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ON BOUNDS FOR THE TORSIONAL STIFFNESS OF SHAFTS OF VARYING CIRC--ETC(U)
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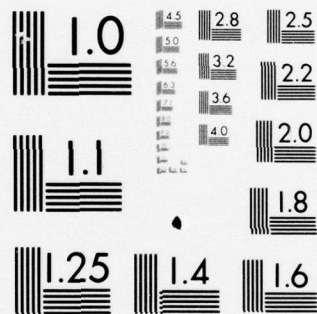
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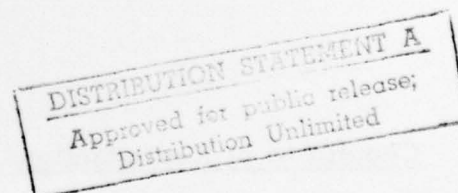
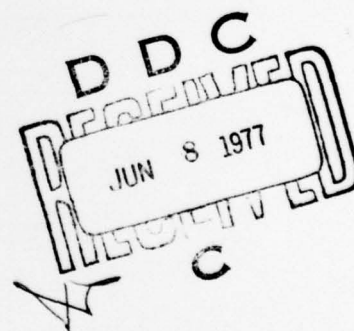
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by E. Reissner



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ON BOUNDS FOR THE TORSIONAL STIFFNESS OF SHAFTS OF
VARYING CIRCULAR CROSS SECTION[†]

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ABSTRACT

We state upper and lower bound formulas for the torsional stiffness of shafts of varying circular cross section, in accordance with the classical Michell formulation of this problem, through use of the principles of minimum potential and complementary energy. The general results are used to obtain explicit first-approximation bounds which, for the limiting case of the cylindrical shaft, reproduce the known elementary exact results. It is conjectured that the first-approximation lower bound is significantly closer to the exact result than the first-approximation upper bound.

[†] A report on work supported by the Office of Naval Research, Washington, D. C.

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ON BOUNDS FOR THE TORSIONAL STIFFNESS OF SHAFTS OF VARYING CIRCULAR CROSS SECTION

E. Reissner

We consider, in terms of cylindrical coordinates r and z , a homogeneous isotropic linear elastic solid bounded by plane surfaces $z = 0$ and $z = L$, and by a surface of revolution $r = r_0(z)$ for $0 \leq z \leq L$. We assume that the surface portion $z = 0$ is held fixed, and that the boundary conditions for the surface portion $z = L$ are mixed conditions, of prescribed circumferential displacement $v = \Theta r$, and of vanishing normal stress σ_z and shear stress τ_{rz} . We further assume that the surface portion $r = r_0(z)$ is traction free.

The solution of this problem can be obtained, as first shown by J. H. Michell, by assuming the vanishing of all radial and axial displacements and of all stresses except $\tau_{r\theta}$ and $\tau_{z\theta}$. With this there remain the stress strain relations $\tau_{r\theta} = G\gamma_{r\theta}$ and $\tau_{z\theta} = G\gamma_{z\theta}$, with strain displacement relations

$$\gamma_{z\theta} = v_{,z}, \quad \gamma_{r\theta} = r(r^{-1}v)_{,r}, \quad (1)$$

where v is independent of θ , and with stress stress-function relations

$$\tau_{z\theta} = r^{-2}\Psi_{,r}, \quad \tau_{r\theta} = -r^{-2}\Psi_{,z}, \quad (2)$$

where Ψ is independent of θ .

While the boundary conditions for $z = 0$ and $z = L$ are now directly given in terms of v , the corresponding conditions of no tractions for $r = r_0(z)$ are readily shown to be equivalent to the one condition $\Psi = \text{const.}$

In what follows we are interested, in particular, in the values of the stiffness coefficient K in the torque-twist relation $T = K\Theta$ where

$$T = 2\pi \int_0^{r_0} \tau_{z\theta} r^2 dr. \quad (3)$$

Specifically, our object is to obtain upper and lower bounds for K , through use of both the principles of minimum potential and minimum complementary energy, analogous to what has recently been done for the problem of the end loaded cantilever beam, treated within the framework of the theory of plane stress [1].

The appropriate basic inequalities for the special case under consideration are readily shown to be of the form

$$\tilde{T}\Theta - 2\pi \int_0^L \int_0^{r_0} \tilde{B} r dr dz \leq \frac{1}{2} T\Theta \leq 2\pi \int_0^L \int_0^{r_0} \tilde{A} r dr dz. \quad (4)$$

In this we have

$$\tilde{A} = \frac{1}{2}G(\tilde{\gamma}_{z\theta}^2 + \tilde{\gamma}_{r\theta}^2), \quad \tilde{B} = \frac{1}{2G}(\tilde{\tau}_{z\theta}^2 + \tilde{\tau}_{r\theta}^2), \quad (5)$$

where $\tilde{\gamma}_{z\theta}$ and $\tilde{\gamma}_{r\theta}$ are given in terms of a function $\tilde{v}$ in accordance with (1), and $\tilde{\tau}_{z\theta}$ and $\tilde{\tau}_{r\theta}$ are given in terms of a function $\tilde{\Psi}$ in accordance with (2). The function $\tilde{v}$ must be differentiable and satisfy the boundary conditions $\tilde{v}(r, 0) = 0$, $\tilde{v}(r, L) = \Theta r$. The function $\tilde{\Psi}$ must also be differentiable and satisfy the boundary condition $\tilde{\Psi}[r_0(z), z] = \text{const.}$ for $0 \leq z \leq L$. Furthermore, $\tilde{v}$ and $\tilde{\Psi}$ must be such that the double integrals in (4) do in fact exist. We note that the equal signs in (4) apply in the event that $\tilde{v}$ and $\tilde{\Psi}$, respectively, are the actual solution functions v and Ψ of the Michell boundary value problem.

First - Approximation Upper Bound. We assume

$$\tilde{v} = r V(z) \quad (6)$$

giving $\tilde{\gamma}_{r\theta} = 0$ and $\tilde{\gamma}_{z\theta} = r V'(z)$ and therewith

$$\tilde{\Pi}_d = 2\pi \int_0^L \int_0^{r_0} \tilde{A} r dr dz = \frac{1}{4}\pi G \int_0^L [V'(z)]^2 r_0^4 dz. \quad (7)$$

We determine $V(z)$ by setting $\delta \tilde{\Pi}_d = 0$, subject to the constraint conditions $V(0) = 0$, $V(L) = \Theta$, and obtain

$$V(z) = \Theta \frac{\int_0^z r_o^{-4} dz}{\int_0^L r_o^{-4} dz}, \quad (8)$$

and therewith

$$\tilde{\Pi}_d = \frac{\pi G \Theta^2}{4 \int_0^L r_o^{-4} dz}, \quad (9)$$

and then, from (4),

$$K_{U1} = \frac{\pi G}{2 \int_0^L r_o^{-4} dz} \equiv K_o. \quad (10)$$

We note that K_o is the approximate value of K which follows by an elementary combination of the solutions for a large number of short shafts of appropriate differing uniform circular cross section.

First-Approximation Lower Bound. We assume

$$\tilde{\Psi} = c(r/r_o)^4, \quad (11)$$

which satisfies $\tilde{\Psi}(r_o, z) = c$, is exact for the uniform diameter case, and gives

$$\tilde{\tau}_{z\theta} = 4cr_o^{-4}r, \quad \tilde{\tau}_{r\theta} = 4cr_o^{-5}r'r^2. \quad (12)$$

Therewith

$$\tilde{\Pi}_s = \tilde{T}\Theta - 2\pi \int_0^L \int_0^{r_o} \tilde{B} r dr dz = 2\pi c \Theta - \frac{4\pi c^2}{G} \int_0^L \left[1 + \frac{2}{3}(r_o')^2\right] \frac{dz}{r_o^4}. \quad (13)$$

We determine the smallest value of $\tilde{\Pi}_s$ by setting $\partial \tilde{\Pi}_s / \partial c = 0$. This gives

$$c = \frac{\Theta G}{4 \int_0^L \left[1 + \frac{2}{3}(r_o')^2\right] r_o^{-4} dz}, \quad \tilde{\Pi}_s = \frac{\pi G \Theta^2}{4 \int_0^L \left[1 + \frac{2}{3}(r_o')^2\right] r_o^{-4} dz}, \quad (14)$$

and then, from (4),

$$K_{L1} = \frac{\pi G}{2 \int_0^L [1 + (2/3)(r'_0)^2] r_0^{-4} dz} . \quad (15)$$

We note that $K_{L1} = K_{U1} = K_0$, as it must be, for the case of the uniform circular cylinder for which $r'_0 = 0$, throughout. For other cases we expect, on the basis of experience with other problems, that K_{L1} will be a better approximation than K_{U1} to the exact value of K .

Results for conical shafts. Setting $r_0 = a + z \tan \alpha$, where a and α are constants we find, from (10) and (15),

$$\frac{K_{L1}}{K_{U1}} = \frac{1}{1 + (2/3) \tan^2 \alpha} , \quad (16)$$

showing a significant difference between K_{L1} and K_{U1} for sufficiently large values of α . For example, when $\alpha = 30^\circ$ then $K_{L1}/K_{U1} = 9/11 \approx .82$, and when $\alpha = 45^\circ$ then $K_{L1}/K_{U1} = .6$. We note that the known closed-form solution for conical shafts, [2], is not an exact solution of the problem as stated here inasmuch as this closed-form solution implies rigid rotations of spherical end surfaces, rather than of plane end surfaces.

Higher-Approximation Bounds. It appears to be easier to derive improved lower bounds of relatively attractive appearance than to do the same for the problem of upper bounds.

In order to obtain a sequence of lower bounds K_{LN} for $N = 1, 2, \dots$ we may set

$$\tilde{\Psi} = (r/r_0)^4 [c_1 + c_2 (r/r_0)^2 + \dots + c_N (r/r_0)^{2N-2}] , \quad (17)$$

and then determine values of the coefficients c_n from the N simultaneous linear equations $\partial \tilde{\Pi}_s / \partial c_n = 0$, $n = 1, 2, \dots, N$.

In order to obtain improved upper bounds K_{UN} we may set, in generalization of (6)

$$\tilde{v} = \Theta [rV_1(z) + r^3V_2(z) + \dots + r^{2N-1}V_N(z)] . \quad (18)$$

In this the functions V_n have to satisfy the constraint boundary conditions $V_n(0) = 0$, $V_1(L) = 1$, $V_2(L) = 0, \dots, V_N(L) = 0$, with the variational equation $\delta\tilde{\Pi}_d = 0$ then leading to a system of simultaneous linear second-order differential equations for the functions V_n .

To illustrate the nature of the problem we consider the case $\tilde{v} = \Theta(rV_1 + r^3V_2)$ where we will assume, additionally, that V_1 coincides with the solution function V in our first-approximation calculation. We now obtain, in place of equation (7)

$$\tilde{\Pi}_d = \frac{\pi}{4} G\Theta^2 \int_0^L [r_o^4(V_1')^2 + \frac{8}{3} r_o^6 V_2^2 + \frac{1}{2} r_o^8 (V_2')^2 + \frac{4}{3} r_o^6 V_1' V_2'] dz , \quad (19)$$

and with this, as differential equation for V_2 ,

$$(r_o^8 V_2')' - \frac{16}{3} r_o^6 V_2 = - \frac{4}{3} (r_o^6 V_1')' = - \frac{8r_o r_o'}{3 \int_0^L r_o^{-4} dz} . \quad (20)$$

The associated minimum value of $\tilde{\Pi}_d$ follows from (19), with the help of (20) and (8), in the form

$$\tilde{\Pi}_d = \frac{G\Theta^2}{4 \int_0^L r_o^{-4} dz} \left[1 - \frac{4}{3} \int_0^L r_o r_o' V_2 dz \right] , \quad (21)$$

and then from (4), and with K_o as in (10),

$$K_{U2} = K_o \left[1 - \frac{4}{3} \int_0^L r_o r_o' V_2 dz \right] . \quad (22)$$

Equation (22) may be written, with the help of (20) and upon making use of the boundary conditions for V_2 , in the alternate form

$$K_{U2} = K_o \left\{ 1 - \frac{8}{3} \int_0^L \left[r_o^8 V_2^2 + \frac{3}{16} r_o^8 (V_2')^2 \right] dz \right\} , \quad (22')$$

which makes it evident that K_{U2} is in fact a better upper bound for K than the first approximation bound K_{U1} .

As regards the determination of the function V_2 we note the possibility of an explicit solution, as a combination of powers of r_0 , for the case $r_0 = a + z \tan \alpha$. The resulting ratio K_{U2}/K_{U1} is a less simple expression than the ratio K_{L1}/K_{U1} in (16), and we refrain from stating it here, in the hope that some alternate, simpler, upper bound improvement might be obtained later, in a different way.

REFERENCES

1. E. Reissner, Upper and Lower Bounds for Deflections of Laminated Cantilever Beams Including the Effect of Transverse Shear Deformation, *J. Appl. Mech.* 40, 988-991, 1973.
2. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd Ed., 341-349, 1970.

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