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ON THE RELATION BETWEEN THE FRACTURE STATISTICS OF VOLUME DISTRIBUTED AND SURFACE DISTRIBUTED CRACKS

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UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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It is shown for both theories that the distribution of crack critical strength can be given in closed form when the failure statistics in simple tension are known. It is also shown that the predicted biaxial failure statistics based on given uniaxial statistics are the same, regardless of which theory is used. Evidence is cited which suggests that the latter result may hold for shear-sensitive as well as shear-insensitive cracks.

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I. INTRODUCTION

Many structural materials, especially those with high strength at high temperature and those that are transparent to electromagnetic radiation, tend to be brittle. Brittle materials exhibit a substantial dispersion in fracture stress which must be taken into account in design. This is done by making use of the measured or theoretically determined fracture statistics for the material under the stress conditions encountered in service.

The statistical fracture theory most frequently employed is that proposed by Weibull[1] in 1939. This theory assumed that the variation in measured strength of nominally identical specimens is due to the presence of invisible flaws. The flaws have a distribution in strength characteristic of the material. As a result, different volume elements have different flaw contents and different strengths. A structure is assumed to fail when the weakest element is overloaded, just as a chain breaks when the strength of its weakest link is exceeded. Weibull worked out the fracture statistics of the weakest link concept, and in addition gave without proof a procedure for finding the fracture statistics for polyaxial stress states when they are known for uniaxial tension. Weibull's rules are very simple to apply when the uniaxial data are represented in terms of Weibull's 2-parameter formula.

A new weakest link theory was proposed recently by Batdorf and Crose.[2] This theory differed from Weibull's in several respects: (1) it identified the flaws as flat cracks, an assumption which implies that flaws have a strength which varies with their orientation with respect to the applied stresses, (2) it generalized Weibull's 3-parameter representation of experimental data and made more accurate fitting possible by the use of a Taylor expansion with an arbitrary number of coefficients, and (3) it assumed that only the component of stress normal to the crack plane contributes to the fracture of the crack. The latter assumption is a fair approximation and has the advantage that use of it avoids the necessity of specifying crack shapes and Poisson's ratio. The theory was formulated for volume distributed cracks in a macroscopically isotropic material, so all crack orientations were considered equally likely.

An analogous theory for surface-distributed cracks was subsequently developed by Batdorf.[3] This theory assumed that the crack plane is always normal to the surface. Thus the orientation of the surface can be specified with a single parameter, whereas the orientation of an interior crack requires two parameters.

In References 2 and 3 the coefficients in the Taylor series used to represent the distribution of crack strengths were determined by inverting a matrix. This implies restricting the series expansion to a limited number of terms, and thus involves an approximation. It was subsequently shown by D. J. Chang that an integral equation approach can be used which is in principle exact. In the following section exact approaches are given for both volume distributed and surface distributed cracks. Also a rather surprising fact relating the two theories is proved. This is, if the statistics of biaxial tension are derived from known statistics of uniaxial tension, the results obtained are the same irrespective of which theory is used. As a result the simpler surface crack theory can be used to find the fracture statistics of biaxially stressed structures with volume distributed cracks.

II. EVALUATION OF CRACK DENSITY FUNCTION

Volume Distributed Cracks

According to the statistical fracture theory outlined in Reference 2, the probability of survival of a volume V of material subjected to a uniform stress state Σ is

$$P_s(\Sigma) = \exp\left[- V \int_0^{\sigma_1} \frac{\Omega(\Sigma,\sigma_{cr})}{4\pi} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} \right] . \quad (1)$$

In this expression σ_1 is the maximum principal tensile stress. $N(\sigma_{cr})$ is the number of cracks per unit volume with a critical stress less than or equal to σ_{cr}, and Ω is a solid angle such that a crack with critical stress σ_{cr} will be fractured if and only if its normal lies within the solid angle. Ω can be calculated for any stress state, so the corresponding probability of failure can be determined if $N(\sigma_{cr})$ is known. $N(\sigma_{cr})$ can be evaluated if $P_s(\sigma)$ is known for any stress state. Generally the available data are for uniaxial tension. In Reference 2 an approximate technique was employed for obtaining $N(\sigma_{cr})$ from data in simple tension, which involved inverting the matrix of an ill conditioned set of simultaneous linear algebraic equations. A better technique is to treat Equation (1) as an integral equation. An elementary method of doing this suggested by B. Budiansky[4] follows.

It is shown in Reference 2 that for the case of simple tension, the assumption that only the component of stress normal to the crack plane contributes to fracture leads to

$$\frac{\Omega}{4\pi} = 1 - \sqrt{\frac{\sigma_{cr}}{\sigma}} . \quad (2)$$

Thus,

$$P_s(\sigma) = \exp\left[-V\int_0^{\sigma}\left(1-\sqrt{\frac{\sigma_{cr}}{\sigma}}\right)\frac{dN(\sigma_{cr})}{d\sigma_{cr}}\,d\sigma_{cr}\right] . \qquad (3)$$

Integrating by parts and making use of the fact that N(0) = 0, the integrated term vanishes and

$$P_s = \exp\left[-V\int_0^{\sigma}\frac{N(\sigma_{cr})}{2\sqrt{\sigma\sigma_{cr}}}\,d\sigma_{cr}\right] . \qquad (4)$$

Thus,

$$\sqrt{\sigma}\,\ln P_s = -\frac{V}{2}\int_0^{\sigma}\frac{N(\sigma_{cr})}{\sqrt{\sigma_{cr}}}\,d\sigma_{cr} \qquad (5)$$

$$\frac{d}{d\sigma}\left(\sqrt{\sigma}\,\ln P_s\right) = -\frac{V}{2}\frac{N(\sigma)}{\sqrt{\sigma}} \qquad (6)$$

from which

$$VN(\sigma) = -2\sqrt{\sigma}\frac{d}{d\sigma}\left\{\sqrt{\sigma}\,\ln P_s\right\} \qquad (7)$$

Surface Distributed Cracks

According to the analogous theory for surface cracks of Reference 3, the probability of survival in any stress state Σ is given by

$$P_s(\Sigma) = \exp\left[-A\int_0^{\sigma_1}\frac{\omega}{\pi}\frac{dN(\sigma_{cr})}{d\sigma_{cr}}\,d\sigma_{cr}\right] . \qquad (8)$$

where A is the area and ω is an angle such that a crack of critical stress σ_{cr} will be fractured if its normal lies in ω. It is shown in Reference 3 that for uniaxial tension

$$P_s(\sigma) = \exp\left[-\frac{2A}{\pi}\int_0^{\sigma}\frac{dN}{d\sigma_{cr}}\cos^{-1}\sqrt{\frac{\sigma_{cr}}{\sigma}}\,d\sigma_{cr}\right] . \qquad (9)$$

Taking the logarithm of both sides and integrating by parts we obtain

$$\ell n\ P_s(\sigma) = -\frac{2A}{\pi}\int_0^{\sigma}\frac{N(\sigma_{cr})}{\sqrt{\sigma_{cr}}\sqrt{\sigma-\sigma_{cr}}}\,d\sigma_{cr}\ . \qquad (10)$$

This is an Abel integral equation with the solution

$$AN(\sigma_{cr}) = -\sqrt{\sigma_{cr}}\int_0^{\sigma_{cr}}\frac{\frac{d}{d\sigma}\ \ell n\ P_s(\sigma)}{\sqrt{\sigma_{cr}-\sigma}}\,d\sigma\ . \qquad (11)$$

If desired, the singularity in the denominator can be removed by integrating by parts, which results in

$$AN(\sigma_{cr}) = 2\sqrt{\sigma_{cr}}\int_0^{\sigma_{cr}}\frac{d^2}{d\sigma^2}\left[\ell n\ P_s(\sigma)\right]\sqrt{\sigma_{cr}-\sigma}\ d\sigma\ . \qquad (12)$$

III. EQUIVALENCE OF THE TWO THEORIES

In this section we assume that the probability of survival is known for simple tension and calculate the probability of survival for biaxial tension. The latter result turns out to be the same for surface distributed cracks as it is for volume distributed cracks.

If the probability of survival in simple tension is a reasonably well-behaved function of the stress level, it can be expressed to arbitrary accuracy by the power series

$$P_s(\sigma) = \exp\left[-\Sigma\, a_i\, \sigma^i\right] \quad . \tag{13}$$

Using the theory of the previous section, we use equation (13) to find $VN(\sigma_{cr})$ and $AN(\sigma_{cr})$, respectively, and then calculate the probability of failure for a biaxial stress state.

<u>Volume Distributed Cracks</u>

Substituting Equation (13) into Equation (7) we obtain

$$VN(\sigma) = 2\sqrt{\sigma}\,\frac{d}{d\sigma}\left\{\sqrt{\sigma}\,\Sigma\, a_i\, \sigma^i\right\} \quad . \tag{14}$$

Carrying out the indicated operations we obtain

$$VN(\sigma_{cr}) = 2\sum_i (i + 0.5) a_i\, \sigma_{cr}^{\ i} \quad . \tag{15}$$

For each of the special cases of simple tension and equibiaxial tension, $\Omega/4\pi$ is a simple function of σ_{cr}/σ. For other cases we can define Ω as

$$\Omega = \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\theta_1}^{\theta_2} \sin\theta\, d\,\theta \quad , \tag{16}$$

when the coordinate system used is shown in Figure 1. The limits φ_1, φ_2, θ_1 and θ_2 are chosen in such a manner that the integral is taken only over the

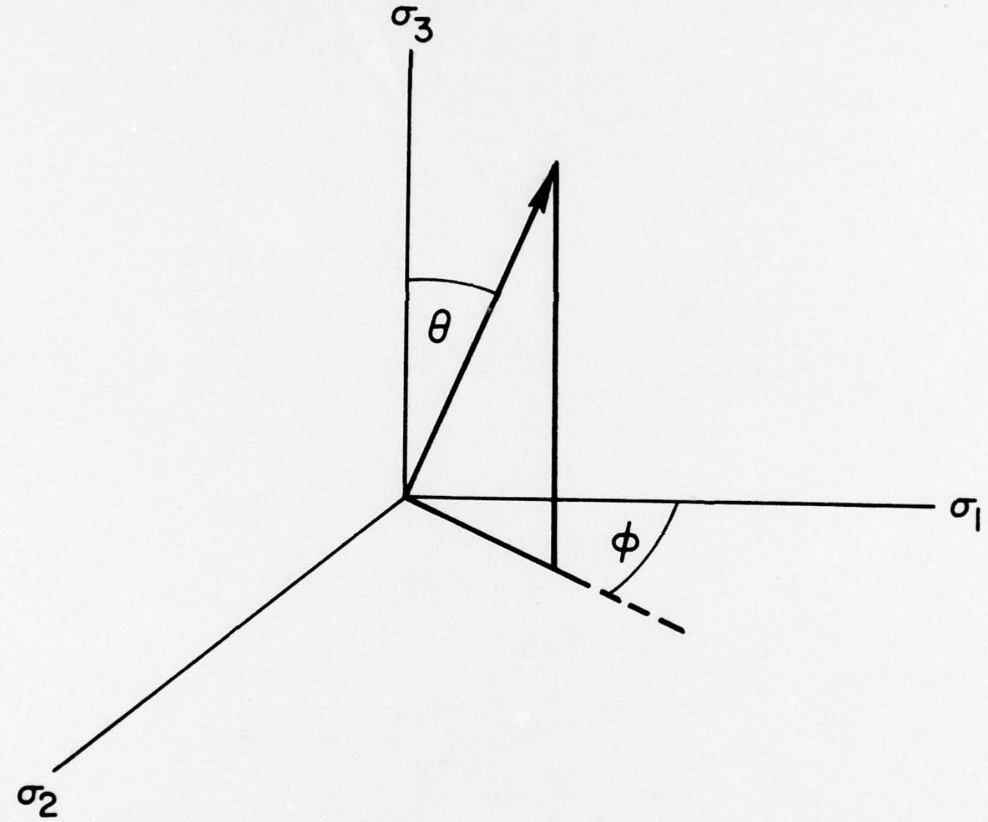

Figure 1. Coordinate System Employed.

range of angles φ and θ within which $\sigma_n > \sigma_{cr}$. Alternatively, we can define Ω by

$$\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, H(\sigma_{cr}, \sigma_n) \, d\theta \quad , \qquad (17)$$

where

$$H(\sigma_{cr}, \sigma_n) = 1 \quad \text{for } \sigma_{cr} < \sigma_n \qquad (18a)$$

$$= 0 \text{ for } \sigma_{cr} > \sigma_n \quad . \qquad (18b)$$

Substituting Equation (17) into Equation (1) and integrating over σ_{cr} we obtain

$$P_s(\Sigma) = \exp\left[- V \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, N(\sigma_n)\right] \quad . \qquad (19)$$

We note in passing that this equation is equivalent to the procedure for polyaxial stress states given without proof by Weibull in Reference 1.

Substituting Equation (15) into Equation (19) and making use of the relation

$$\sigma_n = (\sigma_1 \cos^2\varphi + \sigma_2 \sin^2\varphi) \sin^2\theta = \sigma_1(\cos^2\varphi + K\sin^2\varphi)\sin^2\theta \quad , \qquad (20)$$

where $\quad K \equiv \sigma_2/\sigma_1 \quad ,$ (21)

we obtain

$$\ln P_s = -\frac{1}{2\pi} \sum_i a_i(i + 0.5)\sigma_1{}^i \int_0^{2\pi} (\cos^2\varphi + K \sin^2\varphi)^i \, d\varphi$$

$$\times \int_0^{\pi} \sin^{2i+1} \theta \, d\theta$$

$$= -\frac{1}{2\pi} \sum a_i(i + 0.5)\sigma_1{}^i \, \Phi(i, K) \, B(i + 1, \tfrac{1}{2}) \quad , \qquad (22)$$

where B is the beta function and

$$\Phi(i,\ K) = \int_0^{2\pi} (\cos^2\varphi + K \sin^2\varphi)^i \, d\varphi$$

$$= \int_0^{2\pi} \left[1 - (1-K)\sin^2\varphi\right]^i \, d\varphi \tag{23}$$

The substitution

$$\sin^2\varphi = z \tag{24}$$

leads to

$$\Phi(i,\ K) = 2 \int_0^1 \frac{[1 - (1-K)\ z]}{\sqrt{z}\ \sqrt{1-z}} \, dz\ . \tag{25}$$

Expressing the beta function in terms of gamma functions, we obtain from Equation (22)

$$\ell n\ P_s = - \frac{1}{2\pi} \sum_i \sigma_1^{\ i}\ a_i \frac{\Gamma(i+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i + \frac{1}{2}\right)} \Phi(i,\ K)\ . \tag{26}$$

For present purposes there is no need to evaluate $\Phi(i,\ K)$ since it will appear also in the expression for the probability of failure using the theory for surface distributed cracks.

<u>Surface Distributed Cracks</u>

Substituting Equation (13) into Equation (11) we obtain

$$AN(\sigma_{cr}) = \sqrt{\sigma_{cr}} \int_0^{\sigma_{cr}} \frac{\frac{d}{d\sigma} \Sigma\ a_i \sigma^i}{\sqrt{\sigma_{cr} - \sigma}} \, d\sigma$$

$$= \sqrt{\sigma_{cr}} \int_0^{\sigma_{cr}} \frac{\Sigma i\ a_i\ \sigma^{i-1}\ d\sigma}{\sqrt{\sigma_{cr} - \sigma}}\ . \tag{27}$$

If we let

$$\frac{\sigma}{\sigma_{cr}} = x \quad , \tag{28}$$

then Equation (27) becomes

$$AN(\sigma_{cr}) = \sum_i i\, a_i \sigma_{cr}{}^i \int_0^1 \frac{x^{i-1}\, dx}{\sqrt{1 - x}} \tag{29}$$

$$= \sum_i i\, a_i\, \sigma_{cr}{}^i \frac{\Gamma(i)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i + \frac{1}{2}\right)} \tag{30}$$

$$= \Sigma\, \alpha_i\, \sigma_{cr}{}^i \quad , \tag{31}$$

where

$$\alpha_i = a_i \frac{\Gamma(i+1)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i+\frac{1}{2}\right)} \quad . \tag{32}$$

In the case of biaxial tension, Reference 3 gives for the probability of survival

$$\ell n\ P_s = -\frac{A}{\pi} \int_0^{\sigma_1} 2\theta_{cr} \frac{dN}{d\sigma_{cr}}\, d\sigma_{cr} \quad , \tag{33}$$

where

$$\theta_{cr} = \cos^{-1} \sqrt{1 - \frac{\sigma_1 - \sigma_{cr}}{\sigma_1 (1-K)}} \quad \text{for} \quad K\sigma_1 < \sigma_{cr} < \sigma_1 \tag{34a}$$

$$= \frac{\pi}{2} \qquad \text{for} \quad \sigma_{cr} < K\sigma_1 \quad . \tag{34b}$$

Integrating by parts

$$\ell n\ P_s = -\frac{A}{\pi} \int_{K\sigma_1}^{\sigma_1} \frac{N(\sigma_{cr})\ d\sigma_{cr}}{\sqrt{\frac{\sigma_1 - \sigma_{cr}}{\sigma_1 (1 - K)}} \sqrt{1 - \frac{\sigma_1 - \sigma_{cr}}{\sigma_1 (1 - K)}}\ \sigma_1 (1 - K)} \quad . \tag{35}$$

Making the substitution

$$z = \frac{\sigma_1 - \sigma_{cr}}{\sigma_1(1 - K)} \quad , \tag{36}$$

and substituting Equation (31) into Equation (35) we obtain

$$\ell n\ P_s = -\frac{1}{\pi}\int_0^1 \frac{\Sigma\alpha_i \sigma_{cr}{}^i\ dz}{\sqrt{z}\ \sqrt{1 - z}} \quad . \tag{37}$$

Now from (36) it follows that

$$\sigma_{cr} = \sigma_1 - z(1-K)\sigma_1 \tag{38}$$

Substituting Equations (38) and (32) into Equation (37) we obtain

$$\ell n\ P_s = -\frac{1}{2\pi}\ \Sigma\ a_i(\sigma_1)^i \frac{\Gamma(i+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i + \frac{1}{2}\right)}\ \Phi(i,\ K) \quad , \tag{39}$$

which is identical to Equation (26).

IV. DISCUSSION

What we have just demonstrated is that when the uniaxial stress statistics are known, the biaxial stress statistics turn out to be the same, irrespective of whether we employ the theory for volume distributed cracks or the theory for surface distributed cracks. Since the statistics for any stress ratio are uniquely related to the statistics for any other stress ratio, it is not necessary to use uniaxial stress statistics as the starting point. The statistics for any convenient stress ratio can be taken as known and the statistics for any other stress ratio can be calculated; using either theory, the same result will be obtained. Since surface theory involves an integration over only one variable whereas volume theory requires integration over two, it is generally advantageous to use surface theory.

The preceding remarks apply to the case of shear-insensitive cracks, i.e., to the case in which only the normal stress contributes to fracture. It is known that the shear acting on the crack plane also contributes to fracture, and it would therefore be of considerable interest to know whether the same result holds for shear-sensitive cracks. This is not an easy question to settle in complete generality. First, there is as yet no concensus in the fracture mechanics community as to how to treat mixed mode fracture. Thus, we would have not one but several theories of shear-sensitive cracks to investigate. Second, even the simple case of shear-insensitive cracks required a somewhat involved theoretical analysis, and it is to be expected that the analysis for shear-sensitive cracks, in some cases at least, would be considerably more involved. However, it is easy to show that the same rule applies in a particularly simple special case. This is the case in which $P_f(\sigma, \sigma)$ is calculated from $P_f(\sigma, 0)$ when the effective stress leading to fracture is assumed to be

$$\sigma_{eff} = \sqrt{\sigma_n^2 + \tau^2} \quad , \tag{42}$$

where σ_n and τ are the normal and shear stress acting on the crack plane respectively. A proof for this statement is outlined in the Appendix. Although the matter is not yet settled, it is the present view of the writers that, at least for some shear-sensitive crack models, the same rule applies.

Finally, it should be noted that the equivalence between volume and surface distributed crack theories is restricted to the question of using the known fracture statistics for one stress ratio to calculate the statistics for another stress ratio. The distributions of crack critical stress deduced from a given set of data, i.e., the functional forms found for $N(\sigma_{cr})$, will be quite different for the two theories.

APPENDIX

Let us consider a particular type of shear-sensitive crack, i.e. that for which fracture occurs when the function

$$\sigma_{eff} = \sqrt{\sigma_n^{\ 2} + \tau^2} \tag{A1}$$

reaches a critical value. Since

$$\sigma_n = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta \tag{A2a}$$

$$\tau = (\sigma_1 - \sigma_2)\sin\theta \cos\theta \quad , \tag{A2b}$$

it follows that for uniaxial tension

$$\sigma_{eff} = \sigma \cos\theta \quad . \tag{A3}$$

For the volume distributed crack theory, Equation (1) still applies, but Ω is changed:

$$\frac{\Omega}{4\pi} = 1 - \sigma_{cr}/\sigma \qquad \text{uniaxial tension} \tag{A4a}$$

$$= \sqrt{1 - (\sigma_{cr}/\sigma)^2} \qquad \text{equibiaxial tension} \tag{A4b}$$

When $P_s(\sigma, 0)$ is known we can find $N(\sigma)$ in the following manner. Combining Equations (1) and (A4a),

$$\ell n\ P_s = -V\int_0^{\sigma} \left(1 - \frac{\sigma_{cr}}{\sigma}\right) N'(\sigma_{cr})d\sigma_{cr} \quad . \tag{A5}$$

Integrating by parts we obtain

$$\ell n\ P_s = -\frac{V}{\sigma}\int_0^{\sigma} N(\sigma_{cr})d\sigma_{cr} \quad , \tag{A6}$$

whence

$$V\,N(\sigma) = -\frac{d}{d\sigma}\left[\sigma\ \ell n\ P_s(\sigma,0)\right] \quad . \tag{A7}$$

The plan of attack here is to assume a very general form for $P_s(\sigma, 0)$ and use volume distributed crack theory to obtain $P_s(\sigma, \sigma)$. Using this value of $P_s(\sigma, \sigma)$ as given, we will use surface distributed crack theory to calculate $P_s(\sigma, 0)$, and show that it is the same as what we used at the outset.

With this in mind we assume

$$\ln P_s(\sigma, 0) = -\Sigma\, a_i \sigma^i \quad . \tag{A8}$$

Combining Equations (1), (A4b), (A7), and (A8), we obtain

$$\ln P_s(\sigma, \sigma) = -\sum_i a_i (i + 1) i \int_0^\sigma \sigma_{cr}^{i-1} \sqrt{1 - (\sigma_{cr}/\sigma)^2}\, d\sigma_{cr} \quad . \tag{A9}$$

Substitution of

$$x = (\sigma_{cr}/\sigma)^2 \tag{A10}$$

converts the integral into a standard form for the beta function, and the result expressed in terms of gamma functions is

$$\ln P_s(\sigma, \sigma) = -\Sigma\, \alpha_i \sigma^i \tag{A11}$$

where

$$\alpha_i = a_i (i + 1) i \frac{\Gamma(0.5i)\Gamma(1.5)}{\Gamma(0.5i + 1.5)} \quad . \tag{A12}$$

Now, according to the surface distributed crack theory,

$$N(\sigma) = -\ln P_s(\sigma, \sigma) \quad . \tag{A13}$$

For the shear-sensitive cracks under consideration,

$$\ln P_s(\sigma, 0) = -A \int N'(\sigma_{cr}) \cos^{-1} (\sigma_{cr}/\sigma) d\sigma_{cr} \quad . \tag{A14}$$

Integrating by parts, and using Equations (A10), (A11), and (A13), we obtain

$$\ln P_s(\sigma, 0) = \frac{1}{\pi} \Sigma\, \alpha_i \sigma^i \int_0^1 \frac{y^{0.5(i - 1)}}{\sqrt{1 - y}}\, dy \tag{A15}$$

$$= \frac{1}{\pi} \Sigma \frac{\alpha_i}{i + 1} \sigma^i \frac{\Gamma(0.5i + 0.5)\Gamma(0.5)}{\Gamma(0.5i + 1)} \quad . \tag{A16}$$

Substitution of Equation (A12) into Equation (A16) and some manipulation using the properties of gamma functions leads to the expected result

$$\ln P_s(\sigma, 0) = - \Sigma\, a_i\, \sigma^i \quad . \tag{A17}$$

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