-\alpha} = \chi^2_{.9}$ at low degrees of freedom. As an adjustment, $2\hat{r}$ for 90% LG LCL's was constrained to be at least 2.5; the minimum $2\hat{r}$ for 80% LCL's remains 2.0. The results can be seen in Table VIII.

Consider the classical binomial component LCL procedure on the probability of mission success p . If p is estimated by $\hat{p} = X/N$ where X is the observed number of successes in N trials, then a $100(1-\alpha)\%$ LCL on p , denoted $\hat{p}_{L(\alpha)}$, is the solution for p in the equation

$$\sum_{j=x}^N \binom{N}{j} p^j (1-p)^{N-j} = \alpha \quad (5.4)$$

Under the simulation of Chapter III, if N_1 is the component one sample size and no failures occur in a replica, it can be equivalently stated that N_1 "systems" were tested without failure. Likewise, one component failure implies exactly one system failure. However, when two or more different types of components fail, it cannot be ascertained how many system failures occurred. Figure 3 illustrates this for the two-failure case.

Figure 3
Component Failures Versus System Failures

Component	1	2	K		1	2	K	
N_1 systems tested with one system failure	κ	κ	-	or	N_1 systems tested with two system failures ?	κ	-	-
	-	-	-			-	κ	-
	:						:			
	-	-	-			-	-	-

Treating a "system" as a component, equation (5.4) can be used to calculate $\hat{R}_{S,L}(\alpha)$ when zero or one failures occur.

a. Zero failures:

For $X=N$ successes, $\hat{P}_{L,}(\alpha) = \alpha^{1/N_1}$. Modification two revised this upwards to

$$\hat{R}_{S,L}(\alpha) = N_1 \sqrt{\alpha} + .2(1 - N_1 \sqrt{\alpha}) \quad (5.5)$$

b. One failure:

Equation (5.4) was applied directly for $X=N_1-1$. That is, $\hat{R}_{S,L}(\alpha)$ is the solution for p in the equation

$$\sum_{j=N_1-1}^{N_1} \binom{N_1}{j} p^j (1-p)^{N_1-j} = \alpha \quad (5.6)$$

or
$$N_1 p^{N_1-1} (1-p) + p^{N_1} = \alpha \quad (5.7)$$

c. Two through eight failures:

The LG method was adjusted upwards as follows:

$$\hat{R}'_{S,L}(\alpha) = \frac{1}{2} \left[\hat{R}_{S,L}(\alpha)(N_1, FT) + \hat{R}_{S,L}(\alpha)(N_1, FT-1) \right] \quad (5.8)$$

where $\hat{R}_{S,L}(\alpha)(N_1, FT)$ denotes the LG LCL computed with the observed total replica failures FT when the original sample size is N_1 .

d. More than eight failures:

The original LG procedure is used with exact degrees of freedom. The results are in Table IX.

TABLE IX

Modification Two of the Log-Gamma Method

-use of the classical binomial model for zero and one failure
 -use of the averaged LG method between two and eight failures
 -an adjustment for degrees of freedom in the 90% procedure

CASE	\hat{T}_T	R_s	80% ICL's $A_{1-\alpha/s}$			90% ICL's $A_{1-\alpha/s}$		
			LG	CLG	MLR	LG	CLG	MLR
Case 1: $N_1=20$ $K=15, R_1=.99$ $i=1, 15$ min=0, max=8	2.9	.860	.792/.078	.779/.077	.851/.255	.855/.093	.679/.080	.853/.263
Case 5: $N_1=20$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ min=0, max=11	4.3	.792	.759/.095	.739/.098	.776/.224	.700/.094	.679/.093	.773/.230
Case 10: $N_1=20$ $K=15, R_1=.995$ $i=1, 15$ min=0, max=6	1.5	.928	.938/.075	1.00/.100	.923/.251	.913/.102	1.00/.148	.891/.267

2. Observations and Analysis

Modification two did improve the accuracy of the LG method, while slightly increasing dispersion. The 90% procedure was noticeably improved by the degrees of freedom constraint, although still conservative. There is, however, still too much discreteness in values of $\hat{R}_{S,L}(\alpha)$ when few failures occur. Table X generated at NPS by APL subroutine "Histlist" (see documentation), illustrates this for case five. This discreteness usually causes the procedure to yield more than the target $1-\alpha$ confidence, and is a problem with nearly any method using only attributes data.

This being the case, the next logical step was to fit a continuous curve to the procedure by randomization.

C. MODIFICATION THREE: RANDOMIZATION

1. Randomization of the Log-Gamma Method

Many LCL procedures which use only attributes data are degraded in that they yield conservative bounds. Because only a finite set of outcomes can be observed for any test plan, then only a finite set of LCL's can be realized. Thus, $\hat{R}_{S,L}(\alpha)$ is a discrete random variable which ideally gives the LCL closest to but not greater than the "exact" LCL.

The 1975 article of Barr and Jayachandran [Ref. 7, p. 67-68] developed an exact, randomized binomial component LCL procedure. The same rationale used before (see Figure 3) allows the randomized binomial procedure to be used for

TABLE X
Case 5 Log-Gamma 80 Percent ICL's with Modification Two

SERIAL NUMBER	ORDERED DATA	OCCURENCES	PERCENT
1	.407610	1	.002
2	.444490	1	.002
3	.444960	1	.002
4	.471830	1	.002
5	.475840	2	.004
7	.477390	3	.006
10	.482700	2	.004
12	.484660	1	.002
13	.487430	1	.002
14	.527610	1	.002
15	.534450	3	.006
18	.538120	10 *	.020
28	.538550	1	.002
29	.540650	1	.002
30	.541090	1	.002
31	.542420	2	.004
33	.542860	1	.002
34	.571020	11 *	.022
45	.576210	1	.002
46	.577770	7 *	.014
53	.578300	2	.004
55	.580530	10 *	.020
65	.581070	1	.002
66	.583950	1	.002
67	.584260	2	.004
69	.584910	2	.004
71	.616160	13 **	.026
84	.620600	5 *	.010
89	.622570	14 **	.028
103	.623270	1	.002
104	.624390	24 ***	.048
128	.625090	2	.004
130	.627950	1	.002
131	.628010	1	.002
132	.628080	1	.002
133	.662770	23 ***	.045
156	.666380	1	.002
157	.666490	14 **	.028
171	.667840	1	.002
172	.668570	16 **	.032

TABLE X
(continued)

SERIAL NUMBER	ORDERED DATA	OCCURENCES	PERCENT
188	.669380	21 ***	.042
209	.669530	1	.002
210	.670340	6 *	.012
216	.671090	2	.004
218	.672070	1	.002
219	.673220	1	.002
220	.710230	37 ****	.074
257	.713320	22 ***	.044
279	.714870	25 ***	.050
304	.715380	6 *	.012
310	.716280	2	.004
312	.756600	47 ****	.094
359	.758850	21 ***	.042
380	.759200	21 ***	.042
401	.761320	2	.004
403	.789380	39 ****	.078
442	.791840	27 ***	.054
469	.890140	21 ***	.042
490	.938140	11 *	.022

"system" LCL's when zero or one failure occurs. Equations for the Randomized Log-Gamma method (RLG) are derived in Appendix F. An example is given in Appendix G.

a. Randomized Log-Gamma Procedure

(1) If zero failures occur, generate a uniform (0,1) random number Y . Then

$$\hat{R}_{S,L}(\alpha) = \begin{cases} \left(\frac{\alpha}{1-Y}\right)^{1/N_1} & \text{if } Y \leq 1 - \alpha \\ 1.0 & \text{if } Y > 1 - \alpha \end{cases} \quad (5.9)$$

(2) When one failure is observed, generate a uniform (0,1) random number Y . Then $\hat{R}_{S,L}(\alpha)$ is the solution for p in the equation

$$(1 - Y)N_1 p^{N_1 - 1} (1 - p) + p^{N_1} = \alpha \quad \text{if } Y \leq 1 - \alpha \quad (5.10)$$

This solution can be approximated by

$$\hat{R}_{S,L}(\alpha) = \alpha^{1/N_1} \quad (5.11)$$

for $Y > 1 - \alpha$ (see Appendix F).

(3) Two through four failures inclusive:

Generate a uniform (0,1) random number Y_i for each component that had at least one failure, $i=1,2,\dots,\dots,K$. Then use the LG method with $F_i^* = F_i - Y_i$ instead of F_i . Thus, the RLG formulas become

$$A_i = \frac{2N_i - 3}{2N_i - 2} \quad (5.12)$$

$$B_i = \frac{N_i}{N_i - 1} \quad (5.13)$$

$$\hat{T}_i = A_i \left(\frac{F_i - Y_i}{N_i} \right) + \frac{B_i}{2} \left(\frac{F_i - Y_i}{N_i} \right)^2 \quad (5.14)$$

$$\hat{S} = \sum_{i=1}^K \hat{T}_i \quad (5.15)$$

$$\hat{r} = \frac{\hat{S}^2}{\sum_{i=1}^K \frac{\hat{T}_i}{N_i}} \quad (5.16)$$

and
$$\hat{R}_{S,L(\alpha)} = \text{Exp} \left\{ \frac{-2\hat{r}\hat{S}}{\chi_{1-\alpha, 2\hat{r}}^2} \right\} \quad (5.17)$$

The only differences between the original LG formulas and those above is the computation of \hat{T}_i in equation (5.14), and usage of exact degrees of freedom.

(4) Five or more failures:

Use the non-randomized LG method with exact degrees of freedom. That is, use the formulas in (3) above except use F_i vice $F_i - Y_i$ in the computation of \hat{T}_i .

2. Modification Three Cases

The number of cases simulated under modification three was expanded greatly. Cases one through twelve have an N_1 of either 20 or 50, and cover the range of pertinent system reliability R_g . All twelve cases were first ran with the number of components $K=15$, and then reran with $K=40$. Cases with equal component reliabilities were also ran with $K=4$. The system sample size was reduced to ten, and the applicable twelve cases ran again, with $K=15, 40$, and 4. Modification three results are listed in Table XI

for $K=15$, Table XII for $K=40$, and Table XIII for $K=4$. Table XIV lists results of cases from Ref. 1, which had highly-varied component sample sizes.

3. Results, Analysis, and Conclusions

All three method results were tabled for comparison under modification three, although randomization of the LG method was the only change.

The accuracy of the RLG was comparable to that of the MLR, while having much lower sample standard deviations. Both the RLG and CLG methods had dispersion inversely proportional to sample size, while the MLR continued to consistently have high dispersion regardless of the circumstances.

The most significant improvement in the RLG was its performance with few failures and less testing. Especially in the $K=4$ and $K=15$ cases with small \hat{T} , the RLG was near exact while the CLG sometimes gave $A_{1-\alpha}$ of 1.00 (meaning at least the upper $(100)\alpha\%$ of the simulated distribution was LCL's of 1.00). The MLR procedure was inaccurate for very high reliability systems, such as case 11 with $K=15$, due to the upper limit of its possible LCL values. In case 11, with $N_1=20$, the maximum MLR LCL is .923 with zero failures (see Appendix D), whereas true system reliability is .956. As in the earlier LG results, the 90% RLG procedure was usually less accurate, less stable and slightly more dispersed. The presence of one component with dissimilar

reliability, like case seven with $K = 15$ seemed to degrade accuracy somewhat. Additional testing, however, as in case eight, dampens this out.

Cases one and two of the original LG paper [Ref. 1] were also simulated with modification three to check sensitivity to mixed sample sizes (see Table XIV). Note that randomization was rarely necessary in case two ($\min = 2$), and never in case one ($\min = 5$). The accuracy of all three methods was high due to the large amount of testing. Note the small sample size of the twelfth component in case one. This seemingly had little effect on the RLG method's accuracy, although s is suspiciously large when compared to case two. When R_{12} was set to .85, the effect on the RLG became clearer. When a component twelve failure occurred, its contribution to \hat{S} and \hat{r} so dominates that $\hat{R}_{S,L}(\alpha)$ is nearly halved from what it would have been otherwise.

Figures 4 and 5 give histograms for $\hat{R}_{S,L}(\alpha)$ of the RLG procedure generated under cases one and two respectively. Both have skewness and kurtosis roughly comparable to a normal distribution, and both have medians near-equal to their medians (symmetric). Figure 5 (case two with $N_1 = 50$) has a lower maximum $\hat{R}_{S,L}(\alpha)$ and a higher minimum than Figure 4 (case one with $N_1 = 20$), reflecting the decreased dispersion of the RLG procedure with increased testing.

TABLE XI
 Modification Three Results: The Randomized Log-Gamma Method
 K = 15

CASE	\hat{T}_T	R_8	80% LCL's $A_{1-\alpha}/s$			90% LCL's $A_{1-\alpha}/s$		
			RLG	CIG	MLR	RLG	CIG	MLR
Case 1: $N_1=20$ $K=15, R_1=.99$ $i=1, 15$ $\min=0, \max=8$	2.9	.860	.872/.088	.799/.080	.853/.283	.852/.103	.679/.084	.864/.286
Case 2: $N_1=50$ $K=15, R_1=.99$ $i=1, 15$ $\min=0, \max=15$	7.2	.860	.847/.060	.847/.052	.865/.258	.854/.056	.829/.051	.864/.268
Case 1T: $N_1=10$ $K=15, R_1=.99$ $i=1, 15$ $\min=0, \max=6$	1.4	.860	.861/.123	1.00/.172	.851/.289	.855/.178	1.00/.238	.794/.287
Case 3: $N_1=20$ $K=15, R_1=.99$ $i=1, 14; R_{15}=.995$ $\min=0, \max=8$	2.8	.864	.869/.089	.800/.082	.866/.270	.860/.105	.679/.090	.870/.287
Case 4: $N_1=50$ $K=15, R_1=.99$ $i=1, 14; R_{15}=.995$ $\min=0, \max=15$	6.9	.864	.848/.061	.847/.052	.862/.246	.855/.057	.829/.051	.874/.259
Case 3T: $N_1=10$ $K=15, R_1=.99$ $i=1, 14; R_{15}=.995$ $\min=0, \max=5$	1.4	.864	.864/.122	1.00/.169	.851/.280	.856/.176	1.00/.238	.794/.279

TABLE XI
(continued)

CASE	\hat{T}_T	R_s	80% ICL's $A_{1-\alpha}/s$				90% ICL's $A_{1-\alpha}/s$			
			80% ICL's $A_{1-\alpha}/s$		90% ICL's $A_{1-\alpha}/s$		80% ICL's $A_{1-\alpha}/s$		90% ICL's $A_{1-\alpha}/s$	
			RLG	CLG	MLR		RLG	CLG	MLR	
Case 5: $N_1=20$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ $\min=0, \max=12$	4.3	.792	.783/.100	.736/.083	.776/.225	.685/.090	.659/.072	.765/.229		
Case 6: $N_1=50$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ $\min=3, \max=22$	10.6	.792	.785/.055	.784/.056	.795/.212	.770/.055	.768/.056	.793/.220		
Case 5T: $N_1=10$ $K=15, R_1=.995$ $i=1, 14; R_{15}=.85$ $\min=0, \max=7$	2.1	.792	.794/.137	.642/.153	.790/.251	.794/.169	.471/.182	.793/.248		
Case 7: $N_1=20$ $K=15, R_1=.99$ $i=2, 15; R_{15}=.85$ $\min=1, \max=11$	5.6	.738	.762/.103	.690/.087	.717/.247	.676/.088	.655/.077	.719/.244		
Case 8: $N_1=50$ $K=15, R_1=.85$ $R_{15}=.99, i=2, 15$ $\min=3, \max=26$	13.7	.738	.730/.064	.728/.065	.757/.223	.731/.063	.728/.064	.751/.224		
Case 7T: $N_1=10$ $K=15, R_1=.85$ $R_{15}=.99, i=2, 15$ $\min=0, \max=8$	2.8	.738	.747/.143	.639/.132	.716/.250	.749/.155	.466/.131	.723/.241		

TABLE XI
(continued)

CASE	\hat{T}	R_B	80% ICL's $A_{1-\alpha}/s$			90% ICL's $A_{1-\alpha}/s$		
			RLG	CLG	MLR	RLG	CLG	MLR
Case 9: $N_1=20$ $K=15, R_1=.995$ $i=1,15$ $\min=0, \max=5$	1.5	.928	.933/.076	1.00/.100	.923/.257	.934/.117	1.00/.148	.891/.270
Case 10: $N_1=50$ $K=15, R_1=.995$ $i=1,15$ $\min=0, \max=11$	3.7	.928	.925/.050	.906/.042	.927/.237	.934/.053	.859/.041	.926/.252
Case 9T: $N_1=10$ $K=15, R_1=.995$ $i=1,15$ $\min=0, \max=4$.8	.928	.932/.113	1.00/.192	.851/.205	.926/.159	1.00/.293	.794/.216
Case 11: $N_1=20$ $K=15, R_1=.997$ $i=1,15$ $\min=0, \max=5$.9	.956	.955/.064	1.00/.107	.923/.245	.950/.103	1.00/.176	.891/.261
Case 12: $N_1=50$ $K=15, R_1=.997$ $i=1,15$ $\min=0, \max=8$	2.2	.956	.958/.039	.915/.040	.956/.221	.957/.053	.860/.052	.955/.245
Case 11T: $N_1=10$ $K=15, R_1=.997$ $i=1,15$ $\min=0, \max=3$.4	.956	.952/.096	1.00/.177	.851/.196	.964/.132	1.00/.282	.794/.204

TABLE XII
 Modification Three Results: The Randomized Log-Gamma Method
 K = 40

CASE	\hat{T}	R_8	80% ICL's $A_{1-\alpha}/s$			90% ICL's $A_{1-\alpha}/s$		
			RLG	CLG	MLR	RLG	CLG	MLR
Case 1: $N_1=20$ $K=40, R_1=.99$ $i=1, 40$ $\min=1, \max=15$	7.2	.669	.650/.114	.645/.099	.664/.280	.662/.097	.613/.090	.668/.257
Case 2: $N_1=50$ $K=40, R_1=.99$ $i=1, 40$ $\min=7, \max=28$	18.0	.669	.659/.062	.656/.063	.685/.266	.658/.061	.655/.062	.686/.252
Case 1: $N_1=10$ $K=40, R_1=.99$ $i=1, 40$ $\min=0, \max=8$	3.5	.669	.675/.159	.591/.123	.648/.276	.690/.148	.449/.105	.660/.251
Case 3: $N_1=20$ $K=40, R_1=.99$ $i=1, 39; R_{40}=.995$ $\min=1, \max=15$	7.1	.672	.650/.114	.645/.099	.667/.281	.663/.098	.613/.090	.678/.259
Case 4: $N_1=50$ $K=40, R_1=.99$ $i=1, 39; R_{40}=.995$ $\min=7, \max=28$	17.7	.672	.659/.063	.657/.063	.689/.265	.658/.062	.656/.062	.688/.252
Case 3: $N_1=10$ $K=40, R_1=.99$ $i=1, 39; R_{40}=.995$ $\min=0, \max=9$	3.5	.672	.665/.156	.592/.129	.660/.273	.683/.147	.450/.115	.671/.250

TABLE XII
(continued)

CASE	$\hat{T}T$	R_s	80% ICL's $A_{1-\alpha}/s$				90% ICL's $A_{1-\alpha}/s$			
			RLG		CLG		RLG		CLG	
			RLG	CLG	RLG	CLG	RLG	CLG	RLG	CLG
Case 5: $N_1=20$ $K=40, R_1=.995$ $i=1, 39; R_{40}=.85$ $\min=1, \max=13$	6.5	.699	.712/.111	.683/.094	.701/.245	.617/.086	.669/.097	.617/.086	.711/.240	
Case 6: $N_1=50$ $K=40, R_1=.995$ $i=1, 39; R_{40}=.85$ $\min=6, \max=28$	15.8	.699	.693/.063	.691/.064	.703/.249	.690/.063	.693/.063	.690/.063	.708/.243	
Case 5T: $N_1=10$ $K=40, R_1=.995$ $i=1, 39; R_{40}=.85$ $\min=0, \max=8$	3.2	.699	.733/.163	.595/.146	.680/.259	.451/.144	.729/.163	.451/.144	.675/.245	
Case 7: $N_1=20$ $K=40, R_1=.85$ $i=1, 39; R_{40}=.85$ $\min=0, \max=8$	9.4	.574	.567/.101	.559/.098	.569/.233	.534/.090	.539/.091	.534/.090	.556/.211	
Case 8: $N_1=50$ $K=40, R_1=.85$ $R_1=.99; i=2, 40$ $\min=14, \max=35$	23.9	.574	.573/.062	.568/.063	.551/.230	.565/.062	.571/.061	.565/.062	.567/.209	
Case 7T: $N_1=10$ $K=40, R_1=.85$ $R_1=.99; i=2, 40$ $\min=0, \max=10$	4.7	.574	.614/.162	.521/.126	.572/.247	.412/.103	.451/.130	.412/.103	.561/.221	

TABLE XII
(continued)

CASE	\hat{T}_T	R_s	80% LCL's $A_{1-\alpha}/s$			90% LCL's $A_{1-\alpha}/s$		
			RLG	CLG	MLR	RLG	CLG	MLR
Case 9: $N_1=20$ $K=40, R_1=.995$ $i=1,40$ min=0, max=10	3.9	.818	.811/.109	.777/.089	.830/.288	.830/.107	.679/.083	.822/.287
Case 10: $N_1=50$ $K=40, R_1=.995$ $i=1,40$ min=1, max=20	9.4	.818	.807/.057	.806/.056	.823/.273	.790/.056	.789/.055	.813/.275
Case 9T: $N_1=10$ $K=40, R_1=.995$ $i=1,40$ min=0, max=6	1.9	.818	.851/.135	.643/.161	.816/.287	.824/.187	1.00/.204	.794/.281
Case 11: $N_1=20$ $K=40, R_1=.997$ $i=1,40$ min=0, max=8	2.3	.887	.901/.089	.800/.093	.889/.285	.896/.117	.685/.115	.891/.293
Case 12: $N_1=50$ $K=40, R_1=.997$ $i=1,40$ min=1, max=13	5.8	.887	.909/.058	.867/.046	.878/.265	.858/.052	.847/.044	.880/.279
Case 11T: $N_1=10$ $K=40, R_1=.997$ $i=1,40$ min=0, max=6	1.1	.887	.896/.122	1.00/.188	.851/.254	.887/.176	1.00/.272	.794/.260

TABLE XIII
Modification Three Results: The Randomized Log-Gamma Method
K=4

CASE	\hat{T}	R_s	80% ICL's $A_{1-\alpha/s}$			90% ICL's $A_{1-\alpha/s}$		
			RLG	CLG	MLR	RLG	CLG	MLR
			Case 1: $N_1=20$ $K=4, R_1=.99$ $i=1,4$ $\min=0, \max=4$.8	.961	.954/.062	1.00/.106	.923/.231
Case 2: $N_1=50$ $K=4, R_1=.99$ $i=1,4$ $\min=0, \max=7$	2.0	.961	.961/.035	.915/.039	.960/.163	.959/.051	1.00/.054	.955/.185
Case 1T: $N_1=10$ $K=4, R_1=.99$ $i=1,4$ $\min=0, \max=3$.4	.961	.958/.098	1.00/.178	.851/.200	.936/.123	1.00/.285	.794/.208
Case 9: $N_1=20$ $K=4, R_1=.995$ $i=1,4$ $\min=0, \max=3$.4	.980	.979/.051	1.00/.097	.923/.133	.978/.073	1.00/.171	.891/.148
Case 10: $N_1=50$ $K=4, R_1=.995$ $i=1,4$ $\min=0, \max=5$	1.0	.980	.982/.030	1.00/.046	.968/.147	.982/.049	1.00/.079	.955/.168

TABLE XIII
(continued)

CASE	\hat{T}_T	R_B	80% LCL's $A_{1-\alpha/s}$			90% LCL's $A_{1-\alpha/s}$		
			RLG	CLG	MLR	RLG	CLG	MLR
Case 9T, $N_1=10$ $K=4, R_1=.995$ $i=1,4$ $\min=0, \max=2$.2	.980	.998/.082	1.00/.150	.851/.122	.996/.102	1.00/.245	.794/.134
Case 11, $N_1=20$ $K=4, R_1=.997$ $i=1,4$ $\min=0, \max=3$.2	.988	.982/.042	1.00/.083	.923/.119	.984/.054	1.00/.150	.891/.132
Case 12, $N_1=50$ $K=4, R_1=.997$ $i=1,4$ $\min=0, \max=12$.6	.988	.988/.024	1.00/.044	.968/.135	.985/.037	1.00/.082	.955/.150
Case 11T, $N_1=10$ $K=4, R_1=.997$ $i=1,4$ $\min=0, \max=3$.1	.988	.977/.073	1.00/.121	.851/.116	.973/.089	1.00/.197	.794/.125

TABLE XIV

CASE	\hat{T}_T	R_B	80% ICL's $A_{1-\alpha}/s$			90% ICL's $A_{1-\alpha}/s$		
			RIG	CLG	MLR	RIG	CLG	MLR
Case 1 of original Log-Gamma paper								
$N_1=150$ $R_1=.995$								
$N_2=90$ $R_2=.985$								
$N_3=75$ $R_3=.979$								
$N_4=100$ $R_4=.988$	15.3	.723	.722/.105	.720/.109	.709/.243	.723/.117	.720/.121	.722/.240
$N_5=125$ $R_5=.982$								
$N_6=18$ $R_6=.980$								
$N_7=28$ $R_7=.967$								
$N_8=125$ $R_8=.995$								
$N_9=63$ $R_9=.970$								
$N_{10}=125$ $R_{10}=.995$								
$N_{11}=59$ $R_{11}=.968$								
$N_{12}=5$ $R_{12}=.980$								
$N_{13}=19$ $R_{13}=.900$								
$K=13$								
min=5, max=26								

TABLE XIV
(continued)

CASE	$\hat{\tau}_T$	R_8	80% ICL's $A_{1-\alpha}/s$			90% ICL's $A_{1-\alpha}/s$		
			RLG	CLG	MLR	RLG	CLG	MLR
Case 1 of original Log-Gamma paper except $R_{12}=.85$ min=6, max=31	16.5	.627	.673/.153	.670/.153	.617/.237	.676/.169	.673/.176	.639/.228
Case 2 of original Log-Gamma paper $K=14, i=1, 14$ $R_1=.99, N_2=40$ $N_1=250, N_4=15$ $N_3=120, N_6=65$ $N_5=130, N_8=75$ $N_7=70, N_{10}=90$ $N_9=100, N_{12}=60$ $N_{11}=60, N_{14}=30$ $N_{13}=20$ min=2, max=20	10.2	.869	.870/.063	.869/.063	.868/.242	.866/.071	.865/.072	.866/.256

APPENDIX A

APPROXIMATION OF S BY A TAYLOR SERIES

The Taylor series approximation of $f(x)$ about $x = a$ can be stated as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \dots + \frac{f^i(a)(x-a)^i}{i!} + \dots \quad (\text{A.1})$$

Here, $f(x) = -\ln(x)$, where $x = (1 - Q_i)$. Let $a = 1$.

Then $(x - a) = -Q_i$

$$f(a) = 0$$

$$f'(x) = \frac{1}{1 - Q_i}, \quad f'(a) = -1$$

$$f''(x) = \frac{1}{(1 - Q_i)^2}, \quad f''(a) = 1$$

$$f'''(x) = \frac{-2}{(1 - Q_i)^3}, \quad f'''(a) = -2$$

Equation (A.1) becomes:

$$\begin{aligned} f(x) &= 0 + (-1)(-Q_i) + \frac{(1)(-Q_i)^2}{2!} + \frac{(-2)(-Q_i)^3}{3!} + \dots \\ &= Q_i + \frac{Q_i^2}{2} + \frac{Q_i^3}{3} + \dots + \frac{Q_i^j}{j} + \dots \\ &= \sum_{j=1}^{\infty} \frac{Q_i^j}{j} \end{aligned} \quad (\text{A.2})$$

$$\text{Thus, } S = \sum_{i=1}^K -\ln(1 - Q_i) = \sum_{i=1}^K \sum_{j=1}^{\infty} \frac{Q_i^j}{j} \quad (\text{A.3})$$

Dropping terms of order three and higher,

$$S \approx \sum_{i=1}^K \left[Q_i + \frac{Q_i^2}{2} \right] \quad (\text{A.4})$$

)

APPENDIX B

AN UNBIASED ESTIMATOR FOR T_i

Let $T_i = Q_i + \frac{Q_i^2}{2}$. An unbiased estimator \hat{T}_i is desired for T_i , which is of the form

$$\hat{T}_i = A_i \hat{Q}_i + \frac{B_i}{2} \hat{Q}_i^2 \quad (B.1)$$

where $\hat{Q}_i = \frac{F_i}{N_i}$. The number of failures of i th component F_i is a binomial ($N = N_i$, $p = Q_i$) random variable with mean $N_i Q_i$ and variance $N_i Q_i (1 - Q_i)$. Thus,

$$E(\hat{Q}_i) = \frac{1}{N_i} E(F_i) = Q_i \quad (B.2)$$

$$\text{Var}(\hat{Q}_i) = \frac{1}{N_i^2} \text{Var}(F_i) = \frac{Q_i(1 - Q_i)}{N_i} \quad (B.3)$$

$$\begin{aligned} E(\hat{Q}_i^2) &= \text{Var}(\hat{Q}_i) + (E(\hat{Q}_i))^2 \\ &= \frac{Q_i(1 - Q_i)}{N_i} + Q_i^2 \\ &= \frac{Q_i - Q_i^2 + N_i Q_i^2}{N_i} \end{aligned} \quad (B.4)$$

Substituting

$$\begin{aligned} E(\hat{T}_i) &= A_i E(\hat{Q}_i) + \frac{B_i}{2} E(\hat{Q}_i^2) \\ &= (A_i + \frac{B_i}{2N_i}) Q_i + (B_i - \frac{B_i}{N_i}) \frac{Q_i^2}{2} \end{aligned} \quad (B.5)$$

That \hat{T}_i is unbiased implies

$$E(\hat{T}_i) = T_i = Q_i + \frac{Q_i^2}{2} \quad (B.6)$$

Equating coefficients with equation (B.5)

$$A_i + \frac{B_i}{2N_i} = 1 \quad (\text{B.7})$$

$$B_i - \frac{B_i}{N_i} = 1 \quad (\text{B.8})$$

Solving the above two equations simultaneously gives the desired expressions for A_i and B_i

$$A_i = \frac{2N_i - 3}{2N_i - 2} \quad (\text{B.9})$$

$$B_i = \frac{N_i}{N_i - 1} \quad (\text{B.10})$$

APPENDIX C

DERIVATION OF THE ACTUAL VARIANCE OF \hat{T}_i
AND COMPARISON WITH $\sum_{i=1}^K \frac{\hat{T}_i}{N_i}$

Reference 1 used $\frac{\hat{T}_i}{N_i}$ as a variance estimate for \hat{T}_i .

Then for the method of moments gamma fit,

$$\text{Var}(\hat{S}) = \sum_{i=1}^K \text{Var}(\hat{T}_i) \approx \sum_{i=1}^K \frac{\hat{T}_i}{N_i} \quad (\text{C.1})$$

The following is a derivation of the actual variance of \hat{T}_i , and then a comparison with the above estimator.

$$\text{Since } \hat{T}_i = A_i \hat{Q}_i + \frac{B_i}{2} \hat{Q}_i^2 \quad (\text{C.2})$$

$$\text{then } \text{Var}(\hat{T}_i) = A_i^2 \text{Var}(\hat{Q}_i) + A_i B_i \text{COV}(\hat{Q}_i, \hat{Q}_i^2) + \frac{B_i^2}{4} \text{Var}(\hat{Q}_i^2) \quad (\text{C.3})$$

where $\hat{Q}_i = \frac{F_i}{N_i}$. The total number of failures on the i th component F_i , is a binomial ($N = N_i$, $p = Q_i$) random variable.

Then

$$E(\hat{Q}_i) = \frac{1}{N_i} E(F_i) = Q_i \quad (\text{C.4})$$

$$\text{Var}(\hat{Q}_i) = \frac{1}{N_i^2} \text{Var}(F_i) = \frac{Q_i(1 - Q_i)}{N_i} \quad (\text{C.5})$$

$$\begin{aligned} E(\hat{Q}_i^2) &= \text{Var}(\hat{Q}_i) + (E(\hat{Q}_i))^2 \\ &= \frac{Q_i - Q_i^2}{N_i} + Q_i^2 \end{aligned} \quad (\text{C.6})$$

$$\text{and } \text{COV}(\hat{Q}_i, \hat{Q}_i^2) = E(\hat{Q}_i^3) - E(\hat{Q}_i)E(\hat{Q}_i^2) \quad (\text{C.7})$$

Using the same model as before,

$$\begin{aligned}
 E(F_i^4) &= \sum_{j=k=l=mn} E(I_j^4) + \sum_j \sum_k E(I_j^2) E(I_k^2) + \sum_j \sum_k E(I_j^3) E(I_k) \\
 &\quad \text{(all indexes equal)} \quad \text{(indexes equal in pairs)} \quad \text{(three indexes the same and one distinct)} \\
 &+ \sum_j \sum_k \sum_l E(I_j^2) E(I_k) E(I_l) \\
 &\quad \text{(two indexes the same and two distinct)} \\
 &+ \sum_j \sum_k \sum_l \sum_{mn} E(I_j) E(I_k) E(I_l) E(I_{mn}) \quad (C.13) \\
 &\quad \text{(all four indexes distinct)} \\
 &= N_i Q_i + 3N_i(N_i - 1)Q_i^2 + 4N_i(N_i - 1)Q_i^2 + 6N_i(N_i - 1)(N_i - 2)Q_i^3 \\
 &\quad + N_i(N_i - 1)(N_i - 2)(N_i - 3)Q_i^4
 \end{aligned}$$

Equation (C.13) is used to find

$$\begin{aligned}
 E(\hat{Q}_i^4) &= \frac{1}{N_i^3} \left[Q_i + 7(N_i - 1)Q_i^2 + 6(N_i - 1)(N_i - 2)Q_i^3 \right. \\
 &\quad \left. + (N_i - 1)(N_i - 2)(N_i - 3)Q_i^4 \right] \quad (C.14)
 \end{aligned}$$

Substituting equation (C.14) into equation (C.12),

$$\begin{aligned}
 \text{Var}(\hat{Q}_i^2) &= \frac{Q_i}{N_i^3} \left\{ 1 + (6N_i - 7)Q_i + 4(N_i - 1)(N_i - 3)Q_i^2 \right. \\
 &\quad \left. - 2(N_i - 1)(2N_i - 3)Q_i^3 \right\} \quad (C.15)
 \end{aligned}$$

The final result can now be obtained by substituting equations (C.15), (C.11), and (C.5) into equation (C.3), yielding

$$\begin{aligned} \text{Var}(\hat{T}_i) = & \frac{A_i^2}{N_i}(Q_i - Q_i^2) + A_i B_i \frac{Q_i}{N_i^2} \left[1 + (2N_i - 3)Q_i - 2(N_i - 1)Q_i^2 \right] \\ & + \frac{B_i^2}{4} \frac{Q_i}{N_i^3} \left\{ 1 + (6N_i - 7)Q_i + \left[4(N_i - 1)(N_i - 3) \right] Q_i^2 \right. \\ & \left. - 2(N_i - 1)(2N_i - 3)Q_i^3 \right\}^2 \end{aligned} \quad (\text{C.16})$$

The cumbersome expression above can be greatly simplified by dropping terms with divisors of N_i^2 and higher, yielding

$$\text{Var}(\hat{T}_i) \approx \frac{Q_i}{N_i} \left[(1 - Q_i)(A_i + B_i Q_i)^2 \right] \quad (\text{C.17})$$

$$\text{Then } \widehat{\text{Var}}(\hat{T}_i) = \frac{\hat{Q}_i}{N_i} \left[(1 - \hat{Q}_i)(A_i + B_i \hat{Q}_i)^2 \right] \text{ where } \hat{Q}_i = \frac{F_i}{N_i} \quad (\text{C.18})$$

A comparison was made between $\widehat{\text{Var}}(\hat{T}_i)$ as given by equation (C.18), and that used by Ref. 1. The results are listed for $N_i = 10, 20,$ and 50 in Table XV. The simplistic esti-

mator $\sum_{i=1}^K \frac{\hat{T}_i}{N_i}$ turns out to be an excellent approximation

$$\text{for } \text{Var}(\hat{S}) = \sum_{i=1}^K \text{Var}(\hat{T}_i).$$

TABLE XV
Actual Variance of \hat{T}_i Compared to Estimate
(One Failure Per Component)

FAILURES	$\widehat{\text{Var}}(\hat{S}) = \sum_{i=1}^K \frac{\bar{T}_i}{N_i}$	$\widehat{\text{Var}}(\hat{S}) = \sum_{i=1}^K \frac{\hat{Q}_i}{N_i} (1 - \hat{Q}_i) (A_i + B_i \hat{Q}_i)^2$	
1	$N_1=10$.01000	.01003
	20	.00250	.00250
	50	.00040	.00040
2	$N_1=10$.02235	.02242
	20	.00527	.00527
	50	.00082	.00082
3	$N_1=10$.03469	.03480
	20	.00804	.00805
	50	.00123	.00123
4	$N_1=10$.05032	.05050
	20	.01113	.01113
	50	.00167	.00167
5	$N_1=10$.06594	.06619
	20	.01421	.01422
	50	.00210	.00210
6	$N_1=10$.08635	.08672
	20	.01767	.01769
	50	.00255	.00255
7	$N_1=10$.10676	.10725
	20	.02113	.02115
	50	.00301	.00301
8	$N_1=10$.13454	.13526
	20	.02504	.02506
	50	.00348	.00348
9	$N_1=10$.16231	.16327
	20	.02895	.02897
	50	.00395	.00395
10	$N_1=10$.20231	.20377
	20	.03339	.03342
	50	.00445	.00445

APPENDIX D

DERIVATION OF THE ZERO FAILURE CASE FOR THE MODIFIED LIEBERMAN-ROSS METHOD

The variables U and I are undefined when no failures occur. Consider type III censoring in which N components are placed on simultaneous test, with replacement, until a pre-selected time T_0 [Ref. 8, p. 238]. Type III censoring is a component LCL procedure, and not a system LCL procedure. Now, consider the Modified Lieberman-Ross (MLR) procedure where N_i items of the i th component have been tested with no failures, $i=1,2,\dots,K$. If the N_i are identical (as is the case in this study for zero failures), then the above is equivalent to testing N_i "systems" with no failures until $T_0 = 1$ "system" mission.

The type III $100(1-\alpha)\%$ LCL on the mission probability of success p of a component is

$$\hat{P}_L(\alpha) = \text{Exp} \left(\frac{-\chi^2_{\alpha, 2F+2}}{2NT_0} \right) \quad (\text{D.1})$$

where F is the observed number of failures, N is the number of components on simultaneous test, and T_0 is the preselected test time. Then for $F=0$, $2F+2=2$ and

$$2NT_0 = 2N_i \cdot 1 \text{ system mission unit} = 2N_i$$

A $100(1-\alpha)\%$ LCL on system reliability R_s when no failures occur is

$$\hat{R}_{S,L}(\alpha) = \text{Exp} \left(\frac{-\chi^2_{\alpha, 2}}{2N_i} \right) \quad (\text{D.2})$$

for N_i equal, $i=1,2,\dots,K$. If the N_i are not equal, the minimum N_i should be used.

APPENDIX E

USE OF THE PROBABILITY INTEGRAL TRANSFORM (PIT) IN GENERATING EXPONENTIAL AND WEIBULL FAILURE DATA

Assume an exponential failure process for the i th component with rate λ_i per mission unit. Then the reliability of the i th component for completing one mission unit is:

$$R_i(1) = \text{Exp}(-\lambda_i \cdot 1) \quad (\text{F.1})$$

Utilizing the PIT method, draw a uniform (0,1) random number, denoted "U", and form the equality:

$$U = \text{Exp}(-\lambda_i t) \quad (\text{F.2})$$

Solving for time-to-failure t

$$\ln(U) = -\lambda_i t \quad (\text{F.3})$$

It follows that:

$$t = \frac{-\ln(U)}{\lambda_i} \quad (\text{F.4})$$

The computer program uses the following algorithm:

- 1) Generate a uniform (0,1) random variable
- 2) If $U = \text{Exp}(-\lambda_i t)$ is less than $\text{Exp}(-\lambda_i \cdot 1)$, then $t > 1$ mission unit, and a success has occurred. Truncate the time-on-test for that component trial to 1.

- 3) Otherwise, a failure has occurred at $t = \frac{-\ln(U)}{\lambda_i}$

The simulation was also ran under a Weibull failure process for sensitivity analysis. The Weibull (β, λ) density function is

$$f_T(t) = \beta \lambda t^{\beta-1} \text{Exp}(-\lambda t^\beta) \quad (\text{F.5})$$

and hazard function

$$h(t) = \beta \lambda t^{\beta-1} \quad t \geq 0, \lambda > 0, \beta > 0 \quad (\text{F.6})$$

Using the PIT method as before, draw a uniform (0,1) random number U and set

$$\begin{aligned} U &= \bar{F}(t) \\ &= \text{Exp}(-\lambda_i t^\beta) \end{aligned} \quad (\text{F.7})$$

thus
$$T = \left(-\frac{\ln(U)}{\lambda_i} \right)^{1/\beta} \quad (\text{F.8})$$

Comparing equations (F.4) and (F.8) shows that if $T = \frac{-\ln(U)}{\lambda_i}$ is an exponential (λ_i) component failure time, then $T^{1/\beta}$ is Weibull (β, λ_i). Thus, to perform the Weibull sensitivity analysis, only the above change needs to be made to the simulation algorithm (Chapter III).

Taking the derivative of the Weibull hazard function with respect to time

$$\frac{d(h(t))}{dt} = \lambda_i (\beta^2 - \beta) t^{\beta-2} \quad (\text{F.9})$$

Therefore, any $0 < \beta < 1$ yields a decreasing hazard rate, and any $\beta > 1$ gives an increasing one. Case one was ran with $\beta = .5$ and 3.0 . Note that the comparison logic, used by the simulation algorithm to determine if a failure has occurred, reduces to the original expression. The number of failures is then the same as before. The Weibull results are shown in Table IV.

APPENDIX F

DERIVATION OF THE RANDOMIZED LOG-GAMMA METHOD

The technique for randomization of the classical binomial model is a direct application of Barr and Jayachandran's 1975 article [Ref. 7]. It can be applied directly to obtain a system reliability LCL when a total of zero or one failure occurs among all component tests, because these cases translate into equivalent system failures (see Figure 3, p. 45).

The binomial point estimate for p is $\hat{p} = X/N$, where X is the number of observed successes in N trials. The classical binomial $100(1-\alpha)\%$ LCL for p , denoted $\hat{p}_L(\alpha)$, is the solution for p in the equation

$$\sum_{j=X}^N \binom{N}{j} p^j (1-p)^{N-j} = \alpha \quad (\text{F.1})$$

when no failures occur, or $X = N$, this becomes

$$\hat{p}_L(\alpha) = \alpha^{1/N} \quad (\text{F.2})$$

and for one failure, $\hat{p}_L(\alpha)$ is the solution for p in

$$Np^{N-1}(1-p) + p^N = \alpha \quad (\text{F.3})$$

This $\hat{p}_L(\alpha)$ has the property that $\Pr[\hat{p}_L(\alpha) \leq p] \geq 1 - \alpha$

Reference 7 develops a method of randomizing discrete LCL procedures which are conservative, and illustrates it for the binomial case. The exact randomized $100(1-\alpha)\%$ binomial LCL for p is developed as follows.

a. The number of observed successes X is a binomial random variable with probability mass function $B(x;N,p)$.

Define $h(p)$ as the smallest integer such that

$$\Pr(X > h(p)) \leq \alpha \quad x = 0, 1, \dots, N \quad (F.4)$$

Thus, $h(p)$ denotes an observed value of X .

b. Define

$$\alpha_p \equiv \Pr(X > h(p)) \quad (F.5)$$

and
$$y(p) \equiv F^{-1} \left\{ 1 - (\alpha - \alpha_p) / B(h(p); N, p) \right\} \quad (F.6)$$

where F is the "randomizing" distribution function of Y . Reference 7 shows that F can be the uniform $(0,1)$ distribution function without loss of generality. Then using equation (F.5), $y(p)$ can be simplified to

$$y(p) = 1 - (\alpha - \Pr(X > h(p))) / \Pr(X = h(p)) \quad (F.7)$$

Finally, define

$$g(p) \equiv h(p) + y(p) \quad (F.8)$$

Then the exact binomial randomized LCL for p , $L^*(x+y)$ is given by

$$L^*(x+y) = \begin{cases} g^{-1}(x+y); & 1 - \alpha \leq x+y \leq N+1-\alpha \\ 0 & ; \quad x+y < 1-\alpha \\ 1.0 & ; \quad x+y > N+1-\alpha \end{cases} \quad (F.9)$$

To solve for $L^*(x+y)$, form the equality

$$g(p) = X + Y = h(p) + y(p) \quad (F.10)$$

Since $h(p)$ is an observed value of X , this reduces to

$$Y = y(p) = 1 - (\alpha - \Pr(X > h(p))) / \Pr(X = h(p)) \quad (F.11)$$

By solving for p and applying equation (F.9)

$$L^*(x+y) = \begin{cases} \text{the solution for } p \text{ in} \\ (1-Y)\Pr(X=x) + \Pr(X>x) = \alpha & 1-\alpha \leq x+y \leq N+1-\alpha \\ 0 & x+y < 1-\alpha \\ 1.0 & x+y > N+1-\alpha \end{cases} \quad (\text{F.12})$$

Thus, for no failures, or $X=N$, $L^*(x+y)$ is the solution for p in

$$(1-Y)p^N = \alpha \quad (\text{F.13})$$

$$\text{or } \hat{P}_L(\alpha) = \begin{cases} \left(\frac{\alpha}{1-Y}\right)^{1/N} & y \leq 1-\alpha \\ 1.0 & y > 1-\alpha \end{cases} \quad (\text{F.14})$$

when $X=N-1$, or one failure, $L^*(x+y)$ is the solution for p in

$$(1-Y)\Pr(X=x) + \Pr(X>x) = \alpha \quad (\text{F.15})$$

$$\text{or } (1-Y)Np^{N-1}(1-p) + p^N = \alpha$$

When two or more failures occur, a component LCL procedure such as $L^*(x+y)$ cannot be used. The original Log-Gamma procedure can, however, be "randomized" as follows. Draw a uniform (0,1) random number for each type of component that fails. Denote it Y_i for the i th component, and compute the "randomized" number of failures

$$F^* = F_i - Y_i \quad (\text{F.16})$$

where F_i is the observed number of failures on the i th component.

The Randomized Log-Gamma procedure (RLG) was developed by trial and error via computer simulation, using the above adjustments where most effective. It can be stated as follows:

Randomized Log-Gamma Method

a. Zero failures in all components:

$$\hat{R}_{S,L}(\alpha) = \begin{cases} \left(\frac{\alpha}{1-Y}\right)^{1/N_1} & y \leq 1 - \alpha \\ 1.0 & y > 1 - \alpha \end{cases} \quad (\text{F.17})$$

where $Y \sim \text{uniform}(0,1)$.

b. One failure among all components:

$$\hat{R}_{S,L}(\alpha) = \begin{cases} \text{the solution for } p \text{ in} \\ (1-Y)N_1 p^{N_1-1}(1-p) + p^{N_1} = \alpha \end{cases} \quad (\text{F.18})$$

The use of α^{1/N_1} as an approximation to $\hat{R}_{S,L}(\alpha)$, for $y > 1 - \alpha$, is reasonably accurate for low N_1 and very accurate for large N_1 , as shown below.

TABLE XVI
Accuracy of the Approximation α^{1/N_1}

$Y = .9$

$\alpha = .2$

Actual $\hat{R}_{S,L}(\alpha)$	N =	10	20	50
α^{1/N_1}		.837	.915	.965
		.851	.923	.968

c. Two through four component failures:

For each different type of component that had at least one failure, draw a uniform (0,1) random number.

Denote it Y_i for the i th component, and compute

$$F_i^* = F_i - Y_i \quad (F.19)$$

Compute Log-Gamma method with exact degrees of freedom and F_i^* instead of the observed number of failures F_i .

d. Five or more failures:

Use the Log-Gamma method with exact degrees of freedom.

APPENDIX G

AN EXAMPLE OF THE RANDOMIZED LOG-GAMMA PROCEDURE

Let $N_1 = 20$, $K = 15$, and $\alpha = .2$. Suppose that the replica has no failures. Then a uniform $(0,1)$ random number Y is generated; say $Y = .5$. Then since $Y < 1 - \alpha$,

$$\hat{R}_{S,L(.2)} = \left(\frac{.2}{1-.5}\right)^{\frac{1}{.20}} = .955$$

as opposed to the classical binomial LCL of .923.

Suppose some future replica had one failure in some component, and $Y = .6$ is generated. Then $\hat{R}_{S,L(.2)}$ is the solution for p in the equation

$$20p^{19}(1-p)(1-.6) + p^{20} = .2$$

Therefore, $\hat{R}_{S,L(.2)} = .892$. The classical binomial LCL is .858.

Now, suppose another replica had two failures in the 5th component and one failure in the 15th component, and that $Y_5 = .3$ and $Y_{15} = .8$ were generated. Thus, $F_5^* = 1.7$ and $F_{15}^* = .2$. Applying the LG procedure, $N_5 = 20$, $N_{15} = 19$

$$\hat{S} = \sum_{i=1}^{15} \hat{T}_i = \hat{T}_5 + \hat{T}_{15} = .0969$$

$$\hat{r} = \frac{(\hat{S})^2}{\sum_{i=1}^{15} \frac{\hat{T}_i}{N_i}} = \frac{(.0969)^2}{.0049} = 1.926$$

$$2\hat{r} = 3.853$$

$$\chi_{.8,3.853}^2 \approx 1.56$$

Then $\hat{R}_{S,L(.2)} = \text{Exp} \left\{ \frac{-2\hat{r}\hat{S}}{\chi^2_{.8,3.853}} \right\} = .787$

without randomization, the LCL would have been

$$\hat{R}_{S,L(.2)} = .740$$

APPENDIX H

THE COMPUTER PROGRAM

The following pages list the computer program used for the simulations. Flags were used to designate the portions of the program to be executed, depending on which modifications were desired. Proprietary subroutines [Ref. 9] were utilized for inverse chi square values (MDCHI), sorting the approximate distributions of $\hat{R}_{S,L(\alpha)}$, (VSORTA), generation of uniform (0,1) variates (GGUB), and solving for the RLG procedure (ZREAL2). The APL subroutines HISTLIST and HIST were used in Table X and Figures 4 and 5 respectively [Ref. 10]. The plotting subroutine PLOTP was used in Tables VII and VIII [Ref. 11].

The program was written in the FORTRAN IV language for use on the IBM-360 system at the Naval Postgraduate School.

```

COMMON /EALNI /UUC, ALPHA
DIMENSION FER(500), RI(500), ROR(500), R02(500), RIB(500), RT9(500)
DIMENSION A(40), KFF(40,250), RAMEDA(40), I(40,250)
DIMENSION TLAS(800), ZL(40), KTSUM(40), NN(40), TSUM(40), B(40)
DIMENSION C(40,250), Z(800)
DIMENSION TLSUM(40), TL(40,250), ILROSS(40)
EXTERNAL FCN
FIMAL=1.
BETA=1.
AL=150.
NUSE=NI
NK=1
KKNI=NK-1, NKMI
DC 9090 K=1, NKMI
R(K)=.557
R(K)=.557
R(K)=.557
NN(1)=150
NN(2)=70
NN(3)=75
NN(4)=125
NN(5)=125
NN(6)=18
NN(7)=28
NN(8)=125
NN(9)=63
NN(10)=125
NN(11)=59
NN(12)=5
NN(13)=10
NN(14)=30
R(2)=.585
R(3)=.579
R(4)=.588
F(5)=.682
K(6)=.58
R(7)=.567
R(8)=.595
R(9)=.57
R(10)=.555
R(11)=.568
R(12)=.5
R(13)=.59
WRITE(C(1,103))
WRITE(C(1,100))
WRITE(C(1,101)) AL, NK
WRITE(C(1,1007)) BETA
WRITE(C(1,1009)) I, FINAL
WRITE(C(1,1023)) (K, R(K), F=I, NK)
WRITE(C(1,1020)) (I, JK, AN(I,JK), I, JK=1, NK)
SYSREL=1.0
DC 55 K=1, NK
RAMEDA(K)=-1.*ALCG(R(K))
SYSPFL=SYSPFL*(K)
CONTINUE
KKKMAX=0
KKKMIN=INT(0.5*NI)
KFAIL=0
LC 77 L=1, 500
ZD=C.
ZT=0.
LU=FINAL*FICAL(NI)
ZUVAR=C.
ZVVAR=0.
KUTAL=0

```

```

9C90
C
C
C
55

```

```

TTTT00010
TTTT00020
TTTT00030
TTTT00040
TTTT00050
TTTT00060
TTTT00070
TTTT00080
TTTT00090
TTTT00100
TTTT00110
TTTT00120
TTTT00130
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TTTT00630
TTTT00640
TTTT00650
TTTT00660
TTTT00670
TTTT00680
TTTT00690
TTTT00700

```

```

C INPLTS FOR ZRFALZ
EPS=1.E-03
ETA=1.E-03
NSIG=2
C NPLAC=0 MEANS NI EVEN
NPLAC=0
IF(MOD(NPLAC,2).NE.0) NPLAC=1
NRG1=2
NRG2=3
NRG3=1
IF(NX=0)
  DO 88 K=1,NK
  NTEMP=ANK J
  CALL GGH(MX,NTEMP,Z)
  IFK=EC*ANK) GO TO 58
  DO 44 J=1,NTEMP
    44 U(K,J)=Z(J)
    58 KLT=1
  KCOUNT=0
  TLSUM(K)=0.
  TFSUM(K)=0.
  KFSUM(K)=0.
  LL=99 J=1,NTEMP
  IF(K=EC*NK) GO TO 27
  IF(U(K,J).LE.1.)AND.(U(K,J).GE.R(K)) GO TO 5
  CC=TC 18 J=1,LE-1.)AND.(Z(J).GE.R(K)) GO TO 57
  TLAST(J)=1
  RELAST(J)=0
  GO TO 50
  18 T(K,J)=1
  C FAILURE
  CC=TC 10)
  RELAST(J)=1
  KCOUNT=KCOUNT+1
  TLAST(J)=(1.-ALCG(Z(J)))/RABDA(K))**(1./BETA)
  TLAST(KCOUNT)=TLAST(J)+FLCAT(J-KLT)
  KLT=J+1
  TFSUM(K)=TFSUM(K)+TL(K,KOUNT)
  GO TO 58
  5 KFK=J=1
  KCOUNT=KCOUNT+1
  T(K,J)=(1.-ALCG(U(K,J)))/PABDA(K))**(1./BETA)
  T(K,KOUNT)=T(K,J)+FLCAT(J-KLIF)
  KLF=J+1
  TFSUM(K)=TFSUM(K)+T(K,KOUNT)
  GO TO 10
  58 TFSUM(K)=TFSUM(K)+TLAST(J)
  KFSUM(K)=KFSUM(K)+RELAST(J)
  GO TO 99
  10 T(K,K)=TFSUM(K)+T(K,J)
  KFSUM(K)=KFSUM(K)+KFK(J)
  55 KCOUNT=KCOUNT+1
  T(K,K)=T(K,K)+KFSUM(K)
  IF(KFSUM(K).EQ.0) CC=TC 65
  IF(TFSUM(K).LT.CC) CC=TLSUM(K)
  C FAILURE
  T(K,KOUNT)=T(K,K)+KFSUM(K)
  A(K)=(2.*FLOAT(NTEMP)-3.)/(2.*FLOAT(NTEMP)-2.)
  E(K)=FLOAT(NTEMP)/FLCAT(NTEMP)-1.)
  KFSUM=FLCAT(KFSUM(K))
  ZDK=AKI+(KFSUM/FLCAT(NTEMP))+(B(K)/2.)*(R(K)-SUM/FLCAT(NTEMP))
  1*2
  ZDVAR=ZDVAR+(E(K)/FLCAT(NK))
  ZDT=ZDT+(E(K))
  ZTK=FLCAT(KFSUM(K))/TFSUM(K)
  ZTVF=ZTVF+(T(K)/FLCAT(NK))

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TTTC0710
TTTC0710
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TTTC1390
TTTC1400

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IF (IER.EQ.34) WRITE(6,5557) UNU
IF (IER.EQ.34) GO TO 5103
5553 FORWAT(IX, PNEWT1, 3F8.5)
5100 CONTINUE
DC 5161 IR=1,3
IF (PNEWT1(1R).LT.C.) GO TO 5161
IF (PNEWT1(1R).GT.1.) GO TO 5161
IRSAV=IR
5161 CONTINUE
IF (IP9.EQ.1) GO TO 5162
RDBLL=PNEWT1(MNSAV)
ITMAX=500
ALPHA=.1
IF 5=1
5171 CALL ZREAL2(FCN,EFS,EFS2,ETA,NSIG,NROOT3,PNEWT3,ITMAX,IER)
IF (IER.EQ.34) WRITE(6,5557) UNU
IF (IER.EQ.34) GO TO 5103
ITMAX=500
GO TO 516C
5162 K5(1)=PNEWT3(1KSAV)
GO TO 5103
5150 CONTINUE
C NI EVEN THU POSITIVE PCCTS
IF (LU.LT.8) GO TO 5808
RDBLL=.2*(1.78*ALN1)
IF (ULU.GE.5) R9(L)=1*(1./REALN1)
IF (U9U.GE.9) GO TO 5103
PNEWT1=.52
ALPHA=.1
ITMAX=500
CALL ZREAL2(FCN,EFS,EFS2,ETA,NSIG,NROOT1,PNEWT1,ITMAX,IER)
IF (IER.EQ.34) R9(L)=1*(1./REALN1)
IF (IER.EQ.34) GO TO 5103
R5(1)=PNEWT1
ITMAX=500
GO TO 5103
5808 IF (INDUSE.LE.25) PNEWT(1)=.89
PNEWT(2)=1.1
ALPHA=.2
CALL ZREAL2(FCN,EFS,EFS2,ETA,NSIG,NROOT,PNEWT,ITMAX,IER)
FORWAT(IX, PNEWT1, 2F8.5)
IF (IER.EQ.34) GO TO 5800
GO 5151 MN=1,2 MNSAV=MN
5151 RDBLL=PNEWT(MNSAV)
ITMAX=510
ALPHA=.1
CALL ZREAL2(FCN,EFS,EFS2,ETA,NSIG,NROOT,PNEWT,ITMAX,IER)
IF (IER.EQ.34) WRITE(6,5557) UNU
DC 5152 MN=1,2
IF (PNEWT(MN).LT.1.) MN9SAV=MN9
R9(1)=PNEWT(MN9SAV)
GO TO 5103
5890 CONTINUE
IF 2=2
C FAILED TO CONVERGE
PNEWT1=.55
IF (PNEWT1(1).LT.1.) PNEWT1=PNEWT(1)
IF (PNEWT1(1).GT.1.) GO TO 5881
RDBLL=.2*(1./REALN1)
R9(1)=1*(1./REALN1)
GO TO 5103
5881 RDBLL=PNEWT1
ITMAX=510
ALPHA=.1
CALL ZREAL2(FCN,EFS,EFS2,ETA,NSIG,NROOT1,PNEWT1,ITMAX,IER)
ITMAX=510

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111C2810
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LIST OF REFERENCES

1. Woods, W. M. and Borsting, J. R., A Method for Computing Lower Confidence Limits on System Reliability Using Component Failure Data with Unequal Sample Sizes, p. 1-23, U. S. Naval Postgraduate School, June 1968.
2. Gatcliffe, T. R., Accuracy Analysis for a Lower Confidence Limit Procedure for System Reliability, Master Thesis, U. S. Naval Postgraduate School, September 1976.
3. Lockheed Missiles and Space Company, Inc., Publication LMSC-D368002A, Reliability Evaluation Plan for Trident I (C4) Missile, Appendix C, p. C-1 to C-34.
4. Lieberman, G. and Ross, S., "Confidence Intervals for Independent Exponential Series Systems," Journal of the American Statistical Association, p. 837-840, December 1971.
5. Fishman, G. S., Concepts and Methods in Discrete Event Digital Simulation, p. 168, Wiley, 1973.
6. U. S. Naval Postgraduate School Report NPS-53ZH76047 The Influence of Data Order on the Lieberman-Ross Method, by A. L. Schoenstadt, p. 1-37, April 1976.
7. Barr, D. R. and Jayachandran, R., "Improved Confidence Bounds Applied to Reliability," IEEE Transactions on Reliability, Vol. R-24, No. 1, p. 67-68, 1 April 1975.
8. Bazovsky, I., Reliability Theory and Practice, p. 238, Prentice-Hall, 1961.
9. International Mathematical and Statistical Libraries, Inc., IMSL Library 1, FORTRAN IV, IBM S/370-360, p. MDCHI, VSORTA, ZREAL2, GGUB, IMSL, Houston, 1975.
10. Hutchinson, D. R., APL Histogram, Density Estimation, and Probability Plotting Routines, p. 10, 25, Master Thesis, U. S. Naval Postgraduate School, December 1976.
11. Subroutine Library of the W. R. Church Computer Center, U. S. Naval Postgraduate School, p. PLOTP, "Plot or Printer Multiple Curves," Monterey, 1974.

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