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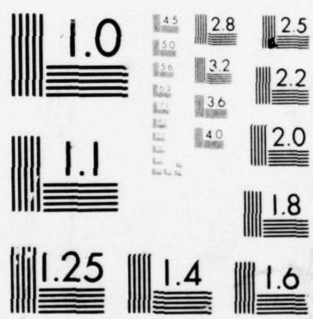

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ESTIMATION AND PREDICTION OF NONLINEAR FUNCTIONALS OF GAUSSIAN PROCESSES

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ABSTRACT

Via the tensor product structure of the nonlinear space we are able to solve the general *estimation problem of nonlinear functionals* of Gaussian processes in the sense that we can reduce the nonlinear problem to a standard linear estimation problem, the theory of which has been well developed. Also we introduce the concept of *super predictor* for a class of prediction problems and derive a lower bound for the mean square error of the nonlinear prediction.

1. BACKGROUND

Let $X = (X_t, t \in T)$ be a zero mean Gaussian process defined on a probability space (Ω, B, P). B is usually taken to be $B(X)$, the σ-field generated by the process X. There are two important Hilbert spaces associated with the (Gaussian) process X. The nonlinear space of X, $L_2(X) = L_2(\Omega, B(X), P)$, consists of all $B(X)$-measurable random variables with finite second moment which are called (nonlinear) L_2-functional of X. The linear space of $X, H(X)$, is the closed subspace of $L_2(X)$ spanned by X_t, $t \in T$, and its elements are called linear L_2-functionals of X. If S is a subset of T, then the nonlinear space and linear space of the Gaussian process $(X_t, t \in S)$ are denoted by $L_2(X;S)$ and $H(X;S)$ respectively. Note that $L_2(X;S)$ is a closed subspace of $L_2(X)$ and $H(X;S)$ a closed subspace of $H(X)$.

Suppose $\xi \in H(X)$ and $E\, \xi^2 = t$. Then ξ is a Gaussian variable with mean zero and variance t. Applying the Gram-Schmidt procedure to orthogonalize the sequence of random variables $1, \xi, \xi^2, \xi^3, \ldots$ in $L_2(X)$, we obtain the orthogonal sequence $H_{o,t}(\xi)$, $H_{1,t}(\xi)$, $H_{2,t}(\xi), \ldots$. $H_{p,t}(\xi)$ is called the Hermite polynomial of degree p with parameter t, and is a polynomial in both variables t and ξ. The first few Hermite polynomials are

$$H_{o,t}(\xi) = 1 \quad H_{1,t}(\xi) = \xi \quad H_{2,t}(\xi) = \xi^2 - t$$

$$H_{3,t}(\xi) = \xi^3 - 3t\xi \ .$$

The Hermite polynomials staisfy the following properties

$$(1) \quad E\ H_{p,t}(\xi) H_{q,t}(\xi) = P!\ \delta_{pq}\, t^P$$

$$(2) \quad \exp\{u\,\xi - \frac{t}{2} u^2\} = \sum_{p \geq 0} H_{p,t}(\xi) \frac{1}{p!} u^p, \quad \forall\, u \in \mathbb{R}$$

$$(3) \quad H_{p,t}(\sigma\xi) = \sigma^P H_{p,\frac{t}{\sigma^2}}(\xi) \quad , \sigma > 0.$$

When $t = 1$, $H_{p,t}(\xi)$ will be written as $H_p(\xi)$.

For each $p = 1, 2, \ldots$, let $H^{\otimes p}(X) = H(X) \otimes \ldots \otimes H(X)$ and respectively $H^{\tilde{\otimes} p}(X) = H_1(X) \tilde{\otimes} \ldots \tilde{\otimes} H(X)$ be the p^{th} power tensor and symmetric tensor products of $H(X)$; for $p = 0$, let $H^{\otimes p}(X) = H^{\tilde{\otimes} p}(X)$ be the space of all constant random variables in $H(X)$. $H^{\otimes p}(X)$ is a Hilbert space and its inner produce is such that

$$(4) \quad \langle \xi_1 \otimes \cdots \otimes \xi_p, \eta_1 \otimes \cdots \eta_p \rangle_{H^{\otimes p}(X)} = \langle \xi_1, \eta_1 \rangle_{H(X)} \cdots \langle \xi_p, \eta_p \rangle_{H(X)}$$

for all ξ's and η's in $H(X)$. $H^{\tilde{\otimes} p}(X)$ is a closed subspace of $H^{\otimes p}(X)$ spanned by all elements of the form

$$(5) \quad \xi_1 \tilde{\otimes} \cdots \tilde{\otimes}\, \xi_p = \frac{1}{p!} \sum_{\pi} \xi_{\pi_1} \otimes \cdots \otimes \xi_{\pi_p}$$

where $\pi = (\pi_1, \cdots, \pi_p)$ runs through all permutations of $(1, \cdots, p)$ and ξ's are elements of $H(X)$. For further properties of tensor and symmetric tensor product spaces see for example [6] and [7].

Our analyses are based on the following tensor product structure of the nonlinear space of a Gaussian process (see [6] and [7]).

THEOREM 1. *Let X be a zero mean Gaussian process. Then there exists a unique isomorphism Φ from* $\bigoplus_{p \geq 0} H^{\tilde{\otimes} p}(X)$ *onto* $L_2(X)$ *such that*

$$(6) \quad \Phi(e^{\tilde{\otimes} \xi}) = e^{\xi - \frac{1}{2} E\, \xi^2}$$

where $e^{\tilde{\otimes}\xi} = \sum_{p \geq 0} (P!)^{-\frac{1}{2}} \xi^{\tilde{\otimes} p}, \xi \in H(X)$.

IF $\xi_1, \cdots, \xi_k \in H(X)$ *are orthogonal then*

$$(7) \quad \Phi(\xi_1^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes}\, \xi_k^{\tilde{\otimes} p_k}) = (P!)^{-\frac{1}{2}} H_{P, E\xi_1^2}(\xi_1) \cdots H_{p_k, E\xi_k^2}(\xi_k)$$

where $p = p + \ldots + p_k$. *IF* $\{\xi_\gamma, \gamma \in \Gamma\}$ (Γ linearly ordered) *is a complete orthonormal set (CONS)* in $H(X)$ *then the family*

$$(8) \quad \left(\frac{P!}{p_{\gamma_1}! \ldots p_{\gamma_k}!}\right)^{\frac{1}{2}} \Phi(\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \cdots \tilde{\otimes}\, \xi_{\gamma_k}^{\tilde{\otimes} p_k})$$

$= (p_\gamma!)^{-1/2} H_{p_{\gamma_1}}(\xi_1) \cdots (p_{\gamma_k}!)^{-1/2} H_{p_{\gamma_k}}(\xi_k),$

$p \geq 0,\ k \geq 1,\ p_{\gamma_1} + \dots + p_{\gamma_k} = p, \gamma_1 < \dots < \gamma_k$, *is a CONS in* $L_2(\lambda)$.

2. NONLINEAR ESTIMATION

Let $X = (X_t,\ t \in T)$ be a second order process with zero mean. Consider the following estimation problem: We observe X_t for $t \in S$, a subset of T, and we want to estimate an L_2-functional θ of X based on the observations. We are interested in finding the best estimate $\hat{\theta}$, an L_2-functional of $(X_t,\ t \in S)$ which minimizes the mean square error of estimation $E(\theta - \hat{\theta})^2$. It is well known that $\hat{\theta}$ can be obtained as the conditional expectation of θ given X_t, $t \in S$; namely

$$\hat{\theta} = E(\theta \mid X_t,\ t \in S).$$

In general, $\hat{\theta}$ is extremely difficult to determine. However, if X is a Gaussian process we have a complete solution.

Let X be a zero mean Gaussian process and $\{\xi_\gamma, \gamma \in \Gamma\}$ (Γ linearly ordered) a CONS in $H(X)$. Then, according to Theorem 1, every L_2-functional of X has the following orthogonal development

$$(9)\quad \theta = \sum_{p \geq 0} \sum_{\substack{p_1 + \dots + p_k = p \\ \gamma_1 < \dots < \gamma_k}} a^{p_1 \dots p_k}_{\gamma_1 \dots \gamma_k} H_{p_{\gamma_1}}(\xi_{\gamma_1}) \cdots H_{p_{\gamma_k}}(\xi_{\gamma_k})$$

THEOREM 2. *Let X be a zero mean Gaussian process and let* $\theta \in L_2(X)$ *have the orthogonal development (9). Then*

$$\hat{\theta} = \sum_{p \geq 0} \sum_{\substack{p_1 + \dots + p_k = p \\ \gamma_1 < \dots < \gamma_k}} a^{p_1 \dots p_k}_{\gamma_1 \dots \gamma_k} H_{p_1, E\hat{\xi}^2_{\gamma_1}}(\hat{\xi}_{\gamma_1}) \cdots H_{p_k, E\hat{\xi}^2_{\gamma_k}}(\hat{\xi}_{\gamma_k})$$

where

$$\hat{\xi}_\gamma = E(\xi_\gamma \mid X_t,\ t \in S) = \mathrm{Proj}_{H(X;S)}\, \xi_\gamma .$$

PROOF: Upon identifying $L_2(X)$ with $\bigoplus_{p \geq 0} H^{\tilde{\otimes} p}(X)$ by virtue of Theorem 1, we have

$$\hat{\theta} = E(\theta \mid X_t,\ t \in S) = \mathrm{Proj}_{L_2(X;S)}\, \theta$$

$$= \sum \sum a^{p_1 \dots p_k}_{\gamma_1 \dots \gamma_k} \mathrm{Proj}_{L_2(X;S)} (\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \xi_{\gamma_k}^{\tilde{\otimes} p_k})$$

Thus to show the theorem it suffices to show

$$(10)\quad \mathrm{Proj}_{L_2(X;S)} (\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \xi_{\gamma_k}^{\tilde{\otimes} p_k}) = \hat{\xi}_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \hat{\xi}_{\gamma_k}^{\tilde{\otimes} p_k}.$$

For each $\rho \in L_2(X;S)$ write

$$\rho = \sum_{q \geq 0} \sum_{\substack{q_1 + \dots + q_j = q \\ \delta_1 < \dots < \delta_j}} b^{q_1 \dots q_j}_{\delta_1 \dots \delta_j} \eta_{\delta_1}^{\tilde{\otimes} q_1} \tilde{\otimes} \dots \tilde{\otimes}\, \eta_{\delta_j}^{\tilde{\otimes} q_j}$$

where $\{\eta_\delta, \delta \in \Delta\}$ (Δ linearly ordered) is a CONS in $H(X;S)$. We have

$$< \xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \xi_{\gamma_k}^{\tilde{\otimes} p_k}, \rho >$$

$$= \sum \sum b^{q_1 \dots q_j}_{\delta_1 \dots \delta_j} < \xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \xi_{\gamma_k}^{\tilde{\otimes} p_k}, \eta_{\delta_1}^{\tilde{\otimes} q_1} \tilde{\otimes} \dots \tilde{\otimes}\, \eta_{\delta_j}^{\tilde{\otimes} q_j} >$$

$$= \sum \sum b^{q_1 \dots q_j}_{\delta_1 \dots \delta_j} < \hat{\xi}_{\delta_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \hat{\xi}_{\gamma_k}^{\tilde{\otimes} p_k}, \eta_{\gamma_1}^{\tilde{\otimes} q_1} \tilde{\otimes} \dots \tilde{\otimes}\, \eta_{\delta_j}^{\tilde{\otimes} q_j} >$$

$$= < \hat{\xi}_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \hat{\xi}_{\gamma_k}^{\tilde{\otimes} p_k}, \rho >$$

where the second equality is a consequence of properties (1), (3) and $<\xi_\gamma, \eta_\delta> = <\hat{\xi}_\gamma, \eta_\delta>$.

Since $\rho \in L_2(X;S)$ is arbitraty and $\hat{\xi}_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \dots \tilde{\otimes}\, \hat{\xi}_{\gamma_k}^{\tilde{\otimes} p_k} \in L_2(X;S)$, (10) follows.

This completes the proof.

COROLLARY 3. *If X is a zero mean Gaussian process and $\xi \in H(X)$, then*

(11) $\quad E(H_{p,E\xi^2}(\xi) \mid X_t, t \in S) = H_{p,E\hat{\xi}^2}(\hat{\xi})$,

(12) $\quad E(\exp\{\xi-\frac{1}{2}E\xi^2\} \mid X_t, t \in S) = \exp\{\hat{\xi}-\frac{1}{2}E\hat{\xi}^2\}$.

If X is a zero mean Gaussian martingale then $Y_t = H_{p,EX_t^2}(X_t)$ *and* $Z_t = \exp\{X_t - \frac{1}{2}EX_t^2\}$ *are martingales.*
(The last statement is well known for X a Weiner process and p = 2.)

PROOF. (11) and (12) follows from properties (2), (3) and Theorem 2. The last assertion is an immediate consequence of (11) and (12).

If X is a zero mean Gaussian process and $T = (-\infty, \infty)$ (or any interval) then by the corollary we have that for all $s \leq t$

(13) $\quad E\{H_{p,EX_t^2}(X_t) \mid X_u, u \leq s\} = H_{p,E\hat{X}^2_{t,s}}(\hat{X}_{t,s})$

where

$$\hat{X}_{t,s} = E(X_t \mid X_u, u \leq s).$$

An expression for $\hat{X}_{t,s}$ can always be obtained via the Cramer - Hida representation of X:

$$X_t = \sum_{n=1}^{N} \int_{-\infty}^{t} f^{(n)}(t,u)\, dz_u^{(n)}$$

Then we have

$$\hat{X}_{t,s} = \sum_{n=1}^{N} \int_{-\infty}^{s} f^{(n)}(t,u)\, dz_u^{(n)}.$$

The case with p = 2, i.e. the L_2-functional $X_t^2 - EX_t^2$, is considered in [2] for a very special class of Gaussian processes X. It should be clear that whenever a simple expression is available for $\hat{X}_{t,s}$, then (13) gives a simple expression for the nonlinear predictor of the L_2-functional $H_{p,EX_t^2}(X_t)$.

We close this section by solving a simple estimation problem. Let $(X_t, 0 \leq t \leq T)$ be a stationary reciprocal Gaussian process with $EX_t = 0, EX_t^2 = 1$, and continuous covariance function $R(t,s) = R(t-s)$. It is known [5] that $R(t)$ must take one of the following forms: e^{-at}, $a > 0$; $\cos at$, $a > 0$ and $T \leq \pi/a$; $1 - at$, $0 \leq a \leq 2/T$. Let $0 < u < t < v < T$ be given. We desire to estimate θ, an L_2-functional of X_t, based on X_s, $s \in S = [0,u] \cup [v,T]$. By reciprocality we have

$$\hat{X}_t = E(X_t \mid X_s, s \in S) = \alpha X_u + \beta X_v;$$

and an easy computation shows that

$$\alpha = \frac{R(u-t)-R(v-t)R(u-v)}{1-R^2(u-v)}, \quad \beta = \frac{R(v-t)-R(u-t)R(u-v)}{1-R^2(u-v)}.$$

Since θ is an L_2-functional of X_t, it has the orthogonal development $\theta = \sum_{p\geq 0} a_p H_{p,E\hat{X}_t^2}(\hat{X}_t)$. Thus by Theorem 2 the best estimate of θ is given by

$$\hat{\theta} = \sum_{p\geq 0} a_p H_{p,E\hat{X}_t^2}(\hat{X}_t) = \sum_{p\geq 0} a_p H_{p,\alpha^2+\beta^2+2\alpha\beta R(u-v)}(\alpha X_u + \beta X_v).$$

3. NONLINEAR PREDICTION

Consider the following prediction problem for a class of processes: Let $X = (X_t, t \in T)$, T an interval, be a second order process and let $Y_t = \theta_t(X_t)$ with θ_t a real function such that $EY_t = 0$ and $EY_t^2 < \infty$ for all $t \in T$. Suppose on the basis of the (past) values of $Y = (Y_s, s \leq t)$ up to time t we want to find the best prediction of the future value of $Y_{t+\tau}$ for fixed $\tau > 0$.

Two predictors are of special interest: the optimal linear predictor $\hat{Y}_\ell(t,\tau)$ and the optimal nonlinear predictor $\hat{Y}_{n\ell}(t,\tau)$. The optimality is in the sense of minimizing the mean square error within the class of all linear and nonlinear predictors respectively. It is well known that

$$\hat{Y}_\ell(t,\tau) = \mathrm{Proj}_{H(Y_s;\, s \leq t)}\, Y_{t+\tau},$$

$$\hat{Y}_{n\ell}(t,\tau) = E(Y_{t+\tau} \mid Y_s, s \leq t).$$

The corresponding mean square prediction errors are denoted by

$$\sigma_\ell^2(t,\tau) = E(Y_{t+\tau} - \hat{Y}_\ell(t,\tau))^2,$$

$$\sigma_{n\ell}^2(t,\tau) = E(Y_{t+\tau} - \hat{Y}_{n\ell}(t,\tau))^2.$$

Now introduce a *super predictor* $\hat{Y}_s(t,\tau)$ to be the nonlinear prediction of $Y_{t+\tau}$ based on X_s, $s \leq t$, i.e.

$$\hat{Y}_s(t,\tau) = (Y_{t+\tau} \mid X_s, s \leq t);$$

its mean square prediction error is denoted by $\sigma_s^2(t,\tau)$. It is clear that

(14) $\quad \sigma_s^2(t,\tau) \leq \sigma_{n\ell}^2(t,\tau) \leq \sigma_\ell^2(t,\tau)$

and thus σ_s^2 provides a lower bound for the mean square errors of linear and nonlinear predictors. If X is a Gaussian process, $\sigma_s^2(t,\tau)$ can be obtained by solving an estimation problem as discussed in Section 2. If, in addition, θ_t happens to be a 1-1 function for each t then the σ-fields generated by X_t and Y_t coincide. In this case $\hat{Y}_{n\ell}(t,\tau) = \hat{Y}_s(t,\tau)$ and the nonlinear predictor can be obtained by solving an estimation problem again.

We now turn to the important special case where $X = (X_t, -\infty < t < \infty)$ is a zero mean stationary Gaussian process with covariance function $R(t,s) = R(t-s)$ and $\theta_t = \theta$ for all t. In this case we can calculate $\sigma_s^2(t,\tau) = \sigma_s^2(\tau)$ as follows. Write

$$(15)\quad Y_t = \theta(X_t) = \sum_{p\geq 1} a_p H_{p,\sigma^2}(X_t)$$

where $\sigma^2 = EX_t^2$. Clearly Y is a stationary process with $EY_t = 0$ and $EY_t^2 = \sum p!a_p^2\sigma^{2p} < \infty$. Since for $\xi,\eta \in H(X)$

$$E\, H_{p,E\xi^2}(\xi) H_{p,E\eta^2}(\eta) = p!\langle\xi^{\otimes p},\eta^{\otimes p}\rangle$$

$$= p!\langle\xi,\eta\rangle^p,$$

and if $p \neq q$

$$E\, H_{p,E\xi^2}(\xi) H_{q,E\eta^2}(\eta) = 0,$$

it follows

$$(16)\qquad E\, Y_t Y_s = \sum p!a_p^2 R^p(t-s).$$

And (16) implies that if X is mean square continuous so is Y.

Let $\hat{X}(t,\tau) = E(X_{t+\tau} \mid X_s, s \leq t)$ be the optimal nonlinear predictor of $X_{t+\tau}$ (which is also the optimal linear predictor since X is Gaussian), and $\sigma_0^2(\tau)$ be the mean square prediction error. Then by Corollary 3

$$(17)\quad \hat{Y}_s(t,\tau) = \sum_{p\geq 1} a_p H_{p,E\hat{X}(t,\tau)}(\hat{X}(t,\tau))$$

and hence

$$(18)\quad \sigma_s^2(\tau) = E(Y_{t+\tau} - \hat{Y}_s(t,\tau))^2$$

$$= EY_{t+\tau}^2 - E\hat{Y}_s^2(t,\tau)$$

$$= \sum_{p\geq 1} p!a_p\sigma^{2p} - \sum_{p\geq 1} p!a_p^2(\sigma^2-\sigma_0^2(\tau))^p$$

$$= \sum_{p\geq 1} p!a_p^2[\sigma^{2p}-(\sigma^2-\sigma_0^2(\tau))^p].$$

It is well known from the general theory of stationary process that $\sigma_0^2(\tau)$ can be obtained analytically (if not explicitly) through the Wiener-Paley factorization theorem if X is regular (i.e. $H(X_s, s \leq t) = \{0\}$). It can be shown that if X is regular so is Y, and therefore $\sigma_\ell^2(\tau)$ can also be obtained analytically.

Jaglom [4] has considered the problem of comparing the performance of optimal linear and nonlinear predictors for polynomial functions of certain stationary Markov processes. Donelson and Maltz [1] studied this problem in detail for polynomial functions of the Ornstein-Uhlenbeck process. The inequality (14) plays a central role in such studies.

As an example consider X the Ornstein-Uhlenbeck process and Y a nonlinear function of X given by (15). Recall that the Ornstein-Uhlenbeck process is a Gaussian process with zero mean and covariance $R(t-s) = e^{-|t-s|}$. By the Markov property we have

$$\hat{X}(t,\tau) = E(X_{t+\tau} \mid X_s, s \leq t) = e^{-\tau}X_t.$$

Thus it follows from (17) and (18) that

$$\hat{Y}_s(t,\tau) = \sum_{p\geq 1} a_p H_{p,e^{-2\tau}}(e^{-\tau}X_t)$$

$$= \sum_{p\geq 1} a_p e^{-p\tau} H_p(X_t),$$

$$\sigma_s^2(\tau) = \sum_{p\geq 1} p!a_p^2(1 - e^{-2p\tau}).$$

This result, with Y a polynomial function of X, has been obtained by Donelson and Maltz using *a different approach.*

Finally, we remark that if $Y_t = H_p(X_t)$ then

$$\hat{Y}_{n\ell}(t,\tau) = \hat{Y}_s(t,\tau) = e^{-p\tau}Y_t,$$

$$\sigma_{n\ell}^2(\tau) = \sigma_s^2(\tau) = 1 - e^{-2p\tau}.$$

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