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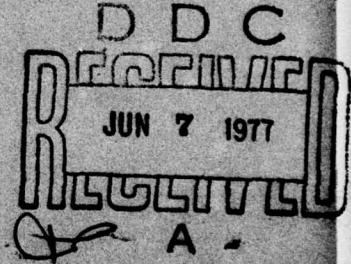
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S. B. BATDORF
H. L. HEINISCH, JR.

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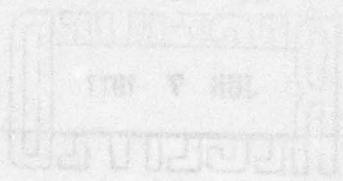
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ARBITRARY FRACTURE CRITERION

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**Weakest Link Theory Reformulated for
Arbitrary Fracture Criterion**

S. B. Batdorf and H. L. Heinisch, Jr.

Abstract

A primary objective of statistical fracture theory is to predict the probability of failure for an arbitrary stress state when the failure statistics are known for a particular stress state, e.g., simple tension. It is proved that the rule for accomplishing this given without proof by Weibull in 1939 is valid for shear-insensitive cracks, i.e., on the assumption that only the component of stress normal to a crack plane contributes to its fracture. Four different failure criteria for shear-sensitive cracks are considered, and the results are compared with test data.

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Introduction

It is widely believed that fracture of brittle materials at stresses far below their theoretical strength is due to the presence of microcracks. According to Griffith,^{1,2} during loading the crack is unaffected until a stress or combination of stresses is reached that causes catastrophic crack growth. Weibull³ introduced a statistical analysis to account for the dispersion in fracture stress and the way in which the average fracture stress and also the dispersion are affected by specimen size. In Weibull's theory the total volume is divided conceptually into many volume elements, each of which has a small probability of failure. The probability of survival of the part as a whole is then found by multiplying together the probabilities of survival of all the elements. The elements are considered to be analogous to the links of a chain, with the weakest link determining the strength of the chain. Thus the properties of a volume element, as inferred from the statistics of fracture in simple tension or bending, played a central role in Weibull's theory. No direct use was made of the hypothesis that fractures are due to crack growth. A rule was given without formal proof for finding the probability of failure under polyaxial stresses when the uniaxial statistics are known.

Recently Batdorf and Crose⁴ proposed a statistical theory in which attention is focussed on the cracks, and their failure under stress. For simplicity the fracture of a crack was assumed to depend only on the component of stress normal to the crack plane. The theory leads in a very straightforward fashion to a technique for finding the fracture statistics for polyaxial stress states when they are known for the uniaxial case.

A number of investigators have questioned the legitimacy of Weibull's rule for polyaxial stress states, and some have advocated the so-called

"postulate of independence".^{5,6,7} Batdorf⁸ showed that use of this postulate is generally unconservative, and compared its predictions with those of Weibull's theory for biaxial tension.

It is shown in the next section that Weibull's rule for treating polyaxial stress states is equivalent to the technique derived in Reference 4. This means that Weibull's rule is correct if cracks obey the assumptions of Reference 4, e.g., independence, random orientation, and shear insensitivity. Since the presence of shear on the crack plane reduces the normal stress needed to produce fracture, it brings the curves for uniaxial tension and biaxial tension closer together. Thus if uniaxial data are used as the basis, biaxial fracture stress predictions will be conservative when shear on the crack plane is neglected. However, better results can be obtained by using a better fracture criterion.

When shear is taken into account a complication arises in that it is necessary to specify the shape of the cracks, and sometimes Poisson's ratio as well. An additional complication is that there is not as yet a consensus regarding how to treat mixed mode fracture. Because of this, in the present paper several criteria are treated. The results obtained are compared with test data. Agreement is best for the relatively shear sensitive crack models arising from strain energy release rate considerations.

Theory

We first derive the fundamental equation of weakest link theory. Let it be assumed that a stressed solid can fail due to any of a number of independent and mutually exclusive mechanisms or causes, each having an infinitesimal probability of failure $(\Delta P_f)_i$. The probability that the i 'th mechanism will

not cause failure is $(P_S)_i = 1 - (\Delta P_f)_i$. The overall probability of survival is the product of the individual probabilities of survival, i.e.,

$$P_S = \prod_i (P_S)_i = \prod_i \left\{ 1 - (\Delta P_f)_i \right\}$$

$$\approx \prod_i \exp \left[- (\Delta P_f)_i \right] = \exp \left[- \sum_i (\Delta P_f)_i \right] \quad (1)$$

The sum of the individual probabilities of failure appearing in the final equality above was called by Weibull the "risk of failure" and was given the symbol B.

The potential causes of failure are the individual cracks. For purposes of analysis it is convenient to group the cracks according to location, the applied stress state, and crack critical stress. We assume that the stress state varies slowly so that within a volume element ΔV all cracks will be subject to the same macroscopic stress. We also assume that the material is macroscopically homogeneous, so that a function $N(\sigma_c)$ can be defined as the number of cracks per unit volume having a critical stress equal to or less than σ_c . The critical stress of a crack is defined as the remote stress which will cause fracture when applied normal to the crack plane. The probability that a crack having a critical stress in the range σ_c to $\sigma_c + d\sigma_c$ exists in volume element ΔV is then $\Delta V [dN(\sigma_c)/d\sigma_c] d\sigma_c$.

If such a crack does exist, the probability that it will fracture depends on its orientation, the stress state, and the fracture criterion. We assume there is a solid angle Ω such that fracture of a crack will occur if and only if its normal lies within Ω . This means that if the normal lies within Ω , $\sigma_e > \sigma_c$, where σ_e is the effective stress corresponding to the fracture

criterion selected. If the cracks are randomly oriented, the probability that a crack will fracture under the applied stress Σ is $\Omega(\Sigma, \sigma_c)/4\pi$.

Now the probability of failure due to a crack in the critical stress range $d\sigma_c$ located in volume element ΔV is the product of the above probabilities, i.e.,

$$(\Delta P_f)_i = \left(\Delta V \frac{dN(\sigma_c)}{d\sigma_c} d\sigma_c \right) \left(\frac{\Omega(\Sigma, \sigma_c)}{4\pi} \right) \quad (2)$$

Substituting equation (2) into equation (1) and changing sums into integrals we obtain

$$P_S = \exp \left[- \int dV \int d\sigma_c \frac{dN}{d\sigma_c} \frac{\Omega}{4\pi} \right] \quad (3)$$

which is the result given previously in Reference 4.

We note in passing that $N(\sigma_c)$ is independent of stress state and depends only on the material. Since σ_c is defined as the stress which causes fracture when applied normal to the crack plane, $N(\sigma_c)$ can be converted directly into crack size distribution when K_{Ic} is known. In this respect it differs from Weibull's n and the $g(S)$ of McClintock et al.^{9,10} These functions represent the number of cracks per unit volume that will be fractured by a particular applied stress state, and they depend on the stress ratio. More precisely, they are essentially equivalent to the second integral in Equation (3). The function $g(S)$ is the same as N for the particular case of hydrostatic tension, because then Ω is 4π or zero depending on the stress level.

A consideration of Ω for simple tension serves to illustrate the dependence of this function on the fracture criterion selected. The normal stress criterion assumes an effective stress

$$\sigma_e = \sigma_n = \sigma \cos^2 \theta \quad (4a)$$

where θ is the angle between the tensile axis and the crack normal. The assumption that fracture occurs when the local stress at some point on the surface of the crack cavity reaches the ultimate strength of the material leads to a result which depends on crack shape. For Griffith cracks and penny-shaped cracks the results are^{11,12,13}

$$\sigma_e = 0.5 \left[\sigma_n + \sqrt{\sigma_n^2 + \tau^2} \right] \text{ (G.C.)} \quad (4b)$$

$$\sigma_e = 0.5 \left[\sigma_n + \sqrt{\sigma_n^2 + \tau^2 / (1-0.5\nu)^2} \right] \text{ (P.S.C.)} \quad (4c)$$

For simple tension the shear on the crack plane is

$$\tau = \sigma \sin \theta \cos \theta \quad (5)$$

A couple of criteria related to strain energy release rate considerations are^{14,15}

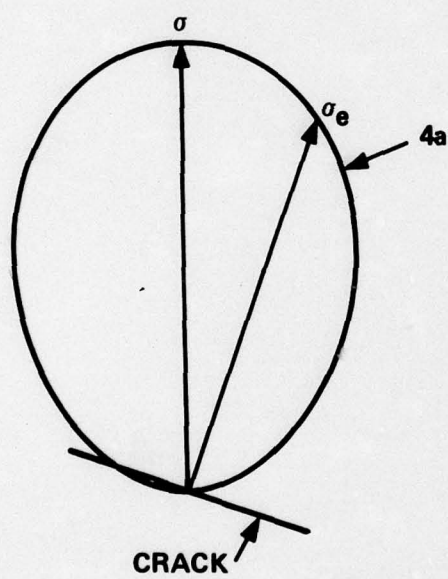
$$\sigma_e = \sqrt{\sigma_n^2 + \tau^2} \quad (4d)$$

and

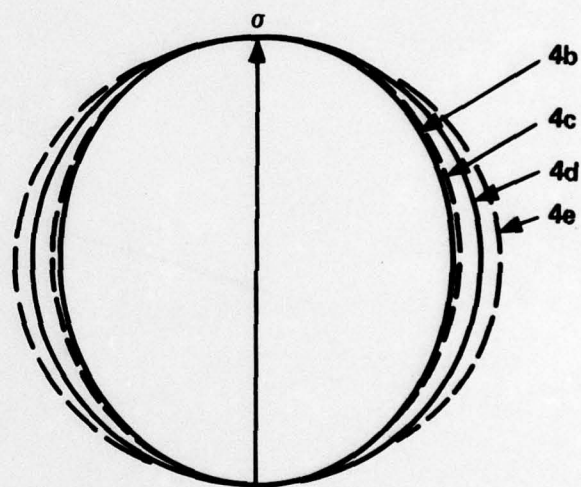
$$\sigma_e = \sqrt{\sigma_n^2 + \tau^2 / (1-0.5\nu)^2} \quad (4e)$$

We note in passing that the preceding criteria are arranged in order of increasing shear sensitivity.

Figure 1a is a polar diagram of effective stress as a function of crack orientation, using the fracture criterion of Equation (4a). It should be understood to be a figure of revolution about the tensile axis. Figure 1b shows the corresponding polar diagrams for the other fracture criteria of Equations (4). The polar diagram for the crack critical stress is a sphere of radius σ_c , since it is defined in such a way as to be independent of orientation. Figure 1c illustrates how the polar diagrams for σ_e and σ_c can be combined to show Ω , the solid angle containing all crack normals such that $\sigma_e > \sigma_c$. Clearly the size of Ω and therefore the probability that a randomly



a



b

Fig. 1a,b. Polar Diagrams of Effective Stress as a Function of Crack Orientation.
 Labels 4a - e are Keyed to Eqs. 4 of the Text. Poisson's Ratio is
 Taken as 0.25

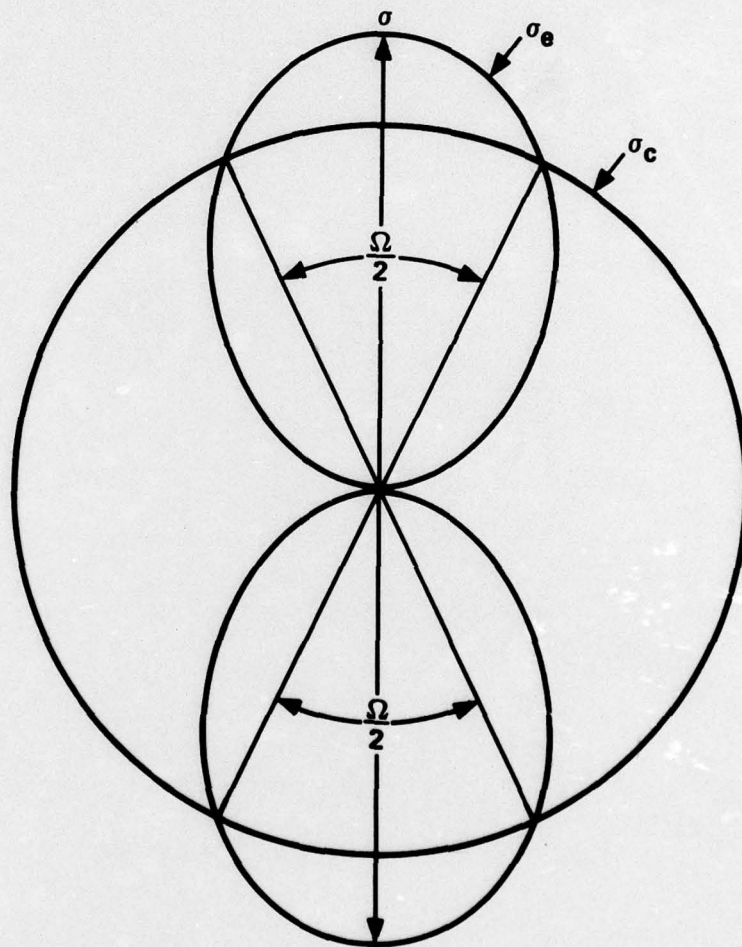


Fig. 1c. Solid Angle Ω Contains all Crack Normals for Which $\sigma_e > \sigma_c$. Polar Diagrams for σ_e Represents Case of Simple Tension

oriented crack of critical stress σ_c will be fractured depends on both the stress level and the fracture criterion.

In simple tension and equibiaxial tension analytical expressions can be found for Ω and used together with Equation (3) to evaluate P_S . In the general case we find Ω by integrating $d\Omega$ over the range in which $\sigma_e > \sigma_c$. One way of accomplishing this is to integrate over the entire angular range but include a suitable operator H in the integral:

$$P_S = \exp \left[- \iiint dVd\Omega d\sigma_c H(\sigma_e, \sigma_c) \frac{dN}{d\sigma_c} \right] \quad (6)$$

where

$$\begin{aligned} H(\sigma_e, \sigma_c) &= 1 \quad \text{when } \sigma_e > \sigma_c \\ &= 0 \quad \text{where } \sigma_e < \sigma_c \end{aligned} \quad (7)$$

We can now carry out the integral over σ_c first, with the result

$$P_S = \exp \left[- \iint dVd\Omega N(\sigma_e) \right] \quad (8)$$

In the case of the normal stress criterion, Equation (4a), σ_n is substituted for σ_e in Equation (8). The resulting equation is completely equivalent to Weibull's unproved rule for treating polyaxial stress states, and shows that this rule is valid if the material obeys the assumptions listed in the Batdorf and Crose paper.⁴

Comparison With Experiment

The theory just outlined is next compared with the results of two experiments. The first, performed by Petrovic and Mendiratta,¹⁶ involved the creation of semicircular cracks in the surface of hot pressed Si_3N_4 . By finding out how the average fracture stress of these controlled cracks subjected to four-point bending varies with crack orientation, a direct test of the

fracture criteria of Equations (4) is possible. This comparison is shown in Figure 2. The theoretical curves are obtained by solving Equations (4) and (5) for σ with σ_e taken as equal to the measured fracture stress when $\theta = 0$. It should be noted that the material fractured because of natural cracks at stresses above approximately 500 MN/mm^2 so that only the data for $\theta = 0^\circ$, 22.5° , and 45° are useful for our purpose. It appears that the second most shear sensitive criterion, Equation (4d), agrees best with the data. However, the criteria do not include corrections for the fact that these were surface cracks. The effect of a free surface is to enhance the influence of normal stress on fracture, i.e., to reduce the relative importance of shear. It is possible that if the criteria were adjusted to take this into account Equation (4e) might turn out to be the best.

The second experiment was conducted by Giovan and Sines¹⁷ on high strength alumina. In this experiment the statistics of failure were taken for rectangular plates in 4-point bending and for disks subjected to concentric ring loading. The scatter of the data points is such that elaborate techniques for closely fitting the data appear unwarranted. Accordingly we employ Weibull's two-parameter representation which has substantial advantages in computational simplicity. Among other things, this allows us to treat bending as though it were uniform tension applied to the reduced volume $V' = V/(m + 1)$, where m is the Weibull exponent.¹⁷

We assume that $V'N(\sigma_c)$ can be expressed in the form

$$V'N(\sigma_e) = K\sigma_c^m \quad (9)$$

Substituting in Equation (3) and assuming the material is uniformly stressed in simple tension we obtain

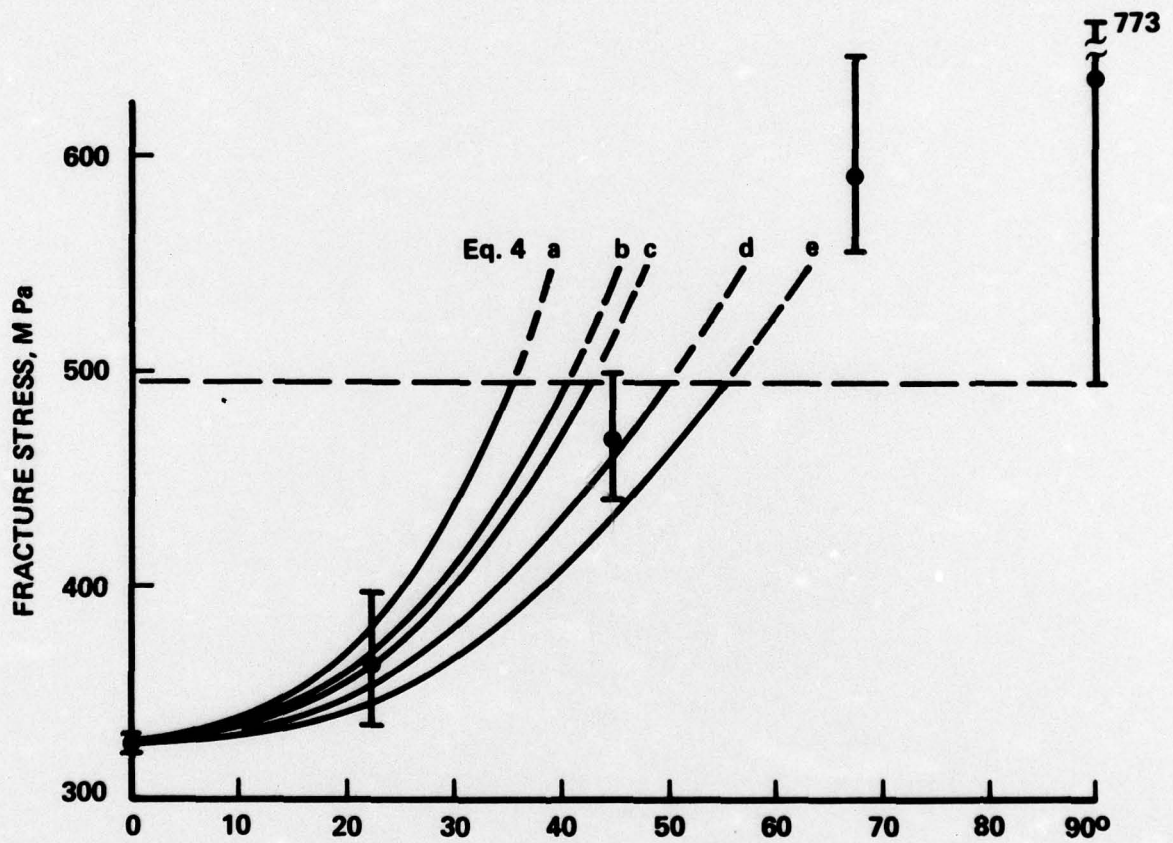


Fig. 2. Fracture Stress in Simple Tension vs Crack Angle (Data of Petrovic and Mendiratta)

$$P_S(\sigma, 0) = \exp \left[- \frac{1}{2\pi} \int_0^\sigma d\sigma_c mK \sigma_c^{m-1} \int_0^{\theta_e} 2\pi \sin \theta d\theta \right] \quad (10)$$

where θ_e is the crack angle for which $\sigma_e = \sigma_c$. Integrating we obtain

$$\begin{aligned} P_S(\sigma, 0) &= \exp \left[- mK \int_0^\sigma (1 - \cos \theta_e) \sigma_c^{m-1} d\sigma_c \right] \\ &= \exp \left[- k_U \sigma^m \right] \end{aligned} \quad (11)$$

where

$$k_U = mK \int_0^1 (1 - \cos \theta_e) x^{m-1} dx \quad (12)$$

and

$$x \equiv \sigma_c / \sigma \quad (13)$$

Similarly for equibiaxial tension

$$\begin{aligned} P(\sigma, \sigma) &= \left[- mK \int_0^\sigma d\sigma_{cr} mK \sigma_c^{m-1} \int_0^{\theta_e} e^{\cos \theta} d\theta \right] \\ &= \exp \left[- k_B \sigma^m \right] \end{aligned} \quad (14)$$

where

$$k_B = mK \int_0^1 \sin \theta_e x^{m-1} dx \quad (15)$$

A convenient way to compare uniaxial and equibiaxial fracture statistics is to plot $\ln P_S(\sigma, \sigma) / \ln P_S(\sigma, 0) = k_B / k_U$ against the Weibull parameter m . This is done in Figure 3. It is evident from this figure that $P_S(\sigma, \sigma)$ differs most strongly from $P_S(\sigma, 0)$ for the case of shear-insensitive cracks, Equation (4a). The curve for this case is the same as that shown in Figure 1 of reference 8. The other four curves correspond to the other fracture criteria listed in Equations (4).

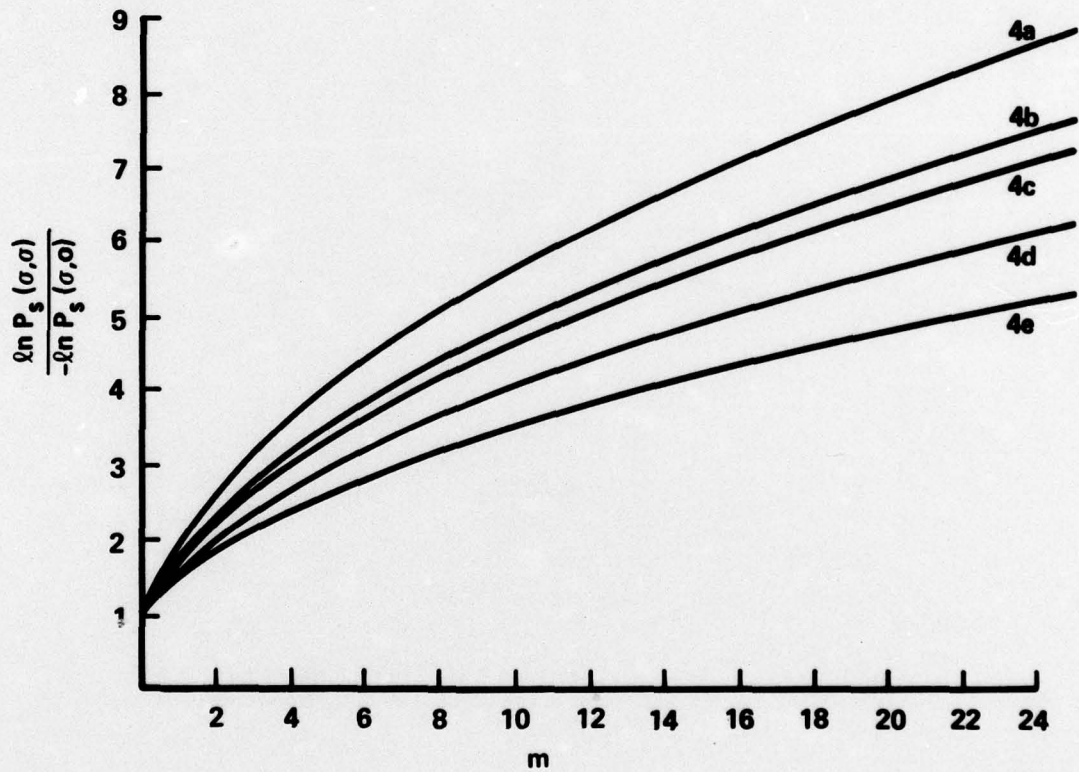


Fig. 3. Relation Between Failure Probability Under Equibiaxial Loading and That for Uniaxial Loading as a Function of Weibull Parameter m . The Curve Labels are Keyed to Eq. (4) of the Text

These curves can be used to analyze the experimental results of Giovan and Sines, which are shown in Figure 4. Either of the curves can be considered as given and used to calculate the other theoretically. Here we take the equibiaxial data as given and by conventional methods find for the best-fitting Weibull formula

$$P_S = \exp [-9.32 \times 10^{-25} \sigma^{15.12}]$$

With the above choice of m we find the k ratios for the various fracture criteria listed in Equations (4). The corresponding uniaxial fracture probabilities are plotted in Figure 4. It is evident that the best agreement is obtained using Equation (4e) as fracture criterion.

Discussion

In this paper two equivalent formulations, Equations (3) and (8), are given of weakest link theory in which the links are considered to be individual cracks. The fracture criterion and the stress state in these formulations are arbitrary. When the cracks are assumed to be shear insensitive, i.e., only the component of applied stress normal to the crack plane contributes to failure, Equation (3) reduces to the theory of Reference 4 and Equation (8) reduces to Weibull's rule for polyaxial stress states.³

A number of fracture criteria for shear sensitive cracks are considered. These criteria are formulated in Equations (4b) to (4e). Generally speaking, penny-shaped cracks are more shear sensitive than Griffith cracks, and criteria resulting from strain energy release rates are more shear sensitive than those based on the assumption that fracture occurs when the local stress exceeds the material strength at some point on the surface of the crack cavity.

The theoretical predictions are compared with two sets of experimental data. The first set (Petrovic and Mendiratta) tested the variation in fracture

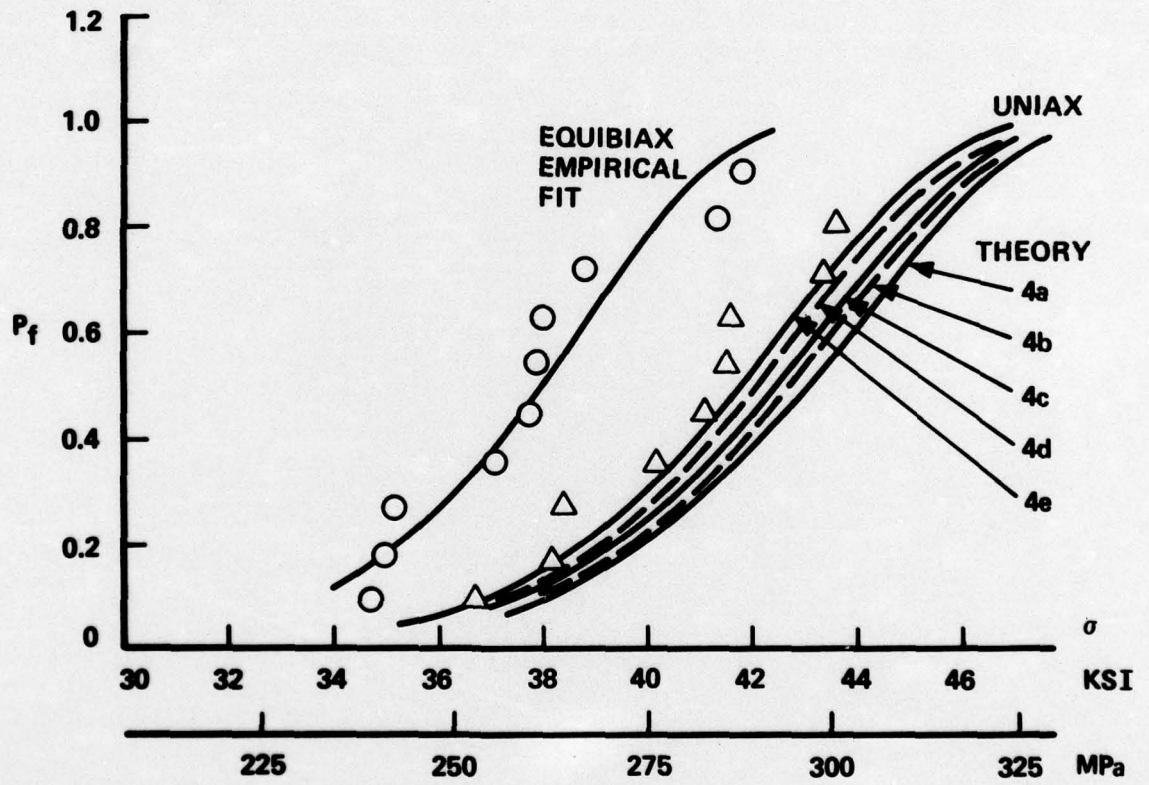


Fig. 4. Theory Compared with Data of Giovan and Sines

strength of surface cracks with angle of inclination to the tensile axis. In this case Equation (4d) appeared to be in best agreement with experiment, and Equation (4e) appeared to be too sensitive to shear. However, the criterion were not adjusted for surface effects which tend to reduce shear sensitivity, so this result may be misleading. In the second set of experiments (Giovan and Sines) Equation (4e) appeared to be in the best agreement. Thus in both experiments the criteria derived from strain energy release rate considerations worked out the best.

It is of some interest that in the second set of experiments a criterion exhibiting greater shear sensitivity than any considered would lead to still better agreement with experiment. The evidence for greater shear sensitivity is not very compelling because the discrepancy between the predicted and measured mean fracture stresses in simple tension is well below the standard deviation that would be calculated for this mean based on the fact that there were only ten test specimens for each stress ratio. However theoretical considerations also suggest greater shear sensitivity, since the strain energy release rate criteria used are based on the simplifying assumption that cracks extend in their own plane, an assumption leading to too high a fracture stress.

It is unfortunate that at the time of writing, substantial differences of opinion exist concerning how to calculate the stresses that cause mixed mode fracture.¹⁸⁻²² Until it becomes clear which criterion is best on theoretical grounds, experimental data cited in the paper suggest it may be satisfactory to use the criterion of Equation (4d). It has the advantage of simplicity, it is probably conservative when uniaxial data are used as the basis, and it is almost certainly preferable to the normal stress criterion employed explicitly by Batdorf and Crose⁴ and implicitly by Weibull. In any event, Equations

(3) and (8) are flexible in the sense that the fracture criterion is arbitrary, so each user can readily insert his own preference.

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