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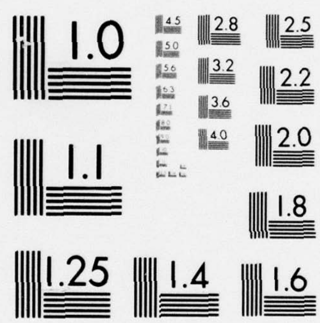

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Interaction Between Steady Non-Uniform Two-Dimensional Currents and Directional Wind-Generated Gravity Waves with Applications for Current Measurements

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Advanced Space Sensing Applications Branch
Space Science Division

May 1977

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Interactions between steady non-uniform two-dimensional currents and directional gravity waves are studied for the case of a random wave field which is actively generated by the wind. Directional wave number spectrum in the form suggested by Phillips, Kitaigorodskii, Pierson-Moskowitz, and Longuet-Higgins-Cartwright-Smith is used as the basic spectral form for zero current conditions. Modified spectral functions under the influence of current, $\vec{U}$, in the form of $\vec{U} = (a|x_2|, bx_1)$ are found by solving the kinematic wave conservation equation and the energy balance equation with the method of characteristics. By using this particular method, one can solve the problems of two-dimensional → next page
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20. Abstract (Continued)

wave-current interactions by either specifying $\vec{U}$ and wave energy, E, or specifying wave number vector, $\vec{k}$, and E, for every point space, $\vec{x}$. The remote sensing techniques are suggested to be the best and the most convenient feasible way of measuring these parameters for inferring the current field. Thus the changes in wave characteristics, in the form of directional wave spectra, can readily be numerically calculated as the directional wind-generated gravity waves encounter the steady non-uniform two-dimensional currents. Using the remote sensing techniques which are capable of observing these changes in waves over a large area, one will be able to infer the magnitude and the direction of the current encountered. This particular method of inferring current field has great potential benefits in global applications.

CONTENTS

INTERACTION BETWEEN STEADY NON-UNIFORM TWO-DIMENSIONAL CURRENTS AND DIRECTIONAL WIND-GENERATED GRAVITY WAVES WITH APPLICATIONS FOR CURRENT MEASUREMENTS

I. INTRODUCTION

In recent years, the interactions between surface waves and variable currents have been studied by many researchers. Unna (1942) and Evans (1955) studied this phenomenon without considering the effects of nonlinear energy transfer between surface waves and currents. The analysis given by Drent (1959) was for plane, long-crested waves and rectilinear shear flow in the absence of surface tension and viscosity. Hughes and Stewart (1961) studied the same case by experiments except the waves are in the gravity-capillary range. Longuet-Higgins and Stewart (1961, 1964) advanced the concept of wave-current interactions by introducing nonlinear energy transfer, between waves and current, in the form of the inner product of the wave-induced radiation stresses and gradients of the current. In their analysis the net total energy change in the dynamic system was assumed to be negligible.

Waves do change their characteristics both kinematically and dynamically whenever they encounter currents. This phenomenon of changes has not only been observed by mariners for centuries but has also been systematically detected from satellite with scanning radiometer as reported by Strong and DeRycke (1973). These observations, combined with the theoretical treatise on wave-current interactions, greatly increase the possibility of inferring currents by using remote sensing as a means to observe the changes in

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ocean wave characteristics. In order to explore this possibility further, Huang, et al. (1972) and Tung and Huang (1973) derived quantitatively the measurable physical parameters such as dispersion relationships, wave spectra, slope spectra, surface roughness, and wave height distribution for wind-generated gravity waves in various steady nonuniform currents. With the progress made recently in the field of remote sensing, these physical parameters should be readily observable from moving platforms such as aircrafts and satellites. Furthermore, based upon the finding of Long and Huang (1976a, 1976b), the same measurement concept and technique can be extended to the range of capillary-gravity waves. Their theoretical predictions were confirmed by the laboratory observations on the changes in kinematic and dynamic wave characteristics for this spectral range of waves with variable currents. Their studies strengthen the concept of determining currents by observing waves remotely.

However, all the studies mentioned above are for the cases of one-dimensional waves. Chen, et al. (1973) investigated the case of directional fetch-limited wind-generated gravity waves on variable shear currents. The continuity equation is identically satisfied in that particular situation where the current, strictly speaking, is still one-dimensional. This paper will discuss the case where the continuity equation is not necessarily identically satisfied and will provide a complete numerical solution of the directional spectrum describing the interactions of wind-generated gravity waves with two-dimensional variable currents in deep water. The method may be extended to include more general cases. Solutions are obtained by the method of characteristics, reducing the partial differential equations to ordinary differential equations, the solutions of which are obtained by numerical integration. The directional wave spectra generated by the wind and measured by Longuet-Higgins, et al. (1963) is used as a guideline for the case of waves with no current.

II. THEORY

Let a component of a general wave field be specified by

$$\zeta(\vec{x},t) = B(\vec{x},t)\ e^{i\chi(\vec{x},t)} \tag{1}$$

where $B(\vec{x},t)$ is the amplitude and $\chi(\vec{x},t)$ the phase function; both are functions, in general, of position vector, $\vec{x}$, and time, t. The wave-number vector, $\vec{k}$, and wave frequency, n, can be defined as

$$\left.\begin{aligned}\vec{k} &= \nabla\chi \\ n &= -\frac{\partial\chi}{\partial t}\end{aligned}\right\} \tag{2}$$

From the two expressions in Equation (2) it implies consequently that the wave number vector is irrotational in space, i.e.,

$$\nabla \times \vec{k} = 0 \tag{3}$$

and

$$\frac{\partial\vec{k}}{\partial t} + \nabla n \equiv 0 \tag{4}$$

which is the kinematic conservation law of waves. The amplitude, $|\vec{k}|$, is defined as $2\pi/\lambda$ where λ is the wavelength. Wave frequency, n, however, is not the same as in an intrinsic oscillatory wave; rather, it represents the total wave frequency which is actually observed. For waves fluctuating on a current, $\vec{U}(\vec{x},t)$, this total wave frequency is composed of two terms, namely, the intrinsic wave frequency, σ, and the convective wave frequency, $\vec{k} \cdot \vec{U}(\vec{x},t)$, as

$$n = \sigma + \vec{k} \cdot \vec{U}(\vec{x},t) \tag{5}$$

where σ and $|\vec{k}|$ satisfy the dispersion relationship, which is defined as

$$\sigma^2 = g|\vec{k}| \tag{6}$$

with g the gravitational acceleration. The weak nonlinear dynamic interaction is neglected in Equation (6).

From Equation (4) it follows immediately that, for steady state,

$$\nabla n = 0 \tag{7}$$

which implies that the total wave frequency is an invariant in steady state condition. With Equations (5) and (7), we will have

$$n = \sigma + \vec{k} \cdot \vec{U}(\vec{x},t) = \sigma_o \tag{8}$$

where the subscript, o, indicates the case when there is no current. From the kinematic influences shown by Equation (8) the "wave frequency dispersion" will produce changes in wave characteristics.

For the current, which is averaged over the depth, the kinematic conservation equation,

$$\nabla \cdot \vec{U} = 0 \tag{9}$$

guarantees the condition of continuity for steady state. Thus, Equations (3), (8), and (9) constitute the kinematic conservation conditions for the combined field of waves with current.

The dynamic influences can be derived and evaluated by the energy balance equation as

$$\nabla \cdot E(\vec{U} + \vec{c}_g) + S_{ij} \frac{\partial U_j}{\partial x_i} = \varepsilon \qquad i,j = 1,2 \tag{10}$$

where E is the wave energy, $\vec{c}_g$ is the group velocity, U_j is the component of $\vec{U}$ in x_j-direction, S_{ij} is the radiation stress and the term, ε, represents the "net" energy gained by the dynamic system. In the case of the assumed steady active wind field the energy gained by waves is more or less equal to the energy lost by waves. Thus it is further assumed here that ε is negligibly small and is set to zero in Equation (10), i.e.

$$\nabla \cdot E(\vec{U} + \vec{c}_g) + S_{ij} \frac{\partial U_j}{\partial x_i} = 0 \tag{11}$$

This equation governs the energy transfer betweeen waves and steady non-uniform current.

III. NUMERICAL SOLUTION

The coordinate system (x_1,x_2) used in the previous section can be arbitrarily chosen so that the coordinates can better describe the physical problem. In other words, the terms are either geometric components or quantities. In forming the numerical solution, the Rectangular Cartesian Coordinates x_1 and x_2 will be assumed for the present, as shown in Figure 1, without loss of generality. Let the x_1-axis be the direction of the wind velocity, $\vec{W}$, and let the wave, whose wave number vector is $\vec{k}$, propagate at an angle, θ, with respect to the x_1-axis. The local current vector, $\vec{U}$, is making an angle, ϕ, with wave number vector, $\vec{k}$, and $\vec{U}$ is also moving at an angle, ξ, with respect to the x_1-axis. All the angles are positive in a counterclockwise direction. With the configuration shown in Figure 1, the formulation of a numerical solution can be derived.

A. Kinematics

Based upon Figure 1, Equations (3), (8), and (9) can be rewritten as

$$\frac{\partial}{\partial x_1} (|\vec{k}| \sin\theta) = \frac{\partial}{\partial x_2} (|\vec{k}| \cos\theta) \quad , \tag{12}$$

$$n = \sigma + |\vec{k}| \; |\vec{U}| \cos\phi = \sigma_o \quad , \tag{13}$$

and

$$\frac{\partial}{\partial x_1} (|\vec{U}| \cos\xi) + \frac{\partial}{\partial x_2} (|\vec{U}| \sin\xi) = 0 \quad . \tag{14}$$

respectively. A solution of Equations (12) and (13) deter mines the local direction of propagation with angle, θ, for wave number vector, $\vec{k}$. Equation (14) describes the characteristics of the current, $\vec{U}$.

Differentiating Equation (13) with respect to x_1 and x_2, solving for $\partial\theta/\partial x_1$ and $\partial\theta/\partial x_2$ in terms of $\partial|\vec{k}|/\partial x_1$ and $\partial|\vec{k}|/\partial x_2$ and combining with Equation (12), it is easy to see that the partial differential equation in the form of,

$$F \frac{\partial\theta}{\partial x_1} + G \frac{\partial\theta}{\partial x_2} = H \quad , \tag{15}$$

for determining $\vec{k}$ can be readily obtained where

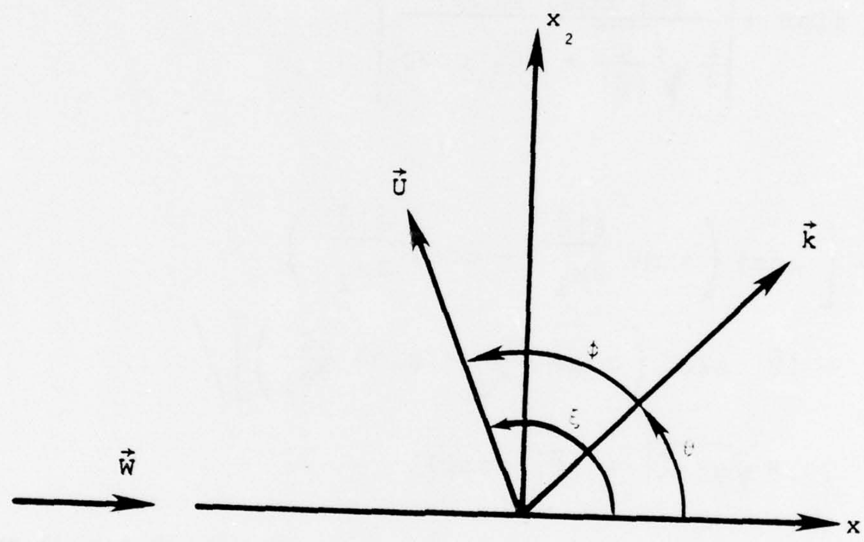

Fig. 1 — Geometric configuration with rectangular Cartesian coordinates

$$F = \cos\theta - \left[\frac{|\vec{U}| \sin\theta \sin\phi}{\frac{1}{2}\sqrt{\frac{g}{|\vec{k}|}} + |\vec{U}| \cos\phi} \right] , \tag{16}$$

$$G = \sin\theta + \left[\frac{|\vec{U}| \cos\theta \sin\phi}{\frac{1}{2}\sqrt{\frac{g}{|\vec{k}|}} + |\vec{U}| \cos\phi} \right] , \tag{17}$$

and

$$H = \left[\cos\phi \left(\sin\theta \frac{\partial|\vec{U}|}{\partial x_1} - \cos\theta \frac{\partial|\vec{U}|}{\partial x_2} \right) + |\vec{U}| \sin\phi \left(\cos\theta \frac{\partial\xi}{\partial x_2} - \sin\theta \frac{\partial\xi}{\partial x_1} \right) \right] \Big/ \left(0.5\sqrt{g/|\vec{k}|} + |\vec{U}| \cos\phi\right) . \tag{18}$$

The wave number $|\vec{k}|$ in the expressions for F, G and H is a function of the angles θ and ϕ and is, in general, a function of the space coordinates x_1 and x_2. Equation (15) is solved by the method of characteristics (see Courant and Hilbert (1966)). By integrating the ordinary differential equations,

$$\frac{dx_1}{ds} = F, \tag{19}$$

$$\frac{dx_2}{ds} = G, \tag{20}$$

and

$$\frac{d\theta}{ds} = H, \tag{21}$$

a family of one-parametered characteristic curves, s, can be obtained. With subscript, o, denoting the initial condition there is a unique curve, in this family of curves, which passes through each point, $(\vec{x}_o, \theta_o)$, on the initial curve, C. This family of curves, s, generates the surface, $\theta = \theta(\vec{x}, \vec{x}_o, \theta_o)$, which is the solution for Equation (15) with initial conditions.

The model of the current is chosen such that $\vec{U}$ is definable in the positive x_1 half space which includes $\vec{x}_o$ or (o,η). To obtain the local direction of wave number vector, $\vec{k}$, at a given $\vec{x}$, Equation (15) is integrated by the Runge-Kutta numerical method, varying the quantity, η, in $\vec{x}_o$ in such a way so that the characteristic curve passes through the given $\vec{x}$. From the solution, $\theta = \theta(\vec{x}, \vec{x}_o, \theta_o)$, the magnitude of the local wave number vector, $|\vec{k}|$, is determined from Equation (13). In other words, in order to obtain the local wave number vector, $\vec{k}$, at a particular space point, $\vec{x}$, an initial value, η, for $\vec{x}_o$ must be determined for each θ_o and $\vec{x}_o = (o,\eta)$. Thus, a set of η values is obtained for a particular space point, $\vec{x}$. This set of η values is used for the solution of the energy balance equation.

The average current field is assumed to be of the form

$$|\vec{U}| \cos\xi = a|x_2|, \tag{22}$$

and

$$|\vec{U}| \sin\xi = bx_1, \tag{23}$$

where a and b are constants and Equations (22) and (23) satisfy the required equation of continuity, Equation (14).

A verification of the numerical formulation is made by assuming only a transverse gadient for the current velocity.

The results show that $|\vec{k}| \sin\theta = |\vec{k}_o| \sin\theta_o$ for each $\vec{x}$ which agrees with the analytical results by Longuet-Higgins and Stewart (1964).

B. Energy Balance Equation

The radiation stress tensor, S_{ij}, has the form of

$$S_{ij} = \begin{pmatrix} \frac{E}{2}\cos^2\theta & \frac{E}{2}\sin\theta\cos\theta \\ \frac{E}{2}\sin\theta\cos\theta & \frac{E}{2}\sin^2\theta \end{pmatrix} \qquad (24)$$

for waves in deep water by referring to Figure 1. Furthermore, Equation (11) can be reduced to

$$L\frac{\partial E}{\partial x_1} + M\frac{\partial E}{\partial x_2} = N \qquad (25)$$

where

$$L = \frac{|\vec{c}|}{2}\cos\theta + |\vec{U}|\cos\xi\ , \qquad (26)$$

$$M = \frac{|\vec{c}|}{2}\sin\theta + |\vec{U}|\sin\xi\ , \qquad (27)$$

and

$$N = -E\left[\frac{1}{2}\left(\frac{\partial|\vec{c}|}{\partial x_1}\cos\theta - |\vec{c}|\sin\theta\frac{\partial\theta}{\partial x_1}\right) + \left(\frac{\partial|\vec{U}|}{\partial x_1}\cos\xi - |\vec{U}|\sin\xi\frac{\partial\xi}{\partial x_1}\right)\left(1 + \frac{\cos^2\theta}{2}\right)\right.$$

$$+ \frac{1}{2}\left(\frac{\partial|\vec{c}|}{\partial x_2} \sin\theta + |\vec{c}| \cos\theta \frac{\partial\theta}{\partial x_2}\right)$$

$$+ \left(\frac{\partial|\vec{U}|}{\partial x_2} \sin\xi + |\vec{U}| \cos\xi \frac{\partial\xi}{\partial x}\right) \left(1 + \frac{\sin^2\theta}{2}\right)$$

$$+ \frac{1}{2} \sin\theta \cos\theta \left(\frac{\partial|\vec{U}|}{\partial x_2} \cos\xi - |\vec{U}| \sin\xi \frac{\partial\xi}{\partial x_2}\right.$$

$$\left.\left. + \frac{\partial|\vec{U}|}{\partial x_1} \sin\xi + |\vec{U}| \cos\xi \frac{\partial\xi}{\partial x_1}\right)\right]. \quad (28)$$

and $|\vec{c}|$ is the phase speed of the wave whose wave number is $|\vec{k}|$.

The solution for Equation (25) is obtained, once again, as for Equation (15), by the method of characteristics. By integrating the ordinary differential equations,

$$\frac{dx_1}{dp} = L, \quad (29)$$

$$\frac{dx_2}{dp} = M, \quad (30)$$

and

$$\frac{dE}{dp} = N \quad (31)$$

the family of one-parametered characteristic curves, p, can be obtained. There is one and only one of this family of curves, p, which will pass through each point $(\vec{x}_o, E_o)$, where the initial wave energy, E_o, is associated with $\vec{x}_o$ and θ_o on the initial curve, C. Using the Runge-Kutta numerical method, Equation (25) is integrated with the same set of initial values of η previously defined by the kinematic consideration. The quanities of θ, $\partial\theta/\partial x_1$, $\partial\theta/\partial x_2$, $|\vec{c}|$, $\partial|\vec{c}|/\partial x_1$ and $\partial|\vec{c}|/\partial x_2$ are determined from the solutions of Equation (15) as functions of $\vec{x}$ in the integration of Equation (25).

C. Directional Wave Spectra

The waves are assumed to be driven by wind with velocity, $\vec{W}$, which is, shown in Figure 1, in the direction of the x_1-axis. The most convenient way of describing directional waves is the directional wave spectra. In this particular study the directional wave number spectra, $\Psi(|\vec{k}|,\theta)$, is used in which wave energy is expressed in terms of a wave whose local wave number is $|\vec{k}|$ and local direction of propagation with respect to wind velocity is the angle of θ. Up until now the theoretical derivation for the directional wave spectra is still not available because the mechanism involved in the generation of waves by wind is not well understood. Nevertheless, for practical reasons, there are some semi-empirical and semi-theoretical directional spectral forms which have been proposed and studied. The majority of them are the mathematical models which describe waves for the region from which data are collected. It is not the purpose here to discuss these directional wave spectra. For more information one may refer to Chen (1972).

The general form which $\Psi(|\vec{k}|,\theta)$ is assumed is based upon the assumption of separation of variables, i.e,

$$\Psi(|\vec{k}|,\theta) = \psi(|\vec{k}|)\ A(|\vec{k}|,\theta) \tag{32}$$

where $\psi(|\vec{k}|)$ is the integrated one-dimensional wave number spectra and $A(|\vec{k}|,\theta)$ is the angular spreading function. By satisfying the dimensional analyses of Kitaigorodskii (1961) and the equilibrium range argument of Phillips (1958), $\psi(|\vec{k}|)$ has the form of

$$\psi(|\vec{k}|) = \frac{\beta}{|\vec{k}|^3} e^{-\alpha\left(|\vec{k}|_m/|\vec{k}|\right)^2} \tag{33}$$

where α and β are coefficients, and $|\vec{k}|_m$ is the peak wave number where wave energy is the maximum. From the Resonant Theory of Phillips (1969) on the generation of waves by wind in deep water

$$|\vec{k}|_m = \frac{g}{|\vec{W}|^2} . \tag{34}$$

Also, based upon Phillips (1969) and Pierson and Moskovitz (1964), the values of α and β can be chosen as 0.74 and $0.40 \cdot 10^{-2}$ respectively for a case of rather long fetch value. In regard to the angular spreading function, Longuet-Higgins, et al. (1963) measure and propose the only form which will allow waves to travel against wind. Chen (1972) evaluates their angular spreading function of the form

$$A(|\vec{k}|,\theta) = \frac{\Gamma(q+1)}{4\sqrt{\pi}\ \Gamma(q+1/2)\ |\vec{k}|} \cos^{2q}\frac{\theta}{2} \tag{35}$$

where

$$q = 10^{\frac{-0.3788\ |\vec{W}|}{|\vec{c}|} + 1.059} \tag{36}$$

and Γ is the Gamma function.

From Equations (33), (34) and (35), the directional wave number spectra can be rewritten as

$$\Psi(|\vec{k}|,\theta) = \frac{0.001\ \Gamma(q+1)}{\sqrt{\pi}\ \Gamma(q+1/2)\ |\vec{k}|^4} \cdot e^{-0.74\left(g/|\vec{W}|^2|\vec{k}|\right)^2} \cos^{2q}\frac{\theta}{2} \tag{37}$$

where the value of q is given by Equation (36). By the definition of wave energy density, the initial wave energy, E_o, for the wave with wave number vector, $\vec{k}_o$, in the case with no current can be obtained by using Equations (36) and (37) as

$$E_o(\vec{k}_o) = E_o(|\vec{k}_o|, \theta_o) = \Psi\ (|\vec{k}_o|, \theta)\ |\vec{k}_o|$$

$$= \frac{0.001\ \Gamma(q+1)}{\sqrt{\pi}\ \Gamma(q+1/2)\ |\vec{k}_o|^3}$$

$$\cdot\ e^{-0.74\left(g/|\vec{W}|^2|\vec{k}_o|\right)^2} \cos^{2q}\frac{\theta_o}{2} \qquad (38)$$

where

$$q = 10^{-0.3788|\vec{W}|\ (g/|\vec{k}_o|)^{1/2}} + 1.059 \qquad (39)$$

Equations (38) and (39) are used to evaluate $E_o(\vec{k}_o)$ for the numerical calculations.

IV. NUMERICAL RESULTS

Contour wave energy plots of the directional wave spectra with $\vec{k}$, or $|\vec{k}|$ and θ, at $\vec{x}$ = (50 km, 5 km) are made for one case of current gradients. Namely, by referring to Equations (22) and (23), current gradients are chosen to have $a = 2 \cdot 10^{-5}$ and $b = 2 \cdot 10^{-5}$. The results produced by the interactions between steady non-uniform currents with these current gradients and wind-generated gravity waves are given in Figure 2. A reference case of no current is also given, but in Figure 3. The active wind speed, $|\vec{W}|$, in all of these cases is chosen to be 10 meters per second. In order to appreciate the results, one has to compare the displacements of the energy contours from one case to another. For this purpose several energy contours of the

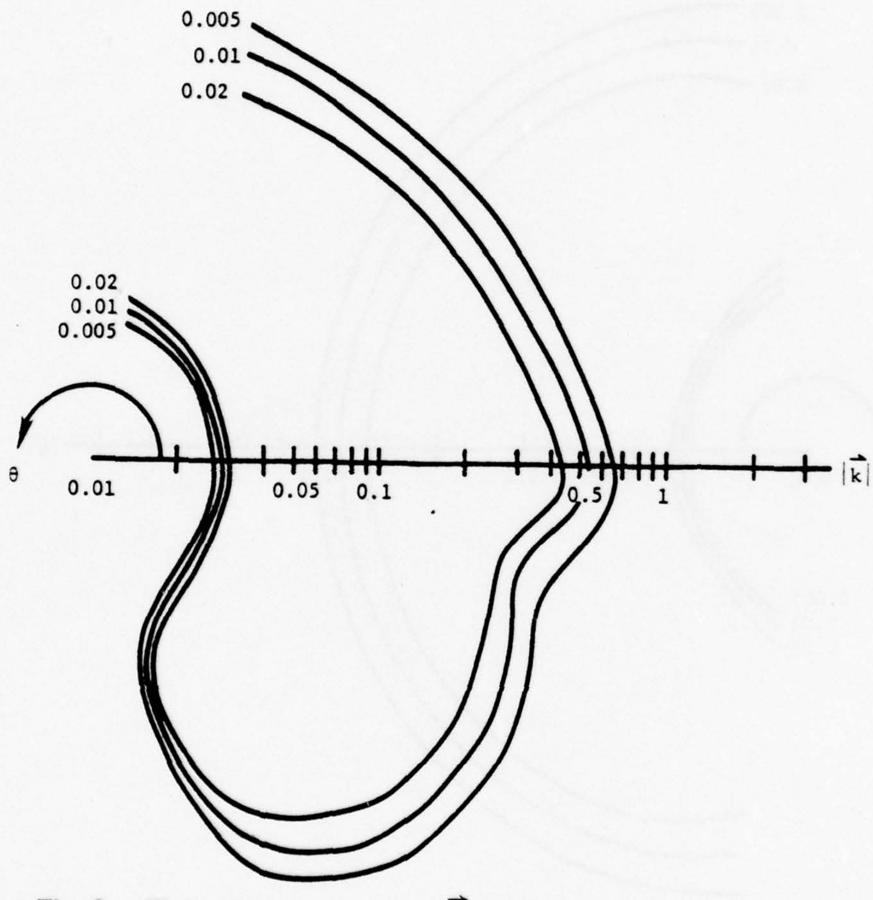

Fig. 2 — Wave energy contours at $\vec{x}$ = (50 km, 5 km) with current, a = $2 \cdot 10^{-5}$/sec and b = $2 \cdot 10^{-5}$/sec, and wind speed, $|\vec{W}|$ = 10 m/sec

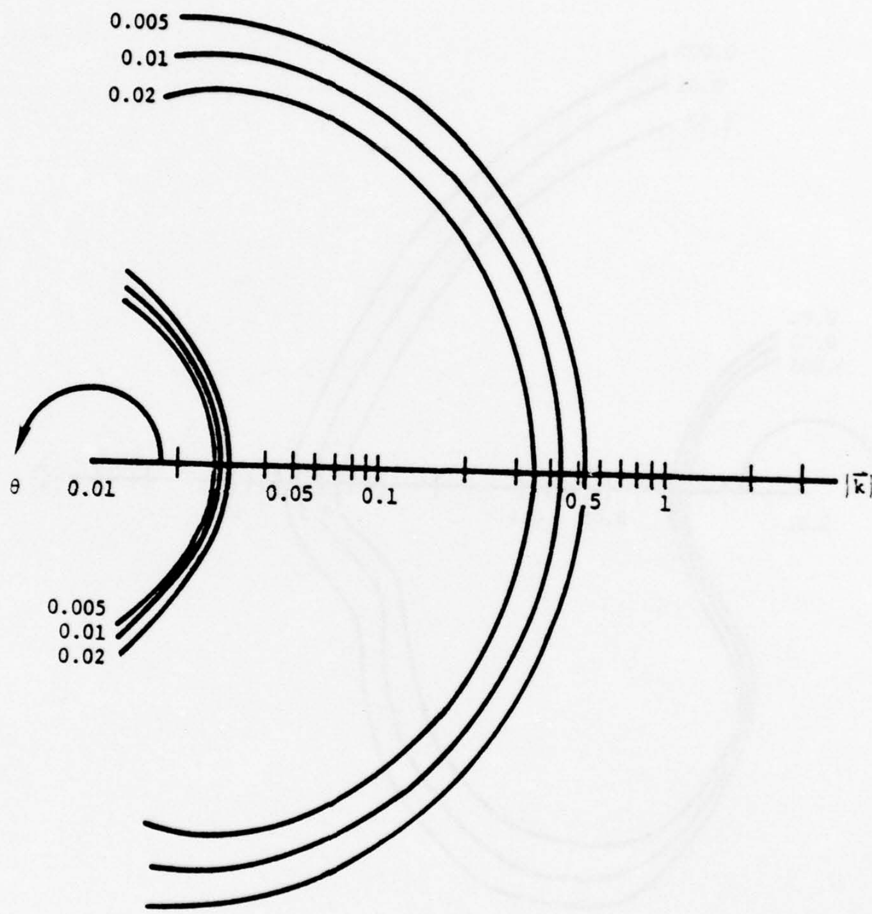

Fig. 3 — Wave energy contours at $\vec{x}$ = (50 km, 5 km) with current, a – b = 0/sec, and wind speed, $|\vec{W}|$ = 10 m/sec

same set of values are plotted on each figure. The results for these plots are obtained by utilizing the CDC 1604 computer.

The most direct computational method to obtain results for the contour plots is the use of root subroutines, as shown in the Appendix, to calculate $E = E(\vec{x}, \vec{k}, \vec{x}_o, \vec{k}_o)$ where $|\vec{k}_o|$ and η in $\vec{x}_o = (o,\eta)$ are the independent variables and E is the solution in Equation (25) with θ, $\partial\theta/\partial x_1$, $\partial\theta/\partial x_2$, $|\vec{c}|$, $\partial|\vec{c}|/\partial x_1$, and $\partial|\vec{c}|/\partial x_2$ determined by the solution of Equation (15), $\theta = \theta(\vec{x}, \vec{x}_o, \theta_o)$. For each value of θ_o, the value of $\vec{x}_o$ is varied in order that the wave energy characteristics of Equation (25) passes through the selected space point, $\vec{x}$, and the value, $|\vec{k}_o|$, is also varied so that wave energy, E, at this space point, $\vec{x}$, lies on the wave energy contour; for each space point, $\vec{x}$, along the characteristic curve of Equation (25) η is varied so that the characteristic curve of Equation (15) passes through the space point, $\vec{x}$, of the integration for Equation (25).

Furthermore, to decrease the time required on the computer, a generalized Newton's method of approximation is used in addition to root subroutines to obtain solutions. A listing of the Fortran program is given in the Appendix. The main routine PDEQ integrates Equation (25) for initial values of η and values of $|\vec{k}_o|$; the subroutine DISP integrates Equation (15), calculating θ, $\partial\theta/\partial x_1$, $\partial\theta/\partial x_2$, $|\vec{c}|$, $\partial|\vec{c}|/\partial x_1$ and $\partial|\vec{c}|/\partial x_2$ at each space point, $\vec{x}$, in the integration of Equation (25) by the main routine.

V. DISCUSSION

In the processes of solving Equations (15) and (25), which are the first order partial differential equations, by the method of characteristics numerically, one notices that the current, $\vec{U}$, is required to be specified initially, on the initial curve, C, as well as along each characteristic curve. Yet this specified current field has to satisfy the condition of continuity, Equation (9). Because of these two conditions which are not the constraints of the method, the first way to deal with the problem will require the computation of wave field for all specifiable current fields in the ocean which satisfy Equation (9) under various steady active wind conditions. This is quite a task; even the initial reference wave field is specified. The second way to deal with the problems will be to specify the wave field and to solve for current field inversely under various steady active wind conditions. However, the processes of solving Equations (15) and (25) by this method of characteristics will require $\vec{k}(\vec{x})$ and wave energy, $E(\vec{k})$, to be specified initially as well as along each characteristic curve. The most convenient way of satisfying these requirements is by using the remote sensing techniques which can provide information over a large area synoptically.

Nevertheless, a few cautions are warranted here. First, through the analysis the linear dispersion relationship as shown in Equation (6) has been used. This implies that the weak nonlinear dynamic interaction is neglected. However, recent theoretical analyses by Huang and Tung (1976a and 1976b) show that the reported 10% increase in wave number, $|\vec{k}|$, as compared with that derived from Equation (6) by Longuet-Higgins, et al. (1963) can be produced by the directional nonlinear wave-wave interactions. Improvements can, of course, be made to include such nonlinear mechanisms in the future.

Second, it is important that within the length scale of wave decay there should be appreciable changes in current velocity, $\vec{U}$. Because of this, capillary waves are not considered in this study. Yet if we concentrate on gravity wave range, the minimum wave length of 0.3 meters will have a decay time of $2.486 \cdot 10^3$ seconds and decay distance of 1700 meters for kinematic viscosity of $9.17 \cdot 10^{-7}$ seconds per meter square.

Finally, the energy balance equation, as shown by Equation (11), is obtained by integrating Navier-Stokes equation over the depth of water without considering the problem of wave breaking. This wave breaking phenomenon will cause serious mathematical difficulties in the case of a saturated ocean encountering opposing currents where the vigorous wave breaking will drain an analytically unaccountable large amount of energy. Extra caution should be taken in dealing with this situation.

VI. CONCLUSION

The changes in wave characteristics, in the form of directional wave spectra, have been numerically calculated as the directional wind-generated gravity waves encounter the steady non-uniform two-dimensional currents. Using the remote sensing techniques which are capable of observing these changes in waves over a large area, one will be able to infer the magnitude and the direction of the current encountered. This particular method of inferring current field has great potential benefits in global applications.

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APPENDIX

```
      PROGRAM PDEQ
      DIMENSION ALPH1(6),ALPH2(6),FOA(7),AKOA(2,16,7)
      DIMENSION THETAV(16)
      DIMENSION F1(10)
      DIMENSION AKO2(8)
      DIMENSION DK1(10)
      DATA((THETAV(J),J=1,16)=10.,20.,30.,40.,50.,60.,70.,80.,
     1-10.,-20.,-30.,-40.,-50.,-60.,-70.,-80.)
      DATA((ALPH2(J),J=1,2)=2.E-5,0.0)
      DATA(YM=5.E+3),(X2=2.E+4)
      DATA(G=9.8),(PI=3.14592...)
      DATA(W=10.)
      DATA((ALPH1(J),J=1,2)=2.E-5,0.)
      DATA((FOA(J),J=1,4)=.005,.01,.02,.03)
      EXTERNAL FOX
      COMMON/FLAG/NFLAG
      COMMON/BLK1/THETAF,W1,FO1F
      COMMON/BLK2/ALPHA1,ALPHA2
      W1=W
      XLB=.007
      XUB=7.
      DO 410 JA=1,3
      DO 410 IA=1,8
      THETAO=THETAV(IA)*PI/180.
      THETAF=THETAO
      FO1F=FOA(JA)
      TOL=1.E-2*FO1F
      CALL FCNZERO(FOX,XLB,XUB,TOL,2,NUMB,AKO2)
      IF(NUMB.EQ.0)430,450
  430 DO 440 IA1=IA,8
      AKOA(1,IA1,JA)=AKOA(1,IA1+8,JA)=AKO2(1)
      AKOA(2,IA1,JA)=AKOA(2,IA1+8,JA)=AKO2(3)
  440 CONTINUE
      GO TO 460
  450 CONTINUE
      AKOA(1,IA,JA)=AKOA(1,IA+8,JA)=AKO2(1)
      AKOA(2,IA,JA)=AKOA(2,IA+8,JA)=AKO2(3)
      PRINT 420,NUMB,AKO2(1),AKO2(3)
  420 FORMAT(1X,6HROTS = ,I4,2X,6HRT1 = ,F10.3,2X,6HRT2 = ,F10.3)
  410 CONTINUE
  460 CONTINUE
      DO 17 JJ=1,2
      ALPHA1=ALPH1(JJ)
      ALPHA2=ALPH2(JJ)
      PRINT 750,ALPHA1,ALPHA2
  750 FORMAT(1H1,1X,8HGRAD1 = ,E7.1,1X,8HGRAD2 = ,E7.1,/)
      PRINT 20
   20 FORMAT(/,3X,5HTHETA,9X,1HK, 12X,1HX, 11X,1HF,10X,2HY0,
     110X,6HTHETAO,6X,2HK0,10X,2HE0,/)
      DO 19 I=1,16
      IF(I.EQ.8)19,910
  910 CONTINUE
  900 CONTINUE
      THETAO=THETAV(I)*PI/180.
      Y0I1=YM-X2*TANF(THETAO)
      THETAF=THETAO
      IFLAG=0
      DO 15 JJ1=1,2
      DO 15 J2=1,3
      FO1F=FOA(J2)
      AKO=AKOA(JJ1,I,J2)
      DKO1=.02*AKO
      L=0
      CALL ENGY(X2,YM,Y0I1,Y0I,IFLAG,THETAO,AKO,AKK,XOUT,THETA,F)
      IF(ALPHA1.EQ.0..AND.ALPHA2.EQ.0.)12,76
   76 IF(NFLAG.EQ.1)15,52
   52 CONTINUE
      IFLAG =1
      TS=ABSF(FO1F-F)
      EX=F
      AKO=AKO+DKO1
      CALL ENGY(X2,YM,Y0I1,Y0I,IFLAG,THETAO,AKO,AKK,XOUT,THETA,F)
      IF(NFLAG.EQ.1)15,54
   54 CONTINUE
      TS1=ABS(FO1F-F)
      IF(TS1.LT.TS)63,64
   63 IF(L.GT.10)68,73
   73 EX=F
      TS=TS1
      AKO=AKO+DKO1
      CALL ENGY(X2,YM,Y0I1,Y0I,IFLAG,THETAO,AKO,AKK,XOUT,THETA,F)
      IF(NFLAG.EQ.1)15,56
   56 CONTINUE
      TS1=ABSF(FO1F-F)
      L=L+1
      IF(TS1.LT.TS)63,69
   69 AKO=AKO-DKO1 $GO TO 68
   64 AKO=AKO-DKO1
   65 IF(L.GT.10)68,75
   75 AKO=AKO-DKO1
      CALL ENGY(X2,YM,Y0I1,Y0I,IFLAG,THETAO,AKO,AKK,XOUT,THETA,F)
      IF(NFLAG.EQ.1)15,58
   58 CONTINUE
      TS1=ABSF(FO1F-F)
      L=L+1
      IF(TS1.LT.TS)66,70
   70 AKO=AKO+DKO1 $GO TO 68
   66 TS=TS1
      EX=F
      GO TO 65
   68 F1(1)=EX
      DKO=.01*AKO
      I1=1
      GO TO 160
  630  IF(I1.GT.20)12,670
  670 I1=I1+1
      IF(I1.EQ.2)615,606
  615 AKO=AKO+DKO $DK1(I1)=DKO $GO TO 617
  606 DK1(I1)=DK1(I1-1)/(F1(I1-1)-F1(I1-2))*(FO1F-F1(I1-1))
      AKO=AKO+DK1(I1)
      IF(I1.GT.5)40,617
   40 IF(ABSF(DK1(I1)/DK1(I1-1)).LT.1.)617,12
  617 CONTINUE
      CALL ENGY(X2,YM,Y0I1,Y0I,IFLAG,THETAO,AKO,AKK,XOUT,THETA,F)
```

```
      IF(NFLAG.EQ.1)15,60
   60 CONTINUE
      F1(I)=F
  160 IF(ABSF((F1(I)-F01F)/F01F).LT.0.02)12,630
   12 TH=THETA*180./PI
      PRINT 153,TH,AKK,XOUT,F,Y01,THETAV(I),AK0,F01F
  153 FORMAT(2X,F7.2,4(2X,F10.3),5X,F7.2,2(2X,F10.3))
   15 CONTINUE
   19 CONTINUE
   17 CONTINUE
      END
      SUBROUTINE ENGY(X2,YM,Y011,Y01,IFLAG,THETA0,AK0,AKK,XOUT,THETA,F)
      DATA(N=2)
      COMMON/FLAG/NFLAG
      COMMON/BLK1/THETAF,W1,F01F
      COMMON/FUNB/DTX,DTY,C,DCX,DCY
      DIMENSION Y(2),F(2),X(10)
      DATA(G=9.8)
      W=W1
      IF(AK0.LT.0.)50,52
   50 NFLAG=1 $ GO TO 13
   52 CONTINUE
      CALL PHI(THETA0,W,AK0,F0)
      K=0
      IF(IFLAG.EQ.0)1,660
    1 Y01=Y011
      Y012=ABSF(Y011)
  660 IF(K.EQ.10)650,880
  880 K=K+1
      NT=0
      IF(K.EQ.1)807,26
   26 IF(K.EQ.2)28,34
   28 IF(Y(1).GT.X2)30,32
   30 Y01=Y01+H/2.
      DY0A=H/2.
      GO TO 807
   32 Y01=Y01-H/2.
      DY0A=-H/2.
      GO TO 807
   34 DY0A=DY0A/(X(K-1)-X(K-2))*(X2-X(K-1))
      Y01=Y01+DY0A
  807 H=(YM-Y01)/10.
      Y(1)=0.
      Y(2)=ALOG(F0)
      Y21=Y01
      DO 100 J=1,10
   10 CALL FUN1(Y,Y21,F,Y01,AK0,THETA0,THETA)
      IF(NFLAG.EQ.1)500,550
  500 IF(THETA0.GT.0.)510,530
  510 IF(Y011.LT.YM)541,930
  930 PRINT 920
      GO TO 13
  541 Y011=Y011+Y012/4.
      Y01=Y011 $K=0 $NFLAG=0 $GO TO 660
  530 IF(Y011.GT.YM)542,910
  910 PRINT 920
  920 FORMAT(2X,11HNO SOLUTION)
      GO TO 13
  542 Y011=Y011-Y012/4.
      Y01=Y011 $K=0 $NFLAG=0 $GO TO 660
  550 CONTINUE
      S=RKLDEQ(N,Y,F,Y21,H,NT)
      IF(S-1.1)10,10,100
  100 CONTINUE
      CALL FUN1(Y,Y21,F,Y01,AK0,THETA0,THETA)
      X(K)=Y(1)
      IF(ABSF((X(K)-X2)/X2).LT.0.02)650,660
  650 F=EXP(Y(2))
      XOUT=X(K)
      AKK=G/C**2
   13 RETURN
      END
      SUBROUTINE FUN1(Y,Y21,F,Y01,AK0,THETA0,THETA)
      DIMENSION Y(2),F(2)
      COMMON/FLAG/NFLAG
      COMMON/FUNB/DTX,DTY,C,DC1,DC2
   10 CALL SUB1(Y(1),Y21,U1,DU1,DU2,DTHX,DTHY,THETAB)
      CALL DISP(Y(1),Y21,Y01,THETA0,AK0,THETA)
      IF(NFLAG.EQ.1)900,910
  910 CONTINUE
      COS2=COS(THETAB)
      SIN2=SIN(THETAB)
      COS1=COS(THETA)
      SIN1=SIN(THETA)
      TS=.5*C
      F1=TS*COS1+U1*COS2
      F2=TS*SIN1+U1*SIN2
  120 F3=.5*(DC1*COS1-C*SIN1*DTX)
     C+(DU1*COS2-U1*SIN2*DTHX)*(1.+COS1**2/2.)
     C+.5*(DC2*SIN1+COS1*C*DTY)+(DU2*SIN2+U1*COS2*DTHY)*(1.+SIN1**2/2.)
     C+SIN1*COS1*(DU2*COS2-U1*SIN2*DTHY+DU1*SIN2+U1*COS2*DTHX)*.5
      F(2)=-F3/F2
  110 F(1)=F1/F2
  900 RETURN
      END
      SUBROUTINE DISP(X,Y2,Y0,THETA0,AK0,THETA)
      DIMENSION Y(2),F(2),X1(10)
      DATA(N=2)
      COMMON/FLAG/NFLAG
      COMMON/FUNB/DTX1,DTY1,C1,DCX,DCY
      IF(X.EQ.0)1,2
    1 Y(1)=0. $Y(2)=THETA0 $Y21=Y0
      GO TO 13
    2 CONTINUE
      NT=0
      K=0
  100 K=K+1
      IF(K.GT.7)13,110
  110 CONTINUE
      IF(K.EQ.1)30,3
   30 Y01=Y0
      GO TO 20
    3 IF(K.EQ.2)5,8
    5 IF(X1(1).GT.X)6,7
```

```
  6 DYO=H/2.
    GO TO 9
  7 DYO=-H/2.
    GO TO 9
  8 DYO=DYO/(X1(K-1)-X1(K-2))*(X-X1(K-1))
  9 YO1=YO1+DYO
 20 H=(Y2-YO1)/5.
    Y(1)=0. $Y(2)=THETA0 $Y21=YO1
    DO 11 J=1,5
 10 CALL FUN(Y,Y21,F,AKO)
    IF(NFLAG.EQ.1)14,15
 15 CONTINUE
    S=RKLDEQ1(N,Y,F,Y21,H,NT)
    IF(S-1.)10,10,11
 11 CONTINUE
    X1(K)=Y(1)
    IF(ABS((X1(K)-X)/X).LT.0.02)13,100
 13 CALL FUN(Y,Y21,F,AKO)
    THETA=Y(2)
 14 RETURN
    END
    SUBROUTINE FUN(Y,Y21,F,AKO)
    COMMON/FUNB/DTX1,DTY1,C1,DCX,DCY
    COMMON/FLAG/NFLAG
    DIMENSION Y(2),F(2)
    DATA(G=9.8)
102 CALL SUB1(Y(1),Y21,U1,DU1,DU2,DTHX,DTHY,THETAB)
    PHI=THETAB-Y(2)
    COS1=COS(Y(2))
    COS2=COS(PHI)
    SIN1=SIN(Y(2))
    SIN2=SIN(PHI)
    IF(U1.EQ.0..OR.COS2.EQ.0.)200,210
200 NFLAG=0
    AK=AKO
*   GO TO 9
210 A=G+4.*SQRT(AKO*G)*U1*COS2
    IF(A.LT.0.)220,230
220 NFLAG=1
    PRINT 240,Y(1),Y21,U1,THETAB
240 FORMAT(1X,16H(X,Y,U,THETA) = ,4F10.3)
    GO TO 100
230 NFLAG=0
    AK=((-SQRT(G)+SQRT(A))/(2.*U1*COS2))**2
  9 TS=1./(.5*SQRT(G/AK)+U1*COS2)
    F1=COS1-TS*U1*SIN1*SIN2
    F2=SIN1+TS*U1*COS1*SIN2
    F3=TS*(SIN1*COS2*DU1-U1*SIN1*SIN2*DTHX-COS1*COS2*DU2+U1*COS1*
   CSIN2*DTHY)
    F(1)=F1/F2
    F(2)=F3/F2
101 C1=SQRT(G/AK)
    TS1=.5*C1*TS
    DTX1=F3/F1 $DTY1=F3/F2
    DCX=TS1*(U1*SIN2*(DTX1-DTHX)+COS2*DU1)
    DCY=TS1*(U1*SIN2*(DTY1-DTHY)+COS2*DU2)
100 RETURN
    END
    SUBROUTINE PHI(THETA,W,AK,EO)
    DATA(G=9.8),(PI=3.141592654)
    C=SQRT(G/AK)
    S=10.**(-0.3788*W/C+1.059)
    S1=S+1 $S2=S+0.5 $S3=0.
    CALL GAMMA(S1,S3,GAM1,TS)
    CALL GAMMA(S2,S3,GAM2,TS)
    EO=COS(THETA/2.)**(2.*S)*GAM1/(4.*SQRT(PI)*GAM2)
    AK1=W**2
    AK2=AK1*AK
    AK3=(G/AK2)**2
    AK4=-0.74*AK3
    AK5=EXPF(AK4)
    AK6=AK**3
    EO=EO*0.4E-2*AK5/AK6
    RETURN
    END
    SUBROUTINE SUB1(X,Y,U1,DU1,DU2,DTHX,DTHY,THETAB)
    DATA(PI=3.141592654)
    COMMON/BLK2/TS1,TS2
    DATA(X0=1.E+3)
    ALPHA1=TS1 $ALPHA2=TS2
    IF(ALPHA1.EQ.0..AND.ALPHA2.EQ.0.)4,6
  4 U1=DU1=DU2=DTHX=DTHY=THETAB=0.
    GO TO 5
  6 CONTINUE
    IF(ALPHA2.EQ.0.)100,110
100 U1=ALPHA1*X
    DU1=ALPHA1
    DU2=DTHX=DTHY=0.
    THETAB=PI/2.
    GO TO 5
110 CONTINUE
    IF(X.LT.X0)1,2
  1 U1=DU1=DU2=DTHX=DTHY=THETAB=0.
    GO TO 5
  2 A1SQ=ALPHA1**2
    A2SQ=ALPHA2**2
    XSQ=X*X
    YSQ=Y*Y
    U1=SQRT(A1SQ*XSQ+A2SQ*YSQ)
    DU1=A1SQ*X/U1
    DU2=A2SQ*ABSF(Y/U1)
 20 THETAB=ATAN(ALPHA1*X/(ALPHA2*ABSF(Y)))
    COSSQ=COS(THETAB)**2
    DTHX=COSSQ*ALPHA1/(ALPHA2*ABSF(Y))
    DTHY=-COSSQ*ALPHA1*X/(ALPHA2*YSQ)
  6 CONTINUE
    RETURN
    END
    FUNCTION FOX(AK02)
    COMMON/BLK1/THETAF,W,E01F
    W1=W
    THETA=THETAF
    CALL PHI(THETA,W1,AK02,E02)
    FOX=E02-E01F
```

```
      RETURN
      END
      FUNCTION RKLDEQ(N.Y.F.X.H.NT)
      DIMENSION Y(10).F(10).Q(10)
      NT=NT+1
      GO TO (1.2.3.4).NT
    1 DO 11 J=1.N
11    Q(J)=0.
      A=.5
      X=X+H/2.
      GO TO 5
2     A=.2928932188I
      GO TO 5
3     A=1.707106781?
      X=X+H/2.
      GO TO 5
4     DO 41 I=1.N
41    Y(I)=Y(I)+H*F(I)/6.-Q(I)/3.
      NT=0
      RKLDEQ=2.
      GO TO 6
5     DO 51 L=1.N
      Y(L)=Y(L)+A*(H*F(L)-Q(L))
51    Q(L)=2.*A*H*F(L)+(1.-3.*A)*Q(L)
      RKLDEQ=1.
6     CONTINUE
      RETURN
      END
      FUNCTION RKLDEQ1(N.Y F.X.H.NT)
      DIMENSION Y(10).F(10).Q(10)
      NT=NT+1
      GO TO (1.2.3.4).NT
1     DO 11 J=1.N
11    Q(J)=0.
      A=.5
      X=X+H/2.
      GO TO 5
2     A=.2928932188I
      GO TO 5
3     A=1.707106781?
      X=X+H/2.
      GO TO 5
4     DO 41 I=1.N
41    Y(I)=Y(I)+H*F(I)/6.-Q(I)/3.
      NT=0
      RKLDEQ1=2.
      GO TO 6
5     DO 51 L=1.N
      Y(L)=Y(L)+A*(H*F(L)-Q(L))
51    Q(L)=2.*A*H*F(L)+(1.-3.*A)*Q(L)
      RKLDEQ1=1.
6     CONTINUE
      RETURN
      END
      SUBROUTINE GAMMA(ZR.ZI.GR.GI)
      IF (ZR) 1.7.7
    1 N=1.-ZR
      FN=N
      ZS=FN+ZR
      IF (ZI) 2.6.6
    2 ZI=-ZI
      CALL GAMMAZ(ZS.ZI.GR.GI)
      GI=-GI
    3 J=0
    4 FJ=J
      ZS=ZR+FJ
      W=ZS*ZS+ZI*ZI
      RE=(ZS*GR+ZI*GI)/W
      GI=(ZS*GI-ZI*GR)/W
      GR=RE
      J=J+1
      IF (N-J)5.5.4
    5 RETURN
    6 CALL GAMMAZ(ZS.ZI.GR.GI)
      GO TO 3
    7 IF (ZI) 8.9.9
    8 ZI=-ZI
      CALL GAMMAZ(ZR.ZI.GR.GI)
      GI=-GI
      RETURN
    9 CALL GAMMAZ(ZR.ZI.GR.GI)
      RETURN
      END
      SUBROUTINE GAMMAZ(ZR.ZI.GR.GI)
C     TO CALCULATE GAMMA(ZR+I(ZI)).WHERE ZR AND ZI ARE POSITIVE
      PI=2.506628274?
      IF (ZI-1.) 1.1.2
    1 T1=ZR
      T2=ZI
      CALL GAMMA1(T1.T2.GR.GI)
      RETURN
    2 M=ZI
      N=M+1
      FN=N
      X=ZR/FN
      Y=ZI/FN
      T1=X
      T2=Y
      CALL GAMMA1(T1.T2.T3.T4)
      S=T3
      T=T4
      J=1
      FJ=J
      FJN=FJ/FN
    3 X=X+FJN
      T1=X
      T2=Y
      CALL GAMMA1(T1.T2.T3.T4)
      U=T3
      V=T4
      R=S*U-T*V
      T=S*V+T*U
      S=R
      J=J+1
```

```
      IF (M-J) 4,3,3
    4 C1=PI**M*SQRTF(FN)
      C2=LOGF(FN)
      C3=ZR*C2
      C4=ZI*C2
      C5=EXPF(C3)/C1
      C=C5*COSF(C4)
      D=C5*SINF(C4)
      P=C*S-D*T
      GI=C*T+D*S
      GR=P
      RETURN
      END
      SUBROUTINE GAMMA1(X,Y,U,V)
C     TO CALCULATE GAMMA(X+IY),WHERE X IS POSIVTIVE AND Y IS LESS THAN 1
      IF (X-1.) 1,1,2
    1 CALL GAMMA2(X,Y,U,V)
      RETURN
    2 N=X
      FN=N
      X=X-FN
      IF (X) 22,21,22
   21 N=N-1
      X=X+1.
   22 S1=X
      S2=Y
      CALL GAMMA2(S1,S2,S3,S4)
      U=S3
      V=S4
      J=0
    3 FJ=J
      P=(X+FJ)*U-Y*V
      V=(X+FJ)*V+Y*U
      U=P
      J=J+1
      IF(N-J) 4,4,3
    4 RETURN
      END
      SUBROUTINE GAMMA2(X,Y,U,V)
C     TO CALCULATE GAMMA(X+IY),WHERE X AND Y ARE NOT GREATER THAN ONE
C     BY PADE-POWER APPROXIMATION OF 1/GAMMA(Z)
      DIMENSION A(9),B(9)
      A(9)=+7.9027636693E-05
      A(8)=-1.7669280217E-03
      A(7)=+1.0021677762E-02
      A(6)=+9.5294080001E-03
      A(5)=-1.9367439704E-01
      A(4)=6.2294126401E-02
      A(3)=+1.2536302998E+00
      A(2)=+1.
      A(1)=0.
      B(9)=+9.9862892410E-06
      B(8)=+9.5129148083E-05
      B(7)=+8.7521996448E-04
      B(6)=+5.1647208257E-03
      B(5)=+2.7018504538E-02
      B(4)=+1.0279994528E-01
      B(3)=+3.2772379961E-01
      B(2)=+6.7641463946E-01
      B(1)=+1.
      E=A(9)*X+A(8)
      F=A(9)*Y
      G=B(9)*X+B(8)
      H=B(9)*Y
      I=7
    1 P=E*X-F*Y
      F=E*Y+F*X
      E=P+A(I)
      Q=G*X-H*Y
      H=G*Y+H*X
      G=Q+B(I)
      I=I-1
      IF(I)2,2,1
    2 W=E*F+F*F
      U=(G*E+H*F)/W
      V=(E*H-G*F)/W
      RETURN
      END
      SUBROUTINE FCNZERO(LC,LB,UB,TL,NR,SC,LST)
C     IDENT NUMBER - 80001002
C     TITLE - REAL ZEROS OF A SINGLE-VALUED FUNCTION
C     IDENT NAME - RD-NRL-FCNZERO
C     LANGUAGE - FORTRAN
C     COMPUTER - CDC-3800
C     ORGANIZATION - RESEARCH COMPUTATION CENTER, MIS
C     DATE - 1 AUGUST 1968, REVISED 1 MAY 1970
C     PURPOSE - TO FIND THE REAL ZEROS OF A SINGLE-VALUED FUNCTION OF
C     ONE REAL VARIABLE BY A MODIFIED METHOD OF FALSE POSITION
      DIMENSION BND(4), LST(2250)
      TYPE REAL LBOUT,INPUT,LB,LST,KA,LC
      TYPE INTEGER TW,FSC,R,S,T,V,SC
      COMMON/BZ/INPUT,OUTPUT,LIM,VAL,Z0,Z1,FZ0,FZ1
      LIM=1000
      DELTAX=0.0
      VAL=0.2
      KA=0.36787944
      TW=FSC=SC=R=V=0
      DX=DELTAX
      IF(DX.EQ.0)1,2
    1 DX=UB-LB
    2 IF(NR.EQ.0)3,4
    3 NR=512
    4 DUMMY=INPUT=LB
      LBOUT=LC(DUMMY)
    5 SC=R=0
      BND(1)=LB
      BND(2)=LBOUT
    6 IF(BND(1).EQ.UB)7,8
    7 IF(ABS(BND(2)).LT.TL)9,20
    9 LST(R+1)=LST(R+3)=BND(1)
      LST(R+2)=LST(R+4)=BND(2)
      SC=SC+1
   10 GO TO 20
    8 INPUT=BND(1)+DX
```

```
      IF(INPUT.GT.0)11,12
   11 INPUT=0
   12 BND(3)=DUMMY=INPUT
      BND(4)=LC(DUMMY)
      IF(BND(2).EQ.0)17,13
   13 IF(BND(2).LT.0)14,15
   14 IF(BND(4).GT.0)17,19
   15 IF(BND(4).LT.0)17,19
   17 DO 18 I=1,4
      LST(R+I)=BND(I)
   18 CONTINUE
      R=R+4
      SC=SC+1
      IF(SC.GE.S1)25,19
   19 BND(1)=BND(3)
      BND(2)=BND(4)
      GO TO 6
   20 IF(SC.EQ.FSC)21,22
   21 TW=TW+1
      GO TO 23
   22 TW=1
   23 IF(SC.LT.NR.AND.TW.LT.10)24,25
   24 DX=K4*DX
      FSC=SC
      GO TO 5
   25 T=(SC-1)*4+1
      DO 32 S=1,T,4
      IF(LST(S+1).EQ.0)26,27
   26 LST(V+1)=LST(S)
      LST(V+2)=LST(S+1)
      GO TO 31
   27 IF(LST(S+1).LT.0)28,29
   28 Z0=LST(S)$    FZ0=LST(S+1)
      Z1=LST(S+2)$  FZ1=LST(S+3)
      GO TO 30
   29 Z1=LST(S)$    FZ1=LST(S+1)
      Z0=LST(S+2)$  FZ0=LST(S+3)
   30 CALL CONVGE1(LC,TL)
      LST(V+1)=INPUT
      LST(V+2)=OUTPUT
   31 V=V+2
   32 CONTINUE
   33 END
      SUBROUTINE CONVGE1(LCN,TLN)
      TYPE REAL K3,INPUT,LCN
      COMMON/B/INPUT,OUTPUT,LIM,VAL,Z0,Z1,FZ0,FZ1
      K2=0
      FOROUT=K3=1.0E100
   34 DUMMY=INPUT=(FZ1*Z0-FZ0*Z1)/(FZ1-FZ0)
      OUTPUT=LCN(DUMMY)
      K2=K2+1
      IF(ABS(OUTPUT).LT.TLN.AND.ABS(Z1-Z0).LT.TLN)50,35
   35 IF(K2.EQ.1) 42,350
  350 CONTINUE
      RATIO=ABS((FORIN-INPUT)/(Z1-Z0))
      IF(RATIO.LT.VAL.AND.FOROUT*OUTPUT.GT.0)36,42
   36 IF(OUTPUT*FZ0.GT.0)51,360
  360 CONTINUE
      DUMMY=(Z0+INPUT)/2
      GO TO 39
   51 IF(OUTPUT*FZ1.GT.0)39,510
  510 CONTINUE
      DUMMY=(Z1+INPUT)/2
   39 SPEC=INPUT
      ALSO=OUTPUT
      INPUT=DUMMY
      OUTPUT=LCN(DUMMY)
      K2 = K2+1
      IF(ALSO.LT.0)40,41
   40 FORIN=Z0=SPEC
      FOROUT=FZ0=ALSO
      GO TO 42
   41 FORIN=Z1=SPEC
      FOROUT=FZ1=ALSO
   42 IF(OUTPUT.EQ.0)50,43
   43 IF(OUTPUT.LT.0)44,45
   44 FORIN=Z0=INPUT
      FOROUT=FZ0=OUTPUT
      GO TO 46
   45 FORIN=Z1=INPUT
      FOROUT=FZ1=OUTPUT
   46 IF(ABS(OUTPUT).LT.TLN.AND.ABS(Z1-Z0).LT.TLN)50,47
   47 IF(Z0.EQ.0.0.AND.K2.LE.LIM)34,470
  470 CONTINUE
      IF(K2.GT.LIM.OR.ABS((Z0-Z1)/Z0).LT.0.000000001)48,34
   48 PRINT 49,K2,Z0,FZ0,Z1,FZ1
   49 FORMAT(1X,I4,4(3X,F17.10))
   50 END
```