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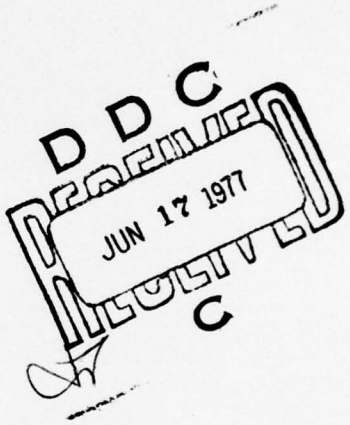
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SOME PROBLEMS IN THE STABILITY OF FLOWS OF VISCOELASTIC FLUIDS

by Pijush K. Kundu



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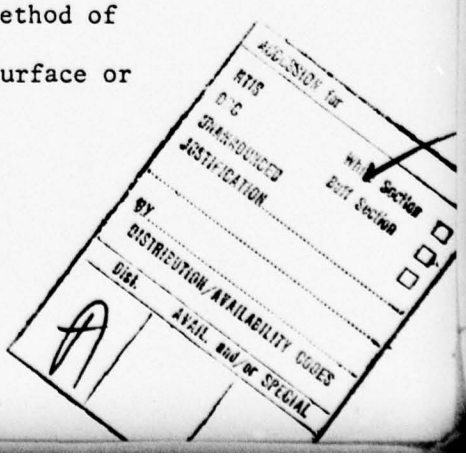
ABSTRACT

Three problems in the stability of viscoelastic flows have been theoretically investigated.

The case of plane Couette flow has been solved by the classical energy method of Orr. For the method to be applicable it has been found that the concept of elastic potential energy of a viscoelastic liquid has to be introduced, and an expression has been found for it. For two-dimensional disturbances of any magnitude the presence of elasticity has been found to stabilize the flow. This result suggests that the viscous sublayer thickness must increase during the so-called Toms phenomenon. It has been found that the results are identical for three fluids, namely the second-order fluid and the Walters fluids A' and B', although the normal stress behavior of these fluids are quite different. This insensitivity to the constitutive equation has been shown to be a result of the two-dimensionality of the problem.

Linear stability theory has been used to investigate the stability of two flows of the Oldroyd fluid, namely the plane Poiseuille flow and the circular Couette flow. For both flows the presence of elasticity has been found to destabilize the flow and to increase the critical wavenumber.

As an aside, it has also been shown that the sign, and magnitude, of the Weissenberg effect may depend on the method of observation, namely through the shape of the free surface or through the height difference of radial tappings.



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LIST OF SYMBOLS

A = Nondimensional relaxation time = $\lambda_1 U_m/\delta$

$A_{ij}^{(N)}$ = N-th order Rivlin-Ericksen tensor

A_{ij} = $A_{ij}^{(1)} = U_{i,j} + U_{j,i} = 2E_{ij}$ = (1) First Rivlin-Ericksen tensor. (2) First Rivlin-Ericksen tensor of basic flow.

a_{ij} = $u_{i,j} + u_{j,i} = 2e_{ij}$ = First Rivlin-Ericksen tensor of disturbance

a = $\Omega_2/\Omega_1 - 1$ = a kinematic parameter in circular Couette flow

B = Nondimensional retardation time = $\lambda_2 U_m/\delta$

b_i^k = Fixed components of the tensor of which the convected components are β_i^k

c = $c_r + ic_i$ = wave velocity

D = Differential operator $d/d\zeta$ or d/dx_2

E = (1) Rate of elongation in spring-dashpot model
(2) Constant in velocity distribution of circular Couette flow

F = Constant in velocity distribution of circular Couette flow

E_{ij} = $\frac{1}{2}(U_{i,j}+U_{j,i})$ = (1) Strain rate. (2) Strain rate of basic flow

e = (1) Internal energy. (2) Elongation in spring-dashpot model

e_{ij} = $\frac{1}{2}(u_{i,j}+u_{j,i})$ = Strain rate of disturbance

G = (1) $\gamma\Gamma/\mu$ = Viscoelastic parameter of Second-order fluid.
(2) Shear modulus. (3) Spring constant

g_{ik} = Metric tensor of fixed coordinates x^i

H = $(3\beta+3\gamma)/4\rho\delta^2$ = Viscoelastic parameter of Second-order fluid

$\underline{\underline{I}}$ = Unit tensor

I = A_{ii} = First invariant of tensor A_{ij}

II = (1) $A_{ij}A_{ij}$ = Second invariant of A_{ij}

(2) $E_{ij}E_{ij}$ = Second invariant of E_{ij}

II' = $a_{ij}a_{ij}$ = Second invariant of a_{ij}

III = (1) $A_{ij}A_{jk}A_{ki}$ = Third invariant of A_{ij}

(2) $E_{ij}E_{jk}E_{ki}$ = Third invariant of E_{ij}

i = $\sqrt{-1}$

k = Nondimensional wavenumber

$N(\tau)$ = Relaxation spectrum

P = (1) Pressure. (2) Pressure of basic flow

p = Disturbance pressure

q_i = Heat flux vector

R = Reynolds number

S_{ij} = (1) Extra-stress. (2) Extra-stress of basic flow

s_{ij} = Extra-stress of disturbance

T = $4\rho^2\delta^4\Gamma^2/\mu^2 = 4\delta^{+4}$ = Nondimensional number

T_{ij} = $-P\delta_{ij} + S_{ij}$ = (1) Stress tensor. (2) Stress tensor of basic flow

t_{ij} = $-p\delta_{ij} + s_{ij}$ = Disturbance stress tensor

U_i = (1) Velocity vector. (2) Velocity vector of basic flow

u_i = Disturbance velocity

u_τ = Friction velocity = $[(\text{wall shear stress})/(\text{density})]^{1/2}$

V_i = $U_i + u_i$ = Velocity vector of disturbed flow

x^i = Fixed coordinates

x_1, x_2, x_3 = Cartesian coordinates

Z = $Z_r + iZ_i = \beta/\gamma$ = A non-Newtonian function

α = (Dimensional) wavenumber

α_{ij} = $V_{i,j} + V_{j,i}$ = First Rivlin-Ericksen tensor of disturbed motion

$\alpha_{ij}^{(2)}$ = Second Rivlin-Ericksen tensor of disturbed motion

β = Non-Newtonian coefficient in Second-order fluid

$\beta(x_2)$ = $1 + ikA(U_1 - c)$ = A non-Newtonian function

Γ = Shear rate = dU_1/dx_2

$\gamma(x_2)$ = $1 + ikB(U - c)$ = A non-Newtonian function

δ = Half the gap between plates in plane Couette or Poiseuille flow; total annular gap in circular Couette flow

δ^+ = $\delta u_\tau/\nu$ = (1) Nondimensional half gap. (2) Non-dimensional viscous sublayer thickness

$\delta(t)$ = Delta function

δ_{ij} = Kronecker delta

ϵ_{ij} = (1) $E_{ij} + e_{ij}$ = Strain rate of disturbed motion. (2) Convective components of E_{ij}

ζ = Nondimensional lateral coordinate x_2

η = Dashpot damping coefficient

λ_1 = Stress relaxation time in Oldroyd and Maxwell fluids

λ_2 = Strain rate relaxation time (or retardation time) in Oldroyd fluid

μ = Dynamic viscosity at zero shear

ν = μ/ρ = Kinematic viscosity

ξ^i = Convective coordinates

π	=	$P + p$ = Pressure of disturbed motion
ρ	=	Density
σ_{ij}	=	(1) $S_{ij} + s_{ij}$ = Extra-stress tensor of disturbed motion.. (2) Convective components of extra-stress S_{ij}
σ_1, σ_2	=	Normal stress functions of viscoelastic fluids
τ_{ij}	=	$T_{ij} + t_{ij} = - \pi\delta_{ij} + \sigma_{ij}$ = Stress tensor of disturbed flow
ϕ	=	Lagrange multiplier satisfying the constraint of incompressibility
Ψ	=	Rate of viscous dissipation
$\Psi_N, \Psi_S, \Psi_W, \Psi_M, \Psi_O$	=	Rate of viscous dissipation in Newtonian, Second-order, Walters, Maxwell, and Oldroyd fluids respectively
ψ	=	Stream function

CHAPTER 1

INTRODUCTION

1.1 The Toms Phenomenon

Great interest was created in the mechanics of non-Newtonian fluids after the discovery of the so called Toms phenomenon. Toms (1948) discovered that when small quantities of long chain linear polymers having molecular weights of the order of five or six figures are added to a turbulent shear flow, substantial reduction (by factors of two or three) in the pressure drop and drag are observed at the same flow rate. There was some conjecture that the reason might be that the addition of the polymer actually stabilized the flow, transforming the turbulent flow into a laminar one, resulting in a reduction in skin friction at the same Reynolds number.

Many theoretical and experimental attempts were subsequently made to investigate the stability characteristics of non-Newtonian fluids. It was found that no great change in the critical Reynolds number occurs due to the non-Newtonian properties of the fluid. Therefore, the conclusion was that the Toms phenomenon cannot be due to the delayed transition of a non-Newtonian fluid [see, for example, Fong & Walters (1965), Tlapa & Bernstein (1970)]. Moreover, it has been observed that the flow does actually remain turbulent during the Toms phenomenon.

No single explanation of the Toms phenomenon has been universally accepted among the scientific community. It was shown by Lumley (1964) that the purely viscous non-Newtonian fluid (Reiner-Rivlin fluid)

cannot cause drag reduction; he concluded that dependence of the stress on the history of the strain rate is necessary, in other words a viscoelastic material is necessary. For speculations as to the mechanism responsible for the Toms phenomenon, see, for example, Lumley (1967, 1969), and Fabula, Lumley & Taylor (1966).

Although the conjecture that the Toms phenomenon is due to delayed transition is now known to be false, the investigation of the instability of a viscoelastic fluid by the so called "energy method" can nevertheless be connected to the phenomenon. For example the experimentally observed thickening of the viscous sublayer during the Toms phenomenon will be explained in the present work by means of a stability analysis.

1.2 Notes on the Two Methods of Stability Analysis

The subject of stability deals with the question of the ability or the inability of a physical system to sustain itself against perturbations to which it is subject. In the area of fluid mechanics it is known that laminar flows occur at low Reynolds numbers and turbulent flows at higher Reynolds numbers. Thus, a typical problem of hydrodynamic stability asks the question: At what Reynolds number does the laminar flow become unstable so that disturbances can grow (and eventually break it up into disorderly turbulent motion)?

There are two different methods for investigating the stability of a disturbed flow - the method of small disturbances (linear theory) and the method of finite disturbances (nonlinear theory). In the method of small disturbances one linearizes the resulting equations by

neglecting terms quadratic or higher in the disturbances and their derivatives. This method has so far received more attention. For example among the major treatises on hydrodynamic stability, Chandrasekhar (1961) deals exclusively with this approach, and Lin (1955) deals very largely with this approach. Although quite successful in a large number of cases "the linear theory suffers from the defect that one cannot, in principle, make judgements regarding the growth potential of finite disturbances. Thus, one cannot say for certain that a given flow will remain stable if disturbed under conditions judged favorable by the linear theory" [Joseph (1966)].

To see this point more clearly, let us suppose that it is possible to decompose the motion into elementary components so that the flow can be considered as a set of interacting elementary nonlinear oscillators [see Monin & Yaglom (1971), page 88]. "In each of these oscillators, due to the influx of energy, self-excited oscillations may arise. The possibility of such oscillations arising is determined by the relationship between the energy E^+ acquired by the oscillator and the energy E^- lost by it for various amplitudes a of the oscillations (Fig. 1.1). If $E^- > E^+$ for all amplitudes (Fig. 1.1a) then the oscillations will evidently be damped for any initial amplitude, and the system will be stable for any disturbances. If $E^- < E^+$ for $a_1 < a < a_0$, but $E^- > E^+$ for $a < a_1$ or $a > a_0$ (Fig. 1.1b), then the oscillations with initial amplitude $a < a_1$ will be damped, but those with initial amplitude $a > a_1$ will increase, until their amplitude attains the equilibrium value a_0. In this case, the system will be stable to small disturbances but unstable to disturbances of sufficiently large amplitude ...

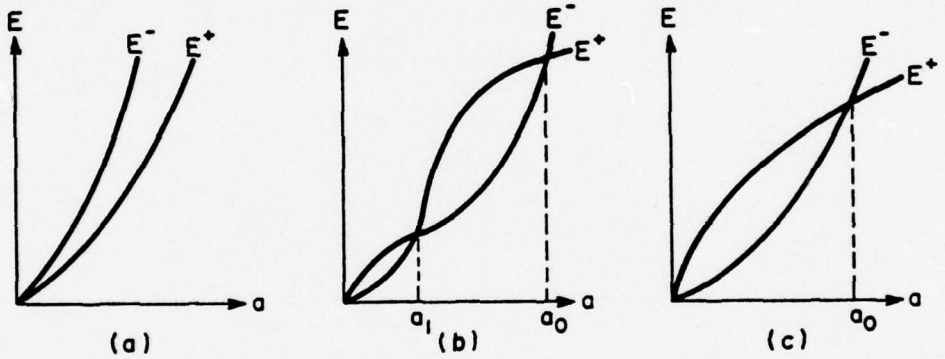


Figure 1.1 Different variants of the dependence of the energy gained and lost by an oscillator on the amplitude of the oscillation [after Monin & Yaglom (1971)].

Finally, if $E^+ > E^-$ for any amplitude below a_0, however small, the system will be unstable to infinitely small disturbances (i.e., absolutely unstable) and will practically always be in the regime of self-excited oscillation with amplitude a_0..."

There are other deficiencies in the linear theory. It is quite clear that the loss of stability does not in itself constitute a transition to turbulence, and the linear theory can at best describe only the very beginning of the process of transition. Moreover, for certain important flows, for example the Couette flow between two parallel planes or Poiseuille flow in a tube, the small disturbance theory predicts that the flow is stable to all infinitesimal disturbances at all Reynolds numbers. Also, in the case of a Poiseuille flow between two parallel planes the linear theory gives a critical Reynolds number which is considerably higher than that at which transition actually occurs in a real flow.

A nonlinear theory of the initiation of turbulence, which takes into account the finite size of the disturbances, has received a lot of attention during the last fifteen years. For a recent review article on this topic see Stuart (1971); see also the books of Betchov & Criminale (1967), Monin & Yaglom (1971) and the earlier review article on general hydrodynamic stability by Stuart (1963).

In addition to the nonlinear theory of the type just mentioned, there is another simple approach to the investigation of the stability of a flow with respect to disturbances of finite amplitude. This is the so called energy method, which is really classical and originated in the writings of Osborne Reynolds (1895). He supposed that a disturbance

motion of finite amplitude is already in existence, and derived the following disturbance energy equation for a Newtonian fluid (for derivation, see Section 3.1):

$$\frac{d}{dt}\int_V \frac{u_i^2}{2}\,dV = -\int_V u_i u_j U_{i,j}\,dV - \nu\int_V u_{i,j}u_{i,j}\,dV \qquad (1.1)$$

where u_i is the disturbance velocity, U_i is the basic flow velocity, ν is the kinematic viscosity, V is the volume of integration, and a comma denotes differentiation with respect to space coordinate. The left hand side of equation (1.1) represents the rate of increase of disturbance kinetic energy throughout the volume V; the first term on the right hand side represents the rate of extraction (or production) of disturbance energy through the interaction of the Reynolds stresses $u_i u_j$ with the gradient $U_{i,j}$ of the basic flow; the last term in (1.1) is always negative and represents viscous dissipation.

Reynolds criterion is that the flow is stable or unstable depending on whether the kinetic energy of the disturbance motion will decrease or increase with time, i.e.,

$$\underset{\text{(Production)}}{-\int_V u_i u_j U_{i,j}\,dV} < \underset{\text{(Dissipation)}}{\nu\int_V u_{i,j}u_{i,j}\,dV} \quad \text{(for stability)}$$

The equality sign in the above relationship corresponds to "neutral stability," for which the rate of change of disturbance kinetic energy is zero.

The first careful use of the energy method was made by Orr (1907). He considered all disturbances which give rise to instantaneous stationarity of disturbance kinetic energy, for which (1.1) reduces to

$$-\int u_i u_j U_{i,j}\,dV = \nu \int u_{i,j} u_{i,j}\,dV \qquad (1.2)$$

For a given function $u_i(\underset{\sim}{x})$, a value of ν can be found from (1.2). The question raised by Orr is the following: "What function u_i makes ν the maximum?" For a larger value of viscosity the motion must be stable. For a Newtonian plane Couette flow having two-dimensional disturbances he found a critical Reynolds number of $\Gamma\delta^2/\nu = 44.3$, where Γ is the shear rate, 2δ is the distance between the plates, and ν is the kinematic viscosity. For plane Poiseuille flow with centerline velocity U_m he obtained a critical Reynolds number $U_m\delta/\nu \simeq 87$.

In a modern formulation of the energy method, Serrin (1959) sought to find conditions under which the kinetic energy of the disturbance motion will surely decrease; that is, he found sufficient conditions for stability under arbitrary three-dimensional disturbances. Serrin's method depends on finding the upper or lower bounds on the various terms in the disturbance energy equation, and the limits of stability given by estimates of this kind do not depend on the specific details of the geometrical configuration or on the distribution of the basic field variables. This generality is obtained at the expense of rather more conservative estimates than can be obtained by using all of the available information.

Joseph (1966) applied the variational method of Orr to a heated plane Couette flow and obtained the stability boundary of $R^2 + Ra = 1708$, where $R = 2\Gamma\delta^2/\nu$ and Ra is the Rayleigh number. For zero Rayleigh number limit, the criterion reduces to $\Gamma\delta^2/\nu = \sqrt{1708}/2 = 20.6$. This value is about one half of that given by Orr; the reason for the discrepancy is that, whereas Orr assumed that the two-dimensional

disturbances are more dangerous than three-dimensional ones, Joseph considered longitudinal roller-type disturbances and proved that they are more unstable.

The energy method has been subject to extensive criticism for quite some time. For example, Lin (1955), Serrin (1959), Joseph (1966) and Schlichting (1968) state that, in this method, one establishes stability relative to arbitrary disturbances while, in reality, only those satisfying the hydrodynamical equations need be considered. Lumley (1971b) in a recent comment on the energy method has shown that this criticism is incorrect, "since functions are admitted for review at a fixed time only; any velocity field at a fixed time satisfies the hydrodynamical equations, which serve to determine its evolution, i.e., the time derivative." In the same paper Lumley also shows that the disturbance motion in the energy method must necessarily be nonlinear, and that the disturbance structure must change because the local growth rate must be different from point to point.

The one particular criticism about the energy method, that it gives low values of the critical Reynolds number, is true. But Lumley (1971b) comments that this is to be expected since the class of disturbances considered is larger than that in the linear theory. Another (perhaps the primary) reason for the lower value is the fact that in this method one does not exclude "the possibility that disturbances which have, at the instant of analysis, positive growth rates, may ultimately decay (Fig. 1.2) ... Thus, one says that at a given Reynolds number the energy method can ascertain stability, but cannot determine instability."

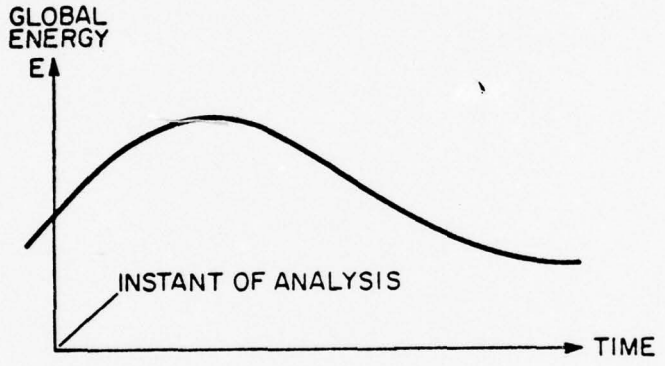


Figure 1.2 Initial growth of a disturbance which ultimately decays.

A great success of the energy method is that it gives a nonzero value of the critical Reynolds number for the plane Couette flow problem, whereas the small disturbance theory predicts this flow to be stable at all Reynolds numbers [see, for example, Petrov (1938)]. Recent work of Joseph & Hung (1971), however, suggests that even the roller-type disturbances in parallel flow, shown by the energy method to be initially unstable, are ultimately stable.

1.3 Survey of Theoretical Work Done on Viscoelastic Fluid Stability

Considerable work has been done on the stability of viscoelastic fluids during the last fifteen years, and a general survey is given below. (For an explanation of terms peculiar to viscoelastic fluids, like "second-order fluid," "Walters fluid," or "Oldroyd fluid" etc., reference may be made to Chapter 2.)

(i) Plane Couette Flow: For a plane Couette flow of a Newtonian fluid the linearized stability equations admit only the trivial (that is, zero) solution for the disturbance quantities at any Reynolds numbers, implying that the flow is always stable. Giesekus (1966) observed that this is not true in presence of viscoelastic parameters, and he obtained a stability criterion for the plane Couette flow of a viscoelastic fluid using the linear stability theory. We however feel that the application of the linear theory to the plane Couette flow may be misleading, even of a viscoelastic flow, and a thorough discussion of this point is given in Section 1.5.

(ii) Horizontal Layer of Fluid Heated from Below (Benard Stability): When a static layer of Newtonian liquid is heated from below, instability sets in at a critical value of the so-called Rayleigh number. It is found that the principle of exchange of stabilities holds for this case, and the neutral state is characterized by steady cellular convection. Intuitively, one may guess that this may not be true in a viscoelastic liquid in which the elastic-like behavior of the fluid may give rise to "overshooting" and an oscillatory type of behavior at the marginal state ("overstability"). Green (1968) considered this possibility for a linearized Oldroyd fluid, and Vest & Arpaci (1969) examined it for a linearized Maxwell fluid. They both came to the conclusion that overstability is possible in a viscoelastic fluid, but that for ordinary thicknesses of common liquids the relaxation times would have to be tremendously high for overstability to take place; therefore, a simple experiment to demonstrate the overstability is not feasible with currently available fluids. They also found that the elasticity of the fluid has a destabilizing effect on a liquid heated from below. This is true both in the sense that oscillatory convection can occur at a lower critical Rayleigh number than does stationary convection, and that this critical number decreases as the viscoelastic parameter increases.

(iii) Plane Couette Flow With Heating: Herbert (1963) considered a very general Oldroyd fluid (which displays variation of viscosity with shear rate) and found that the Rayleigh number is decreased for all nonzero rates of strain; this destabilization was attributed solely due to the variation of the apparent viscosity with shear rate.

McIntire & Schowalter's (1970) analysis is valid for both a "generalized second-order fluid" and an integral equation which has its basis in a molecular model and accounts for network junctions rupturing at a certain critical rate; they found that the flow is stabilized or destabilized according as the second normal stress difference $\sigma_2 = S_{22} - S_{33}$ is negative or positive.

(iv) Circular Couette Flow: This is the stability of flow between two concentric rotating cylinders. Jain (1957) considered a Reiner-Rivlin (or Stokesian) fluid, but errors in Jain's work were pointed out by Graebel (1961) who found that, for fluids with a positive coefficient of cross-viscosity, the critical Taylor number can be appreciably smaller than for the corresponding flow of a Newtonian fluid. Thomas & Walters (1964a) found a decrease in the critical Taylor number due to the non-Newtonian parameter for the Walters liquid B' with short memory, whereas Fong (1965) found that the opposite is true for the Walters liquid A' (without short memory). Thomas & Walters (1964b) also found a destabilization for a Maxwell fluid B. Datta (1964, 1966), Rao (1964) and Ginn & Denn (1969) considered a second-order fluid; the general conclusion was that the second normal stress difference σ_2 (proportional to $\beta + 2\gamma$) and the diameter to gap ratio are crucial: $\sigma_2 > 0$ always results in destabilization, but $\sigma_2 < 0$ gives rise to a destabilization in wide gap and stabilization in narrow gap.

The possibility of overstability of a circular Couette flow was considered by Beard, Davies & Walters (1966). They considered a Maxwell fluid B of stress relaxation time λ_1, viscosity μ, density ρ

and gap-width δ. They concluded that for values of $K = \mu\lambda_1/\rho\delta^2$ up to $K = 0.13$, the only T_c (critical Taylor number) which exists corresponds to a stationary stability mode. In this region, the presence of elasticity in the liquid is a significant destabilizing agent. For higher values of K, a T_c exists for both a stationary and an overstable mode. Since the critical Taylor number associated with the overstable mode was always lower than that associated with the stationary mode, they concluded that overstability sets in at $K = 0.13$ and is present for all higher values of K. The critical Taylor number associated with the overstable mode was found to decrease steadily with increasing K, so that "highly elastic Maxwell liquids can be very unstable indeed."

(v) Flow in a Curved Channel: This is the flow in a curved channel due to a pressure gradient acting along the channel. Thomas & Walters (1963) studied the stability of this flow for a Walters B' fluid (without assuming short memory), but the results were given for a Maxwell fluid B which is a special case of the fluid they considered. They found a destabilization, together with an increase in the critical wavenumber of the convective cells, due to the elasticity of the liquid.

(vi) Plane Poiseuille Flow: This is the stability of flow driven by a pressure gradient between two parallel planes. Fong & Walters (1965) treated it for a Walters B' fluid with short memory, Chun & Schwarz (1968) for a second-order fluid, Tlapa & Bernstein (1968, 1970) for a generalized Maxwell fluid (errors in Tlapa & Bernstein's work will be pointed out in Section 4.11), and Jones & Walters (1968) for a

third-order fluid which exhibits a variation of viscosity with shear rate. All of them concluded a destabilization due to the non-Newtonian parameters.

(vii) Dangers of Using Second-Order Fluid Model: Quite a few articles have pointed out the dangers of using the second-order constitutive equation in a linear stability analysis. Coleman, Duffin & Mizel (1965) have shown that all solutions for unsteady parallel flows of a second-order fluid are unstable. Pipkin (1966) commented that this apparent instability is "an interesting absurdity which arises when the ... [slow flow] approximation is treated as if it were exact." Gupta (1967) showed that a layer of second-order fluid at rest between plane boundaries is in unstable equilibrium; Craik (1968) also reiterated the same fact. Platten & Schechter (1970) showed that a uniform rigid translation (U = constant everywhere) of second-order fluid is unstable at all Reynolds numbers. These results contradict the earlier mentioned works of Fong & Walters (1965) and Chun & Schwarz (1968) who calculated stability envelopes and did not mention any such trouble with second-order fluids. The paradox was explained by McIntire (1971). He considered a plane Couette flow with superposed temperature gradient, and found curves of marginal stability as shown in Fig. 1.3.

Note carefully that for high wavenumbers the flow is unstable at all Rayleigh numbers. If Fong & Walters (1965) and Chun & Schwarz (1968) had examined the very high wavenumber perturbations, they would have found an instability at all Reynolds numbers also. However, it would be quite incorrect to conclude that a real viscoelastic liquid is

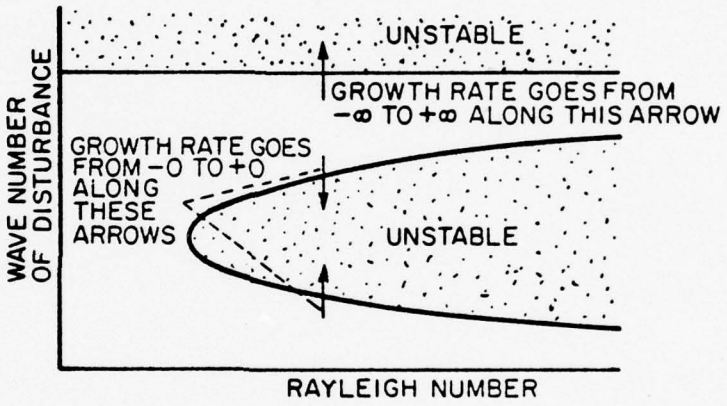


Figure 1.3 Stability diagram of heated Couette flow of second-order fluid, for given values of Reynolds number and non-Newtonian parameters [after McIntire (1971)].

always unstable at high wavenumbers, because the second-order fluid is not applicable near this high wavenumber stability boundary. The reason is that, as noted in Fig. 1.3, the growth rate is continuous through zero across the ordinary (low wavenumber) stability envelope, whereas for the high wavenumber instability the growth rate is discontinuous and very large. In order for the second-order model to be applicable the growth rate of the disturbance must be small compared to a "natural time" of the fluid.

(viii) Application of Nonlinear Methods: In all the works cited so far the linear theory of stability analysis was applied. Feinberg & Schowalter (1969a, 1969b) seem to be the first to consider disturbances of finite magnitude and apply the energy method to the investigation of stability of viscoelastic flows. They followed the technique of Serrin and determined sufficient conditions for stability of fluid motions constitutionally described by the infinitesimal theory of viscoelasticity. Like Serrin's work on Newtonian fluids, their results were also overly conservative.

Denn, Sun & Rushton (1971) carried out a finite amplitude analysis of the secondary motion in a circular Couette flow, using the energy equilibrium approach of Landau and Stuart. Assuming a second-order fluid model, they calculated that the extra torque following the instability is less than that for the Newtonian fluid at the same relative (supercritical) Taylor number. They suggest that this might somehow explain the drag reduction in turbulent pipe flow of polymer solutions.

1.4 Survey of Experimental Work Done on Viscoelastic Stability

All the experimental work on the stability of polymer solutions have been performed on the circular Couette flow set-up, with the outer cylinder stationary. The results have been quoted in terms of the non-dimensional Taylor number $T = \Omega_1^2 \delta^3 R_1/\nu^2$ where Ω_1 and R_1 are respectively the angular velocity and the radius of the inner cylinder, δ is the gap width, and ν is the kinematic viscosity. The results of the various observers are, however, sharply contradictory.

For example Giesekus (1966) found an extremely strong destabilization. For a 4% solution of polyisobutylene in decalin he found instabilities in which the rotational speed was about 600 times lower (*Taylor number approximately* 3×10^5 *times smaller!*) than in a Newtonian liquid. For a 1% solution of Al-naphthenate in decalin he found the critical rotational speed to be about 30 times lower (Taylor number about 900 times smaller) compared to a Newtonian liquid. Giesekus also reported that the critical wavenumber of the convective cells increases (that is, the spacing decreases) due to the addition of polymers. Miller (1967) also found a definite destabilization, but reported that the critical wavenumber decreases with the addition of polymers.

On the other hand Karlsson & Griem (1966) and Rubin & Elata (1966) found definite stabilization even for dilute polymer solutions for which the deviation from Newtonian behavior is otherwise small. Rubin & Elata also reported that the cellular spacing remains relatively constant at its Newtonian value.

Merrill, Mickley & Ram (1962) found that, if ν is taken as the viscosity at the critical shear rate then the solution of polymer

macromolecules does not show any significant change in the critical Taylor number compared to micromolecular Newtonian liquids. However, they also commented that if ν were taken as the viscosity at zero shear rate, then their results would indicate a substantial destabilization. Denn & Roisman (1969) found that for dilute polymer solutions the critical Taylor number may be more or less, depending on the diameter ratio of the cylinders.

1.5 Critical Examination of Giesikus' Work on Plane Couette Flow Instability

It was mentioned in Section 1.3 that Giesekus (1966) analyzed the stability of a viscoelastic plane Couette flow by means of linear stability theory. A critical examination of this work is given below.

Giesekus considered plane Couette flow as a limiting case of circular Couette flow when the radii of the walls go to infinity. That is, he considered that the disturbances are periodic in the x_3-direction and are in the form of "longitudinal rollers" invariant in the x_1-direction (x_1 is taken in the direction of the flow and x_2 is taken perpendicular to the flow). The stability equation obtained was

$$[(D^2-k^2)^2 + k^2\gamma(D^2-k^2) - k^2\gamma^*]\hat{\psi} = 0 \qquad (1.3)$$

with the boundary conditions $\hat{\psi} = D\hat{\psi} = 0$ at $x_2 = 0$ and 1. Here all the variables are dimensionless, $D = d/dx_2$, k is the wavenumber of the disturbance, $\hat{\psi}$ is the amplitude of the stream function in the x_2x_3-plane and $\gamma = -G(F-G)\Gamma^2/2\mu^2$ and $\gamma^* = -G\rho\delta^2\Gamma^2/\mu^2$ are two elastic parameters.

Γ is the shear rate of the basic flow, μ is the viscosity, ρ is the density, δ is the gap width, and F and G are two normal stress functions defined to be related to the normal stresses in the fluid by the equations $S_{11} = -p + \Gamma^2 F$, $S_{22} = -p - \Gamma^2 G$, and $S_{33} = -p$, where p is the pressure.

For a Newtonian fluid $\gamma = \gamma^* = 0$, and the only solution of (1.3) which satisfies the boundary conditions is the trivial solution $\hat{\psi} = 0$. This means that for a Newtonian fluid the transverse yz-flow is always stable at any Reynolds numbers. Giesekus observed that this is not true with γ and γ^* present, and he solved (1.3) and obtained the following stability criterion: for a highly elastic liquid the instability is to be expected if $\gamma \sim 1$, whereas for slightly elastic liquids the instability is to be expected if $\gamma^* = 68$. (It of course follows that G will have to be negative to cause instability.)

It is apparent that Giesekus confined his attention only to the transverse motion. However, the longitudinal disturbance velocity may be crucial in the study of stability. For example, for longitudinal roller type disturbances in a parallel flow of a Newtonian fluid, Joseph & Hung (1971) have shown that the transverse motion monotonically decays, while the longitudinal motion (u_1) may initially increase if the Reynolds number is high enough. Lumley, in a private communication, has shown that the same fact is also true of <u>linear</u> disturbances in a Newtonian fluid. This is explained below.

The linearized equations of motion for the disturbance in this case are

$$\frac{\partial u_1}{\partial t} + \Gamma u_2 = \nu \nabla^2 u_1 \tag{1.4}$$

$$\frac{\partial}{\partial t} \nabla^2 \psi = \nu \nabla^4 \psi \tag{1.5}$$

where $\nabla^2 = \partial^2/\partial x_j \partial x_j$, and ψ is the stream function for the transverse motion. The boundary conditions are $u_1 = \psi = \partial\psi/\partial x_2$ at $x_2 = \pm \delta$. Multiplying Eq. (1.5) by ψ, integrating over the flow volume, and using the divergence theorem and boundary conditions many times, results

$$\frac{1}{2} \frac{d}{dt} \int q^2 dV = - \nu \int \omega^2 dV \tag{1.6}$$

where $q^2 = u_2^2 + u_3^2$ is the transverse energy and $\omega = \nabla^2 \psi$ is the vorticity. This shows that the transverse energy must monotonically decay, which of course must happen since there is no source term in Eq. (1.5).

To investigate the u_1-motion, one must realize that <u>any</u> initial distributions of u_1 and ψ, consistent with boundary conditions, are possible - the equations will determine their evolution. For example, suppose a ψ is found that satisfies the boundary conditions; then initially let us choose $u_1 = - c\psi$ where c is a constant, which also satisfies the boundary conditions on u_1. After multiplying Eq. (1.4) by u_1, integrating over the flow volume, and using the divergence theorem and boundary conditions we get

$$\frac{1}{2}\frac{d}{dt}\int u_1^2 dV = \frac{\Gamma}{c}\int u_1^2 dV - \nu \int \left[\frac{\partial u_1}{\partial x_i}\right]^2 dV \qquad (1.7)$$

It is apparent that although the second term on the right side is always negative, for large enough Γ the growth rate may be made positive initially.

The longitudinal motion having thus been shown to be crucial, the work of Giesekus becomes questionable. It seems that a method which deals with all three components of velocity in a single equation, and treats simultaneously all parts of the flow, would be preferable in this case. The classical energy method is such a one; it has the additional advantage that it is unnecessary to assume the disturbances small at the beginning.

1.6 Estimating Viscous Sublayer Thickness Through Energy Method

G. I. Taylor (1916) suggested using the energy-method type of analysis for the purpose of estimating the thickness of the viscous sublayer next to the wall in a turbulent flow, and Lumley (1971b) recently applied the idea to estimate the thickness of the viscous sublayer in a Newtonian fluid. Their argument is as follows: a viscous sublayer* exists along the walls of a turbulent flow because the proximity of the wall makes the flow here dominated by viscosity. Now imagine a turbulent plane Couette flow (Fig. 1.4); the two viscous

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* Note that, strictly speaking, it is incorrect to speak of a "laminar" sublayer since it consists of unsteady disturbances, although dominated by viscosity.

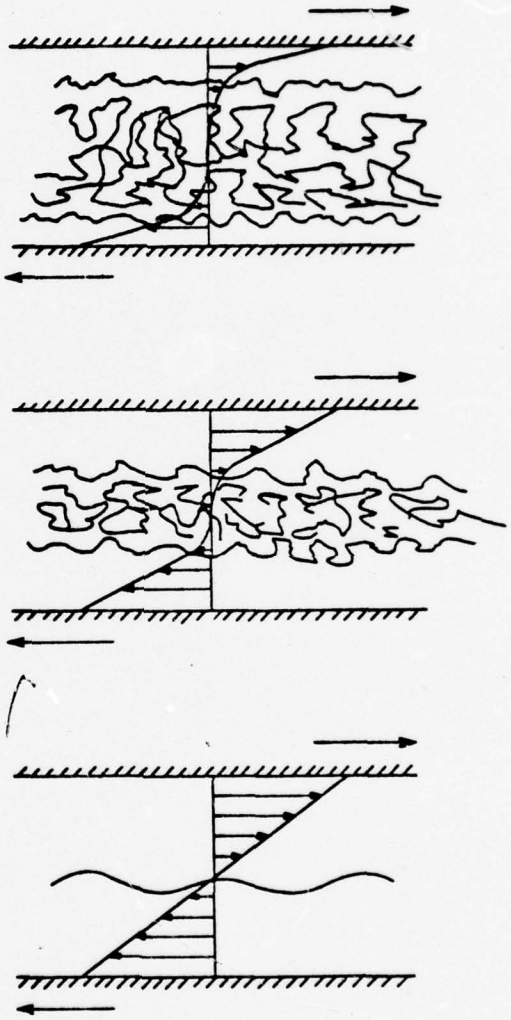


Figure 1.4 Gradual increase of viscosity in a turbulent plane Couette flow.

sublayers in this flow are expected to have the same structure as in any other turbulent flow. If the viscosity of the fluid is gradually increased, other things being equal, the sublayer disturbances will not be able to sustain themselves without increasing in thickness. The sublayers will therefore go on increasing in thickness until they are face to face. This is the so-called "critical state" of stability studies, at which point a further increase in viscosity will damp out all disturbances and produce a steady laminar flow.

These considerations can be phrased as a well defined extremization problem: consider the class of large disturbances having instantaneously stationary total energy; what is the equation obeyed by that member which can exist with the largest viscosity (other things being equal)?

The procedure of estimating the viscous sublayer thickness is now clear: at the neutral or critical state the viscous sublayers are face to face, and half the distance between the plates is the sublayer thickness. Using Orr's solution (which considered only two-dimensional disturbances) for the critical Reynolds number and noting that by definition $u_\tau = \sqrt{\nu\Gamma}$, one gets $\delta^+ = \delta u_\tau/\nu = \sqrt{\Gamma\delta^2/\nu} = \sqrt{44.3} \approx 6.65$; using Joseph's solution (which considered longitudinal roller-type disturbances) one gets $\delta^+ = \sqrt{20.6} = 4.54$. These values are in fairly good agreement with experimental values of a Newtonian viscous sublayer.

In the present work these same ideas will be used in an effort to explain the thickening of the viscous sublayer due to viscoelasticity, as observed in the Toms phenomenon.

1.7 Objectives of the Present Study

From the discussion of Section 1.5 it will be seen that the linear stability analysis is not applicable to plane Couette flow, and a major objective of the present study is to obtain a stability criterion for plane Couette flow by means of the energy method. Since the critical Reynolds number of this flow gives a method of estimating the viscous sublayer thickness of any turbulent flow (as discussed in Section 1.6), one objective of this study would be to predict whether the presence of viscoelasticity should increase or decrease the viscous sublayer thickness. Of course it is known from the Toms phenomenon that the sublayer thickness increases after the addition of polymers to the solution.

From the survey of Section 1.3 it will be seen that the Oldroyd constitutive equation, which is probably the most reliable of all the simple constitutive equations, has not been applied to the investigation of the linear instability of plane Poiseuille flow and circular Couette flow. Results of course exist for the Maxwell fluid for both of these flows, but the Maxwell fluid does not exhibit the experimentally observed effect of strain rate relaxation. Besides, we detected some errors in Tlapa & Bernstein's (1968, 1970) solution of the plane Poiseuille problem, which will be pointed out in Section 4.11. One objective of the present study will therefore be to apply the Oldroyd constitutive equation to the investigation of the instability of the plane Poiseuille flow and the circular Couette flow.

The energy considerations for the various viscoelastic fluid models do not seem to have received any attention in the literature.

One objective of the present study will therefore be to find expressions for the rate of change of elastic potential energy and the rate of viscous dissipation for the commonly used fluid models such as the second-order fluid, Maxwell fluid, Walters fluid and Oldroyd fluid. It will be seen in Chapter 3 that these expressions will be necessary in the application of the energy method to plane Couette flow of a viscoelastic fluid.

As an aside, some unsteady flow problems will be solved for the Oldroyd fluid; these do not seem to have been solved in the literature so far.

CHAPTER 2

VISCOELASTIC CONSTITUTIVE EQUATIONS AND THEIR ENERGY CONSIDERATIONS

2.1 Non-Newtonian Liquids

An incompressible Newtonian fluid is one for which the stress tensor T_{ij} is given by

$$T_{ij} = -P\delta_{ij} + S_{ij} \tag{2.1}$$

with

$$S_{ij} = 2\mu E_{ij} \tag{2.2}$$

where P is an indeterminate hydrostatic pressure, δ_{ij} is the Kronecker delta, S_{ij} is called the extra-stress tensor, $E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the rate of strain tensor, and μ is a constant (viscosity) which can only be a function of temperature. Any liquid whose stress-strain rate relation is different from the linear relation (2.2) is called a non-Newtonian liquid.

For some non-Newtonian liquids it may be possible to explain its properties completely in terms of a single viscosity coefficient, if this is considered to be a function of the local rate of strain of the material:

$$S = \eta(\Gamma)\Gamma \tag{2.3}$$

where S is the shear stress, Γ is the local rate of shear and η is an even function, called the (variable) coefficient of viscosity which may be a function of the rate of shear. A liquid whose behavior at any point where the motion is one of simple shearing (not necessarily

steady) conforms with (2.3) may be called a non-Newtonian viscous liquid. For these liquids the mode of flow under different conditions can be worked out without great difficulty in principle. But there are several rheological phenomena observable in liquids which cannot be explained in terms of this simple theory; in particular the effect of elasticity of shape of the liquid becomes important. These are called viscoelastic liquids, and polymer solutions, molten plastics, biological fluids, protein solutions (e.g., egg white), and most industrial liquids in the form of suspensions and emulsions are viscoelastic. Most viscoelastic liquids also display a variable viscosity with shear rate.

2.2 Some Properties of Viscoelastic Liquids

Some important properties of viscoelastic liquids are described below. For a more complete description with photographs, see the film notes of Markovitz (1968).

(a) Variable Viscosity: For most viscoelastic liquids the shear stress-shear rate relation is nonlinear, implying a coefficient of viscosity which varies with the shear rate. With some liquids such as some polymer solutions the viscosity decreases with increasing shear rate; these are called shear-thinning or pseudoplastic. The decreasing viscosity is thought of as arising from a change in structure of the material. With some suspensions, the viscosity increases with shear rate; this is called shear-thickening or dilatant behavior.

(b) Normal Stress Effects: In steady simple shearing flow between infinite parallel plates, the normal stresses on an infinitesimal volume element in the three orthogonal directions are not equal for a

viscoelastic liquid, i.e., $T_{11} \neq T_{22} \neq T_{33}$; but in a Newtonian liquid, they are equal. The same is also true for the normal stresses in the flow between rotating cylinders or in the torsional shear flow between rotating parallel disks. As a result, the distribution of normal stresses can be quite different for a viscoelastic liquid from what it is for the Newtonian fluid.

For example, in circular Couette flow in the annulus between an outer cylinder and a coaxial rotating shaft, a viscoelastic liquid tends to climb up the inner cylinder; this is called the (positive) Weissenberg effect [Weissenberg (1947)], which is in contrast with the Newtonian case in which the free surface is parabolic. It can be shown [see Truesdell & Noll (1965), p. 453] that the positive Weissenberg effect is possible only if the normal stresses at a point are not all equal.

(c) Expansion of Jet: A jet of viscoelastic liquid swells as it emerges from a conduit; this is called Merrington's effect. It can be shown that [see, for example, Jaunzemis (1967), p. 538] for this to happen $T_{22} > T_{11}$, where 1 is the direction of flow.

(d) Stress Relaxation: In a perfectly viscous liquid the stresses will become hydrostatic as soon as the flow is stopped. In a viscoelastic liquid the stress returns to its hydrostatic value gradually.

(e) Elastic Recovery: If the stress is reduced to its hydrostatic value, the fluid flow will immediately stop in a perfectly viscous liquid; but in a viscoelastic fluid it will do so gradually.

(f) Other Elastic or Memory Effects: In a purely viscous liquid the current value of the stress is determined from a knowledge of the

current state of deformation alone. In a viscoelastic liquid, the stress is a (nonlinear) function of the deformation which the material has experienced previously, i.e., is a function of the deformation gradient history. As a consequence of this memory effect, such liquids show elastic-like behavior. For example, a "silicone-putty ball resting on the ground flows into a puddle even under such a small force as that of gravity, if given sufficient time. But during the short time of a rapid impact, it behaves like an elastic solid because it does not have a chance to "forget" its previous spherical shape. The significant relaxation times for stresses in silicone putty are much less than the time of rapid impact. Thus the putty may exhibit a range of behavior from elastic to viscous, depending on how the characteristic time of the experiment compares with the significant relaxation times" [Markovitz (1968)].

2.3 Simple Fluid

In Sections 2.3 through 2.7, some important constitutive equations will be discussed. We shall always limit ourselves to incompressible fluids, whether or not the incompressibility condition is mentioned explicitly. We shall start with the very general concept of simple fluid.

The concept of simple fluid was originated by Noll (1958), and developed subsequently by Coleman and Noll [see, for example, Coleman & Noll (1961)]. They defined a simple material as a substance for which the stress is determined by a knowledge of the entire history of the deformation gradient (strain). A simple material is called a simple fluid if it has the additional property that all local states are

equivalent in response, with all observable differences in response being due to definite differences in history; that is, it has the property that every configuration is an undistorted configuration.

Consider a material point P which occupies the position $\underset{\sim}{x}$ at the present time t. Let us suppose that at an earlier time τ, the particle occupied the position $\underset{\sim}{\xi}$. For the dependence of $\underset{\sim}{\xi}$ on $\underset{\sim}{x}$, t and τ we write

$$\underset{\sim}{\xi} = \underset{\sim t}{X}(\underset{\sim}{x},\tau)$$

The function $\underset{\sim t}{X}$ may be called the displacement function relative to the configuration at time t. The spatial gradient of $\underset{\sim t}{X}(\underset{\sim}{x},\tau)$ with respect to $\underset{\sim}{x}$ determines the relative deformation gradient tensor at time τ relative to the configuration at time t:

$$\underset{\approx t}{F}(\tau) = \nabla_{x}\underset{\sim t}{X}(x,\tau)$$

We now define a symmetric tensor through the relation

$$\underset{\approx t}{G}(\tau) = \underset{\approx t}{F}^{T}(\tau)\underset{\approx t}{F}(\tau) - \underset{\approx}{I}$$

where the superscript T denotes "transpose" and $\underset{\approx}{I}$ denotes the unit tensor. $\underset{\approx t}{G}(\tau)$ is a measure of the "strain" of the configuration at time τ relative to the configuration at time t; for all τ therefore $\underset{\approx t}{G}(\ )$ may be looked upon as the entire history of the deformation at the material particle P.

It is convenient to regard $\underset{\approx t}{G}(\tau)$ as a function of the time lapse s between τ and t, i.e., $s = t-\tau$ where $o \leq s < \infty$. Noll showed that for

simple fluids the extra stress $\underset{\approx}{S}$ is related to the history of the motion through an equation of the form

$$\underset{\approx}{S}(t) = \underset{\approx}{F}_{s=0}^{\infty} [\underset{\approx t}{G}(s)] \qquad (2.4)$$

where $\underset{\approx}{F}$ is a tensor-valued functional, i.e., an operator (which need not be linear) mapping an entire tensor-valued function $\underset{\approx t}{G}$ into another tensor $\underset{\approx}{S}$.

Simple fluids can exhibit such phenomena as stress-relaxation, non-Newtonian viscosity, and normal stress effects. It may be remarked that a better name for simple fluids would probably be general fluid, because they include perfect fluids, Newtonian fluids, the Reiner-Rivlin fluids, the Rivlin-Ericksen fluids, and the fluids based on the theory of finite linear viscoelasticity as special cases.

In recent years much has been learned about the mechanical behavior of general incompressible simple fluids. It has been found that there are several steady flow problems which can be solved for simple fluids in general, using no special constitutive assumptions beyond the definition (2.4); simple shearing flow, Poiseuille flow and circular Couette flow are examples.

2.4 Rivlin-Ericksen Fluid

Rivlin & Ericksen (1955) considered the theory of isotropic materials for which the extra-stress depends on only the spatial gradients of velocity, acceleration, second acceleration, ... , (N-1)th acceleration. They showed that for an incompressible material of this

type, the extra stress S_{ij} must be given by an isotropic function of the tensors $A_{(M)ij}$, M = 1, 2, ... , N, i.e., an N-th order Rivlin-Ericksen fluid is given by

$$S_{ij} = h_{ij}(A_{(1)k\ell}, A_{(2)k\ell}, \ldots, A_{(N)k\ell}) \tag{2.5}$$

where the $A_{(N)ij}$, called the Rivlin-Ericksen tensors, are given by

$$\begin{aligned} A_{(1)ij} &= U_{i,j} + U_{j,i} \\ A_{(N)ij} &= \frac{DA_{(N-1)ij}}{Dt} + A_{(N-1)ik}U_{k,j} + A_{(N-1)jk}U_{k,i} \end{aligned} \tag{2.6}$$

Here $D/Dt = \partial/\partial t + U_k(\ )_{,k}$. It can be shown [see for example the recent review article of Rivlin & Sawyers (1971)] that the Rivlin-Ericksen constitutive equation is an approximation to Noll's constitutive equation for an incompressible isotropic simple fluid with memory, provided that the Cauchy strain $G_t(t)$ can be expanded as a Taylor series about the time t at which the stress is measured. Many important constitutive equations are special cases of (2.5). A perfect fluid corresponds to the case $S_{ij} = 0$, and for a Newtonian S_{ij} is assumed to depend linearly on $A_{(1)ij}$ only, i.e., $S_{ij} = \mu A_{(1)ij}$, where μ is a constant. In case of the Reiner-Rivlin fluid, the extra stress depends on only $A_{(1)ij}$, but is taken as a general (nonlinear) function of it; for that fluid equation (2.5) and the representation theorem for isotropic tensor functions of one symmetric tensor variable yield

$$S_{ij} = f_1 A_{(1)ij} + f_2 A_{(1)ik} A_{(1)kj} \tag{2.7}$$

where f_1 and f_2 are scalar functions of the invariants of $A_{(1)ij}$. The Reiner-Rivlin constitutive equation (2.7) is purely viscous and has no elastic properties.

Although the general Rivlin-Ericksen fluid (2.5) accounts for shear-dependent viscosity and the normal stress effects, it has the serious shortcoming of not accounting for gradual stress relaxation. When $A_{(M)ij} = 0$ for $M = 1, 2, \ldots, N$, the extra stress S_{ij} of a Rivlin-Ericksen fluid cannot change in time, contrary to the behavior of real viscoelastic fluids in stress-relaxation experiments. Stated differently, "when a Rivlin-Ericksen fluid is brought to rest suddenly its memory does not fade gradually, but rather precipitously."

For steady viscometric flows the Rivlin-Ericksen fluid simplifies considerably. For such flows only the first two tensors $A_{(1)ij}$ and $A_{(2)ij}$ are different from zero. Thus if no other tests beyond those resting upon steady viscometric flows were available, we should be unable to distinguish between the general Rivlin-Ericksen fluid and the special one in which $S_{ij} = h_{ij}(A_{(1)k\ell}, A_{(2)k\ell})$. If we assume that h_{ij} is a polynomial function, by means of the representation theorem of isotropic matrix polynomial of two matrices it can be shown that

$$\underset{\approx}{S} = \alpha_1 \underset{\approx}{A}_1 + \alpha_2 \underset{\approx}{A}_2 + \alpha_3 \underset{\approx}{A}_1^2 + \alpha_4 \underset{\approx}{A}_2^2 + \alpha_5 (\underset{\approx}{A}_1 \underset{\approx}{A}_2 + \underset{\approx}{A}_2 \underset{\approx}{A}_1) + \alpha_6 (\underset{\approx}{A}_1^2 \underset{\approx}{A}_2 + \underset{\approx}{A}_2 \underset{\approx}{A}_1^2)$$

$$+ \alpha_7 (\underset{\approx}{A}_1 \underset{\approx}{A}_2^2 + \underset{\approx}{A}_2^2 \underset{\approx}{A}_1) + \alpha_8 (\underset{\approx}{A}_1^2 \underset{\approx}{A}_2^2 + \underset{\approx}{A}_2^2 \underset{\approx}{A}_1^2)$$

where $\alpha_1, \ldots, \alpha_8$ are polynomials in the ten invariants formed from $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$.

2.5 Second-Order Fluid

Due to the generality of the simple fluid, it is not surprising that not many flow problems can be solved with it. Approximations to the simple fluid then becomes necessary.

To the concept of simple fluid Coleman & Noll (1960) added the notion of gradually fading memory, i.e., the idea that deformations which occurred in the distant past should have less effect on the present value of the stress than the deformations which occurred in the recent past. For this they constructed, for an arbitrary history $\underset{\sim}{G}(\ )$, a new history $\underset{\sim}{G}^{(\alpha)}(\ )$ for each α, $0 < \alpha < 1$, as follows

$$\underset{\sim}{G}^{(\alpha)}(s) = \underset{\sim}{G}(\alpha s) \qquad 0 < s < \infty$$

(Note that we have dropped the subscript t on $\underset{\sim}{G}_t$, which should cause no confusion.) $\underset{\sim}{G}^{(\alpha)}(\ )$ is therefore the same history as $\underset{\sim}{G}(\ )$, but carried out at a slower rate; it can therefore be called the retardation of $\underset{\sim}{G}(\ )$ by the factor α. Coleman & Noll (1960) have shown that the stresses $\underset{\sim}{S}^{(\alpha)}$ corresponding to $\underset{\sim}{G}^{(\alpha)}(\ )$ obey the approximate formulae

$$\underset{\sim}{S}^{(\alpha)} = \mu \underset{\sim}{A}_{(1)}^{(\alpha)} + 0(\alpha^2) \tag{2.8}$$

$$\underset{\sim}{S}^{(\alpha)} = \mu \underset{\sim}{A}_{(1)}^{(\alpha)} + \beta \underset{\sim}{A}_{(1)}^{(\alpha)} \underset{\sim}{A}_{(1)}^{(\alpha)} + \gamma \underset{\sim}{A}_{(2)}^{(\alpha)} + 0(\alpha^3) \tag{2.9}$$

where $\underset{\approx}{A}^{(\alpha)}_{(N)} = \alpha^N \underset{\approx}{A}_{(N)}$; $\underset{\approx}{A}_{(N)}$ are the Rivlin-Ericksen tensors, and μ, β and γ are material constants. Equation (2.8) tells us that, for a simple fluid with fading memory, the Newtonian fluid furnishes a complete first-order correction, for viscoelastic effects, to the theory of perfect fluids in the limit of slow motion. Equation (2.9) exhibits all second-order corrections for viscoelastic effects. The equation obtained by striking out the error term $0(\alpha^3)$ in (2.9) is

$$S_{ij} = \mu A_{(1)ij} + \beta A_{(1)ik} A_{(1)kj} + \gamma A_{(2)ij} \qquad (2.10)$$

This is called a second-order fluid, and can be looked upon as an approximation to the simple fluid in slow motions.

Note that the theory of Reiner-Rivlin fluids (2.7) would not yield the term in (2.9) involving γ; hence that theory cannot give a complete second-order correction to the Newtonian fluid.

The main practical interest in second-order fluids lies in its simplicity, and quite a number of hydrodynamic problems can be solved with it. In several situations where the theory of Newtonian fluids gives rise to linear partial differential equations for the velocity field so also does the theory of second-order fluids. Reference may be made to Markovitz & Coleman (1964) for a good summary of the derivation and applications of the second-order constitutive equation. It is shown there that the fluid displays the normal stress effect and the positive Weissenberg effect. However, the second-order fluid does not display the gradual stress relaxation and the shear-dependent viscosity.

Truesdell (1965) showed that the second-order fluid is

indistinguishable by viscometric experiments from a fluid with convected elasticity. In the theory of fluids with convected elasticity the stress is regarded as a function of the strain of the present configuration with respect to one occupied by the fluid at a certain fixed time t^* before the present time. The time t^*, which is a material property, may be called the response time of such a fluid. Truesdell showed that in viscometric flows a fluid of second order is indistinguishable from a particular fluid of convected elasticity with response time

$$t^* = -\frac{2\gamma}{\mu} \tag{2.11}$$

Since $\mu > 0$, this time is positive if and only if $\gamma < 0$. This fact affords us a provisional interpretation for the coefficient γ. The second-order fluid may be regarded as responding to its past, present, or future experience according as $\gamma < 0$, $\gamma = 0$ or $\gamma > 0$. A different motivation for proving that

$$\begin{aligned} \mu &> 0 \\ \gamma &< 0 \end{aligned} \tag{2.12}$$

can be found in Markovitz & Coleman (1964).

Although the special solutions of the second-order fluid are undoubtedly useful for the interpretation of the behavior of real viscoelastic fluids under appropriate circumstances, Coleman, Duffin & Mizel (1965) have found grave defects in the theory. They have in fact showed that all solutions for unsteady parallel flow of a second-order fluid are unstable. However, it is not certain that these results render this fluid useless. For example, it could be argued that as the

disturbances grow unbounded in time, they are excluded from the class of "slow" flows resulting by retardation of a given flow.

2.6 Oldroyd Fluid

Oldroyd (1950) in a paper entitled "On the Formulation of Rheological Equations of State" laid down what is now known as the "Principle of Material Indifference" or "Material Objectivity": "The form of the completely general equations must be restricted by the requirement that the equations describe properties independent of the frame of reference ... Moreover, only those tensor quantities need be considered which have a significance for the material element independent of its motion as a whole in space." Oldroyd in effect observed that the formulation of all equations in terms of coordinates moving and rotating with the local vorticity suffices to satisfy this requirement.

Let coordinate surfaces ξ^j = constant be chosen as surfaces which move and rotate with the fluid particles; whether in addition these surfaces also deform with the material (i.e., are embedded in the fluid) will not be specified now. In these convected coordinates, ξ^j and time t are taken as independent variables; for example a second order tensor may be specified as $\beta_i^k(\xi,t)$. Having found the constitutive equation in terms of tensors referred to their convected components (i.e., the components referred to a convected system of reference), for the sake of simplicity of applications it is necessary to express all the tensor quantities referred to an arbitrary curvilinear fixed coordinate system

x^i, having a metric tensor of $g_{ik}(\underset{\sim}{x})$. Let the fixed components of $\beta_i^k(\underset{\sim}{\xi},t)$ be $b_i^k(\underset{\sim}{x},t)$. Now in the constitutive equation written in convective frame, there may have been partial time derivatives $\partial\beta_i^k(\underset{\sim}{\xi},t)/\partial t$, holding convected coordinates constant. The question is, how do these "convective derivatives" transform? The answer is provided by the following theorem of Oldroyd:

"The tensor whose convected components are $\partial\beta_{\cdots i\cdots}^{\cdots\ell\cdots}(\underset{\sim}{\xi},t)/\partial t$ has fixed components

$$\frac{d_c b_{..i..}^{..k..}}{dt} = \frac{\partial b_{..i..}^{..k..}}{\partial t} + u^m b_{..i..,m}^{..k..} + \Sigma\, \omega_i^m b_{..m..}^{..k..} + \Sigma'\, \omega_m^k b_{..i..}^{..m..} + \Sigma\, e_i^m b_{..m..}^{..k..} - \Sigma'\, e_m^k b_{..i..}^{..m..} \tag{2.13}$$

where $\Sigma(\Sigma')$ denotes a sum of all similar terms, one for each covariant (contravariant) suffix, a comma preceding a suffix denotes covariant differentiation, and u^i, ω_{ik}, e_{ik} denote the velocity vector, the corresponding vorticity and rate of strain tensors (all functions of x^i and t) associated with the motion in space of an arbitrary <u>coordinate system</u> ξ^j, not necessarily of the <u>fluid</u> unless the ξ^j system is embedded in the fluid." Of course $\omega_{ik} = \frac{1}{2}(u_{k,i} - u_{i,k})$ and $e_{ik} = \frac{1}{2}(u_{k,i} + u_{i,k})$.

The presence of d_c/dt in a constitutive equation referred to fixed coordinates obviously conforms with the principle that the equation must be independent of arbitrary time dependent rigid motion. Whether in addition the coordinates $\underset{\sim}{\xi}$, with respect to which the original

constitutive equation was written, are embedded in the fluid and deform with it cannot be settled by recourse to invariance.

In case the $\underset{\sim}{\xi}$ are embedded in the fluid, Oldroyd calls d_c/dt the "convected derivative" and denotes it by $\delta/\delta t$. Then u_i, e_{ij} and ω_{ij} of the reference frame are equal to the corresponding fluid variables, i.e., $u_i = U_i$, $e_{ij} = E_{ij}$ and $\omega_{ij} = \Omega_{ij}$. From (2.13) therefore the convective derivatives of second-order covariant and contravariant tensors are of the form

$$\frac{\delta b_{ij}(\underset{\sim}{x},t)}{\delta t} = \frac{\partial b_{ij}}{\partial t} + U^m b_{ij,m} + U^m{}_{,i} b_{mj} + U^m{}_{,j} b_{im}$$

$$\frac{\delta b^{ij}(\underset{\sim}{x},t)}{\delta t} = \frac{\partial b^{ij}}{\partial t} + U^m b^{ij}{}_{,m} - U^i{}_{,m} b^{mj} - U^j{}_{,m} b^{im} \qquad (2.14)$$

If the frame $\underset{\sim}{\xi}$ is rigid so that it simply moves and rotates with the fluid but does not deform with it ($e_{ij} = 0$), then Oldroyd denotes d_c/dt by $\mathcal{D}/\mathcal{D}t$ and calls it the "material derivative." From (2.13) the material derivative of b_{ij} in cartesian tensor notation is

$$\frac{\mathcal{D} b_{ij}(\underset{\sim}{x},t)}{\mathcal{D}t} = \frac{\partial b_{ij}}{\partial t} + U_k \frac{\partial b_{ij}}{\partial x_k} + \Omega_{ik} b_{kj} + \Omega_{jk} b_{ik} \qquad (2.15)$$

where of course $\Omega_{ik} = \tfrac{1}{2}(\partial U_k/\partial x^i - \partial U_i/\partial x^k)$.

As a matter of interest it may be mentioned that if we had taken the $\underset{\sim}{\xi}$ to be simply translating with the fluid, neither rotating ($\omega_{ij} = 0$) nor deforming ($e_{ij} = 0$), then from (2.13) d_c/dt would have reduced to the familiar D/Dt which Oldroyd calls the "intrinsic derivative":

$$\frac{Db_i^j(\underset{\sim}{x},t)}{Dt} = \frac{\partial b_i^j}{\partial t} + U^m b^j_{i,m}$$

To summarize in rough language, D/Dt is a measure of the rate of change which allows for the translational motion only, $\mathcal{D}/\mathcal{D}t$ makes allowances for the rotational as well as the translational motion of the element, and $\delta/\delta t$ is the rate of change which takes into account the translational, rotational as well as the deforming motion of the fluid. For a scalar these three derivatives are all equal. This is because the value of a scalar does not depend on any coordinate system, and hence D/Dt of pressure or temperature denotes an intrinsic property of a material element. However, for quantities which are not scalars but are measured in relation to certain directions, it is evident that to get an intrinsic property of a material element the rate of change must be measured in some way related to directions which rotate with the material element.

Since the metric tensor g_{ik} vanishes when operated on by $\partial/\partial t$, D/Dt or $\mathcal{D}/\mathcal{D}t$, it follows that the operations of raising and lowering suffixes commute with these types of differentiation. On the other hand it can be shown that $\delta g_{ik}/\delta t = 2E_{ik}$, $\delta g^{ik}/\delta t = -2E^{ik}$ and therefore the operations of raising or lowering a suffix and of convected differentiation do not commute [if they did, the two equations in (2.14) would be the same].

Having discussed the various types of convective derivatives, Oldroyd illustrates the process of formulating equations of state. It has been found that the behavior of certain viscoelastic liquids at

small rates of strain can be approximately described by a linear equation

$$(1 + \lambda_1 \frac{\partial}{\partial t})S_{ij} = 2\mu(1 + \lambda_2 \frac{\partial}{\partial t})E_{ij} \qquad (2.16)$$

where S_{ij} is the extra stress, $E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the strain rate, and μ is the viscosity; λ_1 and λ_2 are called the "relaxation time" and "retardation time" respectively, with the direct physical significance that if the motion is suddenly stopped the stresses decay as $\exp(-t/\lambda_1)$ and if the stresses are removed the rates of strain decay as $\exp(-t/\lambda_2)$. Equation (2.16) was shown to be associated with a suspension of elastic spherical particles in a Newtonian viscous liquid by Fröhlick & Sack (1946). They showed that the introduction of $\partial S_{ij}/\partial t$ requires a further condition because at certain times, e.g., on application or removal of stress, S_{ij} will be allowed to be a discontinuous function of time; the additional condition is that $\lambda_1 S_{ij} - 2\mu\lambda_2 E_{ij}$ is continuous at a discontinuous change in stress. They also showed that the physical model is such that $\lambda_1 > \lambda_2$. It is apparent that as λ_1 and λ_2 tend to equality, the material must become more and more exactly a Newtonian liquid of viscosity μ.

In addition to being valid only at small rates of strain, equation (2.16) was written in orthogonal cartesian coordinates. It is thus not for general use, and the immediate problem is that of generalizing the equation (i.e., formulate an invariant equation) so that the equations of state can be applied to material in general motion, in which the rates of strain will not be necessarily small. Two possible generalizations of (2.16) are

$$(1 + \lambda_1 \frac{\delta}{\delta t})S_{ij} = 2\mu(1 + \lambda_2 \frac{\delta}{\delta t})E_{ij} \quad \text{(Oldroyd Liquid A)} \qquad (2.17)$$

where S_{ij} and E_{ij} are covariant absolute tensors, and

$$(1 + \lambda_1 \frac{\delta}{\delta t})S^{ij} = 2\mu(1 + \lambda_2 \frac{\delta}{\delta t})E^{ij} \quad \text{(Oldroyd Liquid B)} \qquad (2.18)$$

where S^{ij} and E^{ij} are contravariant components. Lumley (1971a) showed that if the polymer molecules are represented by dumbell models, then a dilute solution of polymer molecules obeys (2.18) with $\lambda_1 = T$, $\lambda_2 = T/(1 + c[\mu])$, $\mu = \eta(1 + c[\mu])$, where T is the terminal relaxation time of the molecules (that is, the initial small departure of the distribution from spherical symmetry relaxes to zero with time constant T), $[\mu]$ is the intrinsic viscosity, c is the concentration of polymer and η is the viscosity of the solvent.

Oldroyd (1958, 1961) showed that the most complete generalization of (2.16) is, in cartesian coordinates,

$$S_{ik} + \lambda_1 \frac{\mathcal{D}S_{ik}}{\mathcal{D}t} + \mu_0 S_{jj}E_{ik} - \mu_1(S_{ij}E_{jk} + S_{jk}E_{ij}) + \nu_1 S_{j\ell}E_{j\ell}\delta_{ik}$$
$$= 2\mu\left(E_{ik} + \lambda_2 \frac{\mathcal{D}E_{ik}}{\mathcal{D}t} - 2\mu_2 E_{ij}E_{jk} + \nu_2 E_{j\ell}E_{j\ell}\delta_{ik}\right) \qquad (2.19)$$

where μ_0, μ_1, μ_2, ν_1 and ν_2 are five more arbitrary constants, each with the dimensions of time. The generalization (2.19) contains terms linear in the stresses and quadratic in stresses and space derivatives of velocity taken together; other generalizations of (2.16) must contain terms of higher degree in the stresses and velocity gradients. The Oldroyd liquid A, represented by (2.17), is a special case of (2.19) with

$$\lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geq 0$$

$$\mu_0 = \nu_1 = \nu_2 = 0$$

It has been shown by Oldroyd (1950) that liquid B would exhibit the positive Weissenberg effect, whereas liquid A would exhibit the negative Weissenberg effect (i.e., sinking down near the inner cylinder).

Oldroyd (1958, 1961) has shown that the constants in (2.19) have the following significance [quoting from Oldroyd (1961)]:

"(a) The behavior at small rates of strain is realistic, provided

$$\mu > 0$$

$$\lambda_1 > \lambda_2 > 0$$

"(b) The variable apparent viscosity is a decreasing function of the rate of shear, varying from the limiting viscosity μ at low rates of shear to a lower limit $\sigma_2\mu/\sigma_1$ at high rates of shear, provided

$$\sigma_1 > \sigma_2 \geq \frac{1}{9}\ \sigma_1 > 0 \ ."$$

where

$$\sigma_1 = \lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2}\nu_1) - \mu_1(\mu_1 - \nu_1)$$

$$\sigma_2 = \lambda_1\lambda_2 + \mu_0\ \mu_2 - \frac{3}{2}\nu_2) - \mu_1(\mu_2 - \nu_2)$$

"(c) The normal stress [effect corresponds to] a simple tension along the streamlines, in excess of the isotropic state of stress in directions normal to the streamlines, in steady shearing flow between a cone and a flat plate, provided

$$\mu_1 = \lambda_1$$

$$\mu_2 = \lambda_2$$

$$\frac{\sigma_1}{\sigma_2} \leq \frac{\lambda_1}{\lambda_2}$$

"(d) A liquid whose constants satisfy (a), (b) and (c) will necessarily show a positive Weissenberg climbing effect, which is that most often observed."

Combining (a), (b) and (c), we are led to the following restrictions on the constants:

$$\begin{gathered} \mu > 0 \\ \lambda_1 > \lambda_2 > 0 \\ \mu_1 = \lambda_1 \\ \mu_2 = \lambda_2 \\ \mu_0\lambda_1 + (\lambda_1 - \tfrac{3}{2}\mu_0)\nu_1 > \mu_0\lambda_2 + (\lambda_1 - \tfrac{3}{2}\mu_0)\nu_2 \geq \tfrac{1}{9}[\mu_0\lambda_1 + (\lambda_1 - \tfrac{3}{2}\mu_0)\nu_1] > 0 \\ (\lambda_1 - \tfrac{3}{2}\mu_0)(\lambda_1\nu_2 - \lambda_2\nu_1) > 0 \end{gathered} \tag{2.20}$$

Observe that any one of the constants μ_0, ν_1 and ν_2 may vanish consistently with these conditions, but if two vanish the only interesting possibility is given by

$$\begin{gathered} \mu > 0 \\ \mu_0 > 0 \\ \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geq \tfrac{1}{9}\lambda_1 > 0 \\ \nu_1 = \nu_2 = 0 \end{gathered} \tag{2.21}$$

which differ from the liquid B only in the presence of a μ_0 term in the equation of state.

A few comments must be made about the experimental measurements of the elastic constants in the Oldroyd equation. Toms & Strawbridge (1953) measured the λ_1 and λ_2 of some polymer solutions in an indirect way. They supposed that the linearized Oldroyd equation (2.16) is valid for small rates of strain, and obtained the theoretical solutions for the amplitude of oscillation of an inner cylinder (of known moment of inertia and held by a torsion wire of known spring constant) when the outer cylinder oscillates at a known amplitude and frequency. They plotted these solutions as a graph of θ, defined as the ratio of the amplitudes of the inner and outer cylinders, against frequency ω, with λ_1 and λ_2 regarded as parameters. Then they obtained the experimental θ-versus-ω curve of a polymer solution, and from a comparison of the experimental and theoretical plots arrived at approximate values of λ_1 and λ_2 of the solution. The important conclusion they arrived at is that while the viscosity μ and the stress relaxation time λ_1 increased steadily with the concentration of the polymer, the strain rate relaxation time λ_2 on the other hand remained independent of the concentration. They suggested that perhaps different molecular mechanisms are responsible for causing λ_1 and λ_2 in a solution.

2.7 Walters Fluids

Let $\Delta\gamma_{ij}(t')$ be the strain at a stationary element of material at time t', which causes an extra-stress $\Delta S_{ij}(t)$ at the same point at time t. Let us assume that the cause and effect are related by

$$\Delta S_{ij}(t) = \psi(t-t')\Delta\gamma_{ij}(t')$$

where ψ may be called the influence function or relaxation function. The above equation simply states that the material under consideration possesses a memory for its past history.

Suppose the material is subjected to a series of step changes in the strains. The resultant stress at time t can be written as

$$S_{ij}(t) = \psi(t-t')\Delta\gamma_{ij}(t') + \psi(t-t'')\Delta\gamma_{ij}(t'') + \ldots$$

if linear (Boltzmann) superposition is valid. If the sequence of strains experienced by the material point is not a series of step functions but rather a continuously varying function of time, we replace the summation by an integration to obtain

$$S_{ij}(t) = \int_{-\infty}^{t} \psi(t-t') \frac{d\gamma_{ij}}{dt'}(t')dt'$$

Denoting $d\gamma_{ij}(t')/dt' = 2E_{ij}(t')$, the strain rate tensor, we get

$$S_{ij}(\underset{\sim}{x},t) = 2\int_{-\infty}^{t} \psi(t-t')E_{ij}(\underset{\sim}{x},t')dt' \qquad (2.22)$$

as the extra-stress at a stationary element of material at $\underset{\sim}{x}$ at time t. Walters (1960) has arrived at the above result by superposing an infinite number of Maxwell elements in parallel. If $N(\tau)d\tau$ represents the total viscosity of all the Maxwell elements with relaxation times between τ and $\tau + d\tau$, then he derives (2.22) with an influence function

$$\psi(t-t') = \int_{0}^{\infty} \frac{N(\tau)}{\tau} e^{-\left(\frac{t-t'}{\tau}\right)} d\tau \qquad (2.23)$$

$N(\tau)$ is called the distribution of relaxation times, or the relaxation spectrum, a characteristic of the material.

Quoting from Walters (1962a), note that in a general flow " ... the equation of state (2.22) must be associated, not with the particular point x^i in space over a period of time, but with the same material element over a period of time: the element which is instantaneously at the point x^i at time t. In consequence, the integral occurring in (2.22) must be regarded as a convected integral following the moving material. Only under the restrictive conditions of small rates of shear in a stationary material element can the convected integral be written as in (2.22).

"The problem is now one of generalizing equation (2.22) into a form that can be applied to materials in general motion, in which the rate of strain is not necessarily small. This of course involves replacing the integral occurring in (2.22) by a convected integral following the moving material. Oldroyd (1950) has included transformation of convected integrals in his analysis." He proved that any integral $\int_{t_o}^{t} \beta^{\cdot\cdot\ell\cdot\cdot}_{\cdot\cdot j\cdot\cdot}(\xi,t')dt'$ transforms into

$$\int_{t_o}^{t} \Pi\left(\frac{\partial x'^m}{\partial x^i}\right) \Pi\left(\frac{\partial x^k}{\partial x'^p}\right) b^{\cdot\cdot p\cdot\cdot}_{\cdot\cdot m\cdot\cdot}(\underset{\sim}{x}',t')dt'$$

when referred to fixed coordinates, where $x'^i = x'^i(\underset{\sim}{x},t,t')$ is the position at time t' of the element which is instantaneously at the point x^i at time t, and $\beta^{\cdot\cdot\ell\cdot\cdot}_{\cdot\cdot j\cdot\cdot}$ are the convected components of $b^{\cdot\cdot\ell\cdot\cdot}_{\cdot\cdot j\cdot\cdot}$. Walters (1962a) points out that "in general, the displacement functions x'^i cannot be expressed as simple functions of the usual kinematic variables and it is necessary to determine them separately in each individual flow problem."

Using Oldroyd's rule and noting that the suffixes may be raised in (2.22) under the restrictive conditions in which the equation is supposed true, the two simple possible generalizations of (2.22) are

$$S_{ik}(\underset{\sim}{x},t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x'^{m}}{\partial x^{i}} \frac{\partial x'^{r}}{\partial x^{k}} E_{mr}(x',t')dt' \quad \text{(Walters Liquid A}') \tag{2.24}$$

$$S^{ik}(\underset{\sim}{x},t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x^{i}}{\partial x'^{m}} \frac{\partial x^{k}}{\partial x'^{r}} E^{mr}(\underset{\sim}{x}',t')dt' \quad \text{(Walters Liquid B}') \tag{2.25}$$

The designations "Liquid A'" and "Liquid B'" were given by Walters (1962a), and have been used by other writers. The liquids designated A and B by Oldroyd (see equations 2.17 and 2.18) are special cases of A' and B' respectively, obtained by substituting

$$N(\tau) = \mu \frac{\lambda_2}{\lambda_1} \delta(\tau) + \mu \frac{\lambda_1-\lambda_2}{\lambda_1} \delta(\tau-\lambda_1)$$

in equations (2.23), (2.24) and (2.25). The Maxwell liquid with one relaxation time at $\tau = \lambda_1$ is also a special case obtained by writing $N(\tau) = \mu\delta(\tau-\lambda_1)$, while the Newtonian liquid is obtained by taking $N(\tau) = \mu\delta(\tau)$.

Some of the simplest consequences of equations of state A' and B' have been investigated by Walters (1962a) for a number of steady flow problems involving simple shearing. No difficulty is encountered in such problems since the displacement functions x'^{i} can be obtained without difficulty and the rate of strain E_{mr} or E^{mr} in any element does not vary with t' and so can be taken outside the integrals.

Walters (1962b) also considered simplifications of his liquids A' and B' for the special case of short memory. The derivation of his simplifications is carried out by referring the equations to a convected coordinate system ξ^j drawn in the material and deforming continuously with it. In the case of liquid A', the equation of state obviously becomes

$$\sigma_{ij}(\xi,t) = 2\int_{-\infty}^{t} \psi(t-t')\varepsilon_{ij}(\xi,t')dt'$$

where σ_{ij} and ε_{ij} are the convected components of S_{ij} and E_{ij} respectively. If the rate of strain tensor is now expanded in a Taylor series about the time t, we have

$$\varepsilon_{ij}(\xi,t') = \varepsilon_{ij}(\xi,t) - (t-t')\frac{\partial}{\partial t}\varepsilon_{ij}(\xi,t) + \ldots$$

where $\partial/\partial t$ denotes a partial differentiation with respect to time holding convected coordinates constant. The constitutive equation then reduces to

$$\sigma_{ij} = 2\varepsilon_{ij}\int_{-\infty}^{t} \psi(t-t')dt' - 2\frac{\partial \varepsilon_{ij}}{\partial t}\int_{-\infty}^{t} (t-t')\psi(t-t')dt' + \ldots$$

Using (2.23) and neglecting terms involving $\int_0^\infty \tau^n N(\tau)d\tau$, $(n \geq 2)$, this can be written as

$$\sigma_{ij} = 2\mu\varepsilon_{ij} - 2K\frac{\partial}{\partial t}\varepsilon_{ij}$$

where $\mu = \int_0^\infty N(\tau)d\tau$ = limiting viscosity at small rates of shear, and $K = \int_0^\infty \tau N(\tau)d\tau$. Transforming the constitutive equation from convective to a fixed coordinate system x^i, we finally obtain

$$S_{ij} = 2\mu E_{ij} - 2K \frac{\delta E_{ij}}{\delta t} \qquad \begin{matrix}(\text{Walters Liquid A}' \\ \text{with short memory})\end{matrix} \qquad (2.26)$$

where $\delta/\delta t$ is Oldroyd's convective derivative.

In a similar manner it can be shown that the simplified form of the equation of state for liquid B' in case of short memory is

$$S^{ij} = 2\mu E^{ij} - 2K \frac{\delta E^{ij}}{\delta t} \qquad \begin{matrix}(\text{Walters Liquid B}' \\ \text{with short memory})\end{matrix} \qquad (2.27)$$

It is interesting to note that we have found that equations (2.26) and (2.27) are also derivable from Oldroyd's equations for slow flows. Let us illustrate for the Oldroyd liquid A:

$$S_{ij} + \lambda_1 \frac{\delta S_{ij}}{\delta t} = 2\mu(E_{ij} + \lambda_2 \frac{\delta E_{ij}}{\delta t})$$

Now for slow flows (or small viscoelasticity, or short memory) the terms multiplied by λ_1 and λ_2 will be smaller compared to the other two (Newtonian) terms. Thus, no great error will be introduced if in the term multiplied by λ_1 we substitute $S_{ij} = 2\mu E_{ij}$. Then Oldroyd's liquid A becomes

$$S_{ij} = 2\mu E_{ij} - 2\mu(\lambda_1 - \lambda_2) \frac{\delta E_{ij}}{\delta t}$$

which is identical to (2.26).

2.8 Normal Stress Functions in Viscoelastic Fluids

Coleman & Noll (1961) have shown that a general memory fluid ('simple fluid') displays the following stress pattern in a simple shearing flow of shear rate Γ:

$$S_{12} = \eta(\Gamma)\Gamma$$

$$S_{11} - S_{33} = \sigma_1(\Gamma) \quad S_{13} = S_{23} = 0 \qquad (2.28)$$

$$S_{22} - S_{33} = \sigma_2(\Gamma)$$

where the direction 1 is taken along the flow and direction 2 is taken perpendicular to the plates. $\eta(\Gamma)$ is the shear dependent viscosity, and σ_1 and σ_2 are the so-called first and second normal stress differences, respectively. The three material functions $\eta(\Gamma)$, $\sigma_1(\Gamma)$ and $\sigma_2(\Gamma)$ are all even functions of the shear rate Γ because of isotropy, and of course $\sigma_1(0) = \sigma_2(0) = 0$ from the hydrostatic condition; the three functions are otherwise quite arbitrary.

Another definition of the first normal stress difference quite widely used [see, for example, McIntire & Schowalter (1970), Ginn & Denn (1969)] is

$$N_1(\Gamma) = S_{11} - S_{22} \qquad (2.29)$$

There have been a number of attempts to measure the normal stress functions of polymer solutions. The experimental difficulties are, however, severe and there is considerable disagreement in the results. Markovitz (1963) found that the Reiner-Rivlin result $\sigma_1 = \sigma_2$ and the Weissenberg conjecture $\sigma_2 = 0$ are both in variance with their experimental data; he also found that $\sigma_1 > \sigma_2 > 0$ in many cases. Denn & Roisman (1969) found that N_1 is always positive and σ_2 is always negative. A negative σ_2 was also reported by Ginn (1968). However, Huppler (1965) reported σ_2 which are not only positive but also far

greater in magnitude, by factors of ten or more, than the results of Denn & Roisman (1969).

The theoretical predictions of the normal stress functions according to the commonly used constitutive equations are definitely of interest. Consequently we have made a list of them, which is displayed in Table 2.1.

2.9 Normal Stresses and Weissenberg Effect

During working with the commonly used constitutive equations, we came to the observation that the sign, and magnitude, of the Weissenberg effect may depend on the method of observation, namely through the shape of the free surface or through the height difference of radial tappings. This will be explained below.

(i) An Apparent Contradiction

Let us consider three well-known constitutive equations, the Oldroyd A fluid

$$S_{ij} + \lambda_1 \frac{\delta S_{ij}}{\delta t} = 2\mu(E_{ij} + \lambda_2 \frac{\delta E_{ij}}{\delta t}), \quad \lambda_1 > \lambda_2 > 0 \tag{2.30}$$

the Walters A' fluid with short memory

$$S_{ij} = 2\mu E_{ij} - 2K \frac{\delta E_{ij}}{\delta t}, \quad K > 0 \tag{2.31}$$

and the second-order fluid

$$S_{ij} = \mu A_{ij} + \beta A_i^k A_{kj} + \gamma \frac{\delta A_{ij}}{\delta t}, \quad \gamma < 0 \tag{2.32}$$

TABLE 2.1

Viscometric Functions of Common Fluids

Fluid	Constitutive Equation	Normal Stresses in Simple Shear	Viscometric Functions
Second-Order	Eq. (2.10)	$S_{11} = \beta\Gamma^2$ $S_{22} = (\beta+2\gamma)\Gamma^2$ $S_{33} = 0$	$\sigma_1 = \beta\Gamma^2$ $\sigma_2 = (\beta+2\gamma)\Gamma^2$ $N_1 = -2\gamma\Gamma^2$
Walters A'	Eq. (2.24)	$S_{11} = 0$ $S_{22} = -2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$ $S_{33} = 0$	$\sigma_1 = 0$ $\sigma_2 = -2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$ $N_1 = 2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$
Walters B'	Eq. (2.25)	$S_{11} = 2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$ $S_{22} = 0$ $S_{33} = 0$	$\sigma_1 = 2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$ $\sigma_2 = 0$ $N_1 = 2\Gamma^2 \int_0^\infty \tau N(\tau) d\tau$
Walters A' (short memory)	Eq. (2.26)	$S_{11} = 0$ $S_{22} = -2K\Gamma^2$ $S_{33} = 0$	$\sigma_1 = 0$ $\sigma_2 = -2K\Gamma^2$ $N_1 = 2K\Gamma^2$
Walters B' (short memory)	Eq. (2.27)	$S_{11} = 2K\Gamma^2$ $S_{22} = 0$ $S_{33} = 0$	$\sigma_1 = 2K\Gamma^2$ $\sigma_2 = 0$ $N_1 = 2K\Gamma^2$
Oldroyd A	Eq. (2.17)	$S_{11} = 0$ $S_{22} = -2\mu(\lambda_1-\lambda_2)\Gamma^2$ $S_{33} = 0$	$\sigma_1 = 0$ $\sigma_2 = -2\mu(\lambda_1-\lambda_2)\Gamma^2$ $N_1 = 2\mu(\lambda_1-\lambda_2)\Gamma^2$
Oldroyd B	Eq. (2.18)	$S_{11} = 2\mu(\lambda_1-\lambda_2)\Gamma^2$ $S_{22} = 0$ $S_{33} = 0$	$\sigma_1 = 2\mu(\lambda_1-\lambda_2)\Gamma^2$ $\sigma_2 = 0$ $N_1 = 2\mu(\lambda_1-\lambda_2)\Gamma^2$

TABLE 2.1 (continued)

Fluid	Constitutive Equation	Normal Stresses in Simple Shear	Viscometric Functions
Maxwell A	Eq. (2.58)	$S_{11} = 0$ $S_{22} = -2\mu\lambda_1\Gamma^2$ $S_{33} = 0$	$\sigma_1 = 0$ $\sigma_2 = -2\mu\lambda_1\Gamma^2$ $N_1 = 2\mu\lambda_1\Gamma^2$
Maxwell B	Eq. (2.59)	$S_{11} = 2\mu\lambda_1\Gamma^2$ $S_{22} = 0$ $S_{33} = 0$	$\sigma_1 = 2\mu\lambda_1\Gamma^2$ $\sigma_2 = 0$ $N_1 = 2\mu\lambda_1\Gamma^2$
Reiner-Rivlin	Eq. (2.7)	$S_{11} = f_2\Gamma^2$ $S_{22} = f_2\Gamma^2$ $S_{33} = 0$	$\sigma_1 = f_2\Gamma^2$ $\sigma_2 = f_2\Gamma^2$ $N_1 = 0$

where we are using general tensor notation and the $\delta/\delta t$ in each of the three equations is given by the first equation of Eq. (2.14). It is apparent that a result for the Walters A' fluid (with short memory) could be deduced either by substituting $\lambda_1 = 0$, $\mu\lambda_2 = -K$ into the corresponding result for Oldroyd A fluid, or by substituting $\beta = 0$, $\gamma = -K$ into a result for the second-order fluid. Using the existing results for the Oldroyd A and second-order fluids, let us therefore predict the sense of the Weissenberg effect for the Walters A' fluid (with short memory) in circular Couette flow - positive (climbing near inner cylinder) or negative (dipping down near inner cylinder).

Let x^1, x^2 and x^3 denote the cylindrical coordinates r, θ and z respectively, with z taken along the axis of the cylinders. The physical components of all tensors will be denoted by using parentheses around the suffixes. Markovitz & Coleman (1964) have shown that ignoring inertia, the difference of physical components of compressive radial stresses $-T_{(11)}$ between two points at the same horizontal level z but at radial positions R_1 and R_2 ($> R_1$) is given by

$$-T_{(11)}^{R_2} + T_{(11)}^{R_1} = \gamma C \qquad (2.33)$$

where terms in β dropped out and C is a positive constant. Since $\gamma < 0$, they conclude that the second-order fluid displays the positive Weissenberg effect. And since β has automatically vanished, it also follows that the same conclusion must also be true of the Walters A' fluid with short memory.

On the other hand Oldroyd (1950) has shown that for the Oldroyd A fluid

$$-T^{R_2}_{(33)} + T^{R_1}_{(33)} = (\lambda_1 - \lambda_2)\, C' \tag{2.34}$$

where C' is a positive constant. Since $\lambda_1 > \lambda_2$, it follows that the Oldroyd A fluid displays the negative Weissenberg effect. (Since a real viscoelastic fluid generally displays the positive Weissenberg effect, Oldroyd therefore discards this constitutive equation and accepts the Oldroyd B fluid which predicts a positive Weissenberg effect.) And by substituting $\lambda_1 = 0$, $\lambda_2 = -K/\mu$ into Eq. (2.34), it follows that the same conclusion must also be true of the Walters A' fluid with short memory.

This reason for this apparent contradiction regarding the sense of the Weissenberg effect of the Walters A' fluid (with short memory) will now be investigated.

(ii) Consideration of Oldroyd A Fluid

Let the physical components of velocity be

$$\begin{aligned} U_{(1)} &= 0 \\ U_{(2)} &= r\omega \\ U_{(3)} &= 0 \end{aligned} \tag{2.35}$$

From Eq. (2.30) it can be deduced that the physical components of extra stress are

$$[S_{(ij)}] = \begin{bmatrix} -2\mu(\lambda_1 - \lambda_2)\left(\frac{r d\omega}{dr}\right)^2 & \mu r \frac{d\omega}{dr} & 0 \\ \mu r \frac{d\omega}{dr} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.36}$$

The equation of motion is

$$\rho \frac{DU^i}{Dt} = f^i - p_{,j} g^{ij} + S^{ij}{}_{,j} \tag{2.37}$$

where f^i is the body force and g^{ij} is the metric tensor. Using (2.36), an integration of the equation of motion in the θ-direction gives the velocity distribution

$$\omega = a - \frac{b}{r^2} \tag{2.38}$$

where a and b can be found in terms of the angular velocities and radii of the boundaries. With Eq. (2.38), an integration of the equation of motion in the radial direction yields the pressure distribution

$$P = \rho(\tfrac{1}{2} a^2 r^2 - \frac{b^2}{2r^2} - 2ab\log r) - 6\mu(\lambda_1 - \lambda_2)\frac{b^2}{r^4} - \rho g z \tag{2.39}$$

The effect of inertia, represented by the first term on the right hand side of the above equation, is, of course, to lower the free surface near the inner cylinder. To find out the sense of the Weissenberg effect, one has to ignore this term and concentrate on the viscoelastic term. The compressive stress on horizontal planes is $-T_{(33)} = P - S_{(33)}$, and since $S_{(33)} = 0$, it follows from Eq. (2.39) that, ignoring inertia,

$$-T^{R_2}_{(33)} + T^{R_1}_{(33)} = 6\mu(\lambda_1 - \lambda_2)\frac{b^2(R_2^4 - R_1^4)}{R_1^4 R_2^4} > 0 \tag{2.40}$$

which shows that the free surface will dip near the inner cylinder. However, let the observation be carried out through radial holes, as shown in Fig. 2.1. Using $-T_{(11)} = P - S_{(11)}$, Eq. (2.39), and $S_{(11)} = 8\mu(\lambda_1 - \lambda_2)b^2/r^4$, it follows that

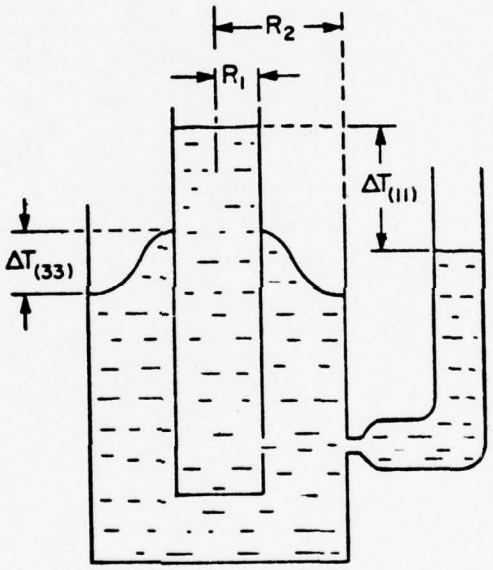


Figure 2.1 Two methods of observing Weissenberg effect.

$$- T_{(11)}^{R_2} + T_{(11)}^{R_1} = - 2\mu(\lambda_1-\lambda_2) \frac{b^2(R_2^4-R_1^4)}{R_1^4 R_2^4} < 0 \qquad (2.41)$$

which shows a positive Weissenberg effect.

The explanation of the apparent contradiction mentioned before is now clear: The Oldroyd A fluid, as well as the Walters A' fluid with short memory, display the negative Weissenberg effect as far as the free surface is concerned, but they would display the positive Weissenberg effect when observed through radial tappings.

As a matter of interest, we may note that for the Oldroyd B fluid, which is widely accepted as one of the most reliable of all simple constitutive equations, it is straightforward to show that the Weissenberg effect is positive, and equal, in both types of measurements. It can also be shown that in a circular Couette flow the Walters A' and B' fluids, <u>without</u> short memory, behave exactly like the Oldroyd A and B fluids respectively.

(iii) <u>Consideration of Second-Order Fluid</u>

Although the apparent contradiction is already resolved, a few more points may be noted from a consideration of the second-order fluid, for which it can be easily shown that

$$[S_{(ij)}] = \begin{bmatrix} (\beta+2\gamma)\left(r\frac{d\omega}{dr}\right)^2 & \mu r\frac{d\omega}{dr} & 0 \\ \mu r\frac{d\omega}{dr} & \beta\left(r\frac{d\omega}{dr}\right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.42)$$

$$\omega = a - \frac{b}{r^2} \tag{2.43}$$

$$P = \rho\left(\tfrac{1}{2} a^2 r^2 - \frac{b^2}{2r^2} - 2ab\log r\right) + \frac{2(2\beta+3\gamma)b^2}{r^4} - \rho gz \tag{2.44}$$

The difference of radial thrusts is then found to be

$$-T^{R_2}_{(11)} + T^{R_1}_{(11)} = \frac{2\gamma b^2 (R_2^4 - R_1^4)}{R_1^4 R_2^4} < 0 \tag{2.45}$$

which shows a positive Weissenberg effect. But if the effect is defined in terms of the free surface, then at a particular depth it is found that

$$-T^{R_2}_{(33)} + T^{R_1}_{(33)} = -2(2\beta+3\gamma)\frac{b^2 (R_2^4 - R_1^4)}{R_1^4 R_2^4} \tag{2.46}$$

which can be of either sign, depending on the relative values of β and γ and would represent a positive Weissenberg effect if $2\beta + 3\gamma > 0$ and a negative Weissenberg effect if $2\beta + 3\gamma < 0$.

Markovitz (1963) has measured γ of a fluid using Eq. (2.45), to which of course a correction due to the presence of inertia was made. They then measured $(3\beta+4\gamma)$ of the fluid from the normal stress measurements in steady torsional flow between parallel plates, and thus arrived at the values of β and γ separately. However, it appears to us that, instead of doing the torsional flow experiment, the $(2\beta+3\gamma)$ of the fluid can be easily measured from Eq. (2.46). Thus, both β and γ of the fluid can be simply derived from the measurements of $\Delta T_{(11)}$ and $\Delta T_{(33)}$ of Fig. 2.1. Of course the normal stress measurements in torsional flow are no more, and no less, reliable than the measurement

of the free surface heights of the circular Couette flow. However, there is an important point in favor of the method we are suggesting. This is the fact that the velocity distribution of a simple torsional flow satisfies the equations of motion only if inertia is neglected, whereas the circular Couette flow satisfies the equations of motion irrespective of the inertia, and a correction for inertia can always be made to Eq. (2.46).

Alternatively, if the fluid is not assumed to follow any particular constitutive equation, then the experiment we are suggesting may be used to determine the normal stress functions $\sigma_1(\Gamma) = S_{(11)} - S_{(33)}$ and $\sigma_2(\Gamma) = S_{(22)} - S_{(33)}$ of a general memory fluid. There is considerable controversy regarding even the sign of these quantities (see Section 2.8).

(iv) Conclusions

That a constitutive equation, or a real fluid, displays the positive or negative Weissenberg effect is not a sufficient statement. The method of observing the effect - through the shape of the free surface or through the height difference of radial tappings - must be mentioned. The Oldroyd A fluid and the Walters A' fluid display the negative Weissenberg effect as far as the free surface is concerned, but they would display the opposite effect when observed through radial tappings. The Oldroyd B fluid as well as the Walters B' fluid display identical positive effect in both types of observations. The second-order fluid would display the positive Weissenberg effect when observed through radial tappings, but it may display either effect when the free surface is observed. A simple experiment, namely that of observing the

Weissenberg effect by the two methods, is suggested for measuring the normal stress functions of a viscoelastic liquid.

2.10 Elastic Energy of Viscoelastic Liquids

For a purely viscous liquid all the work spent in deforming a material particle is dissipated into internal thermal energy. It is expected that in a viscoelastic liquid a part of the shearing deformation work must be stored as elastic potential energy. That this must be the case is evident from the recoil exhibited by such liquids upon release of stress. Although quite a large amount of literature exists on the general thermodynamics of continua, no work seems to have been done on the specific problem of finding expressions for the dissipation and elastic storage of the various constitutive equations. In the remaining sections of this chapter we shall make an attempt in this direction.

The differential form of the first law of thermodynamics for any fluid motion is [for derivation see for example the book of Aris (1962), p. 121]

$$\rho \frac{De}{Dt} = T_{ij} U_{i,j} - q_{i,i} + \rho r \qquad (2.47)$$

where ρ is the density, T_{ij} is the stress tensor, q_i is the conduction heat flux vector, r is the rate of heat addition (e.g., by radiation) per unit mass, and e is the internal energy per unit mass; for our purposes e should include both thermal and elastic energies. $T_{ij} U_{i,j}$ is the work done by stresses in deforming the fluid particles per unit volume per unit time; we shall call it the deformation work. For an

incompressible Newtonian fluid the term $T_{ij}U_{i,j}$ can be easily shown to be equal to the product of the viscosity and a positive quantity depending on the strain rates, so that the entire amount of deformation work goes to viscous dissipation. For a viscoelastic fluid the rate of deformation work is expected to be the sum of the rate of dissipation and the rate of increase of elastic strain energy.

2.11 Elastic Energy of Second-Order Fluid

The constitutive equation of a second-order fluid is

$$T_{ij} = -P\delta_{ij} + \mu A_{ij} + \beta A_{ik}A_{kj} + \gamma\left(\frac{DA_{ij}}{Dt} + A_{ik}U_{k,j} + A_{jk}U_{k,i}\right)$$

where we have dropped the subscript 1 on the first Rivlin-Ericksen tensor $A_{ij} = U_{i,j} + U_{j,i}$. Multiplying $U_{i,j}$ and summing, the deformation work is

$$T_{ij}U_{i,j} = -PU_{i,i} + \mu A_{ij}U_{i,j} + \beta A_{ik}A_{kj}U_{i,j}$$

$$+ \gamma\left[\frac{\partial A_{ij}}{\partial t}U_{i,j} + U_k A_{ij,k}U_{i,j} + (A_{ik}U_{k,j} + A_{jk}U_{k,i})U_{i,j}\right] \qquad (2.48)$$

Now let us simplify the various terms of the above equation. The second and third invariants of A_{ij} will be denoted by $II = A_{ij}A_{ij}$ and $III = A_{ij}A_{jk}A_{ki}$ respectively.

$$A_{ij}U_{i,j} = \tfrac{1}{2}A_{ij}A_{ij} = \tfrac{1}{2}\,II$$

$$A_{ik}A_{kj}U_{i,j} = A_{ik}A_{kj}\,\tfrac{1}{2}(U_{i,j} + U_{j,i}) = \tfrac{1}{2}A_{ik}A_{kj}A_{ij} = \tfrac{1}{2}\,III$$

$$\frac{\partial A_{ij}}{\partial t} U_{i,j} = \tfrac{1}{2} \frac{\partial A_{ij}}{\partial t} A_{ij} = \tfrac{1}{4} \frac{\partial}{\partial t} (A_{ij}A_{ij}) = \tfrac{1}{4} \frac{\partial II}{\partial t}$$

$$U_k A_{ij,k} U_{i,j} = \tfrac{1}{2} U_k A_{ij,k} A_{ij} = \tfrac{1}{4} U_k (A_{ij}A_{ij})_{,k} = \tfrac{1}{4} U_k II_{,k}$$

$$(A_{ik}U_{k,j}+A_{jk}U_{k,i})U_{i,j} = \tfrac{1}{2}(A_{ik}U_{k,j}+A_{jk}U_{k,i})A_{ij} = \tfrac{1}{2}(A_{ik}A_{ij}U_{k,j}+A_{jk}A_{ij}U_{k,i})$$

$$= \tfrac{1}{4} (A_{ik}A_{ij}A_{kj} + A_{jk}A_{ij}A_{ki}) = \tfrac{1}{2} III$$

With these simplications, (2.48) reduces to

$$T_{ij}U_{i,j} = \mu \frac{II}{2} + \beta \frac{III}{2} + \gamma(\tfrac{1}{4} \frac{\partial II}{\partial t} + \tfrac{1}{4} U_k II_{,k} + \tfrac{1}{2} III)$$

which can be rewritten as

$$T_{ij}U_{i,j} = \Psi_S + \frac{\gamma}{4} \frac{DII}{Dt} \tag{2.49}$$

where

$$\Psi_S = \frac{\mu}{2} II + \tfrac{1}{2}(\beta+\gamma)III \tag{2.50}$$

Equation (2.49) has a striking similarity with the corresponding equation for a Newtonian gas which can store deformation work as "elastic energy" of compression. (Note that for an incompressible second-order fluid we are talking of elasticity of <u>shape</u>, whereas for a Newtonian gas we are talking of elasticity of <u>volume</u>.) For a Newtonian gas

$$T_{ij} = - P\delta_{ij} + \mu A_{ij} - \frac{\mu}{3} A_{kk}\delta_{ij}$$

it can be easily shown that

$$\underset{\substack{\text{Rate of}\\ \text{Deformation}\\ \text{Work}}}{T_{ij}U_{i,j}} = \underset{\substack{\text{Rate of}\\ \text{Dissipation}}}{\Psi_N} + \underset{\substack{\text{Rate of Increase}\\ \text{of "Elastic"}\\ \text{Energy}}}{\frac{P}{\rho}\frac{D\rho}{Dt}} \qquad (2.51)$$

where

$$\Psi_N = \frac{\mu}{2} II - \frac{\mu}{6} I^2$$

and $I = A_{ii}$ is the first invariant of A_{ij}. It can be easily shown that the above can be rewritten as

$$\Psi_N = \mu(A_{12}^2+A_{23}^2+A_{31}^2) + \frac{\mu}{6}[(A_{11}-A_{22})^2 + (A_{22}-A_{33})^2 + (A_{33}-A_{11})^2]$$

so that Ψ_N is non-negative. Equation (2.51) has the straightforward interpretation written below it; the elastic energy of compression would result from a density rising either with time or downstream, as in the steady flow through a diffuser. Comparing (2.49) and (2.51) we come to the interpretation that Ψ_S is the rate of dissipation and $(\gamma/4)DII/Dt$ is the rate of increase of elastic energy of a second-order fluid.

The significance of the terms is brought out even more clearly if we consider the first law of thermodynamics (2.47), which for the second-order fluid becomes

$$\frac{D}{Dt}\left(\rho e - \frac{\gamma}{4} II\right) = \Psi_S - q_{i,i} + \rho r \qquad (2.52)$$

which says that the rate of increase of thermal (= total internal, minus elastic) energy is equal to the sum of viscous dissipation, heat conduction, and heat radiation.

However, there is a difficulty. Whereas Ψ_N for a Newtonian gas is positive definite for all motions, there is no way of proving that

$\Psi_S > 0$ in a second-order fluid unless $\beta = -\gamma$ [see equation 2.50]. But experimental data of Markovitz (1963) on polymer solutions do not justify $\beta = -\gamma$. It would probably be a good idea to use quotation marks and say that Ψ_S is the "dissipation." However, the second-order constitutive equation was derived only as a slow flow approximation to real viscoelastic behavior, and for small rates of strain III will be small compared to II. From (2.50) one can see that Ψ_S will be positive for such flows.

Another case in which Ψ_S will always be positive is that of two-dimensional flows. If A_1, A_2 and A_3 are the principal values of A_{ij}, then $III = A_1^3 + A_2^3 + A_3^3$. Using the condition of incompressibility $A_1 + A_2 + A_3 = 0$, it can be easily shown that $III = 3 A_1 A_2 A_3$, which means $III = 0$ for all two-dimensional flows.

Some remarks of Coleman, Duffin & Mizel (1965) may be of interest here: " ... Truesdell suggested that $T_{ij}U_{i,j} \geq 0$ be a restriction not on constitutive equation but rather on allowable motions. Coleman has shown that thermodynamics requires only that $T_{ij}U_{i,j}$ be ≥ 0 for <u>certain</u> motions of incompressible fluids; among these are the substantially stagnant motions. Coleman has taken the point of view that if $T_{ij}U_{i,j}$ be < 0 in one of these special motions, then the constitutive equation should be rejected. Recently Noll (private communication) has shown that, for a fluid obeying [the second-order constitutive equation] exactly, $T_{ij}U_{i,j}$ is ≥ 0 for substantially stagnant motions if and only if both $\mu \geq 0$ and $\beta = -\gamma$."

We would like to make two comments about the above remarks of Coleman, Duffin & Mizel. Firstly, from our findings here it seems that

$T_{ij}U_{i,j} \geq 0$ need not be a restriction on either the constitutive equation or the allowable motions, only the part of $T_{ij}U_{i,j}$ which does not go to elastic energy need be positive; however the second-order fluid fails to satisfy this unless $\beta = -\gamma$. Secondly, we do not know what Coleman, Duffin & Mizel mean by "substantially stagnant motions," but the private communication of Noll referred to in their paper agrees with our findings if it means a "motion with DII/Dt = 0."

Before closing the section it may be interesting to try to separate the viscous and elastic parts of the second-order constitutive equation. According to Truesdell & Toupin (1960), p. 571, the stresses in a viscoelastic fluid can be divided into an elastic and a dissipative part*

$$T_{ij} = T^{E}_{ij} + T^{D}_{ij}$$

The dissipative or non-recoverable part is given by

$$\Psi = T^{D}_{ij}\, U_{i,j}$$

where Ψ is the rate of dissipation, and the elastic or recoverable part is found from the rule that the work of elastic stresses must equal the rate of change of potential energy:

$$\frac{D(PE)}{Dt} = T^{E}_{ij}\, U_{i,j}$$

---

* It is not possible to do this for an Oldroyd or Maxwell fluid in which, due to the presence of the time derivative of the stresses, the stresses cannot be expressed explicitly in the form $T_{ij} = T_{ij}$ (kinematics).

For a second-order fluid we have shown that $T^E_{ij}U_{i,j} = (\gamma/4)DII/Dt$, and a little reflection on its derivation shows that

$$T^E_{ij} = \gamma \frac{DA_{ij}}{Dt}$$

$$T^D_{ij} = \mu A_{ij} + \beta A_{ik}A_{kj} + \gamma(A_{ik}U_{k,j} + A_{jk}U_{k,i})$$

2.12 Elastic Energy of Walters Liquid with Short Memory

The constitutive equation for the Walters liquid B' for short memory was

$$T^{ij} = -Pg^{ij} + 2\mu E^{ij} - 2K\frac{\delta E^{ij}}{\delta t}$$

where

$$\frac{\delta E^{ij}}{\delta t} = \frac{\partial E^{ij}}{\partial t} + U^m E^{ij}_{,m} - U^i_{,m}E^{mj} - U^j_{,m}E^{im}$$

The deformation work is therefore

$$T^{ij}U_{i,j} = 2\mu E^{ij}U_{i,j} - 2K\left[\frac{\partial E^{ij}}{\partial t}U_{i,j} + U^m E^{ij}_{,m}U_{i,j} - (U^i_{,m}E^{mj} + U^j_{,m}E^{im})U_{i,j}\right]$$

$$= 2\mu E^{ij}E_{ij} - 2K\left[\tfrac{1}{2}\frac{\partial}{\partial t}(E^{ij}E_{ij}) + \tfrac{1}{2}U^m(E^{ij}E_{ij})_{,m} - U^i_{,m}E^{mj}E_{ij} - U^j_{,m}E^{im}E_{ij}\right]$$

$$= 2\mu II - 2K\left[\tfrac{1}{2}\frac{DII}{Dt} - 2\,III\right]$$

where the invariants of the strain rate tensor E^i_j are $II = E^{ij}E_{ij}$ and $III = E^i_m E^{mj}E_{ij}$. (Note that in the previous section II and III referred to the invariants of $A_{ij} = 2E_{ij}$). The deformation work for the Walters liquid B' with short memory can therefore be written as

$$T^{ij}U_{i,j} = \Psi_W - K \frac{DII}{Dt} \tag{2.53}$$

where

$$\Psi_W = 2\mu II + 4 K III \tag{2.54}$$

These expressions look very similar to the corresponding equations for a second-order fluid, equations (2.49) and (2.50). In a general motion III can be of either sign so that the dissipation Ψ_W in a Walters liquid with short memory cannot be proved to be positive. However, short memory is the same as slow motion, for which III << II and therefore $\Psi_W > 0$.

2.13 Elastic Energy of Maxwell Liquid

Consider first the Maxwell constitutive equation referred to a convected frame of reference

$$\sigma_{ij} + \lambda_1 \frac{\partial \sigma_{ij}}{\partial t} = 2\mu \varepsilon_{ij} \tag{2.55}$$

where $\sigma_{ij}(\xi,t)$ is the extra-stress and $\varepsilon_{ij}(\xi,t)$ is the rate of strain, both referred to convective coordinates; $\partial/\partial t$ denotes differentiation with the convective coordinates constant. It is well-known that the physical model of the Maxwell constitutive equation is a spring and a dashpot in series (Fig. 2.2). Let e denote the elongation of the endpoints, S is the force, G is the spring constant, η is the dashpot damping coefficient, $E = \dot{e}$ is the rate of elongation (velocity) of the endpoint. Then the elongations of the spring and dashpot are given by $e_G = S/G$ and $\dot{e}_\eta = S/\eta$, so that the total elongation is

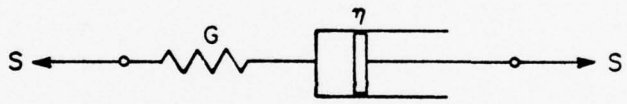


Figure 2.2 Model of Maxwell fluid.

$$e = e_G + e_\eta = \frac{S}{G} + \int_0^t \frac{S}{G}\, dt$$

assuming that $e_\eta = 0$ at $t = 0$. Differentiating the above equation and writing $\lambda_1 = \eta/G$ we get

$$S + \lambda_1 \frac{\partial S}{\partial t} = \eta E$$

which looks similar to the Maxwell constitutive equation (2.55) if force and velocity of the end-points in the model are regarded as analogous to the stresses and strain rates in a viscoelastic material. Let us find the rate of dissipation in the dashpot and the rate of increase of elastic potential energy PE in the spring:

Rate of Dissipation = (Force on Piston) (Velocity of Piston)

$$= S \dot{e}_\eta = \frac{S^2}{\eta}$$

$$\frac{\partial PE}{\partial t} = \text{(Force on Spring) (Rate of Elongation of Spring)}$$

$$= S \dot{e}_G = S(\dot{e}-\dot{e}_\eta) = S(E - \frac{S}{\eta})$$

$$\text{Rate of Total Work} = \text{Rate of Dissipation} + \frac{\partial PE}{\partial t} = \frac{S^2}{\eta} + S(E - \frac{S}{\eta}) = SE$$

With the above analogy in mind, we suggest that the rate of dissipation and the rate of increase of PE for the Maxwell fluid (2.55) are

$$\text{Rate of Dissipation} = \frac{\sigma_{ij}\sigma^{ij}}{2\mu} \tag{2.56}$$

$$\frac{\partial PE(\xi,t)}{\partial t} = \sigma^{ij}(\varepsilon_{ij} - \frac{\sigma_{ij}}{2\mu}) \tag{2.57}$$

Leaving aside the analogy, the above suggestion must be correct because by taking a process sinusoidal in time given by $\varepsilon_{ij} \propto \sin \omega t$, it can be shown that the time integral $\int^t \sigma_{ij}\sigma^{ij}dt/2\mu$ always increases with time, whereas $PE = \int^t \sigma^{ij}(\varepsilon_{ij} - (\sigma_{ij}/2\mu))dt$ may be positive or negative, the integral over a complete period being zero. This is shown rigorously in the next section for a fluid more general than the Maxwell fluid.

Now referring (2.55), (2.56) and (2.57) to fixed coordinates we conclude that for the two kinds of Maxwell fluids

$$S_{ij} + \lambda_1 \frac{\delta S_{ij}}{\delta t} = 2\mu E_{ij} \tag{2.58}$$

and

$$S^{ij} + \lambda_1 \frac{\delta S^{ij}}{\delta t} = 2\mu E^{ij} \tag{2.59}$$

the dissipation and PE are given by

$$\Psi_M = \frac{S_{ij}S^{ij}}{S\mu} \tag{2.60}$$

$$\frac{D(PE)}{Dt} = S^{ij}(E_{ij} - \frac{S_{ij}}{2\mu}) \tag{2.61}$$

In (2.58) and (2.59), $\delta/\delta t$ denotes the convective derivative of Oldroyd, and it has been replaced by $D/Dt = \partial/\partial t + U_k(\ )_{,k}$ in (2.61) because when operating on a scalar $\delta/\delta t = D/Dt$. The deformation work is

$$S^{ij}E_{ij} = \Psi_M + \frac{D(PE)}{Dt} \tag{2.62}$$

as is easily seen by adding (2.60) and (2.61). Equations (2.58) through (2.62) are the results of this section.

2.14 Elastic Energy of Oldroyd Liquid

Consider the Oldroyd equation referred to convected coordinates.

$$\sigma_{ij} + \lambda_1 \frac{\partial \sigma_{ij}}{\partial t} = 2\mu(\varepsilon_{ij} + \lambda_2 \frac{\partial \varepsilon_{ij}}{\partial t}) \tag{2.63}$$

The physical model of this constitutive equation [see Burgers (1935)] is shown in Fig. 2.3. Let suffixes 1, 2 and 3 respectively refer to the series dashpot, the parallel dashpot and the spring, as shown. Then the force S at the end is given by

$$S = S_2 + S_3 = \eta_2 \dot{e}_2 + G_3 e_3$$

Using the conditions $e = e_1 + e_2$, $\dot{e}_2 = e_3$ and $S = \eta_1 \dot{e}_1$, the above becomes

$$S = \eta_2(\dot{e}-\dot{e}_1) + G_3(e-e_1) = \eta_2 \dot{e} - \frac{\eta_2}{\eta_1} S + G_3 e - \frac{G_3}{\eta_1} \int_0^t S dt$$

which can be rewritten as

$$\int_0^t S dt + \frac{\eta_1+\eta_2}{G_3} S = \eta_1 (e + \frac{\eta_2}{G_3} \dot{e})$$

or, differentiating, as

$$S + \lambda_1 \frac{\partial S}{\partial t} = \eta_1 (E + \lambda_2 \frac{\partial E}{\partial t}) \tag{2.64}$$

where we have put $E = \dot{e}$ and

$$\lambda_1 = \frac{\eta_1+\eta_2}{G_3} \tag{2.65}$$

and

$$\lambda_2 = \frac{\eta_2}{G_3} \tag{2.66}$$

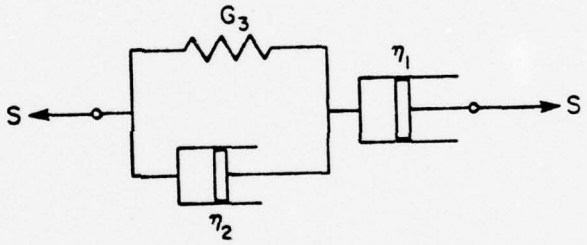


Figure 2.3 Model of Oldroyd fluid.

The model equation (2.64) is analogous to the Oldroyd constitutive equation (2.63) in convected coordinates.

Now consider the rate of dissipation in the two dashpots of the model:

$$\text{Rate of dissipation} = S\dot{e}_1 + S_2\dot{e}_2 = S\dot{e}_1 + S_2(\dot{e}-\dot{e}_1)$$

$$= (S-S_2)\dot{e}_1 + S_2\dot{e} = (S-S_2)\frac{S}{\eta_1} + S_2\dot{e}$$

$$= \frac{S^2}{\eta_1} + S_2(\dot{e} - \frac{S}{\eta_1})$$

Since $S_2 = \eta_2\dot{e}_2 = \eta_2(e-\dot{e}_1) = \eta_2(\dot{e} - S/\eta_1)$, the above becomes

$$\text{Rate of Dissipation} = \frac{S^2}{\eta_1} + \eta_2(E - \frac{S}{\eta_1})^2$$

Now consider the rate of increase of PE in the spring:

$$\frac{\partial(PE)}{\partial t} = S_3\dot{e}_3 = (S-S_2)(\dot{e}-\dot{e}_1) = [S - \eta_2(\dot{e} - \frac{S}{\eta_1}]\ (\dot{e} - \frac{S}{\eta_1})$$

$$= S(E - \frac{S}{\eta_1}) - \eta_2(E - \frac{S}{\eta_1})^2$$

The constant η_2 may be expressed in terms of λ_1 and λ_2 by dividing (2.46) and (2.47), which yields

$$\eta_2 = \frac{\eta_1\lambda_2}{\lambda_1-\lambda_2} \tag{2.67}$$

The expressions for dissipation and elastic energy for the model then become

$$\text{Rate of Dissipation} = \frac{S^2}{\eta_1} + \frac{\eta_1\lambda_2}{\lambda_1-\lambda_2}\left(E - \frac{S}{\eta_1}\right)^2 \tag{2.68}$$

$$\frac{\partial(PE)}{\partial t} = S\left(E - \frac{S}{\eta_1}\right) - \frac{\eta_1\lambda_2}{\lambda_1-\lambda_2}\left(E - \frac{S}{\eta_1}\right)^2 \tag{2.69}$$

Keeping the analogy of (2.63) and (2.64) in mind, let us tentatively assume that for the Oldroyd liquid (2.63)

$$\text{Rate of Dissipation} = \frac{\sigma_{ij}\sigma^{ij}}{2\mu} + \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}\left(\varepsilon_{ij} - \frac{\sigma_{ij}}{2\mu}\right)\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right) \tag{2.70}$$

$$\frac{\partial PE(\xi,t)}{\partial t} = \sigma_{ij}\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right) - \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}\left(\varepsilon_{ij} - \frac{\sigma_{ij}}{2\mu}\right)\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right) \tag{2.71}$$

Since analogy is not proof, some doubt probably remains about this tentative assumption. Let us then forget about the spring-dashpot model, and let us simply split the total deformation work into the following two parts:

$$\sigma_{ij}\varepsilon^{ij} = A + B$$

where

$$A = \frac{\sigma_{ij}\sigma^{ij}}{2\mu} + \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}\left(\varepsilon_{ij} - \frac{\sigma_{ij}}{2\mu}\right)\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right)$$

$$B = \sigma_{ij}\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right) - \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}\left(\varepsilon_{ij} - \frac{\sigma_{ij}}{2\mu}\right)\left(\varepsilon^{ij} - \frac{\sigma^{ij}}{2\mu}\right)$$

We are not saying anything about the significance of A and B now. To find that out, assume that a material particle undergoes a sinusoidal process in its path, represented by

$$\varepsilon_{ij}(\xi,t) = F_{ij}(\xi) \cos \omega t$$

where $F(\xi)$ remains constant along the path of the particle. Now the part A is non-negative, and hence its time integral will always grow with time. If we can show that the time integral of B over a complete period is zero, then we can conclude with some assurance that A is the rate of dissipation of energy and B is the rate of change of potential energy.

Substitution of the process $\varepsilon_{ij} = F_{ij}\cos\omega t$ into the constitutive equation (2.63) results in a linear nonhomogeneous equation for σ_{ij}, whose particular solution is

$$\sigma_{ij} = \frac{2F_{ij}\mu\omega(\lambda_1-\lambda_2)}{1+\lambda_1^2\omega^2} \sin\omega t + \frac{2F_{ij}\mu(1+\lambda_1\lambda_2\omega^2)}{1+\lambda_1^2\omega^2} \cos\omega t$$

and whose homogeneous solution is $\sigma_{ij} = (\text{const.}) \exp(-t/\lambda_1)$; this of course dies out very quickly (it will be zero exactly if the motion started from rest) and therefore will not be considered. With the stresses and strain rates thus determined, a straightforward integration shows that

$$\int_0^{2\pi/\omega} B\,dt = 0$$

We therefore conclude that A is the rate of dissipation and B is the rate of change of PE for the Oldroyd fluid in convected coordinates.

Now referring (2.63), (2.70) and (2.71) to fixed coordinates we conclude that for the Oldroyd fluids

$$S_{ij} + \lambda_1 \frac{\delta S_{ij}}{\delta t} = 2\mu(E_{ij} + \lambda_2 \frac{\delta E_{ij}}{\delta t}) \qquad (2.72)$$

and

$$S^{ij} + \lambda_1 \frac{\delta S^{ij}}{\delta t} = 2\mu(E^{ij} + \lambda_2 \frac{\delta E^{ij}}{\delta t}) \tag{2.73}$$

the dissipation and PE are given by

$$\Psi_0 = \frac{S_{ij}S^{ij}}{2\mu} + \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}(E_{ij} - \frac{S_{ij}}{2\mu})(E^{ij} - \frac{S^{ij}}{2\mu}) \tag{2.74}$$

$$\frac{D(PE)}{Dt} = S_{ij}(E^{ij} - \frac{S^{ij}}{2\mu}) - \frac{2\mu\lambda_2}{\lambda_1-\lambda_2}(E_{ij} - \frac{S_{ij}}{2\mu})(E^{ij} - \frac{S^{ij}}{2\mu}) \tag{2.75}$$

The deformation work is

$$S_{ij}E^{ij} = \Psi_0 + \frac{D(PE)}{Dt} \tag{2.76}$$

as is easily verified by adding (2.74) and (2.75). Equations (2.72) through (2.76) are the results of this section.

CHAPTER 3
STABILITY OF PLANE COUETTE FLOW

In this chapter the stability of a plane Couette flow of the second-order fluid will be studied by the variational technique of the energy method. It is first necessary to derive the energy equation.

3.1 Equation of Global Disturbance Energy

The starting point of the energy method is the equation of disturbance mechanical energy, integrated throughout the flow field. To derive this, let the velocity, pressure and stress tensor of a disturbed motion be broken up into a basic steady motion and a disturbance (not necessarily small):

$$\begin{aligned} V_i &= U_i + u_i \\ \pi &= P + p \\ t_{ij} &= T_{ij} + t_{ij} \end{aligned} \tag{3.1}$$

The disturbances vanish on the boundaries and are either periodic or invariant in the directions in which the flow is unbounded. For example, for a plane Couette or Poiseuille flow, the disturbances are either periodic or invariant in the direction parallel to the walls.

Since both the total and the basic motion must satisfy the equations of motion, we have

$$\begin{aligned} \frac{\partial V_i}{\partial t} + V_j V_{i,j} &= \frac{1}{\rho} \tau_{ij,j} \\ U_j U_{i,j} &= \frac{1}{\rho} T_{ij,j} \end{aligned} \tag{3.2}$$

Subtracting the above equations we get the equation of motion of the disturbance

$$\frac{\partial u_i}{\partial t} = \frac{1}{\rho} t_{ij,j} - u_j U_{i,j} - U_j u_{i,j} - u_j u_{i,j}$$

Multiplying the above by u_i, summing, and integrating over a volume V fixed in space, we get the energy equation

$$\int u_i \frac{\partial u_i}{\partial t} dV = \frac{1}{\rho} \int u_i t_{ij,j} dV - \int u_i u_j U_{i,j} dV - \int u_i U_j u_{i,j} dV - \int u_i u_j u_{i,j} dV$$

Let us choose the volume V so that it coincides with the walls and extends to a length equal to an integral multiple of the wavelength in the direction of periodicity (Fig. 3.1). Then with the help of the equation of continuity ($U_{i,i} = 0$; $u_{i,i} = 0$), and Gauss's theorem relating surface and volume integrals, the terms on the right side of the energy equation can be reduced as follows:

$$\int u_i t_{ij,j} dV = \int (u_i t_{ij})_{,j} dV - \int t_{ij} u_{i,j} dV = \oint (t_{ij} u_i) dS_j - \int t_{ij} u_{i,j} dV$$

$$\int u_i U_j u_{i,j} dV = \tfrac{1}{2} \int U_j (u_i^2)_{,j} dV = \tfrac{1}{2} \int (u_i^2 U_j)_{,j} dV = \tfrac{1}{2} \oint u_i^2 U_j dS_j$$

$$\int u_i u_j u_{i,j} dV = \tfrac{1}{2} \int u_j (u_i^2)_{,j} dV = \tfrac{1}{2} \int (u_i^2 u_j)_{,j} dV = \tfrac{1}{2} \oint u_i^2 u_j dS_j$$

The transport terms (surface integrals) in the above are zero because the integrands are zero on those portions of V coinciding with the walls, and their contributions cancel each other on the open portions of V (see Fig. 3.1 again). The equation of energy therefore reduces to

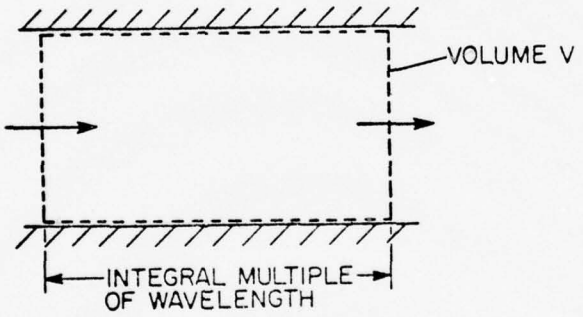
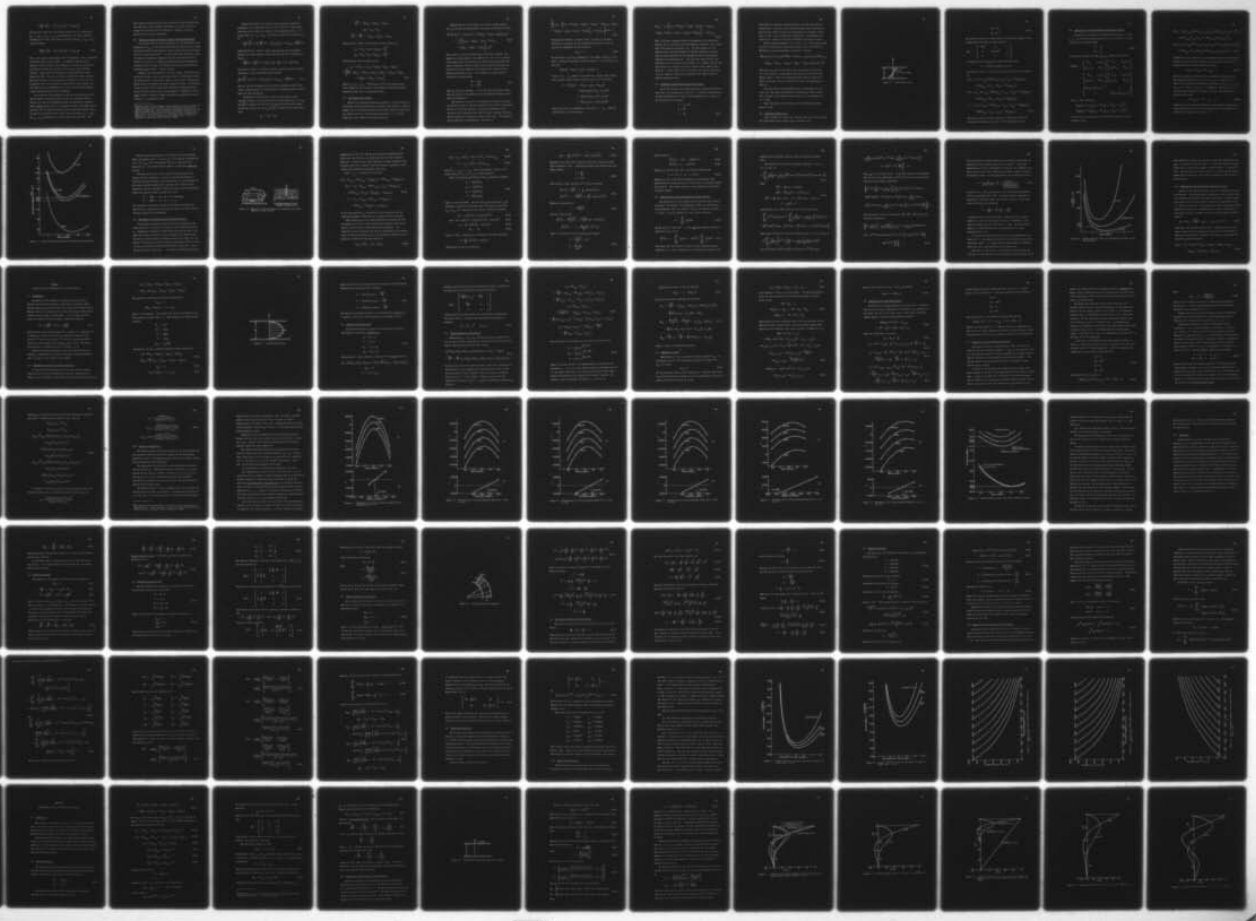


Figure 3.1 Definition of volume of integration.

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$$\tfrac{1}{2}\frac{d}{dt}\int u_i^2 dV = -\frac{1}{\rho}\int t_{ij}u_{i,j}dV - \int u_i u_j U_{i,j}dV$$

[At this point note that the Newtonian equation (1.1) is obtained if $t_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ is substituted in the above equation.] Introducing the strain rate tensor of the basic flow $A_{ij} = U_{i,j} + u_{j,i}$, the above becomes

$$\frac{\rho}{2}\frac{d}{dt}\int u_i^2 dV = -\frac{\rho}{2}\int u_i u_j A_{ij}dV - \int t_{ij}u_{i,j}dV \qquad (3.3)$$

This is the global energy equation for the disturbance, and is attributed to Reynolds (1895) and Orr (1907). It signifies that the rate of change of kinetic energy of the disturbance in the entire field equals the energy extracted from the basic flow by means of the Reynolds stresses $-u_i u_j$, minus the "dissipation" of the disturbance kinetic energy into heat. We have put the word "dissipation" within quotation marks here because from the discussions of Sections 2.10-2.14 it will be apparent that for viscoelastic fluids which can store energy, this term represents the dissipation rate as well as the rate of change of elastic potential energy of the disturbance.

As an aside it may be pointed out that for the application of the energy method to Newtonian fluids one poses the following problem: consider the class of disturbances having instantaneously stationary global kinetic energy; what is the equation obeyed by that member which can exist with the largest viscosity (other things being equal)? Thus, one puts the left hand side of (3.3) to zero, introduces $t_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ and maximizes μ by the calculus of variations. The

Euler equation corresponding to this variational problem represents the most efficient or most dangerous disturbance, one which can extract energy from the basic flow most efficiently, losing as little as possible to the viscous dissipation.

3.2 Equation of Global Disturbance Energy for Second-Order Fluid

It was found that due to the presence of the time derivative in the expression for t_{ij} for the second-order fluid [see equation (3.5) later], the energy method as formulated for a purely viscous fluid is not applicable to a second-order fluid.* As a way around this difficulty, Lumley in a private communication suggested that the time derivative can be avoided if, instead of considering disturbances having instantaneously stationary kinetic energy, one considers those having instantaneously stationary total (kinetic + elastic) energy.

Moreover, the consideration of the total energy, instead of simply the kinetic energy, seems to make better sense in a viscoelastic fluid. The disturbance in a viscoelastic liquid is at all times interchanging energy between its kinetic and elastic (potential) modes. At one moment it may have very little kinetic and a good deal of potential energy, while at another moment it may have the opposite. Therefore it is the total (kinetic + potential) energy that must be significant for a disturbance in a viscoelastic fluid.

---

*Note that even in the stationary neutral state of no rate of change of total disturbance kinetic energy in the flow field, the $\partial/\partial t$ of a quantity is not zero. This is because, although the integral over the entire field is zero, the rate of change of kinetic energy at every point is not; at some points it is increasing, while at others it is decreasing. This must happen due to the nonlinearity of the disturbances in the energy method (Lumley, 1971b).

Recall from Section 2.11 that the elastic potential energy of a disturbance in a second-order fluid is $\gamma II'/4$ where II' is the second invariant for the strain rate tensor of the disturbance motion: $II' = a_{ij}a_{ij}$ where $a_{ij} = u_{i,j} + u_{j,i}$. With this in mind let us rewrite (3.3) as

$$\rho \frac{d}{dt} \int \frac{u_i^2}{2} dV + \frac{\gamma}{4} \int \frac{DII'}{Dt} dV = - \frac{\rho}{2} \int u_i u_j A_{ij} dV - \int (t_{ij} u_{i,j} - \frac{\gamma}{4} \frac{DII'}{Dt}) dV$$

Inserting $DII'/Dt = \partial II'/\partial t + V_k II'_{,k}$ and using continuity and Gauss's theorem, the second integral on the left hand side can be rewritten as follows:

$$\int \frac{DII'}{Dt} dV = \int \frac{\partial II'}{\partial t} dV + \int (V_k II')_{,k} dV = \frac{d}{dt} \int II' dV + \oint V_k II' dS_k$$

The surface integral in the above equation vanishes because of boundary conditions, so that the energy equation becomes

$$\frac{d}{dt} \int (\frac{\rho u_i^2}{2} + \frac{\gamma}{4} II') dV = - \frac{\rho}{2} \int u_i u_j A_{ij} dV - \int (t_{ij} u_{i,j} - \frac{\gamma}{4} \frac{DII'}{Dt}) dV \qquad (3.4)$$

That is, the rate of change of the sum of the disturbance kinetic and elastic energies equals the total production (through Reynolds stresses) minus the total dissipation.

An expression for the stress change t_{ij} which occurs in the equation of energy will now be found for a second-order fluid. Let A_{ij} and $A_{ij}^{(2)}$ be respectively the first and second Rivlin-Ericksen tensors of the basic motion, and let α_{ij} and $\alpha_{ij}^{(2)}$ be those for the total motion. Thus,

$$A_{ij} = U_{i,j} + U_{j,i}$$

$$A_{ij}^{(2)} = U_k A_{ij,k} + A_{ik} U_{k,j} + A_{jk} U_{k,i}$$

$$\alpha_{ij} = V_{i,j} + V_{j,i}$$

$$\alpha_{ij}^{(2)} = \frac{\partial \alpha_{ij}}{\partial t} + V_k \alpha_{ij,k} + \alpha_{ik} V_{k,j} + \alpha_{jk} V_{k,i}$$

Then the stress tensors for the total and basic motions are

$$\tau_{ij} = -\pi \delta_{ij} + \mu \alpha_{ij} + \beta \alpha_{ik} \alpha_{kj} + \gamma \alpha_{ij}^{(2)}$$

$$T_{ij} = -P \delta_{ij} + \mu A_{ij} + \beta A_{ik} A_{kj} + \gamma A_{ij}^{(2)}$$

Subtracting the above equations we get

$$t_{ij} = -(\pi - P)\delta_{ij} + \mu a_{ij} + \beta (a_{ik} A_{kj} + A_{ik} a_{kj} + a_{ik} a_{kj})$$

$$+ \gamma \left(\frac{\partial a_{ij}}{\partial t} + u_k A_{ij,k} + U_k a_{ij,k} + u_k a_{ij,k} + a_{ik} U_{k,j} + A_{ik} u_{k,j} + a_{ik} u_{k,j} \right.$$

$$\left. + a_{jk} U_{k,i} + A_{jk} u_{k,i} + a_{jk} u_{k,i} \right) \qquad (3.5)$$

where $a_{ij} = u_{i,j} + u_{j,i}$. Substitution of the above expression for stress change in (3.4) results in the equation of global disturbance mechanical energy for the second-order fluid.

3.3 The Extremization Problem

Consider the following extremization problem: out of the class of disturbances having instantaneously stationary total (kinetic + elastic) energy, what is the equation obeyed by that member which can exist with the largest viscosity μ (other things being equal), i.e., is most dangerous or most capable of causing instability?

Substituting the stress change (3.5) into the energy equation (3.4), we get for the instantaneous stationarity of disturbance energy

$$
\begin{aligned}
-\frac{\rho}{2}\int u_i u_j A_{ij}\,dV &= \mu\int a_{ij}u_{i,j}\,dV + \beta\int (a_{ik}A_{kj} + A_{ik}a_{kj} + a_{ik}a_{kj})u_{i,j}\,dV \\
&+ \gamma\int\Big[u_k A_{ij,k} + a_{ik}U_{k,j} + A_{ik}u_{k,j} + a_{ik}u_{k,j} \\
&+ a_{jk}U_{k,i} + A_{jk}u_{k,i} + a_{jk}u_{k,i}\Big]u_{i,j}\,dV
\end{aligned}
\tag{3.6}
$$

The viscosity μ will now be extremized in the above equation. The question is, what happens to the coefficients β and γ as μ is varied? If we imagine that the viscosity of a liquid containing polymer molecules is gradually raised, we expect that the polymer molecules will become more and more effective and therefore will increase the non-Newtonian parameters β and γ. It is simplest to assume that the variations are proportional:

$$
\begin{aligned}
\delta\beta &= K_\beta\delta\mu \\
\delta\gamma &= K_\gamma\delta\mu
\end{aligned}
\tag{3.7}
$$

where K_β and K_γ are constants. It will be shown in the next section that the results of the present work can be justified through another type of extremization.

The viscosity μ in (3.6) is extremized by the method of calculus of variations; that is, take the variation of each term of the equation and set $\delta\mu = 0$ for the extremum. During the computation of the variation of the different terms Gauss's theorem, boundary conditions and the equation of continuity are used several times. The algebra is long but perfectly straightforward. The result is

$$\int\int\Big\{\rho A_{ij}u_j - \Big[2\mu a_{ij} + \beta(2a_{ik}A_{kj} + 2A_{ik}a_{kj} + 3a_{ik}a_{kj}) + \gamma(u_k A_{ij,k} + a_{ik}A_{kj} + 2A_{ik}a_{kj} + u_{k,j}A_{ik} + u_{k,i}A_{kj} + 3a_{ik}a_{kj})\Big]_{,j} + \gamma u_{k,j}A_{kj,i}\Big\}\delta u_i dV = 0 \tag{3.8}$$

Following the procedure of Lumley (1971b), note that in the above equation the variations are over incompressible motions so that the δu_i are not independent but are related by

$$\delta u_{i,i} = 0 \tag{3.9}$$

We may specify δu_1 and δu_2 independently, for example, and δu_3 is then given. Denoting the quantity within the braces { } in (3.8) by X_i, we can write

$$\int X_i \delta u_i dV = \int (X_1 \delta u_1 + X_2 \delta u_2 + X_3 \delta u_3) dV = 0$$

If we set $X_3 = -\phi_{,3}$, where ϕ is any arbitrary function, then integration by parts (using boundary conditions) and use of (3.9) gives

$$\begin{aligned} 0 = \int X_i \delta u_i dV &= \int (X_1 \delta u_1 + X_2 \delta u_2 + \phi \delta u_{3,3}) dV \\ &= \int (X_1 \delta u_1 + X_2 \delta u_2 - \phi[\delta u_{1,1} + \delta u_{2,2}]) dV \\ &= \int (X_1 \delta u_1 + X_2 \delta u_2 + \phi_{,1}\delta u_1 + \phi_{,2}\delta u_2) dV \\ &= \int [(X_1 + \phi_{,1})\delta u_1 + (X_2 + \phi_{,2})\delta u_2] dV \end{aligned}$$

Since δu_1 and δu_2 are independent we must have $X_i = -\phi_{,i}$. From the definition of X_i we therefore get

$$\rho A_{ij}u_j = -\phi_{,i} + \Big[2\mu a_{ij} + \beta(2a_{ik}A_{kj} + 2A_{ik}a_{kj} + 3a_{ik}a_{kj}) + \gamma(u_k A_{ij,k} + a_{ik}A_{kj} + 2A_{ik}a_{kj} + u_{k,j}A_{ik} + u_{k,i}A_{kj} + 3a_{ik}a_{kj})\Big]_{,j} - \gamma u_{k,j}A_{kj,i} \tag{3.10}$$

This is the Euler-Lagrange equation corresponding to our variational problem, and is satisfied by the most dangerous disturbance in a second-order fluid obeying assumptions (3.7). The same equation is also obtained if one uses the technique of Lagrange multiplier, in which case ϕ would be identified as the Lagrange multiplier satisfying the constraint of incompressibility. Also note that except for the last term which is not a derivative with respect to x_j, equation (3.10) looks very similar to an equation of motion, in which case ϕ appears like a "pressure." Lumley (1971b) for the Newtonian case uses the expression "ϕ is the 'pressure' which must be applied to the fluid in order to satisfy incompressibility."

3.4 Extremizing Concentration Instead of Viscosity

In the last section μ was extremized and the variations of β and γ were taken to be proportional to that of μ. It would be interesting to see what would happen if we took a linear dependence of μ, β and γ on the concentration C of polymer

$$\begin{aligned} \mu &= \mu_0 + \bar{\mu}C \\ \beta &= \bar{\beta}C \\ \gamma &= \bar{\gamma}C \end{aligned} \tag{3.7}'$$

where $\bar{\mu},\bar{\beta},\bar{\gamma}$ are constants of proportionality, and then maximized the concentration. Physically this variational problem would mean that as more and more polymer molecules are being added to a turbulent Couette flow μ, β and γ are all increasing in magnitude; a concentration will eventually be reached at which the disturbances just begin to vanish.

With the assumed dependence of μ, β and γ on C, if the stress change of (3.5) is substituted in the energy equation (3.4), we get for the disturbances having instantaneous stationarity of energy

$$-\frac{\rho}{2}\int u_i u_j A_{ij}\,dV=\mu_0\int a_{ij}u_{i,j}\,dV+C\int[\bar{\mu}a_{ij}+\bar{\beta}(a_{ik}A_{jk}+A_{ik}a_{jk}+a_{ik}a_{jk})+\bar{\gamma}(u_k A_{ij,k}$$

$$+a_{ik}U_{k,j}+A_{ik}u_{k,j}+a_{ik}u_{k,j}+a_{jk}U_{k,i}+A_{jk}u_{k,i}+a_{jk}u_{k,i})]u_{i,j}\,dV$$

Taking the variation of each term of the above equation and setting $\delta C = 0$ for maximum C, we have found that equation (3.10) of the last section results. This is not surprising since both (3.7) and (3.7)' amount to the fact that the variations of β and γ are proportional to that of μ.

Note also that if we had regarded β and γ as constants as μ was varied, we would again have obtained the same result, that is (3.10). This is because at the critical state we set $\delta\mu = 0$, which makes $\delta\beta = \delta\gamma = 0$ also.

Thus, the results of this chapter can be justified from three points of view.

3.5 Rectilinear Couette Flow

Let us consider the special case when the basic flow is in the form of a plane Couette flow of shear rate Γ (see Fig. 3.2):

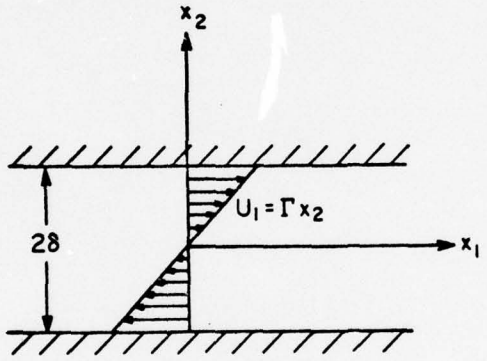


Figure 3.2 Plane Couette flow.

$$U_1 = \Gamma x_2$$
$$U_2 = U_3 = 0 \tag{3.11}$$

For this basic motion, it follows from the constitutive equation of the second-order fluid that the stress tensor is

$$[\underset{\sim}{T}] = \begin{bmatrix} -P+\beta\Gamma^2 & \mu\Gamma & 0 \\ \mu\Gamma & -P+(\beta+2\gamma)\Gamma^2 & 0 \\ 0 & 0 & -P \end{bmatrix}$$

An examination of the equation of motion then shows that

$$P = \text{constant throughout the flow}$$

For the basic motion of (3.11), the three equations in (3.10) simplify to

$$\rho\Gamma u_2 = -\phi_{,1} + 2\mu\, u_{1,jj} + \beta[2\Gamma(3u_{1,12} + u_{2,11} + u_{2,jj}) + 3(a_{1k}a_{kj})_{,j}]$$
$$+ \gamma[\Gamma(3u_{2,jj} + 4u_{1,12} + 2u_{2,11}) + 3(a_{1k}a_{kj})_{,j}]$$

$$\rho\Gamma u_1 = -\phi_{,2} + 2\mu\, u_{2,jj} + \beta[2\Gamma(3u_{2,12} + u_{1,22} + u_{1,jj}) + 3(a_{2k}a_{kj})_{,j}]$$
$$+ \gamma[\Gamma(3u_{1,jj} + 4u_{2,12} + 2u_{1,22}) + 3(a_{2k}a_{kj})_{,j}]$$

$$0 = -\phi_{,3} + 2\mu\, u_{3,jj} + \beta[2\Gamma(2u_{3,12} + u_{1,23} + u_{2,13}) + 3(a_{3k}a_{kj})_{,j}]$$
$$+ \gamma[2\Gamma(u_{3,12} + u_{1,23} + u_{2,13}) + 3(a_{3k}a_{kj})_{,j}] \tag{3.12}$$

Two special cases will now be considered, namely that of the two-dimensional disturbances and that of the "longitudinal rolls."

3.6 Formulation of the Two-Dimensional Disturbance Problem

Consider only two-dimensional disturbances for which $u_3 = 0$ and $(\ )_{,3} = 0$, so that the third equation of (3.12) is absent. A stream function can be defined through

$$\begin{aligned} u_1 &= \psi_{,2} \\ u_2 &= -\psi_{,1} \end{aligned} \tag{3.13}$$

Let us first evaluate the terms $(a_{ik}a_{kj})_{,j}$ which occur in (3.12)

$$[a_{ik}a_{kj}] = \begin{bmatrix} 2u_{1,1} & u_{1,2}+u_{2,1} \\ u_{2,1}+u_{1,2} & 2u_{2,2} \end{bmatrix} \begin{bmatrix} 2u_{1,1} & u_{1,2}+u_{2,1} \\ u_{2,1}+u_{1,2} & 2u_{2,2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\psi_{,12} & \psi_{,22}-\psi_{,11} \\ \psi_{,22}-\psi_{,11} & -2\psi_{,12} \end{bmatrix} \begin{bmatrix} 2\psi_{,12} & \psi_{,22}-\psi_{,11} \\ \psi_{,22}-\psi_{,11} & -2\psi_{,12} \end{bmatrix}$$

$$= \begin{bmatrix} 4\psi_{,12}\psi_{,12}+(\psi_{,22}-\psi_{,11})^2 & 0 \\ 0 & 4\psi_{,12}\psi_{,12}+(\psi_{,22}-\psi_{,11})^2 \end{bmatrix}$$

For i = 1 and 2 this gives

$$(a_{1k}a_{kj})_{,j} = (a_{1k}a_{k1})_{,1} = [4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,1}$$

$$(a_{2k}a_{kj})_{,j} = (a_{2k}a_{k2})_{,2} = [4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,2}$$

In terms of the stream function the first two equations of (3.12) therefore become

$$-\rho\Gamma\psi_{,1} = -\phi_{,1} + 2\mu\psi_{,2jj} + \beta\{2\Gamma(3\psi_{,122} - \psi_{,111} - \psi_{,1jj}) + 3[4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,1}\}$$

$$+ \gamma\{\Gamma(-3\psi_{,1jj} + 4\psi_{,122} - 2\psi_{,111}) + 3[4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,1}\}$$

$$\rho\Gamma\psi_{,2} = -\phi_{,2} - 2\mu\psi_{,1jj} + \beta\{2\Gamma(-3\psi_{,112} + \psi_{,222} + \psi_{,2jj}) + 3[4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,2}\}$$

$$+ \gamma\{\Gamma(3\psi_{,2jj} - 4\psi_{,112} + 2\psi_{,222}) + 3[4\psi_{,12}\psi_{,12} + (\psi_{,22} - \psi_{,11})^2]_{,2}\} \quad (3.14)$$

Elimination of the "pressure" ϕ by cross-differentiation and subtraction between the above equations results in the cancellation of all the non-linear terms and the terms containing β, giving

$$\rho\Gamma\psi_{,12} + \mu\psi_{,iijj} - 2\gamma\Gamma\psi_{,12jj} = 0 \quad (3.15)$$

The final equation is therefore linear, although the problem is not. The boundary conditions are $u_1 = u_2 = 0$ at $x_2 = \pm\,\delta$ (Fig. 3.2). In terms of the stream function the boundary conditions read $\psi_{,1} = \psi_{,2} = 0$ at $x_2 = \pm\,\delta$. But $\psi_{,1} = 0$ at $x_2 = \pm\,\delta$ implies ψ = constant at $x_2 = \pm\,\delta$; for convenience we take the constant to be zero. The boundary conditions are therefore

$$\psi = \psi_{,2} = 0 \quad \text{at} \quad x_2 = \pm\,\delta \quad (3.16)$$

Equation (3.15) with boundary conditions (3.16) define a characteristic value problem which will now be solved following the procedure of Orr (1907) who had equation (3.15) with $\gamma = 0$.

3.7 Solution of the Two-Dimensional Disturbance Problem

To solve the characteristic value problem (3.15)-(3.16), note that since the disturbances are periodic in the x_1-direction

$$\psi = \hat{\psi}(x_2)\, e^{i\alpha x_1} \tag{3.17}$$

where the wavenumber α is real. Substituting the above in (3.15) and expressing in terms of non-dimensional distance and wavenumber

$$\begin{aligned} \zeta &= \frac{x_2}{\delta} \\ k &= \alpha\delta \end{aligned} \tag{3.18}$$

we get (writing $D = d/d\zeta$)

$$(D^2-k^2)^2\hat{\psi} + ikRD\hat{\psi} - 2ikGD(D^2-k^2)\hat{\psi} = 0 \tag{3.19}$$

where we have also introduced the non-dimensional numbers

$$\begin{aligned} R &= \text{Reynolds Number} = \frac{\rho\delta^2\Gamma}{\mu} = \frac{\rho^2 u_\tau^2 \delta^2}{\mu^2} = (\delta^+)^2 \\ G &= \text{Elasticity Parameter} = \frac{\gamma\Gamma}{\mu} = \frac{\text{time scale of fluid}}{\text{time scale of flow}} \end{aligned} \tag{3.20}$$

In the above definition of Reynolds number, μ_τ is called the "shear velocity" and is defined as

$$u_\tau^2 = \frac{\text{wall shear stress}}{\text{density}} = \frac{\mu\Gamma}{\rho}$$

and $\delta^+ = \rho\delta u_\tau/\mu$ is the non-dimensional half-width of the channel.

Since the coefficients of (3.19) are independent of ζ, the solution of the equation can be expressed as a superposition of solutions of the form $\hat{\psi} = e^{im\zeta}$. With this substitution (3.19) becomes

$$m^4 - 2Gkm^3 + 2k^2m^2 - k(2Gk^2 + R)m = 0 \qquad (3.21)$$

Denoting the roots of this equation by m_1, m_2, m_3 and m_4, the general solution is

$$\hat{\psi} = Ae^{im_1\zeta} + Be^{im_2\zeta} + Ce^{im_3\zeta} + De^{im_4\zeta}$$

Because of the boundary conditions $\hat{\psi} = D\hat{\psi} = 0$ at $\zeta = \pm 1$, the condition that not all of A, B, C and D are zero is the vanishing of the determinant

$$\begin{vmatrix} e^{im_1} & e^{im_2} & e^{im_3} & e^{im_4} \\ e^{-im_1} & e^{-im_2} & e^{-im_3} & e^{-im_4} \\ m_1e^{im_1} & m_2e^{im_2} & m_3e^{im_3} & m_4e^{im_4} \\ m_1e^{-im_1} & m_2e^{-im_2} & m_3e^{-im_3} & m_4e^{-im_4} \end{vmatrix} = 0 \qquad (3.22)$$

By expanding the determinant the above condition can be shown to reduce to

$$(m_3m_4+m_1m_2)\sin(m_1-m_2)\sin(m_3-m_4)+(m_2m_4+m_1m_3)\sin(m_2-m_4)\sin(m_3-m_1) + (m_2m_3 + m_1m_4)\sin(m_2 - m_3)\sin(m_1 - m_4) = 0 \qquad (3.23)$$

The procedure of solving the characteristic value problem is now clear: for an assumed value of k and G, find R so that the four roots of (3.21) satisfy (3.23); this gives a point on the stability diagram. The procedure was repeated for several values of k and G on the computer and the results are shown in Fig. 3.3. G, the ratio of the time scales of the fluid and flow, was given a value of G = -0.5 and -0.05. Since γ (and hence G) is negative on thermodynamic grounds, the curve for G = 0.5 shown dotted in Fig. 3.3 has no physical meaning.

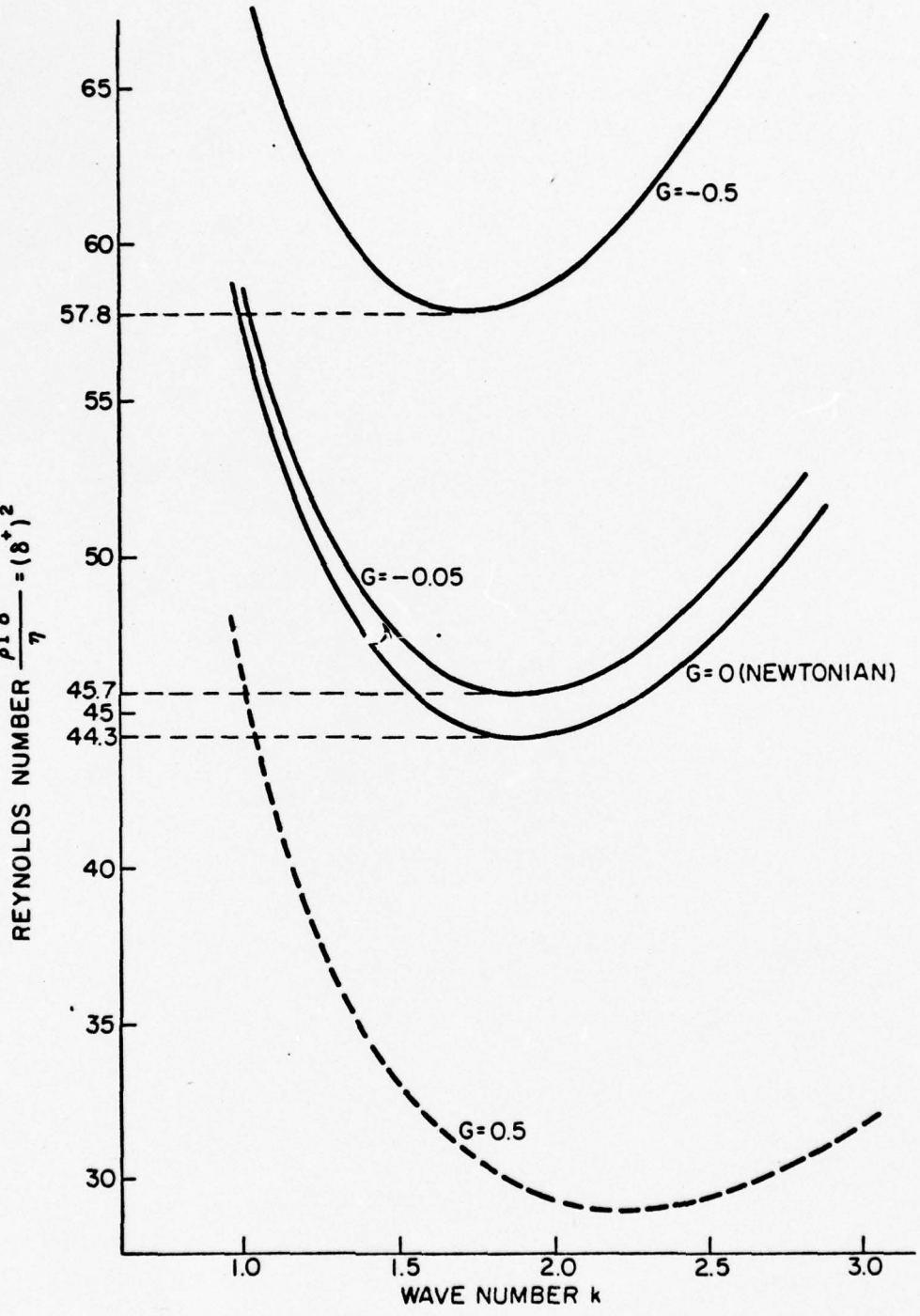


Figure 3.3 Neutral stability curves for two-dimensional disturbances.

From the stability diagram it is seen that the critical Reynolds number corresponding to G = - 0.5 is R_{cr} = 57.8, and that corresponding to G = 0 is R_{cr} = 44.3, which agrees with Orr's solution of the Newtonian case. This shows that the presence of elasticity stabilizes the flow.

From the point of view of the viscous sublayer thickness, the results are very interesting. It will be recalled from Section 1.6 that half the channel width at the critical state is a good estimate of the viscous sublayer thickness of a turbulent flow. Since the non-dimensional channel half width δ^+ equals the square root of the Reynolds number [see equation (3.20)], we have the results

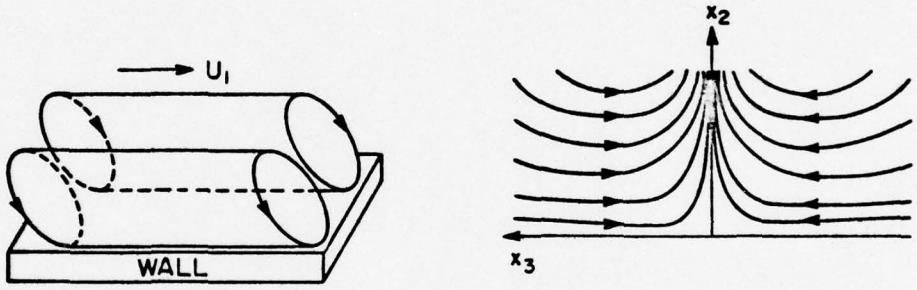
$$\delta^+ = \sqrt{44.3} = 6.65 \quad \text{for} \quad G = 0 \text{ (Newtonian)}$$

$$\delta^+ = \sqrt{57.8} = 7.6 \quad \text{for} \quad G = - 0.5$$

This represents an increase of 14.3% in the sublayer thickness and agrees with the experimentally observed fact that the sublayer thickness increases during the Toms phenomenon.

3.8 Formulation of Three-Dimensional Disturbance Problem

Now consider disturbances which have velocity components in all three directions, but which are invariant in the streamwise direction x_1. Experiments of Bakewell & Lumley (1967) indicate that this is the type of disturbance prevalent in the wall region of a turbulent flow. The motion is of the nature of streamwise vortex pairs (Fig. 3.4), the planes of circulation of which are tipped so that their normals lie more or less in the direction of the maximum positive strain rate. Due to the mean vorticity, these normals sweep from an orientation more nearly



(a) COMPOSITE REPRESENTATION

(b) STREAMLINES IN A PLANE NORMAL TO MEAN FLOW

Figure 3.4 Eddy pair in the wall region of a turbulent flow [after Bakewell & Lumley (1967)].

perpendicular to the wall, through the direction of maximum positive strain rate, and finally to an orientation nearly in the streamwise direction. As they do this, their efficiency of energy extraction from the mean motion first increases, and then decreases, resulting in first a rapid growth followed by relatively slow decay.

With $(\ )_{,1} = 0$, equations (3.12) become

$$\rho\Gamma u_2 = 2\mu u_{1,jj} + \beta[2\Gamma u_{2,jj} + 3(a_{1k}a_{kj})_{,j}] + \gamma[3\Gamma u_{2,jj} + 3(a_{1k}a_{kj})_{,j}]$$

$$\rho\Gamma u_1 = -\phi_{,2} + 2\mu u_{2,jj} + \beta[2\Gamma(u_{1,22} + u_{1,jj}) + 3(a_{2k}a_{kj})_{,j}]$$

$$+ \gamma[\Gamma(3u_{1,jj} + 2u_{1,22}) + 3(a_{2k}a_{kj})_{,j} - u_{k,j}a_{kj,2}] \qquad (3.24)$$

$$0 = -\phi_{,3} + 2\mu u_{3,jj} + \beta[2\Gamma u_{1,23} + 3(a_{3k}a_{kj})_{,j}]$$

$$+ \gamma[2\Gamma u_{1,23} + 3(a_{3k}a_{kj})_{,j} - u_{k,j}a_{kj,3}]$$

In the above equations $\phi_{,1}$ has been set to zero because we are considering disturbances which cause no streamwise "pressure" gradient.

After elimination of ϕ by cross-differentiation between the last two equations of (3.24), it was found that the nonlinear terms do not cancel, and the solution of the problem seems formidable. For simplicity we have solved the case when the disturbances can be assumed small, so that the nonlinear terms in the disturbance quantities can be neglected. Equations (3.24) then simplify to

$$\rho\Gamma u_2 = 2\mu\nabla^2 u_1 + \Gamma(2\beta + 3\gamma)\nabla^2 u_2 \qquad (3.25a)$$

$$\rho\Gamma u_1 = -\phi_{,2} + 2\mu\nabla^2 u_2 + \Gamma(2\beta + 3\gamma)\nabla^2 u_1 + 2\Gamma(\beta+\gamma)u_{1,22} \tag{3.25b}$$

$$0 = -\phi_{,3} + 2\mu\nabla^2 u_3 + 2\Gamma(\beta+\gamma)u_{1,23} \tag{3.25c}$$

where $\nabla^2 = (\ )_{,22} + (\ )_{,33}$. The set of equations (3.25) is to be solved subject to $u_1 = u_2 = u_3 = 0$ at the boundaries.

Since the solutions are periodic in the x_3-direction, introduce

$$\begin{aligned} u_1 &= u(x_2)\cos\alpha x_3 \\ u_2 &= v(x_2)\cos\alpha x_3 \\ u_3 &= w(x_2)\sin\alpha x_3 \\ \phi &= \hat{\phi}(x_2)\cos\alpha x_3 \end{aligned} \tag{3.26}$$

where α is the wavenumber. The sine with u_3 and cosine with u_2 are necessary in order to be consistent with the equation of continuity $u_{2,2} + u_{3,3} = 0$. Equations (3.25) and the continuity then become

$$\rho\Gamma v = 2\mu(d^2-\alpha^2)u + \Gamma(2\beta+3\gamma)(d^2-\alpha^2)v \tag{3.27}$$

$$\rho\Gamma u = -d\hat{\phi} + 2\mu(d^2-\alpha^2)v + \Gamma(2\beta+3\gamma)(d^2-\alpha^2)u + 2\Gamma(\beta+\gamma)d^2u \tag{3.28}$$

$$0 = \alpha\hat{\phi} + 2\mu(d^2-\alpha^2)w - 2\Gamma(\beta+\Gamma)\alpha du \tag{3.29}$$

$$dv = -\alpha w \tag{3.30}$$

where $d \equiv d/dx_2$. Elimination of w between (3.29) and (3.30) gives

$$\hat{\phi} = \frac{2\mu}{\alpha^2}(d^2-\alpha^2)dv + 2\Gamma(\beta+\gamma)du$$

Substitution of this in (3.28) gives

$$\rho\Gamma u = -\frac{2\mu}{\alpha^2}(d^2-\alpha^2)^2 v + \Gamma(2\beta + 3\gamma)(d^2-\alpha^2)u \qquad (3.31)$$

Equations (3.27) and (3.31) constitute a pair which should be solved for the unknowns u and v. Introducing nondimensional distance and wave-number through

$$\zeta = \frac{x_2}{2\delta}$$
$$k = \alpha 2\delta \qquad (3.32)$$

and writing $D \equiv d/d\zeta$, equations (3.27) and (3.31) become

$$(D^2-k^2)u = \frac{2\rho\delta^2\Gamma}{\mu} v - \frac{\Gamma}{2\mu}(2\beta+3\gamma)(D^2-k^2)v$$
$$(D^2-k^2)^2 v = -\frac{2\rho\delta^2\Gamma k^2}{\mu} u + \frac{\Gamma k^2}{2\mu}(2\beta+3\gamma)(D^2-k^2)u \qquad (3.33)$$

Making the transformation

$$v \to \frac{2\rho\delta^2\Gamma k^2}{\mu} v \qquad (3.34)$$

the pair (3.33) becomes

$$(D^2-k^2)u = \frac{4\rho^2\delta^4\Gamma^2k^2}{\mu^2} v - \frac{\rho\delta^2\Gamma^2k^2}{\mu^2}(2\beta + 3\gamma)(D^2-k^2)v$$
$$(D^2-k^2)^2 v = -u + \frac{(2\beta + 3\gamma)}{4\rho\delta^2}(D^2 - k^2)u$$

which, on introduction of the non-dimensional numbers

$$T = \frac{4\rho^2\delta^4\Gamma^2}{\mu^2} = 4\delta^{+4}$$
$$H = \frac{2\beta + 3\gamma}{4\rho\delta^2}, \qquad (3.35)$$

can be written as

$$(D^2-k^2)u = Tk^2v - Tk^2H(D^2-k^2)v \tag{3.36}$$

$$(D^2-k^2)^2v = -u + H(D^2-k^2)u \tag{3.37}$$

Measuring x_2 from the lower wall, the boundary conditions are

$$u = v = Dv = 0 \quad \text{at} \quad \zeta = 0 \text{ and } 1 \tag{3.38}$$

Equations (3.36), (3.37) and (3.38) define a characteristic value problem which has solutions only if T varies with k in a definite manner (for fixed H). The critical value of T will correspond to the minimum of the T-k diagram.

3.9 Solution of the Three-Dimensional Disturbance Problem

The characteristic value problem (3.36)-(3.38) will now be solved according to the procedure of Chandrasekhar (1954); see also the book of Chandrasekhar (1961), pp. 300-303. Since u is required to vanish at $\zeta = 0$ and 1, it can be expanded in a sine series of the form

$$u = \sum_{m=1}^{\infty} C_m \sin m\pi\zeta \tag{3.39}$$

Having chosen u in this manner, we next solve the equation obtained by substitution into (3.37):

$$(D^2-k^2)\,v = -\sum_{m=1}^{\infty} C_m \sin m\pi\zeta + H(D^2-k^2)\sum_{m=1}^{\infty} C_m \sin m\pi\zeta \tag{3.40}$$

and arrange that the solution satisfies the four remaining boundary conditions on v. With v determined in this fashion and u given by the

assumed series, equation (3.36) will lead to the secular equation for T.

The solution of (3.40) with the boundary conditions v = Dv = 0 at $\zeta = 0$ and 1 is

$$v = \sum_{m=1}^{\infty} \frac{C_m}{m^2\pi^2+k^2}\left(H+\frac{1}{m^2\pi^2+k^2}\right)\left\{B_1^{(m)}\sinh k\zeta + A_2^{(m)}\zeta\cosh k\zeta + B_2^{(m)}\zeta\sinh k\zeta - \sin m\pi\zeta\right\} \tag{3.41}$$

where

$$\begin{aligned} B_1^{(m)} &= -\frac{m\pi}{\Delta}\left[k + (-1)^m\sinh k\right] \\ A_2^{(m)} &= \frac{m\pi}{\Delta}\left[\sinh^2 k + (-1)^m k\sinh k\right] \\ B_2^{(m)} &= -\frac{m\pi}{\Delta}\left[\sinh k\cosh k - k + (-1)^m(k\cosh k - \sinh k)\right] \\ \Delta &= \sinh^2 k - k^2 \end{aligned} \tag{3.42}$$

Substituting u and v from (3.39) and (3.41) into (3.36) we get

$$-\sum_{m=1}^{\infty}(m^2\pi^2+k^2)C_m\sin m\pi\zeta = Tk^2\sum_{m=1}^{\infty}\frac{C_m}{m^2\pi^2+k^2}\left(H+\frac{1}{m^2\pi^2+k^2}\right)\left\{-2HkB_2^{(m)}\cosh k\zeta\right.$$

$$\left. + (B_1^{(m)} - 2HkA_2^{(m)})\sinh k\zeta + A_2^{(m)}\zeta\cosh k\zeta + B_2^{(m)}\zeta\sinh k\zeta - [1+H(m^2\pi^2+k^2)]\sin m\pi\zeta\right\}$$

Multiplying the above by $\sin m\pi\zeta$ and integrating over $\zeta = 0$ to 1 results in

$$-Tk^2\sum_{m=1}^{\infty}\frac{C_m}{m^2\pi^2+k^2}\left\{(m^2\pi^2+k^2)\frac{\delta_{mn}}{2Tk^2} + \left(H+\frac{1}{m^2\pi^2+k^2}\right)\left(\frac{n\pi}{n^2\pi^2+k^2}\right)\left\{-2HkB_2^{(m)}\right.\right.$$

$$\times\left[(-1)^{n+1}\cosh k+1\right] + (B_1^{(m)} - 2HkA_2^{(m)})(-1)^{n+1}\sinh k + (-1)^{n+1}A_2^{(m)}[\cosh k$$

$$- \frac{2k}{k^2+n^2\pi^2} \sinh k] + B_2^{(m)} [(-1)^{n+1} \sinh k - \frac{2k}{k^2+n^2\pi^2} \left\{ (-1)^{n+1} \cosh k + 1 \right\}] \Bigg\}$$

$$- [1 + H(m^2\pi^2 + k^2)] \frac{\delta_{mn}}{2} \Bigg) \Bigg\} = 0$$

where δ_{mn} is the Kronecker delta. In the above equation the requirement that not all of the coefficients $C_m/(m^2\pi^2+k^2)$ are zero is the vanishing of the following infinite determinent

$$\Bigg\| (m^2\pi^2+k^2) \frac{\delta_{mn}}{2Tk^2} + (H + \frac{1}{m^2\pi^2+k^2}) \Bigg(\frac{n\pi}{n^2\pi^2+k^2} \Bigg\{ -2HkB_2^{(m)} [(-1)^{n+1} \cosh k + 1]$$

$$+ (B_1^{(m)} - 2HkA_2^{(m)}) (-1)^{n+1} \sinh k + (-1)^{n+1} A_2^{(m)} [\cosh k - \frac{2k}{k^2+n^2\pi^2} \sinh k]$$

$$+ B_2^{(m)} [(-1)^{n+1} \sinh k - \frac{2k}{k^2+n^2\pi^2} \left((-1)^{n+1} \sinh k + 1 \right)] \Bigg\} - [1 + H(m^2\pi^2+k^2)] \frac{\delta_{mn}}{2} \Bigg) \Bigg\| = 0$$

Substituting the values of the constants $B_1^{(m)}$, $A_2^{(m)}$, $B_2^{(m)}$ from (3.42), the above simplifies to

$$\Bigg\| (H + \frac{1}{m^2\pi^2+k^2}) \Bigg(- \frac{2kmn\pi^2}{(n^2\pi^2+k^2)(\sinh^2 k - k^2)} (H + \frac{1}{n^2\pi^2+k^2}) \Bigg\{ (\sinh k \cosh k - k)$$

$$\times [1 + (-1)^{n+m}] + (\sinh k - k \cosh k) [(-1)^{n+1} + (-1)^{m+1}] \Bigg\} + [1 + H(m^2\pi^2+k^2)] \frac{\delta_{mn}}{2} \Bigg)$$

$$- \tfrac{1}{2}(n^2\pi^2+k^2)^2 \frac{\delta_{mn}}{k^2 T} \Bigg\| = 0 \qquad (3.43)$$

This is the secular equation which must be satisfied in order that our characteristic value problem should have a solution. A very good approximation to (3.43) is obtained by setting the (1,1) element of the matrix equal to zero; this is known to give results within 1% accuracy for Newtonian fluids. To this approximation the solution of (3.43) is

$$T = \frac{1}{[1+H(\pi^2+k^2)]^2} \frac{1}{k^2} \frac{(\pi^2+k^2)^3}{1 - \frac{16k\pi^2\cosh^2(k/2)}{(\pi^2+k^2)^2(\sinh k+k)}} \qquad (3.44)$$

A plot of T vs. k for various values of the non-Newtonian parameter H will now be obtained from (3.44). To choose realistic values for $H = (2\beta+3\gamma)/4\rho\delta^2$, note that β and γ are of order $\mu\tau$ where τ is some time scale of the fluid, and therefore

$$H \sim \frac{\tau\mu}{2\rho\delta^2} = \frac{\tau\Gamma}{2} \frac{\nu}{\delta^2\Gamma} = \frac{\tau\Gamma}{2\delta^{+2}}$$

$\tau\Gamma$ being the ratio of the time scales of fluid and flow is taken of order unity, and δ^+ is known to be of order 5. Therefore $H \sim 1/50$. Figure 3.5 shows a plot of (3.44) for $H = \pm 0.01$. From the stability diagram it is seen that the flow is stabilized for $H < 0$, and destabilized for $H > 0$.

It may be noted that the factor involved in the definition of H, namely $2\beta + 3\gamma$, also appears in Section 2.9 to control the sign of the Weissenberg effect: $H > 0$ gives positive (free surface) Weissenberg effect and also a destabilization in the present case.

Note that the curve for $H = 0$ (Newtonian) is valid even for disturbances of large magnitude since the nonlinear terms in (3.24), which

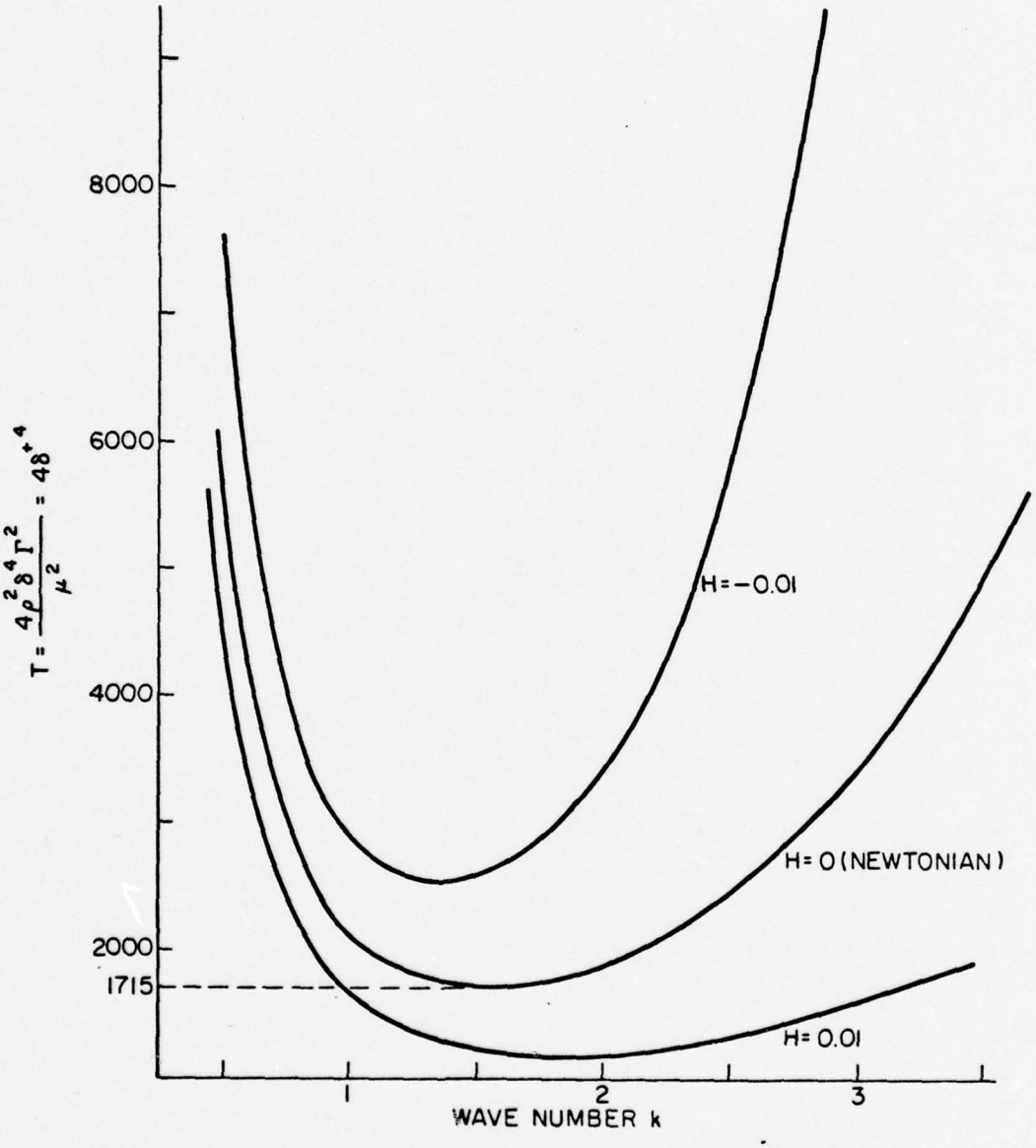


Figure 3.5 Neutral stability curves for disturbances invariant in the flow direction.

were subsequently dropped, were involved in the terms containing β and γ only. As a matter of fact the curve H = 0 yields the solution of Joseph (1966) which in terms of the sublayer thickness reads $\delta^+ = (1715/4)^{1/4} = 4.54$. The agreement with Joseph's solution also shows that the approximation made by setting the (1,1) element of equation (3.43) to zero is quite accurate.

3.10 Stability of Plane Couette Flow of Walters B' Fluid

We have so far assumed the fluid to obey the second-order constitutive equation. For two-dimensional disturbances we have seen that terms in β vanished and the results indicated stabilization. Now the second-order fluid with the β-term removed is identical to the Walters fluid A'. However, the Walters A' fluid predicts a negative Weissenberg effect (when the free surface is considered), and it would be interesting to investigate the present problem assuming the fluid to obey the Walters fluid B':

$$\sigma_{ij} = 2\mu\epsilon_{ij} - 2K\left(\frac{D\epsilon_{ij}}{Dt} - \epsilon_{kj}V_{i,k} - \epsilon_{ik}V_{j,k}\right) \quad [K > 0] \tag{3.45}$$

which shows a positive Weissenberg effect. Repeating the entire procedure, namely that of maximizing the viscosity for all disturbances which have instantaneously stationary total energy, we found that the disturbances obey the equation

$$\rho E_{ij}u_j = -\phi_{,i} + [2\mu e_{ij} + K(-u_k E_{ij,k} + 4e_{ik}E_{kj} + 2E_{ik}e_{kj} + E_{kj}u_{i,k} + E_{ik}u_{j,k} + 6e_{ik}e_{kj})]_{,j} + KE_{kj,i}u_{k,j} \tag{3.46}$$

where E_{ij} and e_{ij} are the strain rates of the basic and disturbance motions. We have only considered the special case of plane Couette flow of shear rate Γ in presence of two-dimensional disturbances. Defining a stream function $u_1 = \psi_{,2}$ and $u_2 = -\psi_{,1}$, elimination of ϕ by cross-differentiation between the 1 and 2 components of equation (3.46) results in

$$\rho\Gamma\psi_{,12} + \mu\nabla^4\psi + 2K\Gamma\psi_{,12jj} = 0 \qquad (3.47)$$

which is identical to equation (3.15) derived for the second-order fluid. Therefore, the Walters liquid B' also predicts a stabilization of a plane Couette flow with two-dimensional disturbances.

3.11 Discussion

The fact that all the three constitutive equations, namely the second-order fluid as well as the Walters fluid A' and B', which display different types of normal stresses and Weissenberg effects, should lead to identical results in two-dimensions is interesting. Walters (1962b) has shown that the two-dimensional linear stability equations for the Walters liquids A' and B' are identical. To explain all this, we substituted the constitutive equations for the two Walters fluids into the equations of motion for the total (disturbed) flow, and eliminated the pressure. In both cases we arrived at the equation

$$\rho \frac{D}{Dt} \nabla^2\psi = \mu\nabla^4\psi - K \frac{D}{Dt} \nabla^4\psi \qquad (3.48)$$

where ψ is the stream function. Moreover, Chun & Schwarz (1968) have derived equation (3.48) for a second-order fluid in two-dimensions, in which case the terms involving β dropped out.

We therefore arrive at the general conclusion that in two dimensions the second-order fluid, as well as the Walters fluids A' and B', all give identical results concerning the velocity distribution, vorticity distribution, or stability characteristics; their normal stress behavior is, however, quite different from each other.

The assurance, that our demonstration of stabilization of a plane Couette flow due to elasticity is not merely a result of a particular constitutive equation, is gratifying.

CHAPTER 4

STABILITY OF PLANE POISEUILLE FLOW OF OLDROYD FLUID

4.1 Introduction

The problem of the stability of a plane Poiseuille flow of a Newtonian fluid has been solved by, among others, Nachtsheim (1964). The same problem has also been solved for a second-order fluid by Chun & Schwarz (1967), for a Maxwell fluid by Tlapa & Bernstein (1970) and for a Walters fluid by Fong & Walters (1965). In this chapter we shall treat this problem for the more general constitutive equation

$$\sigma^{ij} + \lambda_1 \frac{\delta \sigma^{ij}}{\delta t} = 2\mu(\varepsilon^{ij} + \lambda_2 \frac{\delta \varepsilon^{ij}}{\delta t}) \tag{4.1}$$

which has been called "Oldroyd Liquid B" in Section 2.6. Equation (4.1) is identical to equation (2.18); we are using Greek letters for the stresses σ^{ij} and the strain rates ε^{ij} since the corresponding Roman letters are being preserved for the basic steady motion. The Maxwell fluid equation is obtained from (4.1) by setting $\lambda_2 = 0$.

Unlike the case of plane Couette flow, the method of small disturbances is applicable to the case of stability of a plane Poiseuille flow. We shall use this method in this chapter.

4.2 Nondimensional Equations of State and Motion

In the rest of this chapter we shall use only cartesian tensors. Using the rule $(2.14)_2$ for expanding the convective derivative of a doubly contravariant tensor, the equation of state (4.1) can be written as

$$\sigma_{ij} + \lambda_1(\sigma_{ij,t} + V_k\sigma_{ij,k} - \sigma_{kj}V_{i,k} - \sigma_{ik}V_{j,k})$$

$$= 2\mu\varepsilon_{ij} + 2\mu\lambda_2(\varepsilon_{ij,t} + V_k\varepsilon_{ij,k} - \varepsilon_{kj}V_{i,k} - \varepsilon_{ik}V_{j,k})$$

The equations of continuity and motion are respectively

$$V_{i,i} = 0$$

$$\rho(V_{i,t} + V_kV_{i,k}) = -\pi_{,i} + \sigma_{ik,k}$$

where π is the pressure. Let 2δ denote the width of the channel and U_m the centerline velocity (Fig. 4.1). Then introducing the nondimensional variables

$$\tilde{x}_i = x_i/\delta$$

$$\tilde{V}_i = V_i/U_m$$

$$\tilde{t} = tU_m/\delta$$

$$\tilde{\pi} = \pi/\rho U_m^2$$

$$\tilde{\sigma}_{ij} = \sigma_{ij}/\rho U_m^2$$

the equations of state, continuity and motion become respectively

$$\sigma_{ij} + A(\sigma_{ij,t} + V_k\sigma_{ij,k} - \sigma_{kj}V_{i,k} - \sigma_{ik}V_{j,k})$$

$$= \frac{2}{R}\varepsilon_{ij} + \frac{2B}{R}(\varepsilon_{ij,t} + V_k\varepsilon_{ij,k} - \varepsilon_{kj}V_{i,k} - \varepsilon_{ik}V_{j,k}) \tag{4.2}$$

$$V_{i,i} = 0 \tag{4.3}$$

$$V_{i,t} + V_kV_{i,k} = -\pi_{,i} + \sigma_{ij,j} \tag{4.4}$$

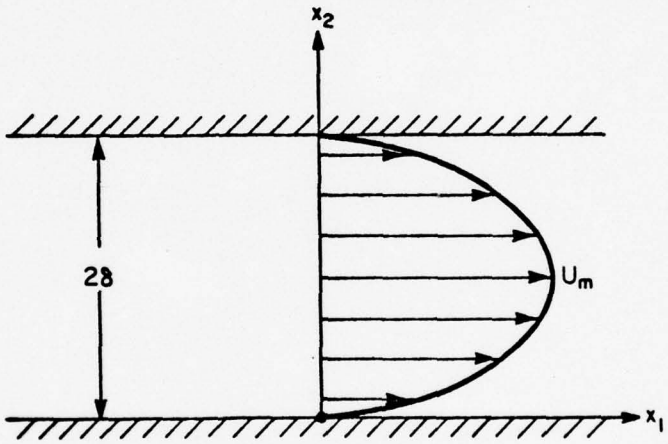


Figure 4.1 Plane Poiseuille flow.

where the tilde (~) on all symbols have been omitted, and the following nondimensional numbers have been introduced

$$\begin{aligned} R &= \text{Reynolds number} = \frac{\rho U_m \delta}{\mu} \\ A &= \text{Relaxation number} = \frac{\lambda_1 U_m}{\delta} \\ B &= \text{Retardation number} = \frac{\lambda_2 U_m}{\delta} \end{aligned} \tag{4.5}$$

The effect of nondimensionalization was to eliminate the appearance of ρ, replace λ_1 and λ_2 by A and B respectively, and replace μ by R^{-1}.

4.3 Solution of the Basic Flow .

Let the disturbed flow be written as a steady basic flow plus a time dependent disturbance, assumed small:

$$\begin{aligned} V_i &= U_i + u_i \\ \pi &= P + p \\ \sigma_{ij} &= S_{ij} + s_{ij} \\ \varepsilon_{ij} &= E_{ij} + e_{ij} \end{aligned} \tag{4.6}$$

The equations of state, continuity and motion for the basic motion are

$$S_{ij} + A(U_k S_{ij,k} - S_{kj} U_{i,k} - S_{ik} U_{j,k}) = \frac{2}{R} E_{ij} + \frac{2B}{R} (U_k E_{ij,k} - E_{kj} U_{i,k} - E_{ik} U_{j,k})$$

$$U_{i,i} = 0$$

$$0 = -P_{,i} + S_{ij,j}$$

Assuming a parallel basic flow of the form $[U_1(x_2),0,0]$, a solution of the equation of state yields the following stress matrix

$$[S_{ij}] = \begin{bmatrix} \frac{2(A-B)}{R}(U_{1,2})^2 & \frac{U_{1,2}}{R} & 0 \\ \frac{U_{1,2}}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Introduction of this stress-tensor into the equation of motion gives the result that $P_{,1}$ = constant, and that the velocity profile is parabolic:

$$U_1 = 2x_2 - x_2^2 \qquad 0 \leq x_2 \leq 2 \tag{4.7}$$

4.4 Stress Change Due to Disturbance

Substituting $\sigma_{ij} = S_{ij} + s_{ij}$ into the constitutive equation (4.2) for the total motion, subtracting the constitutive equation of the basic motion, and neglecting nonlinear terms, we get

$$s_{ij}+A(s_{ij,t}+u_k S_{ij,k}+U_k s_{ij,k}-s_{kj}U_{i,k}-S_{kj}u_{i,k}-s_{ik}U_{j,k}-S_{ik}u_{j,k}) \tag{4.8}$$
$$= \frac{2e_{ij}}{R} + \frac{2B}{R}(e_{ij,t}+u_k E_{ij,k}+U_k e_{ij,k}-e_{kj}U_{i,k}-E_{kj}u_{i,k}-e_{ik}U_{j,k}-E_{ik}u_{j,k})$$

We shall assume here that Squire's theorem, namely that two-dimensional disturbances are more dangerous than three-dimensional ones, is true for this problem; Tlapa & Bernstein (1970) have proved it for plane Poiseuille flow of a Maxwell fluid. Assuming therefore only two-dimensional disturbances $[u_3=0, (\ )_{,3}=0]$, the significant components of (4.8) are

$$s_{11} + A(s_{11,t} + U_1 s_{11,1})$$

$$= \frac{2u_{1,1}}{R} - Au_2 S_{11,2} + 2AU_{1,2} s_{12} + 2A(S_{11} u_{1,1} + S_{12} u_{1,2})$$

$$+ \frac{2B}{R} [u_{1,1t} + U_1 u_{1,11} - (u_{1,2} + u_{2,1}) U_{1,2} - U_{1,2} u_{1,2}]$$

$$s_{12} + A(s_{12,t} + U_1 s_{12,1})$$

$$= \frac{u_{1,2} + u_{2,1}}{R} - Au_2 S_{12,2} + AU_{1,2} s_{22} + AS_{11} u_{2,1} \qquad (4.9)$$

$$+ \frac{2B}{R} [\tfrac{1}{2} (u_{1,2t} + u_{2,1t}) + \tfrac{1}{2} u_2 U_{1,22} + \tfrac{1}{2} U_1 (u_{1,21} + u_{2,11}) - U_{1,2} u_{2,2}]$$

$$s_{22} + A(s_{22,t} + U_1 s_{22,1}) = 2AS_{12} u_{2,1} + \frac{2u_{2,2}}{R}$$

$$+ \frac{2B}{R} [u_{2,2t} + U_1 u_{2,21} - U_{1,2} u_{2,1}]$$

Now assume that the disturbances are periodic in the x_1-direction:

$$u_i = \hat{u}_i(x_2)\, e^{ikx_1 - ikct}$$

$$s_{ij} = \hat{s}_{ij}(x_2)\, e^{ikx_1 - ikct} \qquad (4.10)$$

$$p = \hat{p}(x)\, e^{ikx_1 - ikct}$$

where k is a real constant — the (nondimensional) wavenumber of the disturbance, $c = c_r + ic_i$ is the complex wavespeed, and quantities with (^) are complex amplitudes; as usual only the real part of the right hand side of equation (4.10) represents the physical quantity. The motion is stable or unstable according as $c_i < 0$ or $c_i > 0$.

Substituting (4.10) in (4.9), and defining

$$\beta(x_2) = 1 + ikA(U_1 - c) \tag{4.11}$$

we find that the stress amplitudes are related by

$$\begin{aligned}
\beta\hat{s}_{11} &= \frac{2ik\hat{u}_1}{R} - A\hat{u}_2 S_{11,2} + 2AU_{1,2}\hat{s}_{12} + 2A(ikS_{11}\hat{u}_1 + S_{12}D\hat{u}_1) \\
&\quad - \frac{2B}{R}\left[k^2\hat{u}_1(U_1 - c) + U_{1,2}(2D\hat{u}_1 + ik\hat{u}_2)\right] \\
\beta\hat{s}_{12} &= \frac{D\hat{u}_1 + ik\hat{u}_2}{R} - \frac{A\hat{u}_2 U_{1,22}}{R} + AU_{1,2}\hat{s}_{22} + ikAS_{11}\hat{u}_2 \\
&\quad + \frac{2B}{R}\left[\tfrac{1}{2}ik(D\hat{u}_1 + ik\hat{u}_2)(U_1 - c) + \tfrac{1}{2}U_{1,22}\hat{u}_2 - U_{1,2}D\hat{u}_2\right] \\
\beta\hat{s}_{22} &= \frac{2\beta_{,2}}{R}\hat{u}_2 + \frac{2D\hat{u}_2}{R} + \frac{2B}{R}\left[ikD\hat{u}_2(U_1 - c) - ikU_{1,2}\hat{u}_2\right]
\end{aligned} \tag{4.12}$$

where $D = d/dx_2$ is a differential operator.

4.5 Equations of Motion

Substituting $V_i = U_i + u_i$ into the continuity equation $V_{i,i} = 0$, and subtracting from it the continuity equation for the basic flow $U_{i,i} = 0$, we get

$$u_{i,i} = 0 \tag{4.13}$$

for the continuity equation of the disturbance. Similarly substituting (4.6) into the equation of motion (4.4), and subtracting the equation of motion of the basic flow from it, we get

$$u_{i,t} + u_k U_{i,k} + U_k u_{i,k} = - p_{,i} + s_{ij,j} \tag{4.14}$$

as the equation of motion of the disturbance. Introducing periodicity (4.10) into the continuity (4.13) and the two equations of motion (4.14), we get

$$ik\hat{u}_1 + D\hat{u}_2 = 0$$

$$ik\hat{u}_1(U_1 - c) + \hat{u}_2 U_{1\,2} = -ik\hat{p} + ik\hat{s}_{11} + D\hat{s}_{12} \tag{4.15}$$

$$ik\hat{u}_2(U_1 - c) = -D\hat{p} + ik\hat{s}_{12} + D\hat{s}_{22}$$

Substituting the stress amplitudes from (4.12) into the above, the two equations of motion finally become, after some tedious algebra which results from the coupling of the stress components in equation (4.12),

$$ikR\beta[-(U_1 - c)Dv + vU_{1,2}]$$

$$= k^2 \beta R\hat{p} - \{\gamma(D^3 v - k^2 Dv) + (2\beta_{,2} - \gamma_{,2})D^2 v + (\beta_{,22} + \gamma_{,22})Dv$$

$$+ 2\beta_{,2}(\beta_{,2} - \gamma_{,2})\,Dv\} + k^2 \gamma_{,2} v + \frac{\beta_{,2}}{\beta}\{\gamma(D^2 v - k^2 v) + 2(\beta_{,2} - \gamma_{,2})Dv$$

$$+ (\beta_{,22} - \gamma_{,22})\,v + 2v\beta_{,2}(\beta_{,2} - \gamma_{,2})\} + \frac{2\beta_{,22}\gamma}{\beta} Dv$$

$$- v(\beta_{,222} - \gamma_{,222}) + \frac{2\beta_{,22}(\beta_{,2} - \gamma_{,2})}{\beta}\, v \tag{4.16}$$

$$ik\beta R(U_1 - c)v = -\beta R D\hat{p} + \gamma(D^2 v - k^2 v) + 2(\beta_{,2} - \gamma_{,2})Dv$$

$$+ (\beta_{,22} - \gamma_{,22})v + 2v\beta_{,2}(\beta_{,22} - \gamma_{,22}) \tag{4.17}$$

where we have written u for $\hat{u}_1$, v for $\hat{u}_2$, and defined

$$\gamma(x_2) = 1 + ikB(U_1 - c) \tag{4.18}$$

4.6 Generalized Orr-Sommerfeld Equation

We now have to eliminate $\hat{p}$ between (4.16) and (4.17). To do this, divide (4.16) by $\beta(x_2)$, differentiate with respect to x_2, multiply it by β again, and add it to k^2 times (4.17). This yields, after some lengthy algebra, a generalized Orr-Sommerfeld equation

$$\begin{aligned} &ikR\beta[(U_1 - c)(D^2 v - k^2 v) - vU_{1,22}] \\ &= \gamma D^4 v + b_3 D^3 v + b_2 D^2 v + b_1 Dv + b_0 v \end{aligned} \tag{4.19}$$

where the coefficients are given by

$$b_3 = 2\beta_{,2}\left(1 - \frac{\gamma}{\beta}\right) \tag{4.20}$$

$$b_2 = -2\gamma k^2 + 3\beta_{,22}\left(1 - \frac{\gamma}{\beta}\right) + 2\beta_{,2}(\beta_{,2} - \gamma_{,2})\left(1 - \frac{1}{\beta}\right) + \frac{2\beta_{,2}^2}{\beta}\left(1 - \frac{\gamma}{\beta}\right) \tag{4.21}$$

$$\begin{aligned} b_1 = {} & 2\beta_{,222}\left(1 - \frac{\gamma}{\beta}\right) - 2k^2\beta_{,2}\left(1 - \frac{\gamma}{\beta}\right) + 4\beta_{,2}\left(1 - \frac{1}{\beta}\right)(\beta_{,22} - \gamma_{,22}) \\ & - \frac{4\beta_{,2}\beta_{,22}}{\beta}\left(1 - \frac{\gamma}{\beta}\right) - \frac{4\beta_{,2}^2}{\beta}\left(1 - \frac{1}{\beta}\right)(\beta_{,2} - \gamma_{,2}) \end{aligned} \tag{4.22}$$

$$\begin{aligned} b_0 = {} & \gamma k^4 + \beta_{,2222} - \gamma_{,2222} - k^2\beta_{,22}\left(1 - \frac{\gamma}{\beta}\right) - 2k^2\beta_{,2}(\beta_{,2} - \gamma_{,2}) \\ & + \frac{2k^2\beta_{,2}}{\beta}(\gamma_{,2} - \beta_{,2}\frac{\gamma}{\beta}) - \frac{4\beta_{,2}}{\beta}(\beta_{,222} - \gamma_{,222}) - \frac{3\beta_{,22}}{\beta}(\beta_{,22} - \gamma_{,22}) \\ & + \frac{4\beta_{,2}^3}{\beta^2}(\beta_{,2} - \gamma_{,2}) - \frac{6\beta_{,2}^2}{\beta}\left(1 - \frac{1}{\beta}\right)(\beta_{,22} - \gamma_{,22}) \end{aligned} \tag{4.23}$$

In the limiting case of the Newtonian fluid, we have $\lambda_1 = \lambda_2$, which means A = B and therefore $\beta = \gamma$; the four coefficients then simplify to

$$b_3 = 0$$

$$b_2 = -2\beta k^2$$

$$b_1 = 0$$

$$b_0 = \beta k^4$$

and (4.19) reduces to the Newtonian Orr-Sommerfeld equation

$$ikR[(U_1 - c)(D^2 v - k^2 v) - vU_{1,22}] = D^4 v - 2k^2 Dv + k^4 v$$

However, on substitution of $\gamma = 1$ (Maxwell fluid), our coefficients b_i turn out to be slightly different from those of Tlapa & Bernstein (1970); we however think that there is no algebraic mistake in our calculations.

4.7 Simplification for Small Relaxation Times

It has been verified by Tlapa & Bernstein (1968, 1970) that for values of the stress relaxation number $A = \lambda_1 U_m/\delta$ in the range of 0.0 to 0.1, the stability diagram is virtually unchanged if we neglect terms of order A^2 and higher in the generalized Orr-Sommerfeld equation. This way they achieved considerably greater speed in computation without sacrificing any accuracy.

It should be mentioned here that Tlapa & Bernstein (1968, 1970) have computed, on the basis of the experimental data of Virk et al (1967), that $A = \lambda_1 U_m/\delta$ is of order 10^{-2} at the onset of the Toms phenomenon. But an examination of the procedure of Tlapa & Bernstein (1968) revealed that it was erroneous; what they computed was $\lambda_1 U_\infty/D$ where D is the

diameter of the pipe and U_∞ is the average velocity in turbulent flow. However, D/U_∞ in turbulent flow has the significance of the time scale of the large eddies, which cannot have any fixed relation with the time scale λ_1 of the molecules.

The proper time scale here is that of the "wall flow" (all evidence indicates that the interaction producing the Toms phenomenon occurs in the wall region), which is ν/u_τ^2, where ν is the kinematic viscosity and u_τ is the friction velocity given by u_τ^2 = (wall shear stress)/ρ. For the two fluids that Tlapa & Bernstein consider, Virk et al (1967) give $\lambda_1 u_\tau^2/\nu = 0.256$ and 0.733. In considering the stability of a laminar motion, the time scale ratio should also be based on the wall shear stress. In a plane Poiseuille flow this gives $2\lambda_1 U_m/\delta = 0.256$ and 0.733. Thus, A is considerably larger than 10^{-2} at the onset of the Toms phenomenon.

In any case, in the present work we shall confine ourselves to values of A and B in the range 0.0 to 0.1, so that terms of order equal or higher than A^2, B^2, AB are negligible. Under this assumption the coefficients given by (4.20)-(4.23) simplify to

$$\begin{aligned} b_3 &\simeq 0 \\ b_2 &\simeq -2\gamma k^2 \\ b_1 &\simeq 0 \\ b_0 &\simeq \gamma k^4 \end{aligned} \tag{4.24}$$

and therefore (4.19) reduces to

$$ikRZ[(U_1-c)(D^2v-k^2v)-U_{1,22}v] = D^4v - 2k^2Dv + k^4v \tag{4.25}$$

where

$$Z(x_2) = \frac{\beta}{\gamma} = \frac{1+ikA(U_1-c)}{1+ikB(U_1-c)} \tag{4.26}$$

Equation (4.25) obviously reduces to the Newtonian Orr-Sommerfeld equation if Z = 1, and to the simplified Maxwellian Orr-Sommerfeld equation of Tlapa & Bernstein if Z = β.

Equation (4.25) is true for the stability of all two-dimensional parallel flows of the Oldroyd fluid. For the particular case of Poiseuille flow the basic flow is given by $U_1 = 2x_2 - x_2^2$.

The boundary conditions are that the disturbance velocity components $u_1 = u_2 = 0$ at the boundaries, which imply $v = Dv = 0$ at $x_2 = 0$ and $x_2 = 2$. However, since the velocity profile is an even function of x_2 about the line $x_2 = 1$, the disturbance can be separated into even and odd function parts. The former, which has a simpler flow pattern, usually gives a lower critical Reynolds number; hence the second boundary condition at $x_2 = 2$ is replaced by a condition at $x_2 = 1$ that $Dv = D^3v = 0$. The four boundary conditions are therefore

$$\begin{aligned} v = Dv &= 0 \quad \text{at} \quad x_2 = 0 \\ Dv = D^3v &= 0 \quad \text{at} \quad x_2 = 1 \end{aligned} \tag{4.27}$$

Equation (4.25) with the boundary conditions (4.27) define an eigenvalue problem; for given k, R, A and B a solution (eigenfunction) is possible only for proper values (eigenvalues) of c. For constant A and B, it is of interest to determine the minimum of R in a R-k diagram on which $c_i = 0$, which defines the margin between stability and instability; this is called the critical Reynolds number.

The eigenvalue problem will now be solved by the numerical technique developed by Nachtsheim (1964), who solved the same problem for a Newtonian fluid.

4.8 Initial Value Technique of Nachtsheim

The approach to the eigenvalue problem for fixed k, R, A and B is to find values of $c = c_r + ic_i$ for which (4.25) has solutions that satisfy the boundary conditions.

Trial solutions are obtained by step-by-step numerical integration of the differential equation for the assumed initial values and an assumed value of c. The proper initial values and c are determined by an iterative process that selects the one solution that satisfies the boundary conditions.

The solution is started at $x_2 = 0$ with the proper boundary values and then integrated forward. Another solution is started at $x_2 = 1$ with the proper boundary values and then integrated backward. Next it is necessary to perform a matching in the middle, say at $x_c = 0.5$.

For computational purposes the solution is carried out in terms of the disturbance vorticity amplitude and the x_2-velocity amplitude. A measure of the vorticity amplitude is the quantity s defined by $s = (D^2 - k^2)v$. To show this, note that the vorticity is given by

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial}{\partial x_1}\left[v(x_2)e^{ikx_1 - ikct}\right] - \frac{\partial}{\partial x_2}\left[u(x_2)e^{ikx_1 - ikct}\right]$$

$$= (ikv - Du)e^{ikx_1 - ikct}$$

The vorticity amplitude is therefore really $\hat{\omega} = ikv - Du$; the quality s is related to $\hat{\omega}$ through

$$s = D^2 v - k^2 v = ik(-Du + ikv) = ik\hat{\omega}$$

where the continuity equation $iku + Dv = 0$ has been used.

Instead of solving the fourth order equation (4.25), we can therefore solve a system of two second-order equations

$$\begin{aligned} v'' &= s + k^2 v \\ s'' &= k^2 s + ikRZ[(U_1 - c)s + 2v] \end{aligned} \tag{4.28}$$

where instead of D we are now using prime (') to denote differentiation with x_2; we have also used the condition $U_{1,22} = -2$ for Poiseuille flow. The boundary conditions are $v = v' = 0$ at $x_2 = 0$, and $v' = s' = 0$ at $x_2 = 1$. For details of the numerical procedure, see Nachtsheim (1964).

4.9 Equations in Real Form

The real differential equations are obtained by separating the original equations into real and imaginary parts. For example, (4.28) written in real form are as follows:

$$\begin{aligned} v_r'' &= s_r + k^2 v_r \\ v_i'' &= s_i + k^2 v_i \\ s_r'' &= k^2 s_r - kR\{Z_r[(U_1 - c_r)s_i - c_i s_r + 2v_i] + Z_i[(U_1 - c_r)s_r + c_i s_i + 2v_r]\} \\ s_i'' &= k^2 s_i + kR\{Z_r[(U_1 - c_r)s_r + c_i s_i + 2v_r] - Z_i[(U_1 - c_r)s_i - c_i s_r + 2v_i]\} \end{aligned} \tag{4.29}$$

The differential equations for the variation with respect to the

eigenvalue c_r for both the forward and backward solutions are needed in this method. Differentiating equations (4.29), these are

$$v''_{r,c_r} = s_{r,c_r} + k^2 v_{r,c_r}$$

$$v''_{i,c_r} = s_{i,c_r} + k^2 v_{i,c_r}$$

$$\begin{aligned} s''_{r,c_r} = k^2 s_{r,c_r} &- aR\{Z_r[(U_1-c_r)s_{i,c_r} - c_i s_{r,c_r} + 2v_{i,c_r} - s_i] \\ &+ Z_{r,c_r}[(U_1-c_r)s_i - c_i s_r + 2v_i] \\ &+ Z_i[(U_1-c_r)s_{r,c_r} + c_i s_{i,c_r} + 2v_{r,c_r} - s_r] \\ &+ Z_{i,c_r}[(U_1-c_r)s_r + c_i s_i + 2v_r]\} \end{aligned} \tag{4.30}$$

$$\begin{aligned} s''_{i,c_r} = k^2 s_{i,c_r} &+ aR\{Z_r[(U_1-c_r)s_{r,c_r} + c_i s_{i,c_r} + 2v_{r,c_r} - s_r] \\ &+ Z_{r,c_r}[(U_1-c_r)s_r + c_i s_i + 2v_r] \\ &- Z_i[(U_1-c_r)s_{i,c_r} - c_i s_{r,c_r} + 2v_{i,c_r} - s_i] \\ &- Z_{i,c_r}[(U_1-c_r)s_i - c_i s_r + 2v_i]\} \end{aligned}$$

In equations (4.29) and (4.30), the real and imaginary parts and their derivatives with c_r of the non-Newtonian function $Z(x_2)$ are

$$Z_r = \frac{(1+kAc_i)(1+kBc_i) + k^2 AB(U_1-c_r)^2}{(1+kBc_i)^2 + k^2 B^2 (U_1-c_r)^2}$$

$$Z_i = \frac{k(A-B)(U_1-c_r)}{(1+kBc_i)^2+k^2B^2(U_1-c_r)^2}$$

$$Z_{r,c_r} = \frac{-2k^2B(A-B)(U_1-c_r)(1+kBc_i)}{[(1+kBc_i)^2+k^2B^2(U_1-c_r)^2]^2} \qquad (4.31)$$

$$Z_{i,c_r} = -k(A-B)\frac{(1+kBc_i)^2-k^2B^2(U_1-c_r)^2}{[(1+kBc_i)^2+k^2B^2(U_1-c_r)^2]^2}$$

4.10 Results and Comparisons

The computer program developed by Nachtsheim for the Newtonian case was modified to take into account the presence of the non-Newtonian parameters A and B. The program has been stored in the computer library of The Pennsylvania State University.*

The computation was limited only to the region near the minimum critical numbers R_{cr}. The forward solutions (started at $x_2 = 0$) were matched with the backward solutions (started at $x_2 = 1$) at $x_2 = 0.5$. The range $x_2 = 0$ to 1 was divided into 128 steps for integration. Iterations were stopped when all values for two consecutive iterations agreed to four decimal places.

The Newtonian case (A = B) was programmed first to see whether the results agreed with those of Nachtsheim. The eigenvalues calculated at k = 1.00, 1.01, 1.02, 1.03 and 1.04 for R = 5780, 5770 and 5760 agreed

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*Off campus inquiries requesting source decks or other information may be directed to: Program Librarian, Computer Building, The Pennsylvania State University, University Park, Pennsylvania 16802.

almost exactly with those of Nachtsheim. Fig. 4.2a shows c_i plotted against k for various values of R; Fig. 4.2b gives c_i versus R corresponding to the peaks of Fig. 4.2a. Interpolation gives a minimum critical Reynolds number of R_{cr} = 5767 at k = 1.02, which agrees exactly with Nachtsheim's result.

Eigenvalues were also computed for five non-Newtonian liquids having (A,B) of (0.02, 0.01), (0.02, 0.0), (0.01, 0.0), (0.2, 0.1) and (0.1, 0.0). Figures 4.3-4.7 show the determination of the critical Reynolds number from these eigenvalues.

The critical Reynolds numbers were also computed for the same fluids by drawing the curves of neutral stability (c_i = 0). To draw these curves (Fig. 4.8), the values of c_i were written in a diagram of R vs. k, and points of c_i = 0 were found by interpolation. From Fig. 4.8 the following conclusions can be drawn:

(i) The presence of viscoelasticity destabilizes the flow.

(ii) Comparing the curves for (0.02, 0.01) with (0.02, 0.0), it seems that while the effect of stress relaxation alone is to destabilize the flow compared to a Newtonian fluid, the effect of strain rate relaxation at a fixed stress relaxation time is to stabilize the flow. This, of course, may be expected since the Newtonian results should be approached as $\lambda_2 \to \lambda_1$. However, it should be realized that the effect of strain rate relaxation alone cannot be compared to a Newtonian fluid since the inequality $\lambda_1 > \lambda_2$ must always be satisfied.

(iii) The curves of (0.02, 0.01) and (0.01, 0.0) coincide almost exactly, so that at these low values of A and B it is only (A-B) which is important, not A and B separately. This fact could also be brought

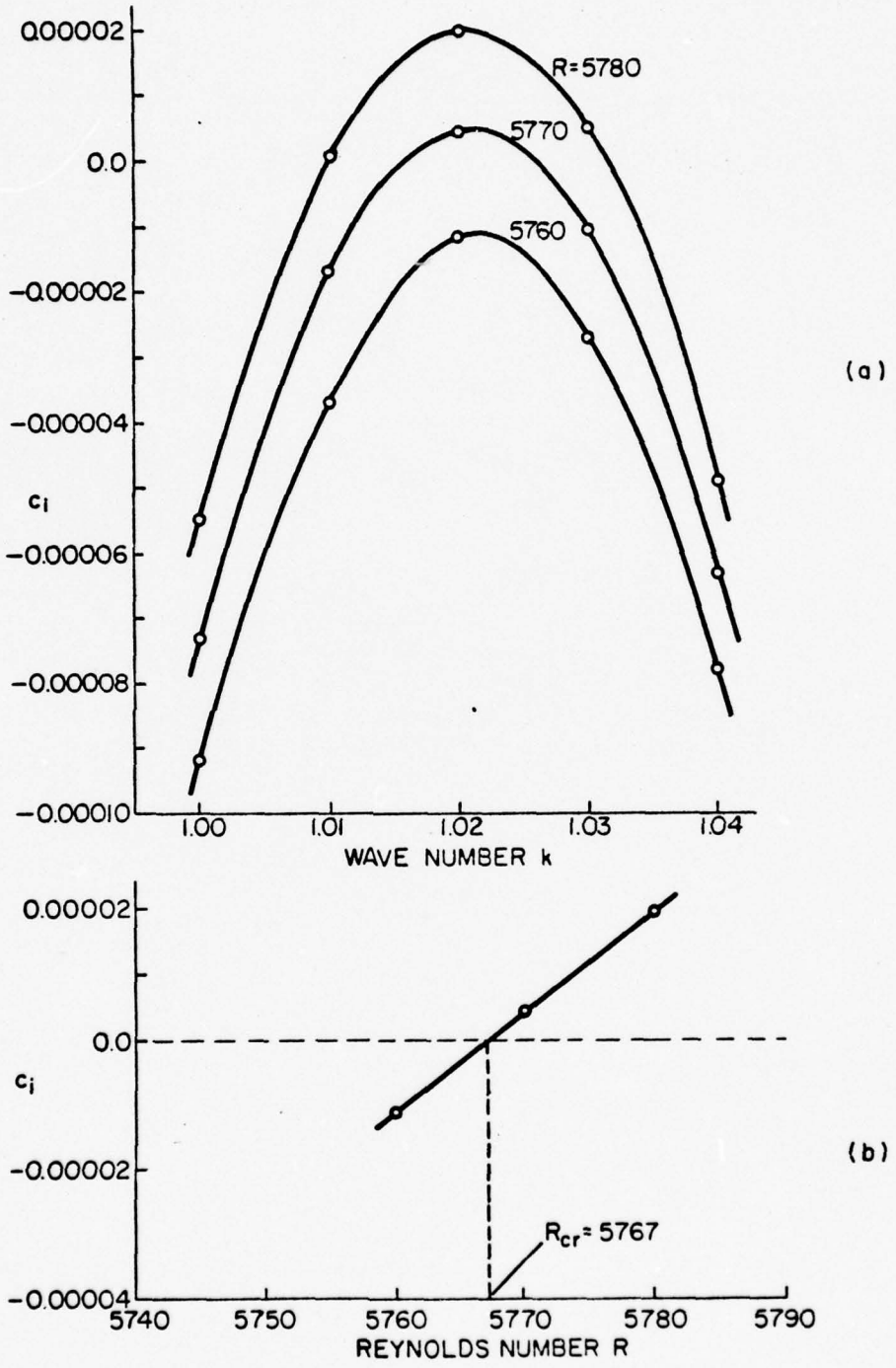


Figure 4.2 Determination of critical Reynolds number for the Newtonian case (A = B).

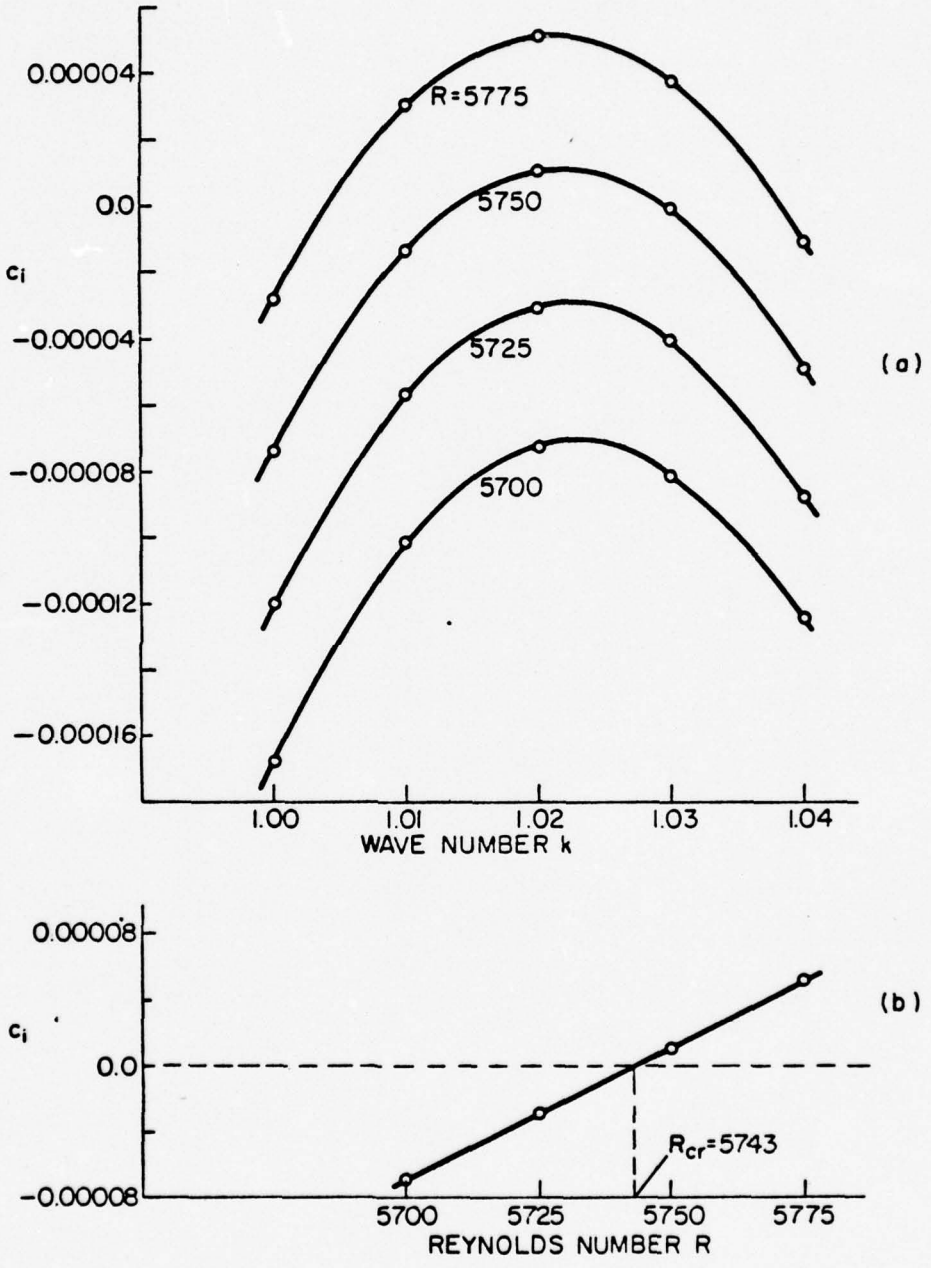


Figure 4.3 Determination of critical Reynolds number for A = 0.02, B = 0.01.

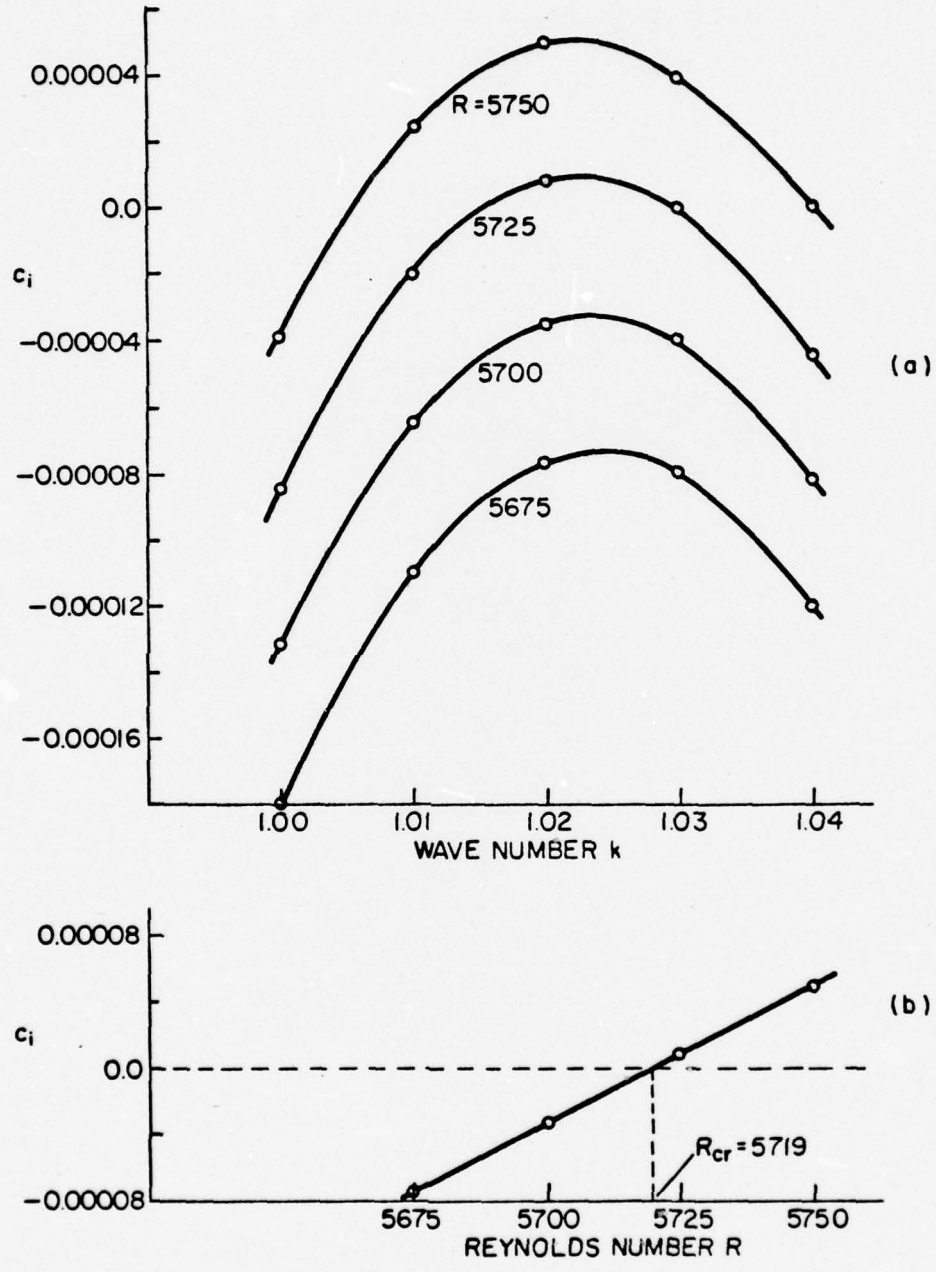


Figure 4.4 Determination of critical Reynolds number for A = 0.02, B = 0.0.

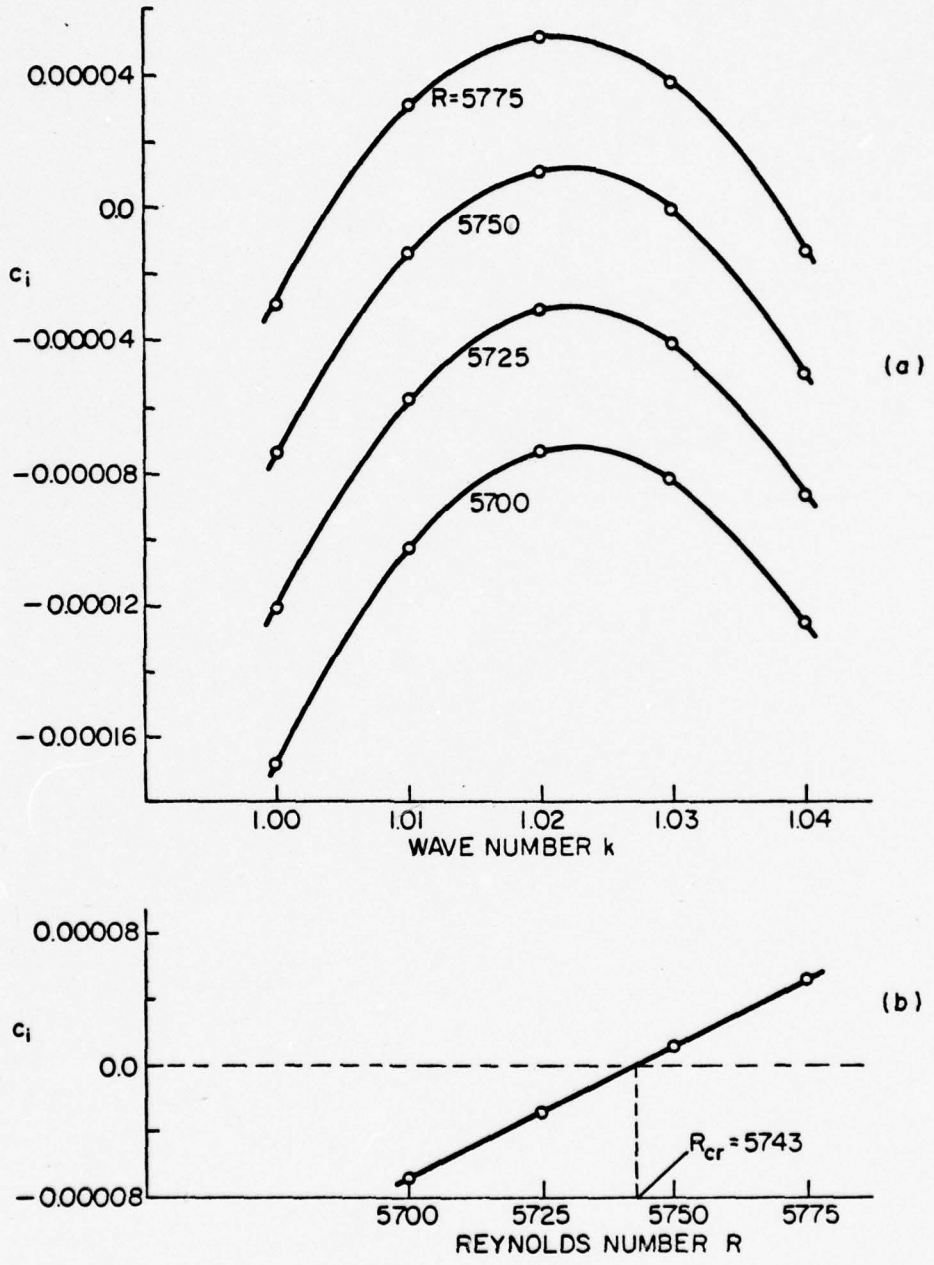


Figure 4.5 Determination of critical Reynolds number for A = 0.01, B = 0.0.

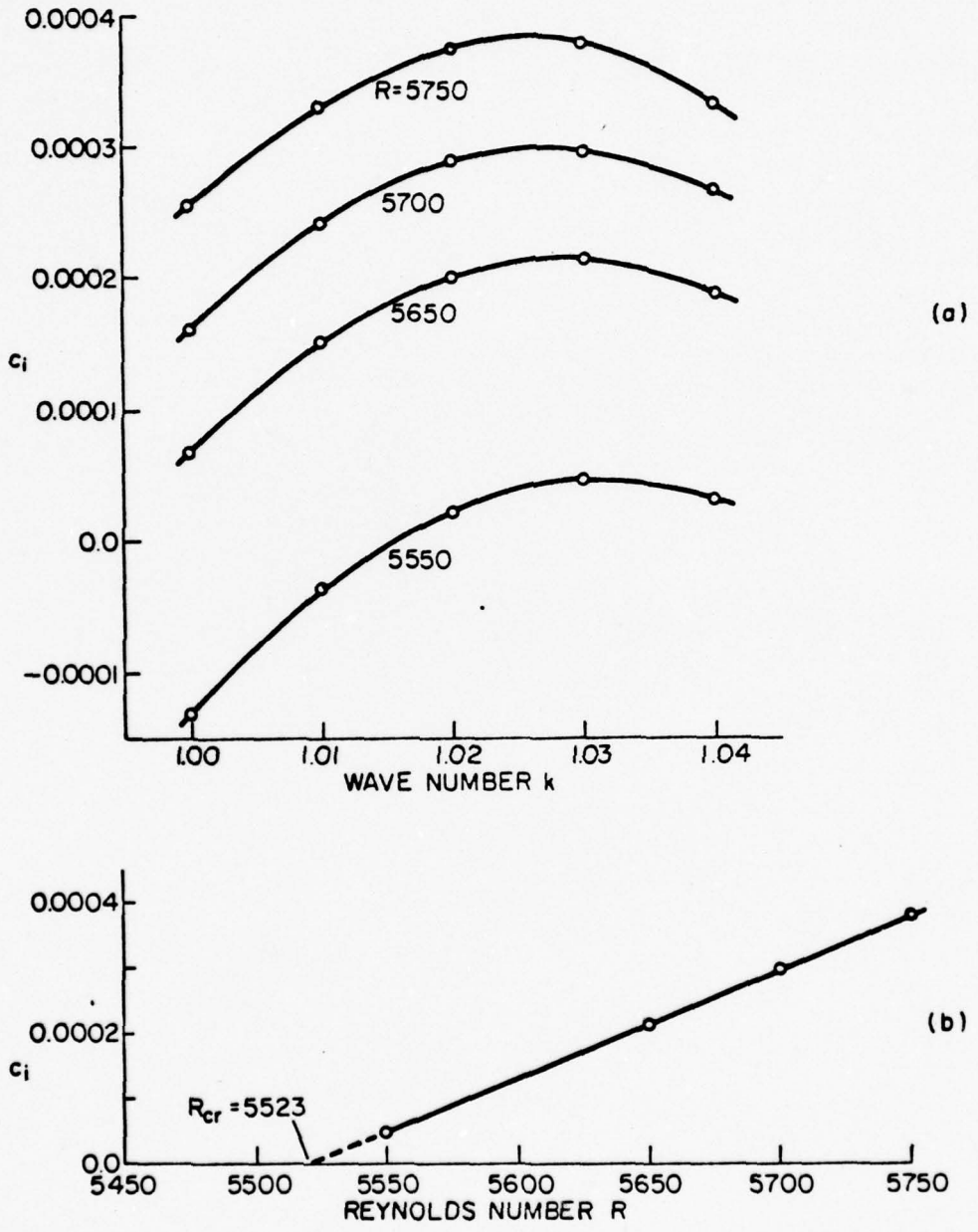


Figure 4.6 Determination of critical Reynolds number for A = 0.2, B = 0.1.

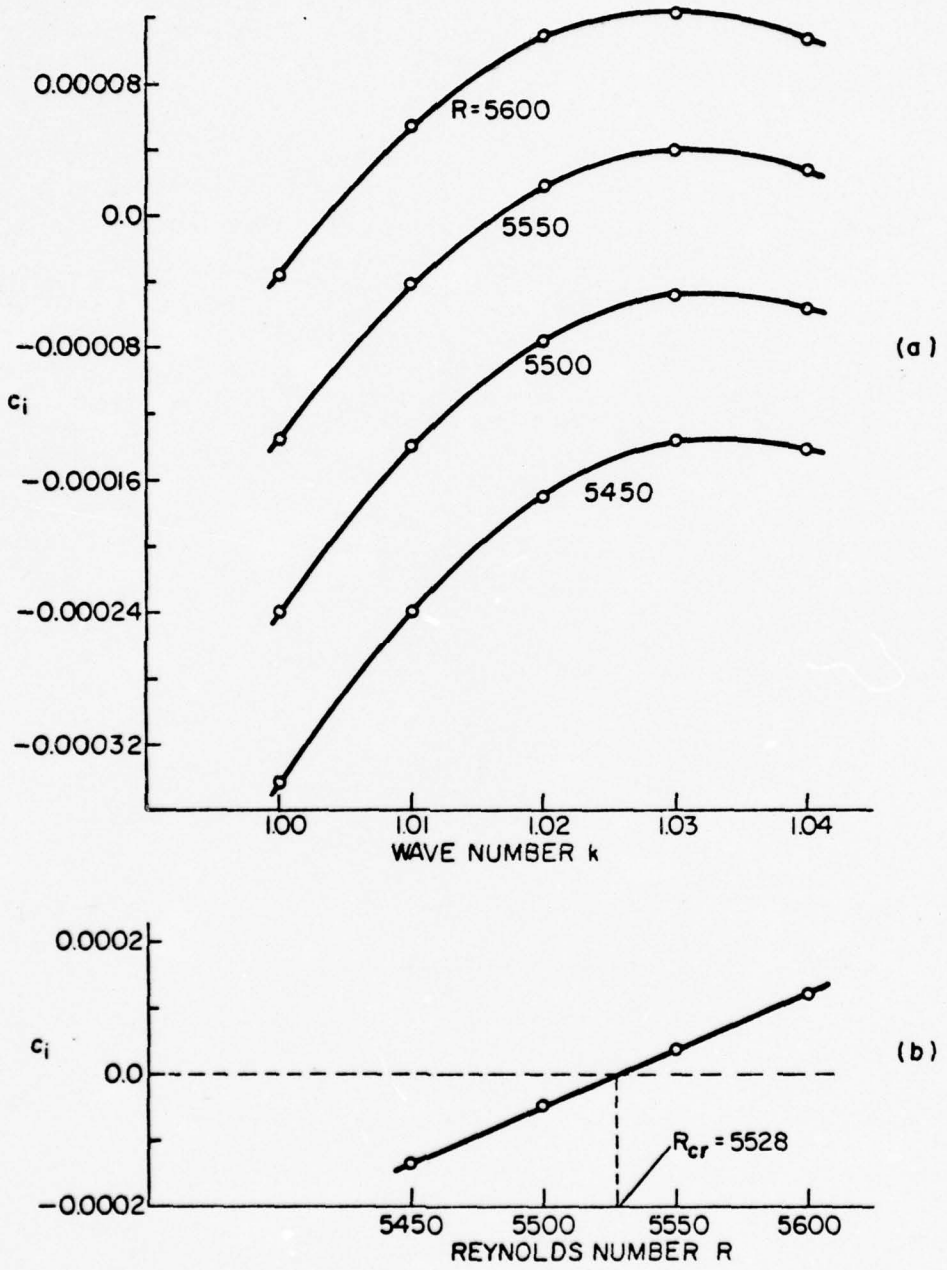


Figure 4.7 Determination of critical Reynolds number for A = 0.1, B = 0.0.

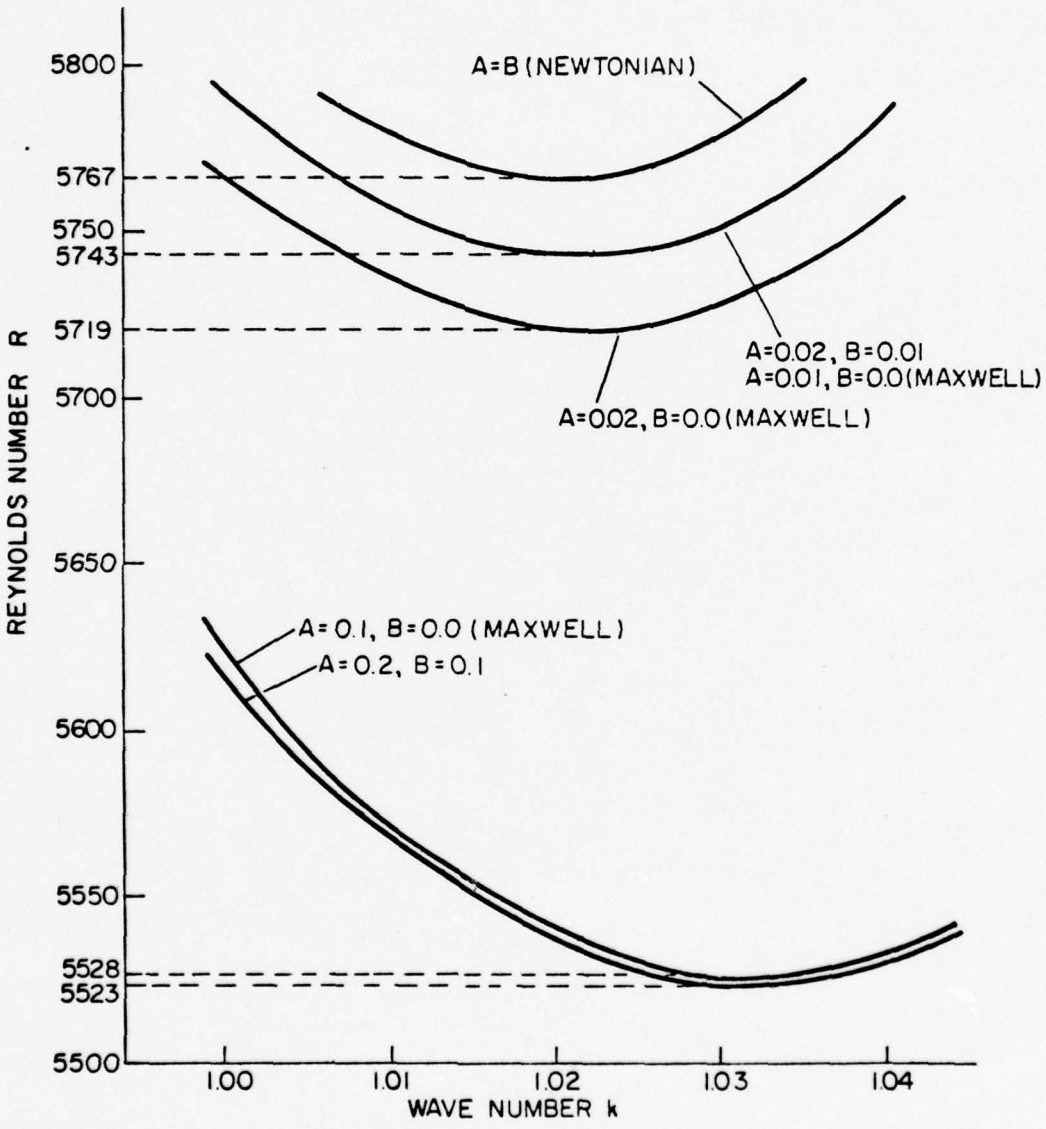


Figure 4.8 Neutral stability curves for various values of A and B.

out more explicitly if, consistent with our earlier linearization in A and B, we had also linearized equation (4.26) to read $Z = 1 + ikX(A-B)(U_1-c)$.

(iv) The critical wavenumber increases (that is, the waves become shorter) as the fluid elasticity is increased.

The conclusions are therefore same as that of the instability of the circular Couette flow of an Oldroyd fluid, treated in the next chapter.

It should be mentioned here that our results for the Maxwell fluid do not agree with those of Tlapa & Bernstein (1968, 1970). For example, their critical Reynolds numbers for A = 0.01 (B = 0) and A = 0.1 (B = 0) are reported [Tlapa & Bernstein (1968)] as 5767 and 5670 respectively, whereas our results for the same fluids are 5743 and 5528 respectively. An examination of their computer program [Tlapa & Bernstein (1968)] revealed that there are at least two errors in their program: (i) the presence of an extra factor of 2 in their Runge-Kutta integration formula [the 22nd card of their subroutine INTEGR version 2 should read CALL F(X+H, P, R, C3) instead of CALL F(X+H/2., P, R, C3)], and (ii) the absence of an exponent 2 in the denominator of their definition of variable DGR (the 25th card of their subroutine EQUTNS). These errors are also reflected in the small but significant inconsistencies between Fig. 2 of Tlapa & Bernstein (1968) [their Newtonian case] and Fig. 1 of Nachtsheim (1964). On the other hand our Fig. 4.2 agrees exactly with Nachtsheim's Fig. 1.

In addition to the above errors in Tlapa & Bernstein's work, there are other inconsistencies between our algebra and theirs, as already

mentioned in Section 4.6. However, these inconsistencies vanish for the simplified case of small A and B, for which their results as well as ours are valid.

4.11 Discussion

From the present as well as the other studies mentioned in Section 1.3(vi), it appears that all the stability investigations on a plane Poiseuille flow have predicted a destabilization due to elasticity, although the constitutive equations assumed display different types of normal stress behavior and the Weissenberg effect. This seems to be a result of the two-dimensionality of the problem, for it was demonstrated in Section 3.11 that in two dimensions the vorticity equations for the entire (disturbed) flow are identical for the second-order fluid and the Walters short memory fluids A' and B'. Although this fact cannot probably be demonstrated in general for the Oldroyd fluids (for which the stresses cannot be explicitly expressed in terms of the strain rates), it must be true at least for the small relaxation times assumed in the present study; this is because at small relaxation times the Oldroyd A and B fluids can be approximated by the Walters short memory fluids A' and B' respectively, as shown in Section 2.7.

CHAPTER 5

STABILITY OF CIRCULAR COUETTE FLOW OF OLDROYD FLUID

5.1 Introduction

The stability of circular Couette flow has always been a happy hunting ground of stability investigators. One reason is that analytical solutions can be obtained without a great deal of difficulty. Moreover the theoretical results can be easily compared with experiments for this geometry. From the discussion of Section 1.3 it can be seen that this problem has not been treated for the Oldroyd fluid. However we think that the Oldroyd constitutive equation is the most reliable of all the simple constitutive equations commonly used. For this reason the stability of a circular Couette flow of the Oldroyd fluid B is investigated in this chapter. The small disturbance theory is applied.

5.2 Some Properties of Cylindrical Coordinates and General Tensors

In this chapter we shall use the notation of general tensors, with covariant suffixes written below and contravariant suffixes above, and a comma shall denote a covariant derivative instead of simply a space derivative.

The cylindrical polar coordinates chosen are denoted by

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = z \tag{5.1}$$

whose metric tensors are

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.2}$$

The scale factors defined by $h_i = \sqrt{g_{ii}}$ (no sum) are therefore

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= 1 \end{aligned} \tag{5.3}$$

and the Christoffel tensors can be shown to be

$$\Gamma^1_{22} = -r \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^i_{jk} = 0 \text{ otherwise} \tag{5.4}$$

The physical components of a vector or tensrr will be noted by suffixes enclosed within parentheses. The rule for transforming a mixed second-order tensor b^i_j into its physical components $b_{(ij)}$ is given by

$$b_{(ij)} = \frac{h_i b^i_j}{h_j} \text{ (no sum)} \tag{5.5}$$

and the rules for transforming other types of tensors can be guessed from the above. Thus, for example, $b_{(i)} = h_i b^i = b_i/h_i$ for a vector. The covariant derivative of a mixed second-order tensor is

$$b^i_{j,k} = \frac{\partial b^i_j}{\partial x^k} + b^p_j \Gamma^i_{pk} - b^i_q \Gamma^q_{jk} \tag{5.6}$$

and the expressions for covariant derivatives of other types of tensors can be deduced from this.

All the above rules of tensor analysis will be used in the next few sections. For a simple treatment of these rules see, for example, the book of Aris (1962).

5.3 Governing Equations

The equations of continuity, motion and state are respectively

$$V^i_{,i} = 0 \tag{5.7}$$

$$\rho\left(\frac{\partial V^i}{\partial t} + V^j V^i_{,j}\right) = -\pi_{,j} g^{ij} + \sigma^{ij}_{\ ,j} \tag{5.8}$$

$$\sigma^{ij} + \lambda_1 \frac{\delta \sigma^{ij}}{\delta t} = 2\mu\left(\varepsilon^{ij} + \lambda_2 \frac{\delta \varepsilon^{ij}}{\delta t}\right) \tag{5.9}$$

Here V^i is the velocity, $\varepsilon_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i})$ is the strain rate, σ^{ij} is the extra stress, π is the pressure, ρ is the density, μ is the viscosity, λ_1 is the stress relaxation time and λ_2 is the strain rate relaxation time (also called 'retardation time'). The notation $\delta/\delta t$ denotes the convective derivative and is given by the rules (2.14), that is, operating on a mixed tensor b^k_i, it is given by

$$\frac{\delta b^k_i}{\delta t} = \frac{\partial b^k_i}{\partial t} + V^m b^k_{i,m} + V^m_{\ ,i} b^k_m - V^k_{\ ,m} b^m_i \tag{5.10}$$

Substituting the rule (5.6) for the covariant derivative into (5.10), it can be easily shown that the terms involving the Christoffel tensors vanish, giving simply

$$\frac{\delta b_i^k}{\delta t} = \frac{\partial b_i^k}{\partial t} + V^m \frac{\partial b_i^k}{\partial x^m} + \frac{\partial V^m}{\partial x^i} b_m^k - \frac{\partial V^k}{\partial x^m} b_i^m \qquad (5.10)'$$

in any coordinate system. The Oldroyd constitutive equation (5.9) therefore becomes

$$\begin{aligned} &\sigma^{ij} + \lambda_1 \left(\frac{\partial \sigma^{ij}}{\partial t} + V^k \frac{\partial \sigma^{ij}}{\partial x^k} - \frac{\partial V^i}{\partial x^k} \sigma^{kj} - \frac{\partial V^j}{\partial x^k} \sigma^{ik}\right) \\ &= 2\mu\varepsilon^{ij} + 2\mu\lambda_2 \left(\frac{\partial \varepsilon^{ij}}{\partial t} + V^k \frac{\partial \varepsilon^{ij}}{\partial x^k} - \frac{\partial V^i}{\partial x^k} \varepsilon^{kj} - \frac{\partial V^j}{\partial x^k} \varepsilon^{ik}\right) \end{aligned} \qquad (5.9)'$$

5.4 Solution of the Basic Flow

Let the disturbed flow be written as a steady basic flow plus a time dependent disturbance, assumed small:

$$\begin{aligned} V^i &= U^i + u^i \\ \pi &= P + p \\ \sigma^{ij} &= S^{ij} + s^{ij} \\ \varepsilon^{ij} &= E^{ij} + e^{ij} \end{aligned} \qquad (5.11)$$

Assume a basic flow with the physical components of velocity as

$$\begin{aligned} U_{(r)} &= 0 \\ U_{(\theta)} &= U(r) \\ U_{(z)} &= 0 \end{aligned} \qquad (5.12)$$

From these the covariant and contravariant components of velocity can be determined from the rule (5.5), giving

$$U_r = 0 \qquad U^r = 0$$

$$U_\theta = rU \qquad U^\theta = \frac{U}{r} \tag{5.13}$$

$$U_z = 0 \qquad U^z = 0$$

The covariant components of the strain rate tensors $E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ are then determined as

$$[E_{ij}] = \begin{bmatrix} 0 & \frac{r^2}{2}\frac{d}{dr}\left(\frac{U}{r}\right) & 0 \\ \frac{r^2}{2}\frac{d}{dr}\left(\frac{U}{r}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which the contravariant components $E^{ij} = g^{ik}g^{jm}E_{km}$ are found to be

$$[E^{ij}] = \begin{bmatrix} 0 & \frac{1}{2}\frac{d}{dr}\left(\frac{U}{r}\right) & 0 \\ \frac{1}{2}\frac{d}{dr}\left(\frac{U}{r}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.14}$$

Substitution of the above into the constitutive equation of the basic flow

$$S^{ij} - \lambda_1\left(\frac{\partial U^i}{\partial x^k}S^{kj} + \frac{\partial U^j}{\partial x^k}S^{ik}\right) = 2\mu E^{ij} - 2\mu\lambda_2\left(\frac{\partial U^i}{\partial x^k}E^{kj} + \frac{\partial U^j}{\partial x^k}E^{ik}\right)$$

yields the stress components

$$[S^{ij}] = \begin{bmatrix} 0 & \mu\frac{d}{dr}\left(\frac{U}{r}\right) & 0 \\ \mu\frac{d}{dr}\left(\frac{U}{r}\right) & 2\mu(\lambda_1 - \lambda_2)\left[\frac{d}{dr}\left(\frac{U}{r}\right)\right]^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.15}$$

Substitution of the above stress matrix into the equation of motion

$$0 = - P_{,j} g^{ij} + S^{ij}{}_{,j}$$

yields the velocity distribution

$$U = Er + \frac{F}{r} \tag{5.16}$$

where

$$E = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}$$
$$F = \frac{R_1^2 R_2^2(\Omega_1 - \Omega_2)}{R_2^2 - R_1^2} \tag{5.17}$$

Here R_1 and R_2 are the radii of the inner and outer cylinders respectively, and Ω_1 and Ω_2 are their angular velocities (Fig. 5.1).

5.5 Stress Change Due to Disturbance

Let us confine our discussion to the state of neutral stability so that $\partial/\partial t = 0$, and let the physical components of the velocity of the disturbed motion be

$$V_{(r)} = u$$
$$V_{(\theta)} = U + v \tag{5.18}$$
$$V_{(z)} = w$$

where u, v and w are functions of r and z. Substituting $\sigma^{ij} = S^{ij} + s^{ij}$ into the constitutive equation (5.9)' for the total motion, subtracting the constitutive equation of the basic motion, and neglecting the nonlinear terms we get

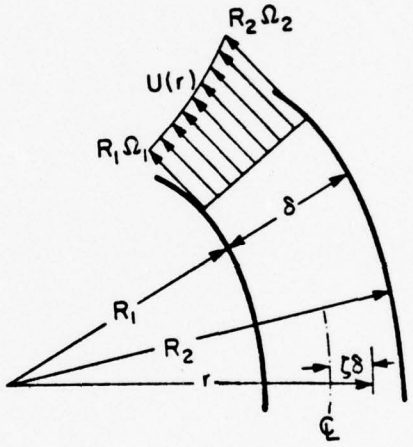


Figure 5.1 Circular Couette flow geometry.

$$s^{ij} + \lambda_1 [u^k \frac{\partial S^{ij}}{\partial x^k} - \frac{\partial u^i}{\partial x^k} S^{kj} - \frac{\partial U^i}{\partial x^k} s^{kj} - \frac{\partial u^j}{\partial x^k} S^{ik} - \frac{\partial U^j}{\partial x^k} s^{ik}]$$

$$= 2\mu e^{ij} + 2\mu\lambda_2 [u^k \frac{\partial E^{ij}}{\partial x^k} - \frac{\partial u^i}{\partial x^k} E^{kj} - \frac{\partial U^i}{\partial x^k} e^{kj} - \frac{\partial u^j}{\partial x^k} E^{ki} - \frac{\partial U^j}{\partial x^k} e^{ik}] \qquad (5.19)$$

From this equation the various components of the disturbance extra stress are found as

$$s^{11} = 2\mu \frac{\partial u}{\partial r}$$

$$s^{12} = \mu \frac{\partial}{\partial r}(\frac{v}{r}) - \frac{6F\mu(\lambda_1-\lambda_2)}{r^3}(\frac{\partial u}{\partial r} + \frac{u}{r})$$

$$s^{13} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r})$$

$$s^{22} = \frac{2\mu u}{r^3} - \frac{8\mu(\lambda_1-\lambda_2)F}{r^3}\frac{\partial}{\partial r}(\frac{v}{r}) + \frac{24F^2\mu\lambda_1(\lambda_1-\lambda_2)}{r^6}(\frac{\partial u}{\partial r} + \frac{3u}{r}) \qquad (5.20)$$

$$s^{23} = \frac{\mu}{r}\frac{\partial v}{\partial z} - \frac{2\mu(\lambda_1-\lambda_2)F}{r^3}(\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial r})$$

$$s^{33} = 2\mu \frac{\partial w}{\partial z}$$

5.6 Equations of Motion for the Disturbance

The equation of continuity for the disturbance is $u^i{}_{,i} = 0$, which becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad (5.21)$$

Subtracting the equation of motion for the basic flow from that of the disturbed total flow, neglecting the nonlinear terms and setting $\partial/\partial t=0$ for neutral stability we get the equation of motion of the disturbance:

$$\rho(U^j u^i{}_{,j} + u^j U^i{}_{,j}) = - p_{,j} g^{ij} + s^{ij}{}_{,j} \tag{5.22}$$

The three components of the above equation give

$$-2\rho(E + \frac{F}{r^2})v = - \frac{\partial p}{\partial r} + \frac{\partial s^{11}}{\partial r} + \frac{s^{11}}{r} + \frac{\partial s^{13}}{\partial z} - rs^{22} \tag{5.23}$$

$$\frac{2\rho Eu}{r} = \frac{\partial s^{12}}{\partial r} + \frac{3s^{12}}{r} + \frac{\partial s^{23}}{\partial z} \tag{5.24}$$

$$0 = - \frac{\partial p}{\partial z} + \frac{\partial s^{13}}{\partial r} + \frac{s^{13}}{r} + \frac{\partial s^{33}}{\partial z} \tag{5.25}$$

Using the expressions (5.20) for the disturbance stresses, the above equations respectively reduce to

$$-2\rho(E + \frac{F}{r^2})\, v = - \frac{\partial p}{\partial r} + \mu[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}]$$

$$+ \frac{8\mu(\lambda_1-\lambda_2)F}{r^2} \frac{\partial}{\partial r}(\frac{v}{r}) - \frac{24F^2\mu\lambda_1(\lambda_1-\lambda_2)}{r^5}(\frac{\partial u}{\partial r} + \frac{3u}{r}) \tag{5.23}'$$

$$2\rho Eu = \mu[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}] - \frac{2\mu(\lambda_1-\lambda_2)F}{r^2}[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 v}{\partial z^2}] \tag{5.24}'$$

$$0 = - \frac{\partial p}{\partial z} + \mu[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}] \tag{5.25}'$$

To simplify the analysis we shall make the usual assumption that the gap δ between the cylinders is small compared with their radii. We now introduce the nondimensional radial coordinate measured from the center of the channel (see Fig. 5.1)

$$\zeta = \frac{r-R_1}{\delta} - \tfrac{1}{2} \tag{5.26}$$

and the kinematic parameter

$$a = \frac{\Omega_2}{\Omega_1} - 1 \tag{5.27}$$

Assuming that $\frac{\delta}{r}$ is small, it can be shown from the expression (5.17) for the constants in the basic velocity distribution that

$$E = \frac{R_1 a\Omega_1}{2\delta}$$

$$F = -\frac{R_1^3 a\Omega_1}{2\delta}$$

$$E + \frac{F}{r^2} = \Omega_1[1 + a(\zeta + \tfrac{1}{2})]$$

Under this small gap approximation, equations (5.21), (5.23)'-(5.25)' become

$$\frac{1}{\delta}\frac{\partial u}{\partial \zeta} + \frac{\partial w}{\partial z} = 0 \tag{5.28}$$

$$-2\rho\Omega_1[1+a(\zeta+\tfrac{1}{2})]v = -\frac{1}{\delta}\frac{\partial p}{\partial \zeta} + \mu\left(\frac{1}{\delta^2}\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{4\mu(\lambda_1-\lambda_2)\Omega_1 a}{\delta^2}\frac{\partial v}{\partial \zeta} - \frac{6\mu\lambda_1(\lambda_1-\lambda_2)\Omega_1^2 R_1 a^2}{\delta^3}\frac{\partial u}{\partial \zeta} \tag{5.29}$$

$$\frac{\rho R_1\Omega_1 a u}{\delta} = \mu\left(\frac{1}{\delta^2}\frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{\mu(\lambda_1-\lambda_2)R_1\Omega_1 a}{\delta}\left(\frac{1}{\delta^2}\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{5.30}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{\delta^2}\frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{5.31}$$

5.7 Stability Equations

Now assume that the disturbances are periodic in the z-direction with wavenumber α

$$\begin{aligned} u &= \hat{u}(\zeta)\sin\alpha z \\ v &= \hat{v}(\zeta)\sin\alpha z \\ w &= \hat{w}(\zeta)\cos\alpha z \\ p &= \hat{p}(\zeta)\sin\alpha z \end{aligned} \tag{5.32}$$

Introducing the nondimensional wavenumber

$$k = \alpha\delta \tag{5.33}$$

the equation of continuity (5.28) gives

$$\hat{w} = \frac{1}{k} D\hat{u} \tag{5.34}$$

and equation (5.31) gives the pressure

$$\hat{p} = \frac{\mu}{\delta k^2} D(D^2-k^2)u \tag{5.35}$$

where $D = d/d\zeta$. The remaining equations (5.29) and (5.30) then become

$$\begin{aligned} \frac{2\rho\Omega_1\delta^2k^2}{\mu}[1+a(\zeta+\tfrac{1}{2})]\hat{v} &= (D^2-k^2)^2\hat{u} + 4(\lambda_1-\lambda_2)\Omega_1 ak^2 D\hat{v} \\ &+ \frac{6\lambda_1(\lambda_1-\lambda_2)\Omega_1^2R_1a^2k^2}{\delta} D\hat{u} \end{aligned} \tag{5.36}$$

$$\rho R_1\Omega_1 a\hat{u} = \frac{\mu}{\delta}(D^2-k^2)\hat{v} + \frac{\mu(\lambda_1-\lambda_2)R_1\Omega_1 a}{\delta^2}(D^2-k^2)\hat{u} \tag{5.37}$$

Making the transformation

$$\hat{v} \rightarrow \frac{2\rho\Omega_1\delta^2k^2}{\mu}\hat{v}$$

equations (5.36) and (5.37) finally become

$$[1+a(\zeta+\tfrac{1}{2})]\hat{v} = (D^2-k^2)^2\hat{u}+2(A-B)aD\hat{v}-3A(A-B)ak^2TD\hat{u} \tag{5.38}$$

$$(D^2-k^2)\hat{v} = -Tk^2[\hat{u} - (A-B)(D^2-k^2)\hat{u}] \tag{5.39}$$

where the following nondimensional numbers have been defined

$$T = \text{Taylor number} = -\frac{2a\Omega_1{}^2\delta^3R_1\rho^2}{\mu^2}$$

$$A = \text{Nondimensional relaxation time} = \frac{\mu\lambda_1}{\rho\delta^2} \tag{5.40}$$

$$B = \text{Nondimensional retardation time} = \frac{\mu\lambda_2}{\rho\delta^2}$$

The boundary conditions are

$$\hat{v} = \hat{u} = D\hat{u} = 0 \qquad \text{at} \quad \zeta = \pm\tfrac{1}{2} \tag{5.41}$$

Equations (5.38) and (5.39) subject to (5.41) constitute a characteristic value problem; that is, a nontrivial solution is possible only if T varies with k in a special manner (for given A, B, and a).

Equations (5.38) and (5.39) were also derived by Thomas & Walters (1963) for the case of Walters fluid B', given by equation (2.25), of which the Oldroyd fluid B is a special case. However, the general solution was not given.

5.8 Solution of the Characteristic Value Problem

The characteristic value problem will now be solved as follows: assume first an expansion for $\hat{u}$ in terms of a set of orthogonal functions which automatically satisfy the four boundary conditions $\hat{u} = D\hat{u} = 0$ at $\zeta = \pm\frac{1}{2}$. This expansion is then substituted into (5.39), which then can

be solved for $\hat{v}$, subject to the remaining two boundary conditions on $\hat{v}$. These results are finally substituted into (5.38), from which we obtain that a secular determinant of infinite order must vanish in order to have nontrivial solutions.

The crucial point in this method lies in the choice of the initial approximation for $\hat{u}$. A good choice of the orthogonal functions used in the expansion of $\hat{u}$ will mean a smaller size of the secular determinant that must be taken in order to achieve a desired accuracy. Chandrasekhar & Reid (1957) [see also the book of Chandrasekhar (1961), p. 634] suggested the use of the orthogonal functions

$$C_n(\zeta) = \frac{\cosh\lambda_n\zeta}{\cosh\tfrac{1}{2}\lambda_n} - \frac{\cos\lambda_n\zeta}{\cos\tfrac{1}{2}\lambda_n}$$

$$S_n(\zeta) = \frac{\sinh\mu_n\zeta}{\sinh\tfrac{1}{2}\mu_n} - \frac{\sin\mu_n\zeta}{\sin\tfrac{1}{2}\mu_n} \qquad (5.42)$$

where λ_n and μ_n are the positive roots of the equations

$$\tanh\frac{\lambda}{2} + \tan\tfrac{1}{2}\lambda = 0$$

$$\coth\tfrac{1}{2}\mu - \cot\tfrac{1}{2}\mu = 0 \qquad (5.43)$$

The functions have the orthogonality properties

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(\zeta)C_n(\zeta)d\zeta = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_m(\zeta)S_n(\zeta)d\zeta = \delta_{mn}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(\zeta)S_n(\zeta)d\zeta = 0 \qquad (5.44)$$

They form, therefore, a complete set of orthogonal functions in the interval $(-\frac{1}{2}, \frac{1}{2})$.

These functions have been used by Reid (1958) for solving the stability of a Newtonian fluid flowing through a curved channel due to a pressure gradient; Thomas & Walters (1963) extended that work to a Maxwell fluid. Thomas & Walters (1964) also applied these functions for solving the instability of circular Couette flow of a Maxwell fluid. We shall apply the method here to the Oldroyd fluid. It may be mentioned here that since Thomas & Walters (1964) omitted all the analytical steps and reported only the resulting graphs, we shall not have a check until we arrive at the final numerical results.

We therefore assume a general expansion

$$\hat{u} = \sum_{m=1}^{\infty} [A_m C_m(\zeta) + B_m S_m(\zeta)] \tag{5.45}$$

Substituting this in (5.39) we get

$$(D^2 - k^2)\hat{v} = -Tk^2 \sum_{m=1}^{\infty} \{A_m[C_m - (A-B)(C_m'' - k^2 C_m)] + B_m[S_m - (A-B)(S_m'' - k^2 S_m)]\} \tag{5.46}$$

which has to be solved subject to $\hat{v} = 0$ at $\zeta = \pm \frac{1}{2}$. The homogeneous solution is obviously

$$\hat{v}_H = K_1 \cosh k\zeta + K_2 \sinh k\zeta$$

If a particular solution is assumed as

$$\hat{v}_p = \sum_{m=1}^{\infty} \{E_m C_m + F_m C_m' + G_m C_m'' + H_m C_m''' + e_m S_m + f_m S_m' + g_m S_m'' + h_m S_m'''\}$$

then substitution into the differential equation and comparison of coefficients gives ($k \neq \lambda_m$ or μ_m)

$$E_m = - \frac{Tk^2 A_m}{\lambda_m^4 - k^4} \left[k^2 - (A-B)\left(\lambda_m^4 - k^4\right)\right]$$

$$F_m = 0$$

$$G_m = - \frac{Tk^2 A_m}{\lambda_m^4 - k^4}$$

$$H_m = 0$$

$$e_m = - \frac{Tk^2 B_m}{\mu_m^4 - k^4} \left[k^2 - (A-B)\left(\mu_m^4 - k^4\right)\right]$$

$$f_m = 0$$

$$g_m = - \frac{Tk^2 B_m}{\mu_m^4 - k^4}$$

$$h_m = 0$$

The total solution is therefore

$$\hat{v} = K_1 \cosh k\zeta + K_2 \sinh k\zeta - Tk^2 \sum_{m=1}^{\infty} \frac{A_m}{\lambda_m^4 - k^4} \left\{C_m\left[k^2 - (A-B)\left(\lambda_m^4 - k^4\right)\right] + C_m''\right\} - Tk^2 \sum_{m=1}^{\infty} \frac{B_m}{\mu_m^4 - k^4} \left\{S_m\left[k^2 - (A-B)\left(\mu_m^4 - k^4\right)\right] + S_m''\right\}$$

The constants K_1 and K_2 are now determined from the boundary conditions $\hat{v} = 0$ at $\zeta = \pm\frac{1}{2}$. The solution finally becomes

$$\hat{v} = Tk^2 \sum_{m=1}^{\infty} \frac{A_m}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2}{\cosh \tfrac{1}{2}k} \cosh k\zeta - C_m [k^2 - (A-B)(\lambda_m^4 - k^4)] - C_m'' \right\}$$

$$+ Tk^2 \sum_{m=1}^{\infty} \frac{B_m}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2}{\sinh \tfrac{1}{2}k} \sinh k\zeta - S_m [k^2 - (A-B)(\mu_m^4 - k^4)] - S_m'' \right\} \tag{5.47}$$

Substituting the expressions for $\hat{u}$ and $\hat{v}$ given by (5.45) and (5.47) into equation (5.38) we get

$$(1 + \tfrac{1}{2}a + a\zeta) Tk^2 \sum_{m=1}^{\infty} \left[\frac{A_m}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2}{\cosh \tfrac{1}{2}k} \cosh k\zeta - C_m [k^2 - (A-B)(\lambda_m^4 - k^4)] - C_m'' \right\} \right.$$

$$\left. + \frac{B_m}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2}{\sinh \tfrac{1}{2}k} \sinh k\zeta - S_m [k^2 - (A-B)(\mu_m^4 - k^4)] - S_m'' \right\} \right]$$

$$= \sum_{m=1}^{\infty} \left[A_m \{ (\lambda_m^4 + k^4) C_m - 2k^2 C_m'' \} + B_m \{ (\mu_m^4 + k^4) S_m - 2k^2 S_m'' \} \right.$$

$$- 3A(A-B)aTk^2 (A_m C_m' + B_m S_m')$$

$$+ 2(A-B)aTk^2 \left(\frac{A_m}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2 k}{\cosh \tfrac{1}{2}k} \sinh k\zeta - C_m' [k^2 - (A-B)(\lambda_m^4 - k^4)] - C_m''' \right\} \right.$$

$$\left. \left. + \frac{B_m}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2 k}{\sinh \tfrac{1}{2}k} \cosh k\zeta - S_m' [k^2 - (A-B)(\mu_m^4 - k^4)] - S_m''' \right\} \right) \right] \tag{5.48}$$

Now multiply the above equation by $C_n(\zeta)$ and $S_n(\zeta)$ in turn, integrate between $\zeta = -\tfrac{1}{2}$ to $\tfrac{1}{2}$, and use the orthogonality properties (5.44). The following two algebraic equations result:

$$\sum_{m=1}^{\infty} A_m \left[\frac{1+\frac{1}{2}a}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2}{\cosh \frac{1}{2}k} P - [k^2 - (A-B)(\lambda_m^4 - k^4)]\delta_{mn} - X_{mn} \right\} \right.$$

$$\left. - \frac{1}{Tk^2} \left\{ (\lambda_m^4 + k^4)\delta_{mn} - 2k^2 X_{mn} \right\} \right]$$

$$+ \sum_{m=1}^{\infty} B_m \left[\frac{a}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2}{\sinh \frac{1}{2}k} U - [k^2 - (A-B)(\mu_m^4 - k^4)] J_{01} - J_{21} \right\} \right.$$

$$\left. + 3A(A-B)aI_{10} - \frac{2(A-B)a}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2 k}{\sinh \frac{1}{2}k} P - [k^2 - (A-B)(\mu_m^4 - k^4)I_{10} - I_{30} \right\} \right] = 0$$

(5.49)

and

$$\sum_{m=1}^{\infty} A_m \left[\frac{a}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2}{\cosh \frac{1}{2}k} V - [k^2 - (A-B)(\lambda_m^4 - k^4)] I_{01} - I_{21} \right\} + 3A(A-B)aJ_{10} \right.$$

$$\left. - \frac{2(A-B)a}{\lambda_m^4 - k^4} \left\{ \frac{2\lambda_m^2 k}{\cosh \frac{1}{2}k} Q - [k^2 - (A-B)(\lambda_m^4 - k^4)] J_{10} - J_{30} \right\} \right]$$

$$+ \sum_{m=1}^{\infty} B_m \left[\frac{1 + \frac{1}{2}a}{\mu_m^4 - k^4} \left\{ \frac{2\mu_m^2}{\sinh \frac{1}{2}k} Q - [k^2 - (A-B)(\mu_m^4 - k^4)]\delta_{mn} - Y_{mn} \right\} \right.$$

$$\left. - \frac{1}{Tk^2} \left\{ (\mu_m^4 - k^4)\delta_{mn} - 2k^2 Y_{mn} \right\} \right] = 0 \qquad (5.50)$$

where we have defined the following functions of k:

$$P(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh k\zeta C_n d\zeta \qquad Q(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sinh k\zeta S_n d\zeta$$

$$U(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta \sinh k\zeta C_n d\zeta \qquad V(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta \cosh k\zeta S_n d\zeta$$

and the following constants independent of k

$$I_{01} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta C_m S_n d\zeta \qquad J_{01} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta S_m C_n d\zeta$$

$$I_{21} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta C_m'' S_n d\zeta \qquad J_{21} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta S_m'' C_n d\zeta$$

$$I_{10} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_m' C_n d\zeta \qquad J_{10} = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_m' S_n d\zeta$$

$$I_{30} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_m''' C_n d\zeta \qquad J_{30} = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_m''' S_n d\zeta$$

$$X_{mn} = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_m'' C_n d\zeta \qquad Y_{mn} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_m'' S_n d\zeta$$

(Note that the symbols for these integrals are not really meaningful or systematic; some of them correspond to those used by earlier workers.) The integrals P(k), Q(k), U(k) and V(k) can be evaluated explicitly with the following results:

$$P(k) = \frac{1}{\cosh \frac{1}{2}\lambda_n} \left[\frac{\sinh \frac{1}{2}(\lambda_n - k)}{\lambda_n - k} + \frac{\sinh \frac{1}{2}(\lambda_n + k)}{\lambda_n + k} \right]$$

$$- \frac{2}{\cos \frac{1}{2}\lambda_n} \left[\frac{k \sinh \frac{1}{2}k \cos \frac{1}{2}\lambda_n + \lambda_n \cosh \frac{1}{2}k \sin \frac{1}{2}\lambda_n}{\lambda_n^2 + k^2} \right] \qquad (5.51)$$

$$Q(k) = \frac{1}{\sinh\tfrac{1}{2}\mu_n}\left[\frac{\sinh\tfrac{1}{2}(\mu_n+k)}{\mu_n+k} - \frac{\sinh\tfrac{1}{2}(\mu_n-k)}{\mu_n-k}\right]$$

$$- \frac{2}{\sin\tfrac{1}{2}\mu_n}\left[\frac{k\cosh\tfrac{1}{2}k\sin\tfrac{1}{2}\mu_n - \mu_n\sinh\tfrac{1}{2}k\cos\tfrac{1}{2}\mu_n}{\mu_n^2 + k^2}\right] \qquad (5.52)$$

$$U(k) = \frac{1}{\cosh\tfrac{1}{2}\lambda_n}\left[\frac{\cosh\tfrac{1}{2}(k+\lambda_n)}{2(k+\lambda_n)} + \frac{\cosh\tfrac{1}{2}(k-\lambda_n)}{2(k-\lambda_n)}\right.$$

$$\left. - \frac{\sinh\tfrac{1}{2}(k+\lambda_n)}{(k+\lambda_n)^2} - \frac{\sinh\tfrac{1}{2}(k-\lambda_n)}{(k-\lambda_n)^2}\right]$$

$$+ \frac{1}{\cos\tfrac{1}{2}\lambda_n}\left[\frac{2(k^2-\lambda_n^2)\cos\tfrac{1}{2}\lambda_n\sinh\tfrac{1}{2}k + 4k\lambda_n\sin\tfrac{1}{2}\lambda_n\cosh\tfrac{1}{2}k}{(k^2+\lambda_n^2)^2}\right.$$

$$\left. - \frac{k\cos\tfrac{1}{2}\lambda_n\cosh\tfrac{1}{2}k + \lambda_n\sin\tfrac{1}{2}\lambda_n\sinh\tfrac{1}{2}k}{k^2+\lambda_n^2}\right] \qquad (5.53)$$

$$V(k) = \frac{1}{\sinh\tfrac{1}{2}\mu_n}\left[\frac{\cosh\tfrac{1}{2}(\mu_n+k)}{2(\mu_n+k)} + \frac{\cosh\tfrac{1}{2}(\mu_n-k)}{2(\mu_n-k)}\right.$$

$$\left. - \frac{\sinh\tfrac{1}{2}(\mu_n+k)}{(\mu_n+k)^2} - \frac{\sinh\tfrac{1}{2}(\mu_n-k)}{(\mu_n-k)^2}\right]$$

$$+ \frac{1}{\sin\tfrac{1}{2}\mu_n}\left[\frac{2(k^2-\mu_n^2)\sin\tfrac{1}{2}\mu_n\cosh\tfrac{1}{2}k - 4k\mu_n\cos\tfrac{1}{2}\mu_n\sinh\tfrac{1}{2}k}{(k^2+\mu_n^2)^2}\right.$$

$$\left. - \frac{k\sin\tfrac{1}{2}\mu_n\sinh\tfrac{1}{2}k - \mu_n\cos\tfrac{1}{2}\mu_n\cosh\tfrac{1}{2}k}{k^2+\mu_n^2}\right] \qquad (5.54)$$

Equations (5.49) and (5.50) can be written in the abbreviated form

$$\sum_{m=1}^{\infty} \left\{ A_m \left[A_{mn} - \frac{1}{Tk^2} B_{mn} \right] + B_m C_{mn} \right\} = 0 \qquad (5.49)'$$

$$\sum_{m=1}^{\infty} \left\{ A_m D_{mn} + B_m \left[E_{mn} - \frac{1}{Tk^2} F_{mn} \right] \right\} = 0 \qquad (5.50)'$$

where we have defined the following functions of k:

$$A_{mn} = \frac{1+\frac{1}{2}a}{\lambda_m^4 - k^4} \left(\frac{2\lambda_m^2}{\cosh \frac{1}{2}k} P - [k^2 - (A-B)(\lambda_m^4 - k^4)] \delta_{mn} - X_{mn} \right)$$

$$B_{mn} = (\lambda_m^4 + k^4) \delta_{mn} - 2k^2 X_{mn}$$

$$C_{mn} = \frac{a}{\mu_m^4 - k^4} \left(\frac{2\mu_m^2}{\sinh \frac{1}{2}k} U - [k^2 - (A-B)(\mu_m^4 - k^4)] J_{01} - J_{21} \right)$$

$$+ 3A(A-B)a I_{10} - \frac{2(A-B)a}{\mu_m^4 - k^4} \left(\frac{2\mu_m^2 k}{\sinh \frac{1}{2}k} P - [k^2 - (A-B)(\mu_m^4 - k^4)] I_{10} - I_{30} \right)$$

(5.55)

$$D_{mn} = \frac{a}{\lambda_m^4 - k^4} \left(\frac{2\lambda_m^2}{\cosh \frac{1}{2}k} V - [k^2 - (A-B)(\lambda_m^4 - k^4)] I_{01} - I_{21} \right)$$

$$+ 3A(A-B)a J_{10} - \frac{2(A-B)a}{\lambda_m^4 - k^4} \left(\frac{2\lambda_m^2 k}{\cosh \frac{1}{2}k} Q - [k^2 - (A-B)(\lambda_m^4 - k^4)] J_{10} - J_{30} \right)$$

$$E_{mn} = \frac{1+\frac{1}{2}a}{\mu_m^4 - k^4} \left(\frac{2\mu_m^2}{\sinh \frac{1}{2}k} Q - [k^2 - (A-B)(\mu_m^4 - k^4)] \delta_{mn} - Y_{mn} \right)$$

$$F_{mn} = (\mu_m^4 + k^4) \delta_{mn} - 2kY_{mn}$$

It is apparent that as n varies from 1 to ∞, equations (5.49)' and (5.50)' represent a doubly infinite system of linear homogeneous equations for the coefficients A_m and B_m. The condition that the constants A_m and B_m do not all vanish is that the determinant of the system should vanish, that is

$$\begin{vmatrix} A_{mn} - \frac{1}{Tk^2} B_{mn} & C_{mn} \\ D_{mn} & E_{mn} - \frac{1}{Tk^2} F_{mn} \end{vmatrix} = 0 \qquad (5.56)$$

This is the required characteristic equation from which the curve of neutral stability can be obtained. Equation (5.56) yields multiple roots of T, but only the smallest positive root of T need be considered.

5.9 Numerical Calculations

The simplest approximation to the characteristic equation (5.56) is obviously a second order determinant, which corresponds to taking m = n = 1, that is when one uses one even and one odd function in the initial representation for $\hat{u}$. For Newtonian fluids this approximation is known to give results within 1% of the exact values, and we shall assume that the accuracy of the method will remain good for our viscoelastic case since we shall confine ourselves with small values of the viscoelastic parameters A and B.

We shall therefore approximate (5.56) by

$$\begin{vmatrix} A_{11} - \frac{1}{Tk^2} B_{11} & C_{11} \\ D_{11} & E_{11} - \frac{1}{Tk^2} F_{11} \end{vmatrix} = 0$$

or

$$(A_{11}E_{11} - C_{11}D_{11})k^4T^2 - (B_{11}E_{11} + A_{11}F_{11})k^2T + B_{11}F_{11} = 0 \qquad (5.57)$$

The two roots of T as a function of k were calculated from the above equation, and the smaller positive root of T(k) defines the neutral stability curve.

The values of the various constants for m = n = 1 are

$$\lambda_1 = 4.730041 \qquad \mu_1 = 7.853205$$

$$I_{01} = 0.1478245 \qquad J_{01} = 0.1478245$$

$$I_{21} = -0.00060475 \qquad J_{21} = -6.682854$$

$$I_{10} = 3.34202 \qquad J_{10} = -3.34202$$

$$I_{30} = -122.06503 \qquad J_{30} = 122.06503$$

$$X_{11} = -12.30262 \qquad Y_{11} = -46.05012$$

Some of these values were already evaluated by Reid (1957) and Thomas & Walters (1963). The new ones were evaluated by direct integration and then checked by numerical evaluation of the integrals in the computer.

5.10 Results and Conclusions

For computing numerical results two cases were considered:

(i) Cylinders rotating in the same direction with the same angular

velocity, i.e., a = 0, and (ii) Outer cylinder stationary, i.e., a = -1. The values of the viscoelastic parameters A and B were varied from 0.0 to 0.1. A few sample neutral stability curves are displayed in Figs. 5.2 and 5.3. All the results are finally shown in compact form in Figs. 5.4 and 5.5, which plot the critical Taylor number T_c against A, with B regarded as a parameter. Figures 5.6 and 5.7 likewise display the variation of the critical wavenumber k_c as a function of A and B. The results for B = 0 agree with those of Thomas & Walters (1963) for the Maxwell fluid.

From the above-mentioned figures the following observations can be made:

(i) The presence of viscoelasticity destabilizes the flow.

(ii) At a fixed λ_1, an increase of $\lambda_2 (< \lambda_1)$ stabilizes the flow. This of course must happen since the results should approach their Newtonian values as $\lambda_2 \to \lambda_1$.

(iii) From Fig. 5.4 (a = 0) it is seen that each curve for B = constant is exactly identical to the others, only displaced horizontally. (Numerical values show it more explicitly, e.g., T_c for A = 0.02, B = 0.01 is exactly the same as the T_c for A = 0.03, B = 0.02 - both being equal to 1426.14.) This means that it is only (A-B) which counts, not A and B separately. That this must happen can be seen from an examination of the stability equations (5.38) and (5.39), which for a = 0 show that only the viscoelastic terms in (A-B) remain in the equations.

From Fig. 5.5 (a = -1) it can be seen that although the curves do not exactly coincide when displaced horizontally, they are however close to being so. (For example T_c for A = 0.02, B = 0.01 is 2821.44,

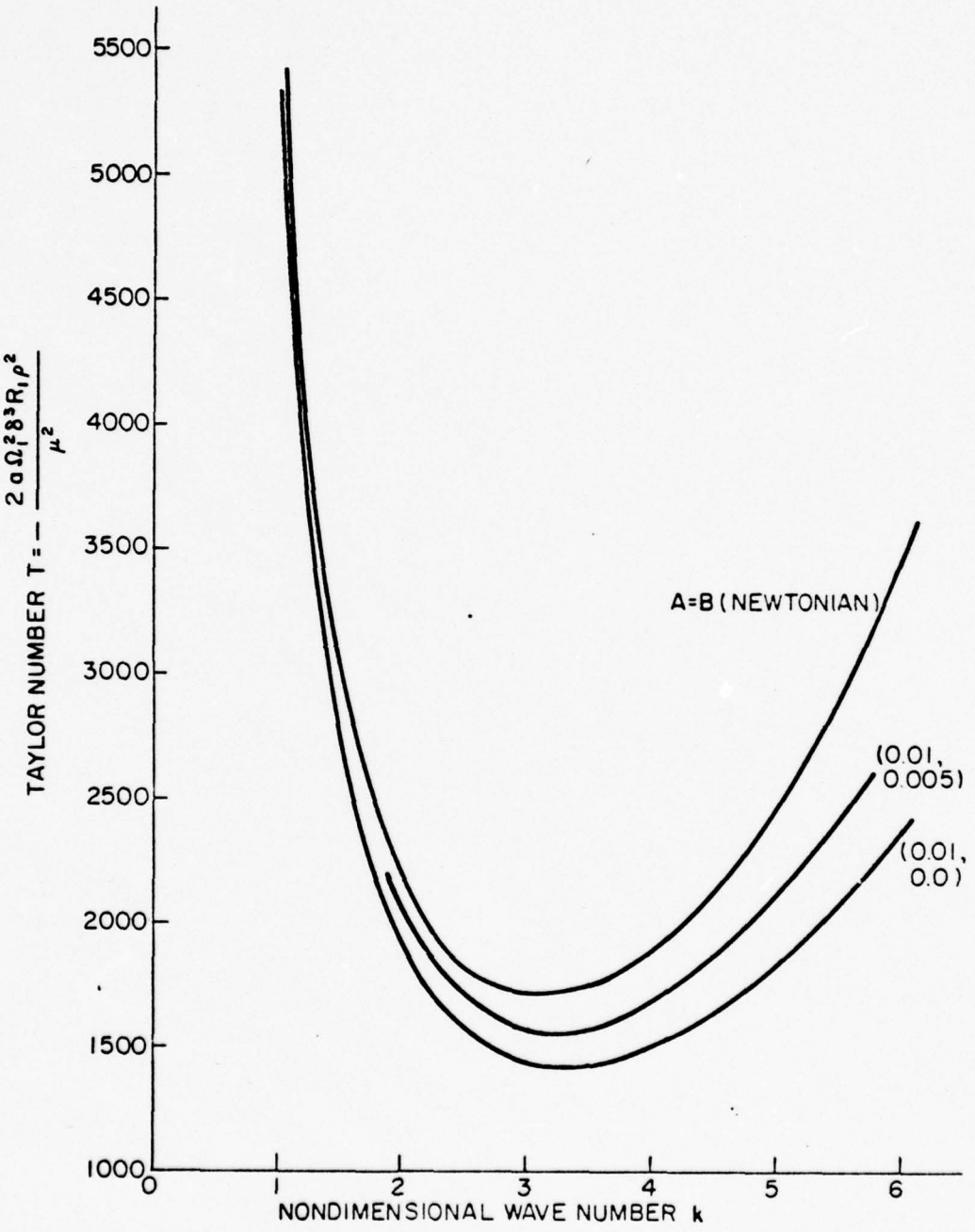


Figure 5.2 Neutral stability curves for some pairs of values of (A,B), with a = 0.0.

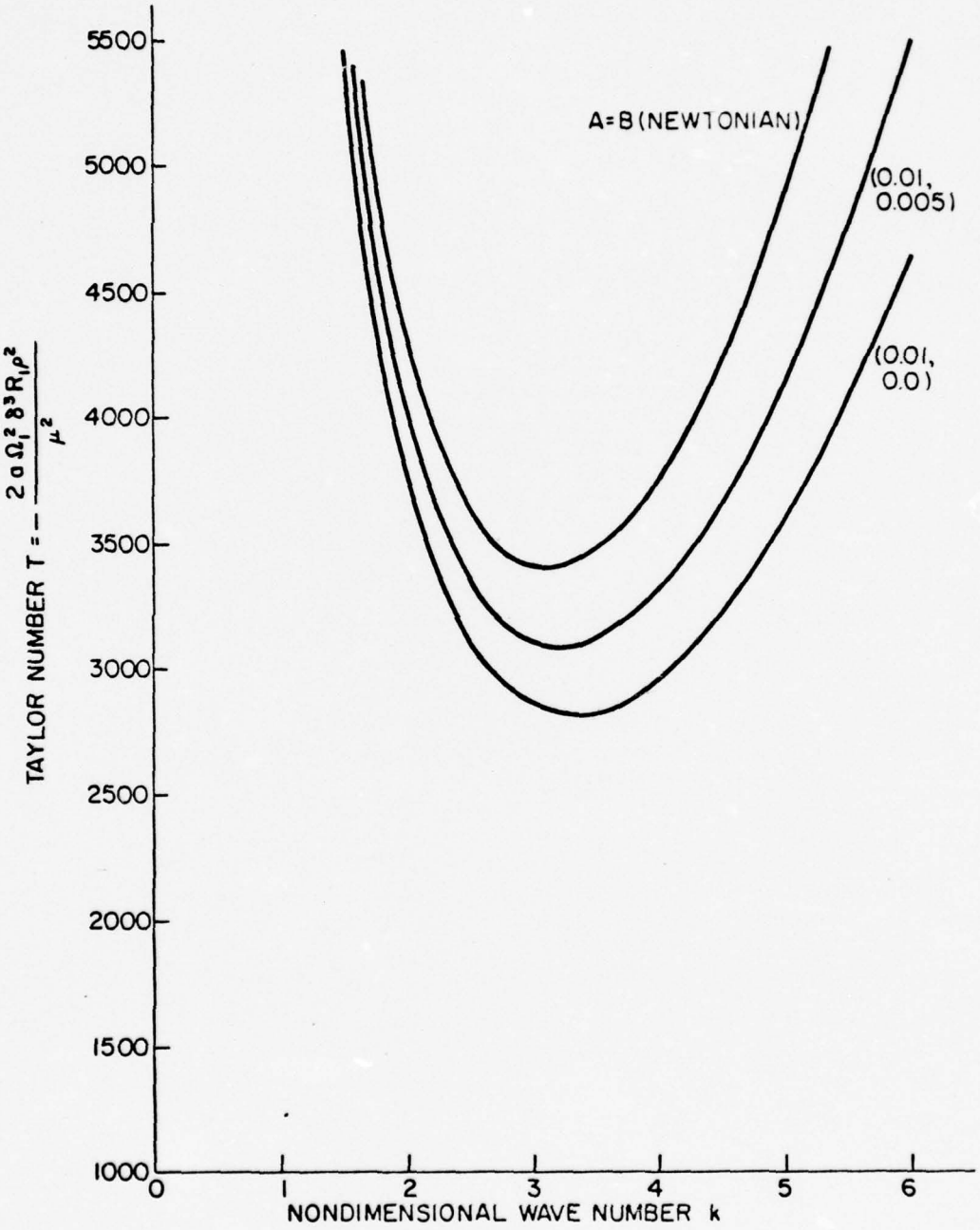


Figure 5.3 Neutral stability curves for some pairs of values of (A,B), with a = -1.0.

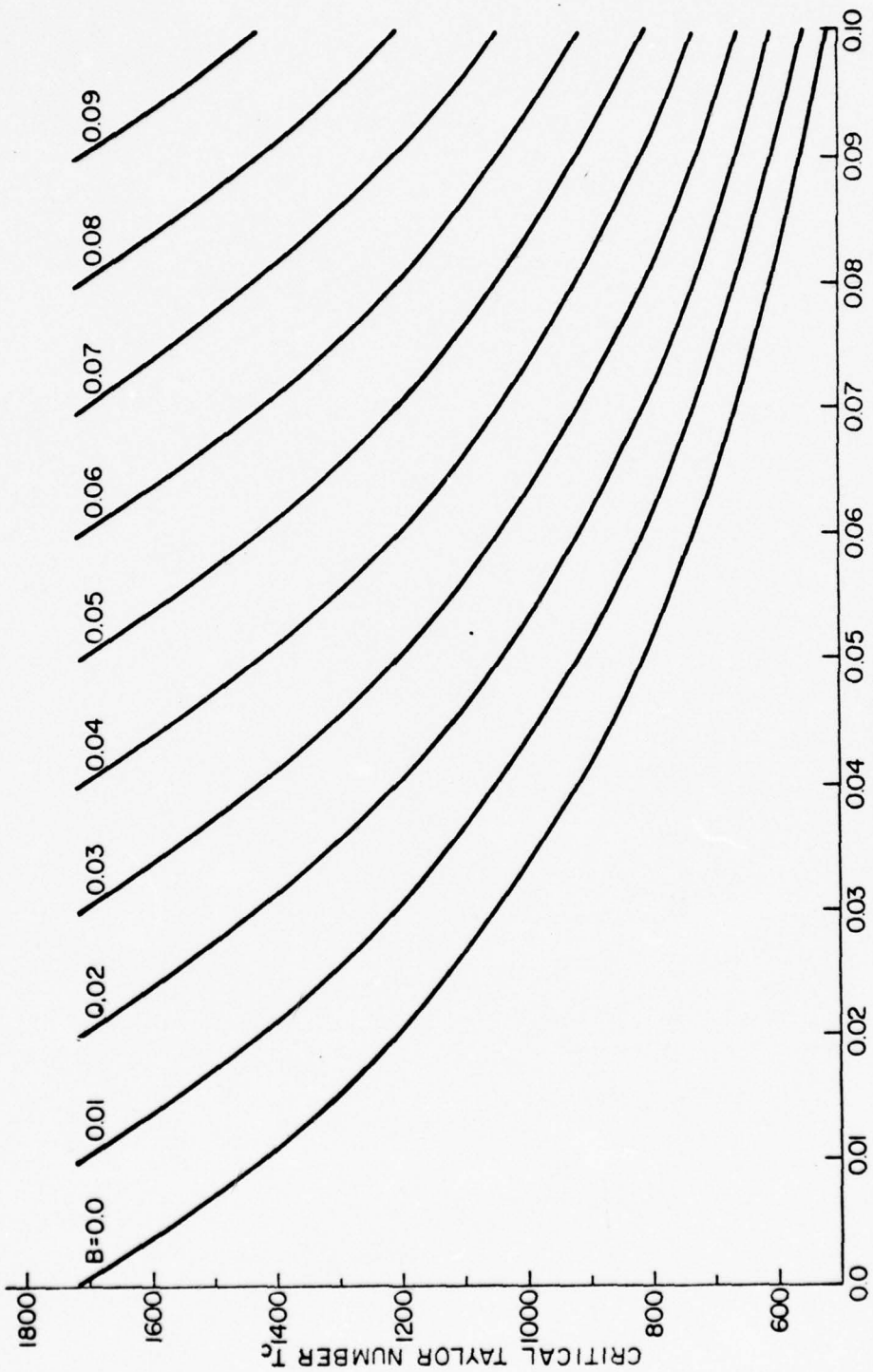


Figure 5.4 Critical Taylor number against A for various values of B, with a = 0.0.

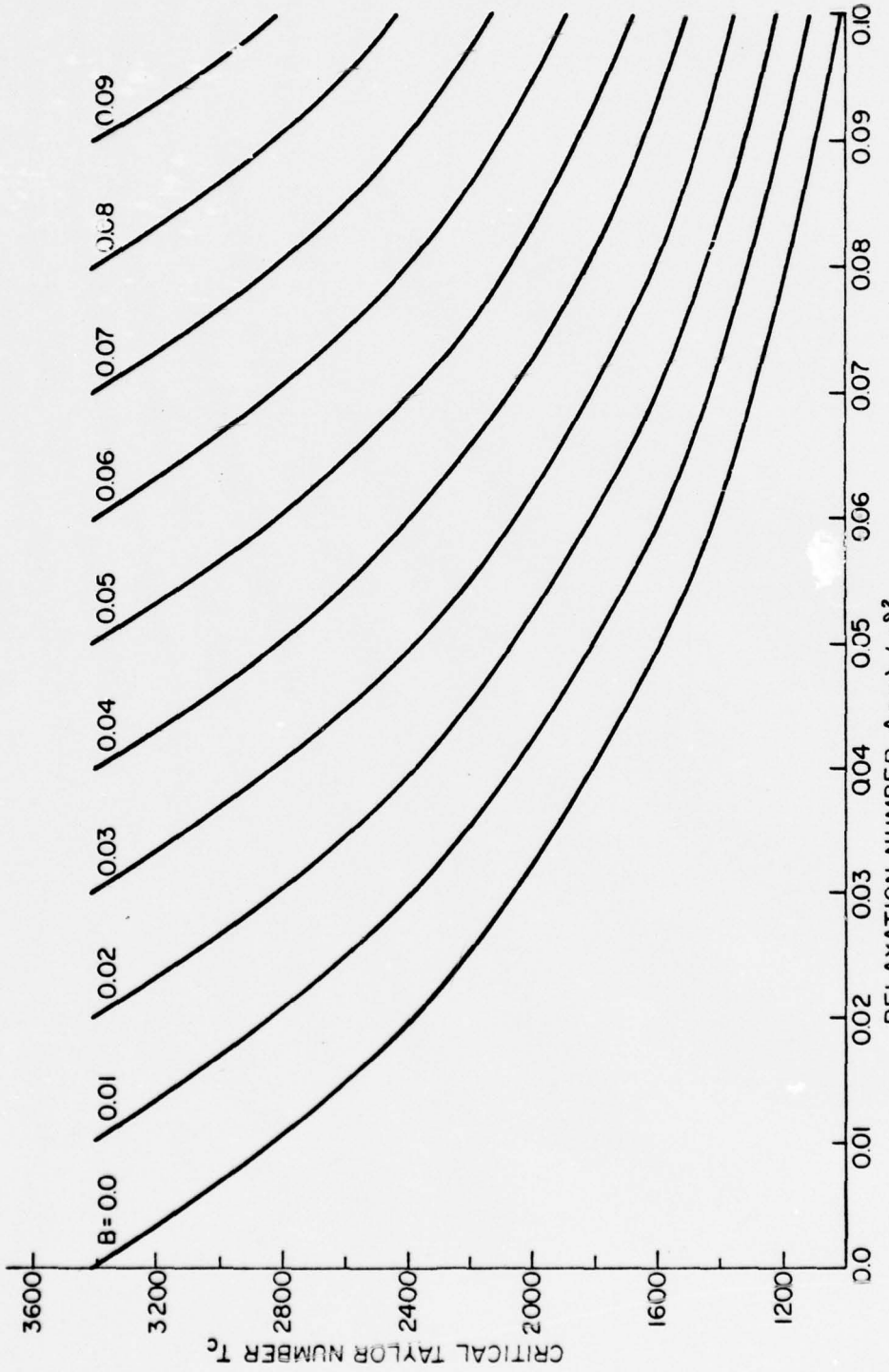


Figure 5.5 Critical Taylor number against A for various values of B, with a = -1.0.

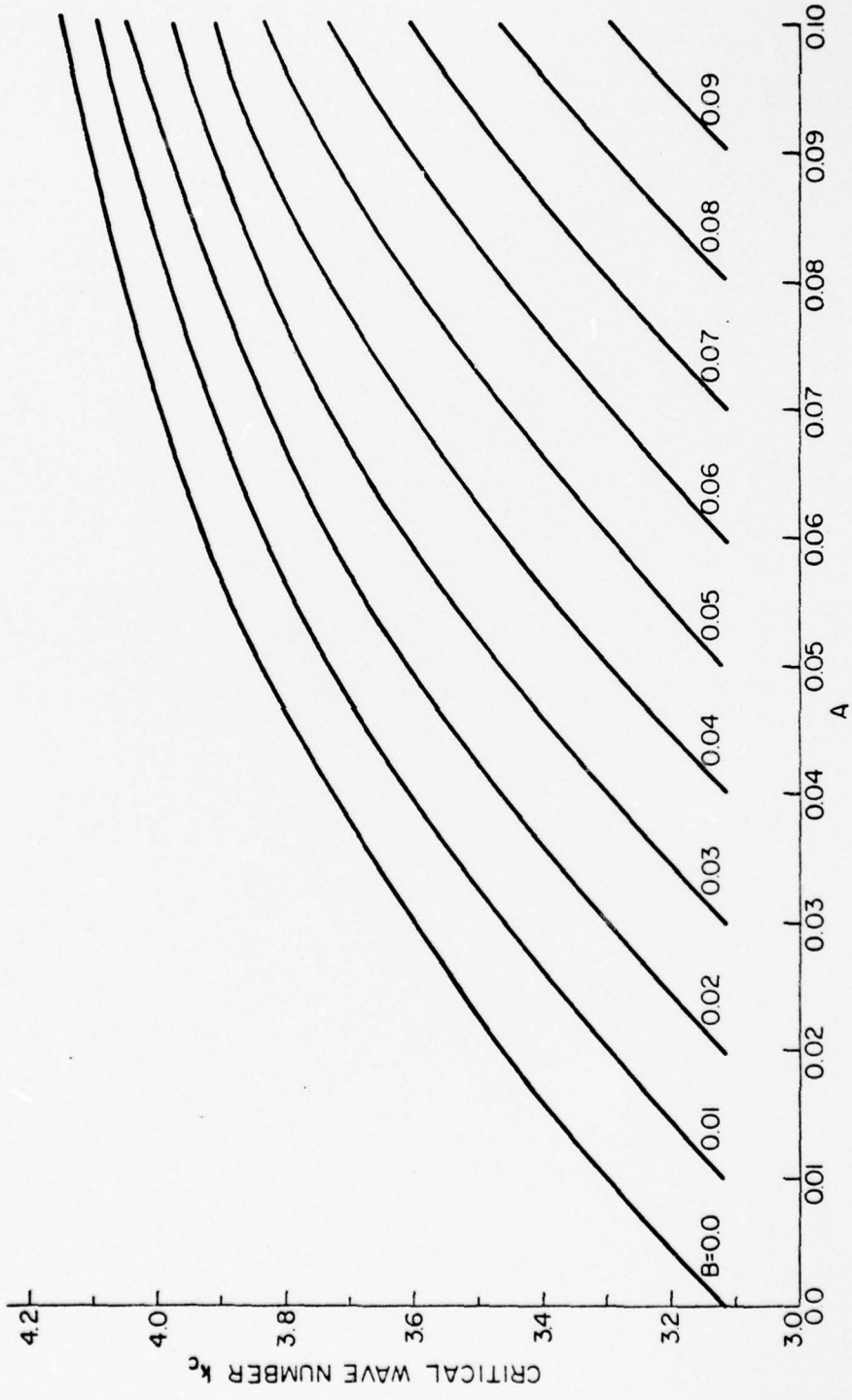


Figure 5.6 Critical wavenumber against A for various values of B, with a = 0.0.

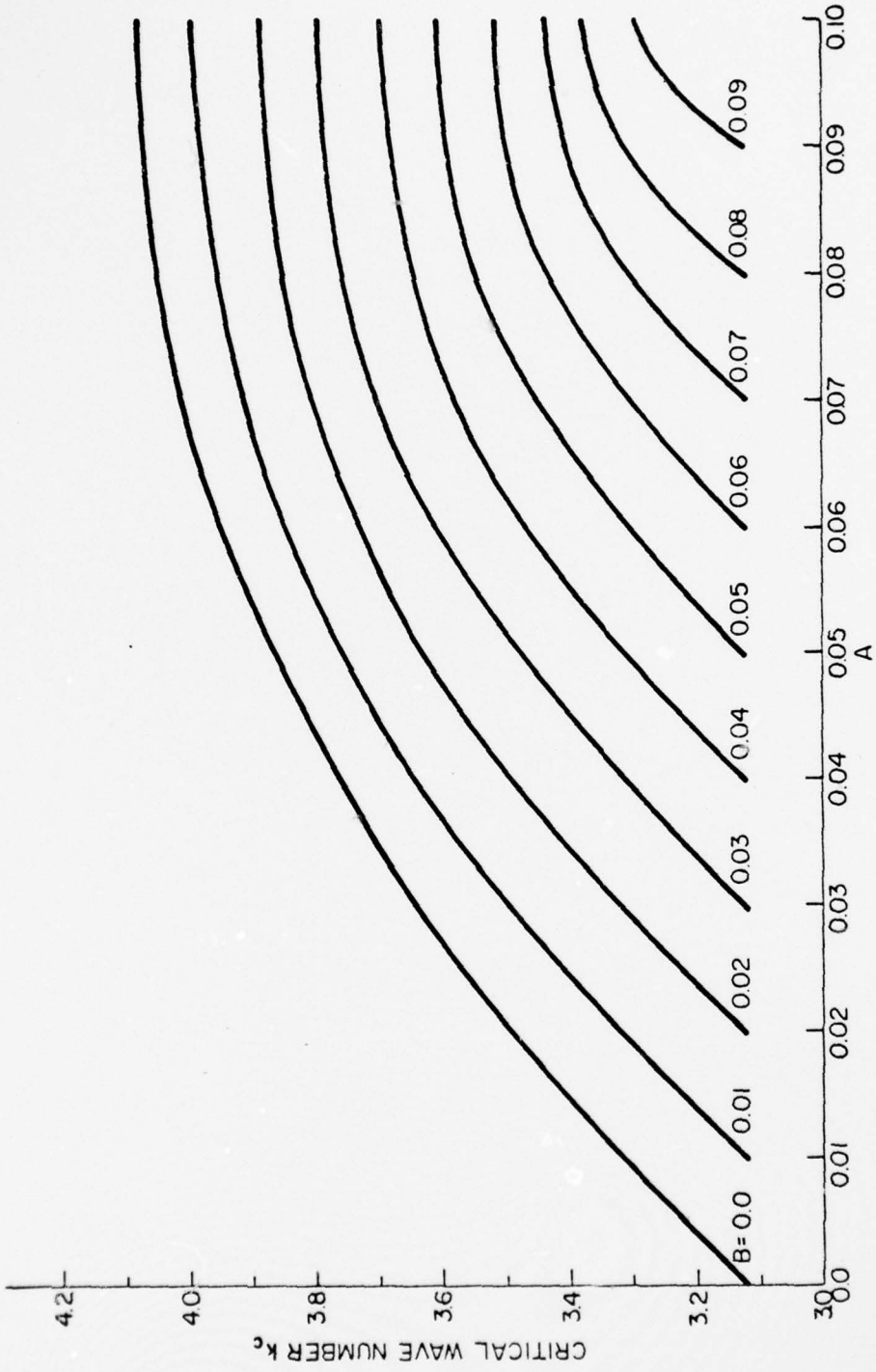


Figure 5.7 Critical wavenumber against A for various values of B, with a = -1.0.

while T_c for A = 0.03, 0.02 is 2817.99.) This means that, even for $a \neq 0$, for all practical purposes it is only (A-B) which is important, not A and B separately. However, this fact is not expected to be true for higher values of A and B when the last term on the right side of equation (5.38) will become larger.

(iv) From Figs. 5.6 and 5.7 it is apparent that the presence of viscoelasticity increases the critical wavenumber, that is decreases the cell size, at the onset of instability.

All the above theoretical conclusions are therefore identical to those of the plane Poiseuille flow of an Oldroyd fluid, studied in the last chapter.

A discussion of the experimental observations of the Couette flow instability was given in Section 1.4, from which it would be seen that the predictions of the destabilization and the increase of critical wavenumber due to viscoelasticity agree with at least half the observers.

CHAPTER 6

SOME UNSTEADY PARALLEL FLOWS OF OLDROYD FLUID

6.1 Introduction

This chapter is not directly related to the rest of the dissertation. For the sake of curiosity some unsteady parallel flows of the Oldroyd fluid were solved, and these will be recorded in this chapter. Sometimes no great effort will be made to interpret the results.

Some unsteady parallel flows of the second-order fluid have been solved by Ting (1963) and Markovitz & Coleman (1964); Tlapa & Bernstein (1968) have solved the problem of the suddenly accelerated plate (Stokes' first problem) in a Maxwell fluid; however, no such works seem to have been done for Oldroyd fluids.

6.2 Governing Equation

The differential equation governing the velocity distribution in unsteady parallel flows of Oldroyd's liquid B will be derived in this section. The velocity components in a parallel flow are

$$\begin{aligned} U_1 &= U_1(x_2,t) \\ U_2 &= U_3 = 0 \end{aligned} \tag{6.1}$$

The constitutive equation for Oldroyd liquid B was given in equation (2.18); in cartesian coordinates it is

$$S_{ij} + \lambda_1(S_{ij,t} + U_k S_{ij,k} - S_{kj}U_{i,k} - S_{ik}U_{j,k})$$

$$= 2\mu E_{ij} + 2\mu\lambda_2(E_{ij,t} + U_k E_{ij,k} - E_{kj}U_{i,k} - E_{ik}U_{j,k}) \qquad (6.2)$$

where S_{ij} is the extra-stress, and $E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the rate of strain. For the case of parallel flow (6.1), the various components of the constitutive equation (6.2) give

$$S_{11} + \lambda_1(S_{11,t} + U_1 S_{11,1}) = 2\lambda_1 S_{12} U_{1,2} - 2\mu\lambda_2 (U_{1,2})^2 \qquad (6.3a)$$

$$S_{12} + \lambda_1(S_{12,t} + U_1 S_{12,1}) = \mu U_{1,2} + \mu\lambda_2 U_{1,2t} + \lambda_1 S_{22} U_{1,2} \qquad (6.3b)$$

$$S_{13} + \lambda_1(S_{13,t} + U_1 S_{13,1}) = \lambda_1 S_{23} U_{1,2} \qquad (6.3c)$$

$$S_{22} + \lambda_1(S_{22,t} + U_1 S_{22,1}) = 0 \qquad (6.3d)$$

$$S_{23} + \lambda_1(S_{23,t} + U_1 S_{23,1}) = 0 \qquad (6.3e)$$

$$S_{33} + \lambda_1(S_{33,t} + U_1 S_{33,1}) = 0 \qquad (6.3f)$$

Using the transformation

$$\xi = x_1 - U_1(x_2)t$$

$$\eta = t$$

equations (6.3d), (6.3e) and (6.3f) can be expressed as

$$S_{ij} + \lambda_1 S_{ij,\eta} = 0 \qquad (ij = 22, 23, 33)$$

whose solution is

$$S_{ij} = A(\xi)e^{-\eta/\lambda_1} = A(x_1 - U_1 t)e^{-t/\lambda_1}$$

If we imagine that the flow started from rest*, then $A(x_1) = 0$ and, therefore,

$$S_{22} = S_{23} = S_{33} = 0$$

Equation (6.3c) then shows $S_{13} = 0$, so that the stress tensor has the form

$$[\underset{*}{S}] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From physical considerations, the derivatives of the above extra-stresses with respect to x_1 are zero.

Now examine the equation of motion

$$\rho U_{i,t} = -P_{,i} + S_{ij,j} \tag{6.4}$$

Consideration of the x_2- and x_3-components of this equation shows that P cannot vary in these directions, i.e., $P = P(x_1,t)$. Next consider the x_1-component

$$-P_{,1} = \rho U_{1,t} - S_{12,2}$$

The left hand side can be a function of (x_1,t) only, while the right hand side can be a function of (x_2,t) only. Therefore we must have

$$-P_{,1} = \rho U_{1,t} - S_{12,2} = f(t) \tag{6.5}$$

Differentiating equation (6.3b) with respect to x_2, setting $S_{12,1} =$

---

*Alternatively, S_{22} etc. being proportional to $\exp(-t/\lambda_1)$ will decay very rapidly because λ_1 is a very short time.

$S_{22} = 0$, and using (6.5), there results the following differential equation governing the velocity distribution:

$$\rho U_{1,t} + \rho\lambda_1 U_{1,tt} = \mu U_{1,22} + \mu\lambda_2 U_{1,22t} + f + \lambda_1 f_{,t} \tag{6.6}$$

If there is no pressure gradient in the direction of flow, then $f = 0$ and therefore

$$\rho \frac{\partial U_1}{\partial t} + \rho\lambda_1 \frac{\partial^2 U_1}{\partial t^2} = \mu \frac{\partial^2 U_1}{\partial x_2^2} + \mu\lambda_2 \frac{\partial^3 U_1}{\partial x_2^2 \partial t} \tag{6.7}$$

If $\lambda_1 = \lambda_2$, then the above reduces to the Newtonian diffusion equation

$$\frac{\partial U_1}{\partial t} = \nu \frac{\partial^2 U_1}{\partial x_2^2}$$

where $\nu = \mu/\rho$. Equation (6.7) may be compared with the corresponding equation for the second-order fluid

$$\rho \frac{\partial U_1}{\partial t} = \mu \frac{\partial^2 U_1}{\partial x_2^2} + \gamma \frac{\partial^3 U_1}{\partial x_2^2 \partial t}$$

derived by Ting (1963), and Markovitz & Coleman (1964). Note that in both these cases the order of the differential equation has increased due to the presence of viscoelasticity.

6.3 Oscillation of One Plate Parallel to Another

Consider an Oldroyd fluid undergoing a nonsteady shearing flow between two parallel plates (Fig. 6.1), one of which is at rest at $y = 0$, while the other located at $y = \delta$ undergoes sinusoidal oscillations in the x-direction with velocity $U_0 \cos\omega t$. The pressure gradient in the direction of flow will be assumed zero. This problem has been solved by Markovitz & Coleman (1964) for a second-order fluid.

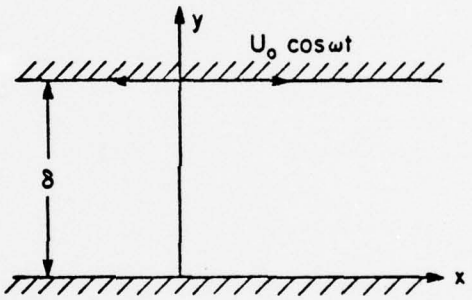


Figure 6.1 Oscillation of one plate parallel to another.

We seek a solution of equation (6.7) of the form

$$U_1(y,t) = \hat{U}(y)e^{i\omega t} \tag{6.8}$$

where we are using y in place of x_2. Actually only the real part of the right hand side is meant, i.e.,

$$U_1 = \hat{U}_R\cos\omega t - \hat{U}_I\sin\omega t \tag{6.9}$$

where the complex amplitude is $\hat{U}(y) = \hat{U}_R + i\hat{U}_I$. The boundary conditions read

$$\begin{aligned} \hat{U}(0) &= 0 \\ \hat{U}(\delta) &= U_0 \end{aligned} \tag{6.10}$$

Substituting (6.8) into (6.7), and using the boundary conditions (6.10), the solution obtained is

$$\hat{U} = U_0 \frac{\sinh ky}{\sinh k\delta} \tag{6.11}$$

where

$$k = \left[\frac{i\omega(1+i\omega\lambda_1)}{\nu(1+i\omega\lambda_2)}\right]^{1/2} \tag{6.12}$$

Denoting k = a + ib, the above gives

$$\begin{aligned} a &= \left\{\frac{\omega}{2\nu(1+\omega^2\lambda_2^2)}\left[\sqrt{1+\omega^4\lambda_1^2\lambda_2^2+\omega^2(\lambda_1^2+\lambda_2^2)} - \omega(\lambda_1-\lambda_2)\right]\right\}^{1/2} \\ b &= \left\{\frac{\omega}{2\nu(1+\omega^2\lambda_2^2)}\left[\sqrt{1+\omega^4\lambda_1^2\lambda_2^2+\omega^2(\lambda_1^2+\lambda_2^2)} + \omega(\lambda_1-\lambda_2)\right]\right\}^{1/2} \end{aligned} \tag{6.13}$$

From (6.12), the real and imaginary parts of $\hat{U}$ become

$$\begin{aligned} \hat{U}_R &= \frac{U_0}{q}[\sinh ay\cos by\sinh a\delta\cos b\delta + \cosh ay\sin by\cosh a\delta\sin b\delta] \\ \hat{U}_I &= \frac{U_0}{q}[\cosh ay\sin by\sinh a\delta\cos b\delta - \sinh ay\cos by\cosh a\delta\sin b\delta] \end{aligned} \tag{6.14}$$

where

$$q = \sinh^2 a\delta \cos^2 b\delta + \cosh^2 a\delta \sin^2 b\delta \qquad (6.15)$$

The solution is therefore (6.9), along with (6.13)-(6.15). For different values of the non-Newtonian parameters $\omega\lambda_1$ and $\omega\lambda_2$, and of the quantity $\delta\sqrt{\omega/2\nu}$, the solutions at various nondimensional times ωt are shown in Figs. (6.2)-(6.6). The quantity $\delta\sqrt{\omega/2\nu}$ can be rewritten as $[(\delta^2/2\nu)/(1/\omega)]^{1/2}$, and therefore has the interpretation of square root of the ratio (viscous diffusion time)/(time scale of flow); for small values of this quantity, Fig. 6.4 shows that the velocity distribution is linear.

Compare the velocity distributions for a fluid with small elasticity [say $\omega\lambda_1 = 0.2$, $\omega\lambda_2 = 0.1$ (Fig. 6.3)] and a fluid with high elasticity [$\omega\lambda_1 = 2.0$, $\omega\lambda_2 = 0.2$ (Fig. 6.6)]. In the former case the velocity distributions for either $\omega t = 0$ or $\omega t = \pi/2$ cross the y-axis only once, whereas in the latter case they cross the y-axis two times; the highly elastic case is therefore beginning to resemble a wave propagation through an elastic solid.

By solving (6.3a) and (6.3b), and of course assuming that the flow started from rest, it can be shown that the stress components are given by

$$S_{11} = \frac{2\mu(\lambda_1-\lambda_2)e^{2i\omega t}}{(1+i\omega\lambda_1)(1+2i\omega\lambda_1)} \left[\frac{U_0 k \cosh ky}{\sinh k\delta}\right]^2$$

$$S_{12} = \mu U_0 k \frac{1+i\omega\lambda_2}{1+i\omega\lambda_1} e^{i\omega t} \frac{\cosh ky}{\sinh k\delta}$$

Thus, the shear stress oscillates with the same frequency (ω) as that of the velocity, while the normal stress oscillates with 2ω; this fact is also true for a second-order fluid [Markovitz & Coleman (1964)].

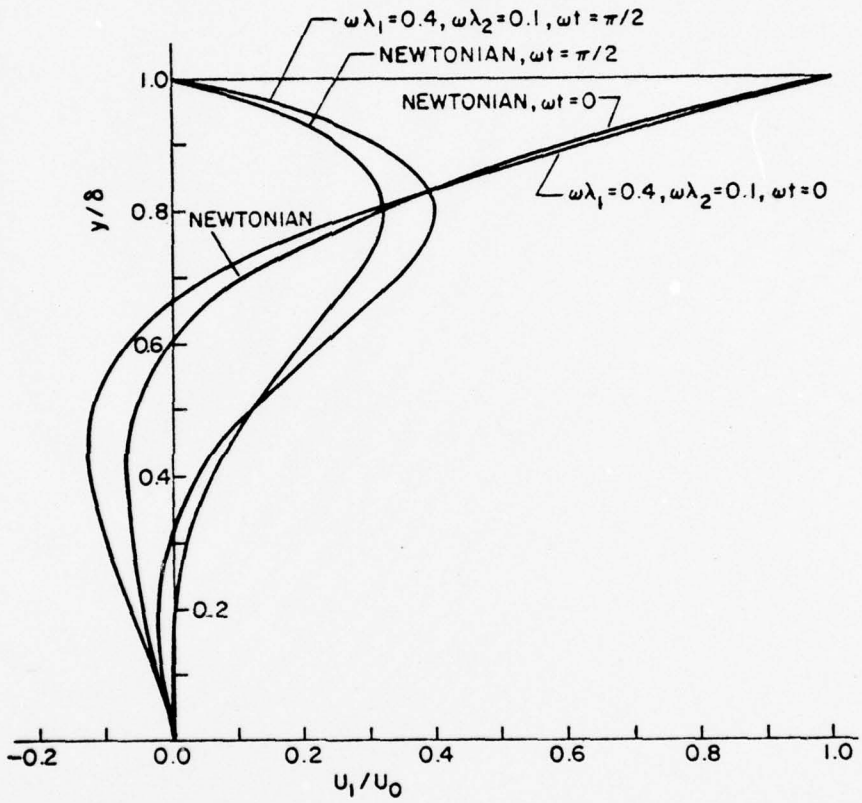


Figure 6.2 Comparison of velocity profiles for Newtonian and viscoelastic fluids, both for $\delta\sqrt{\omega/2\nu} = 4.0$.

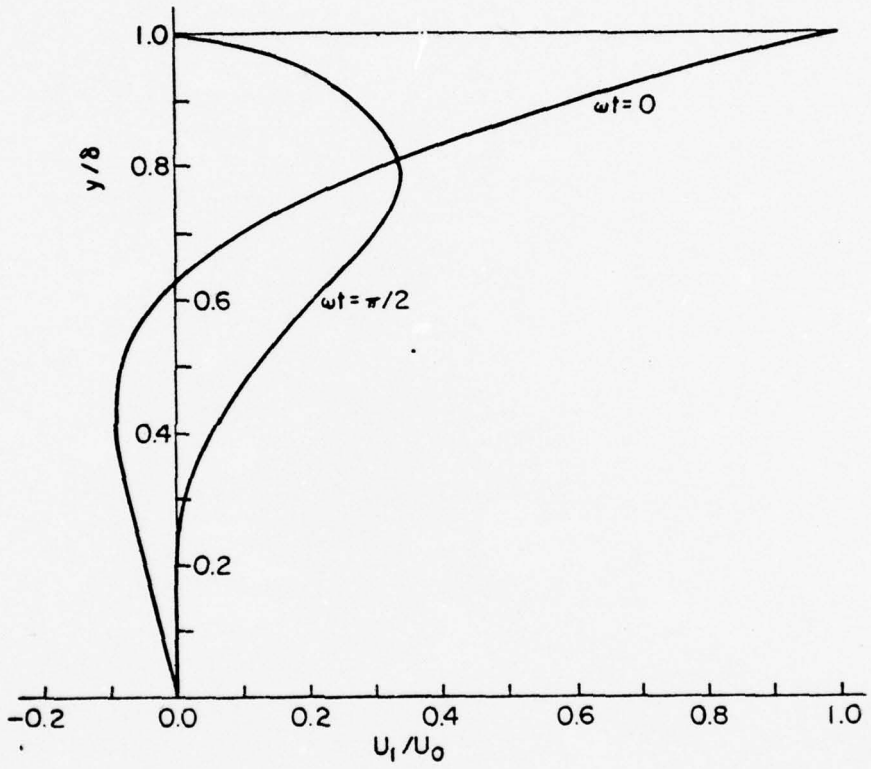


Figure 6.3 Velocity profiles for $\omega\lambda_1 = 0.2$, $\omega\lambda_2 = 0.1$, $\delta\sqrt{\omega/2\nu} = 4.0$

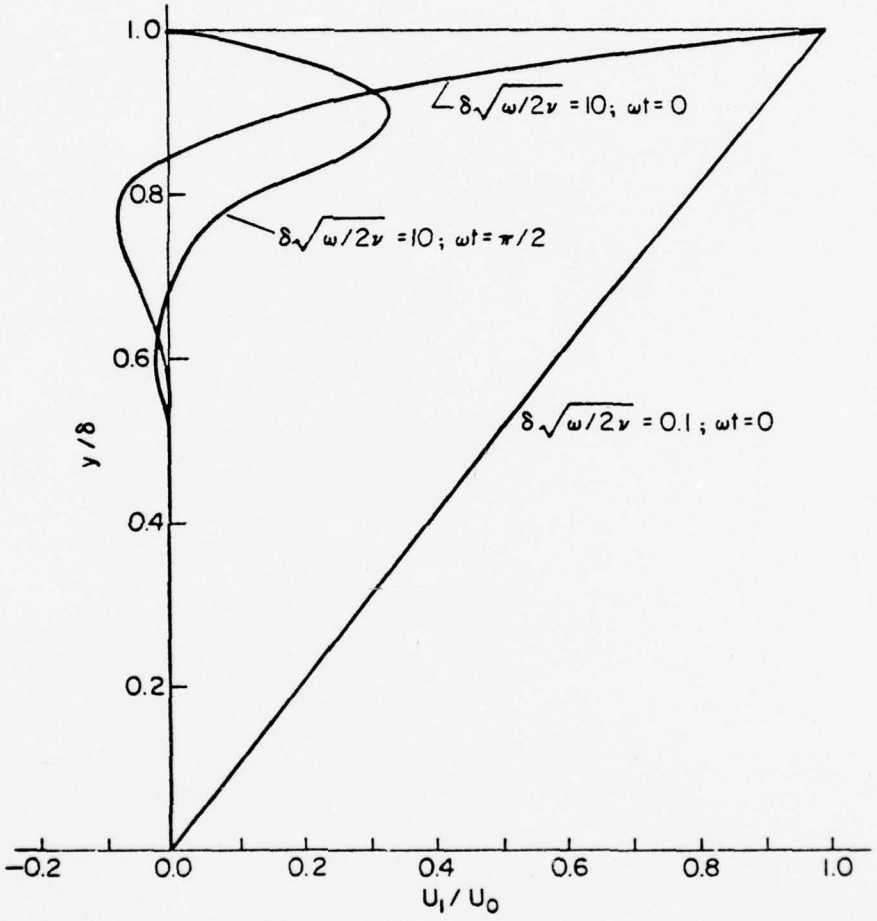


Figure 6.4 Velocity profiles for $\omega\lambda_1 = 0.2$, $\omega\lambda_2 = 0.1$. Compare the profiles at $\omega t = 0$ for the larger and smaller values of $\delta\sqrt{\omega/2\nu}$.

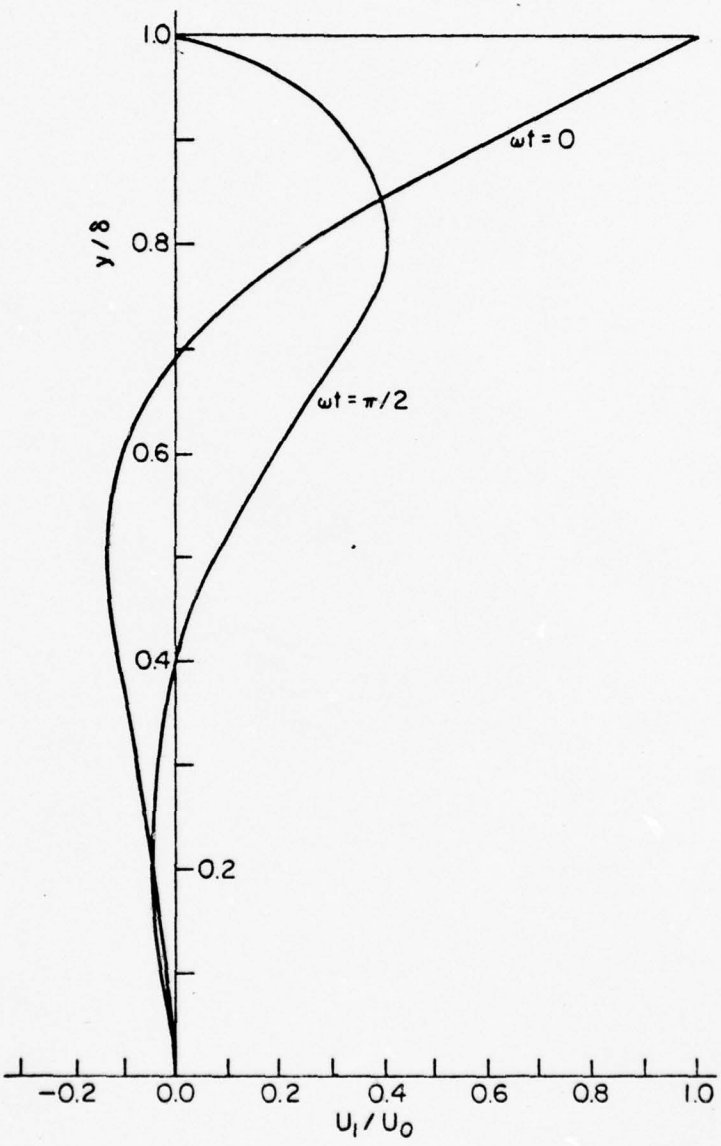


Figure 6.5 Velocity profiles for $\omega\lambda_1 = 1.0$, $\omega\lambda_2 = 0.5$, $\delta\sqrt{\omega/2\nu} = 4.0$.

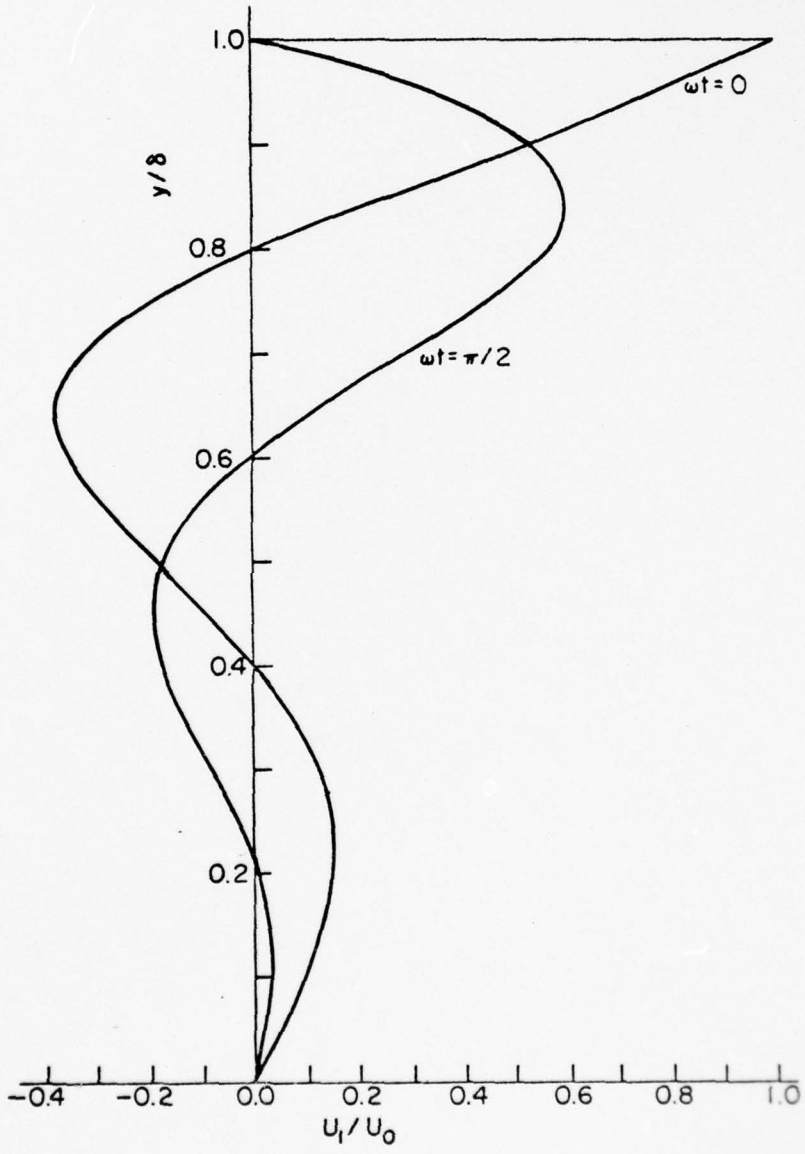


Figure 6.6 Velocity profiles for $\omega\lambda_1 = 2.0$, $\omega\lambda_2 = 0.2$, $\delta\sqrt{\omega/2\nu} = 4.0$.

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SOME PROBLEMS IN THE STABILITY OF FLOWS OF VISCOELASTIC FLUIDS.(U)
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Now let us consider the special case of $\delta\sqrt{\omega/2\nu}$ very small, that is, when the viscous diffusion time is much smaller than the time scale of the flow; this case will correspond to any of the following: (i) small gap, (ii) slow oscillation, or (iii) highly viscous liquid. Under this condition $a\delta$, $b\delta$, and hence $k\delta$, will all be much smaller than unity. Then the hyperbolic sines in (6.11) can be expanded according to the rule $\sinh x = x + x^3/3! + x^5/5! + \ldots$, and keeping only one term of the series we get $\sinh ky \simeq ky$, $\sinh k\delta \simeq k\delta$, so that

$$\hat{U} \simeq \frac{U_0 y}{\delta}$$

Therefore, from (6.9), we get

$$U(y,t) \simeq \frac{U_0 y}{\delta} \cos\omega t$$

which agrees with the linear profile of Fig. 6.4.

The shear stress in this limiting case, found from (6.36), is

$$S_{12} = \frac{U_0 \mu(1+\omega^2\lambda_1\lambda_2)}{\delta(1+\omega^2\lambda_1^2)} (\cos\omega t + \phi)$$

where

$$\phi = -\tan^{-1}\left[\frac{\omega(\lambda_1-\lambda_2)}{1+\omega^2\lambda_1\lambda_2}\right]$$

The shear stress therefore oscillates with the same frequency as that of the velocity, but is out of phase with it by an angle ϕ which approaches zero as $\lambda_1 \to \lambda_2$. It is likely that this phase difference could be related to the stored energy of the fluid.

6.4 Oscillation of a Single Plate in an Infinite Fluid

Consider a flat plate oscillating harmonically parallel to itself in a semi-infinite fluid; this is sometimes called "Stokes' Second Problem," and its Newtonian solution is treated in standard texts [see, for example, Schlichting (1968), p. 85]. We shall however consider an Oldroyd fluid, for which the governing differential equation is (6.7), subject to

$$U_1(o,t) = U_0 \cos\omega t \tag{6.16}$$

and also that $U_1 \to 0$ as $y \to \infty$. Keeping the Newtonian solution in mind, let us seek a solution of the form

$$U_1(y,t) = U_0 e^{-ky} \cos[\omega t - \phi(y)] \tag{6.17}$$

where ϕ is the phase difference at height y; we are going to determine k and $\phi(y)$. Substituting (6.17) into (6.7), we get

$$\left[-\omega - \nu \frac{d^2\phi}{dy^2} + 2\nu k \frac{d\phi}{dy} - \nu\lambda_2\omega \left\{\frac{d\phi}{dy}\right\}^2 + \nu\lambda_2\omega k^2\right] \sin(\omega t - \phi)$$

$$+ \left[-\lambda_1\omega^2 - \nu k^2 + \nu \left\{\frac{d\phi}{dy}\right\}^2 - \nu\lambda_2\omega \frac{d^2\phi}{dy^2} + 2\nu\lambda_2\omega k \frac{d\phi}{dy}\right] \cos(\omega t - \phi) = 0$$

which can be true for all t only if

$$-\omega - \nu \frac{d^2\phi}{dy^2} + 2\nu k \frac{d\phi}{dy} - \nu\lambda_2\omega \left\{\frac{d\phi}{dy}\right\}^2 + \nu\lambda_2\omega k^2 = 0 \tag{6.18}$$

$$-\lambda_1\omega^2 - \nu k^2 + \nu \left\{\frac{d\phi}{dy}\right\}^2 - \nu\lambda_2\omega \frac{d^2\phi}{dy^2} + 2\nu\lambda_2\omega k \frac{d\phi}{dy} = 0 \tag{6.19}$$

Elimination of $(d\phi/dy)^2$ between these two equations gives

$$\frac{d^2\phi}{dy^2} - 2k \frac{d\phi}{dy} = -\frac{\omega}{\nu} \frac{1+\lambda_1\lambda_2\omega^2}{1+\lambda_2^2\omega^2} \tag{6.20}$$

The solution of this equation is

$$\phi = A + B\, e^{2ky} + \frac{\omega}{2k\omega} \frac{1 + \lambda_1\lambda_2\omega^2}{1 + \lambda_2^2\omega^2}\, y$$

where A and B are constants. The condition $\phi = 0$ at $y = 0$ requires $B = -A$, so that the solution is

$$\phi = A(1 - e^{2ky}) + \frac{\omega}{2k\nu} \frac{1 + \lambda_1\lambda_2\omega^2}{1 + \lambda_2^2\omega^2}\, y \qquad (6.21)$$

To determine A, we require that $\lambda_1 = \lambda_2$ should result in the Newtonian expression for ϕ, which is [see, for example, Schlichting (1968), p. 85)] $\phi_{New} = \omega y/2\nu k$. This requires $A = 0$ in equation (6.21), giving

$$\phi = \frac{\omega}{2k\nu} \frac{1 + \lambda_1\lambda_2\omega^2}{1 + \lambda_2^2\omega^2}\, y \qquad (6.22)$$

To determine k, the above expression for ϕ is substituted into (6.19), whose solution which agrees with the Newtonian solution at $\lambda_1 = \lambda_2$ is given by

$$k = \left\{ \frac{-\omega^2(\lambda_1 - \lambda_2) + \omega\sqrt{\omega^2(\lambda_1 - \lambda_2)^2 + (1 + \lambda_1\lambda_2\omega^2)^2}}{2\nu(1 + \lambda_2^2\omega^2)} \right\}^{1/2} \qquad (6.23)$$

The positive sign outside the braces { } was chosen so as to insure $U_1 \neq \infty$ at $y = \infty$. Using $\lambda_1 > \lambda_2$, it can be easily shown that $k^2 > 0$. The velocity distribution given by (6.17), (6.22) and (6.23) is plotted in Figs. 6.7-6.9 for different values of the non-Newtonian parameters and for various nondimensional times ωt. (Note that our Newtonian solution does not agree with the numerical values in Schlichting's Fig. 6.8. His ordinates seem to be $\sqrt{2}$ times their correct values.)

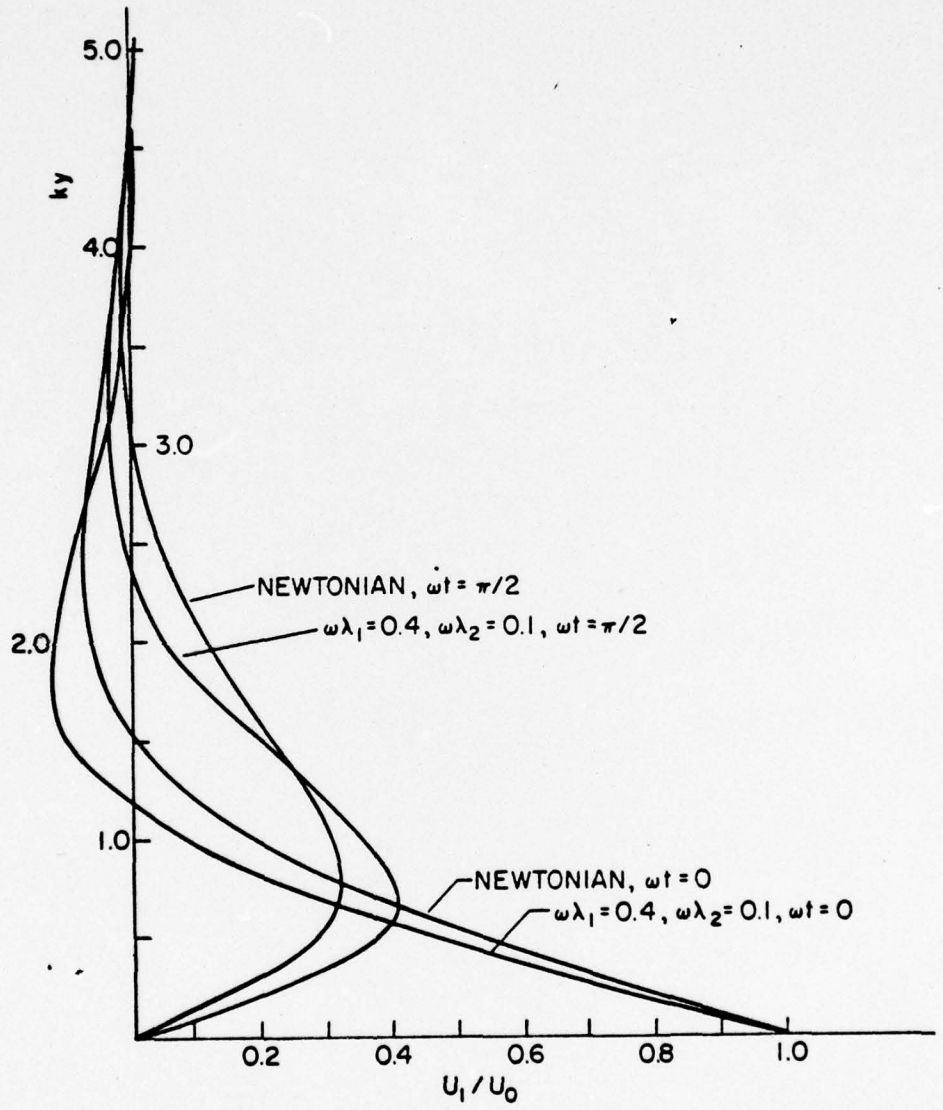


Figure 6.7 Velocity distribution in the neighborhood of an oscillating wall. Comparison of a Newtonian fluid with one having $\omega\lambda_1 = 0.4$, $\omega\lambda_2 = 0.1$.

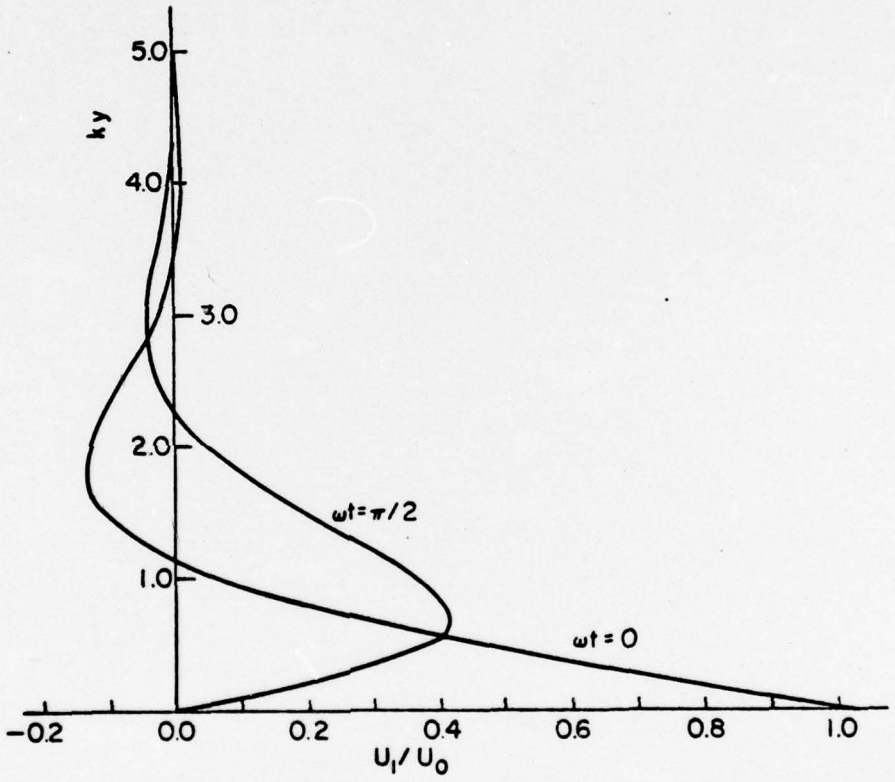


Figure 6.8 Velocity distribution in the neighborhood of an oscillating wall in a fluid having $\omega\lambda_1 = 1.0$, $\omega\lambda_2 = 0.5$.

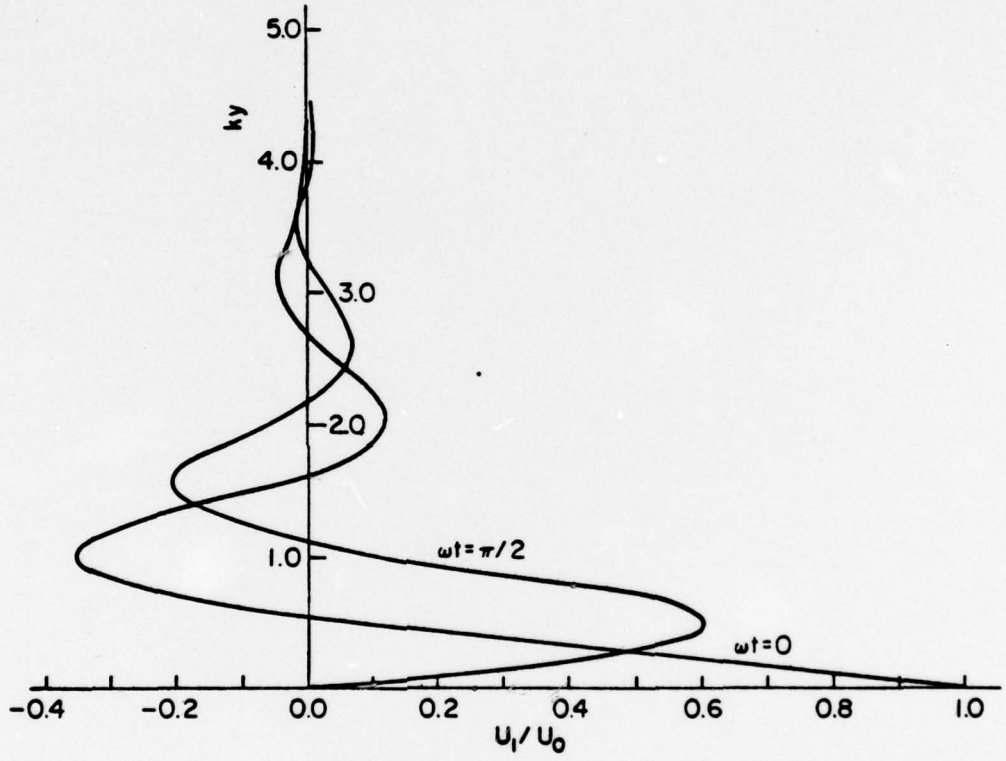


Figure 6.9 Velocity distribution in the neighborhood of an oscillating wall in a fluid having $\omega\lambda_1 = 2.0$, $\omega\lambda_2 = 0.2$.

Compare the velocity distributions for a fluid with small elasticity [say $\omega\lambda_1 = 0.4$, $\omega\lambda_2 = 0.1$ (Fig. 6.7)] and a fluid with high elasticity [$\omega\lambda_1 = 2.0$, $\omega\lambda_2 = 0.2$ (Fig. 6.9)]. In the former case the velocity distributions for either $\omega t = 0$ or $\omega t = \pi/2$ cross the y-axis only once, whereas in the latter case they cross the y-axis three times; the highly elastic case is therefore beginning to resemble the wave propagation through an elastic solid.

6.5 Shear Waves in Oldroyd Fluid

It is well-known that in incompressible fluids without a free surface, the only type of possible wave motion is that of the so-called "shear waves" or "transverse waves" which propagate perpendicular to the fluid velocity. To see this, assume a wave motion given by the velocity vector

$$U_j = C_j e^{i(\alpha_m x_m - \omega t)}$$

where α_m denotes the components of the wavenumber vector. The divergence of the velocity is then

$$U_{j,j} = iC_j\alpha_j e^{i(\alpha_m x_m - \omega t)} = iU_j\alpha_j$$

The condition of incompressibility $U_{j,j} = 0$ therefore requires that the vectors $\underset{\sim}{U}$ and $\underset{\sim}{\alpha}$ must be perpendicular (Fig. 6.10).

Now assume a shear wave in an Oldroyd fluid, of the form

$$U_1 = Ae^{i(\alpha y - \omega t)} = A\, e^{i\alpha y - i\omega_r t}\, e^{\omega_i t} \qquad (6.24)$$

where α is real but $\omega = \omega_r + i\omega_i$. Substituting (6.24) into the governing equation (6.7), we get

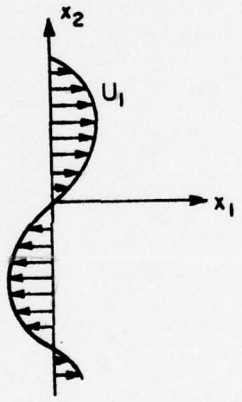


Figure 6.10 Shear waves.

$$i\omega + \omega^2\lambda_1 - \nu\alpha^2 + i\nu\alpha^2\lambda_2\omega = 0 \tag{6.25}$$

This gives $\omega = 0$ (and hence the wave speed $c = 0$) if $\nu = 0$, confirming the fact that shear waves cannot propagate in an ideal liquid. The solution of (6.25) is

$$\omega = \frac{-i(1+\nu\alpha^2\lambda_2) + \sqrt{4\lambda_1\nu\alpha^2-(1+\nu\alpha^2\lambda_2)^2}}{2\lambda_1} \tag{6.26}$$

Two cases are possible depending on the sign of the quantity under the radical sign.

$$\text{Case I:} \quad 4\lambda_1\nu\alpha^2 > (1 + \lambda_2\nu\alpha^2)^2$$

Then (6.26) shows that $\omega_r > 0$ and $\omega_i < 0$, so that the waves are propagated as well as damped in time.

$$\text{Case II:} \quad 4\lambda_1\nu\alpha^2 < (1 + \lambda_2\nu\alpha^2)^2$$

Then (6.26) shows that $\omega_r = 0$, $\omega_i < 0$, so that the waves do not propagate but are "standing" and damped, as in Newtonian flows.

It is not difficult to find a meaning of the inequality $4\lambda_1\nu\alpha^2 >$ or $< (1 + \lambda_2\nu\alpha^2)^2$. We have come across this inequality in many other parallel flows of the Oldroyd fluid that we tried, even in motions not apparently of the wave propagation type (in which cases α signified the Fourier components into which a general velocity distribution can be expanded). We suggest the following interpretation of the inequality, which for Case II is easily recast in the form $4\nu\alpha^2(\lambda_1-\lambda_2) < (1-\lambda_2\nu\alpha^2)^2$. First note that the Newtonian fluid satisfies this inequality. As λ_1 is gradually increased at constant λ_2, this inequality is satisfied in the

beginning but not for large $\lambda_1 \lambda_2$. Thus, Case II represents small elasticity, whereas Case I represents large elasticity.

We can therefore conclude that for small elasticity the waves are purely diffusive and are therefore simply damped in time. On the other hand, a fluid having large elasticity develops some "elastic solid" like characteristics, so that the waves are also propagated.

For the special case of a Maxwell fluid ($\lambda_2 = 0$), the above phenomenon can be given a more complete explanation as follows. From the model of the Maxwell fluid of Section 2.12 it is easily seen that the shearing modulus* of the fluid is $G = 2\mu/\lambda_1$. The speed of propagation of elastic waves is then $C_E = \sqrt{G/\rho} = \sqrt{2\nu/\lambda_1}$. On the other hand the speed of viscous diffusion is $C_V \sim \nu\alpha$. One can consider that the viscoelastic disturbances are propagated if $C_E > C_V$, and are simply damped if $C_E < C_V$. This gives $\lambda_1 \nu\alpha^2 > 2$ for propagation and $\lambda_1 \nu\alpha^2 < 2$ for simple damping, which agree with the inequalities under Cases I and II, except for a numerical factor which could be avoided by defining $C_V = 2\sqrt{2}\,\nu\alpha$.

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*Lumley, in a private communication, has shown an alternative way of finding an expression for it. For a rapidly applied stress the Maxwell fluid displays the characteristics of a purely elastic solid, and gives $\lambda_1 \dot{S} = 2\mu e$, where S is stress and e is displacement. Integration gives $\lambda_1 S = 2\mu e$, from which $G = 2\mu/\lambda_1$.

CHAPTER 7

SUMMARY AND CONCLUSIONS

Three problems on the stability of viscoelastic flows have been solved. In addition, some consideration is given to the energy significance of the common viscoelastic constitutive equations, some interesting observations have been made on the influence of the method of measurement on the sign of the Weissenberg effect, and some unsteady parallel flow problems of viscoelastic fluids have been solved. A summary of the results and conclusions is given below.

7.1 Plane Couette Flow

It has been argued that the application of the small-disturbance linear theory to the investigation of the stability of a plane Couette flow of a viscoelastic fluid is incorrect. Consequently the energy method, following the classical variational technique of Orr, has been used for solving the plane Couette flow problem of a second-order fluid obeying the constitutive equation

$$S_{ij} = \mu A_{ij} + \beta A_{ik}A_{kj} + \gamma A_{ij}^{(2)}$$

where S_{ij} is the extra-stress, A_{ij} is twice the strain rate tensor, $A_{ij}^{(2)}$ is the second Rivlin-Ericksen tensor, μ is the viscosity in simple shear, and β and $\gamma(<0)$ are two elastic properties of the fluid.

It has been found that in order for the energy method to become applicable to a viscoelastic fluid, a new concept, namely that of the elastic potential energy of a viscoelastic fluid, has to be introduced and an expression for it has to be discovered.

No assumption regarding the size of the disturbances was made for the case of two-dimensional disturbances, for which the nonlinear terms vanished from the stability equations and a simple analytical solution could be obtained. The results (Fig. 3.3) indicate that the presence of viscoelasticity stabilizes a plane Couette flow of this fluid. For example, for a Newtonian fluid the critical Reynolds number is found to be $R_{cr} = \rho\Gamma\delta^2/\mu = 44.3$, where Γ is the shear rate of the basic flow, ρ is the density, and δ is half the distance between the plates. On the other hand for a viscoelastic fluid with $G = \gamma\Gamma/\mu$ (which is really the negative of the ratio of the time scale of the fluid to the time scale of the flow) = -0.5, it is found that $R_{cr} = 57.8$.

It was suggested by G. I. Taylor that half the distance between the plates at the critical state in a plane Couette flow is a good measure of the thickness of the viscous sublayer in any turbulent flow. Noting that $R = \rho\Gamma\delta^2/\mu = u_\tau\rho^2\delta^2/\mu^2 = (\delta^+)^2$, where $u_\tau = \sqrt{u\Gamma/\rho}$ is the friction velocity, we get the result that the nondimensional viscous sublayer thickness in a Newtonian fluid is $\delta^+ = \sqrt{44.3} = 6.65$, whereas for a viscoelastic fluid with G = -0.5, it is $\delta^+ = \sqrt{57.8} = 7.6$; this corresponds to an increase of 14.3% in the sublayer thickness. We have thus suggested an explanation of the experimentally observed fact that when small quantities of polymer molecules are added to a turbulent pipe flow of an ordinary Newtonian liquid, a thickening of the viscous sublayer occurs; this in turn results in an increase of the flow rate at the same wall stress (the Toms phenomenon).

For the two-dimensional case mentioned above, the terms involving β had dropped out of the final equations, and therefore the results

would also be valid for the Walters fluid A' (which is in fact identical to the second-order fluid with the β-term removed). However, we also solved the two-dimensional problem for the Walters fluid B', which displays a completely different (and more realistic) type of normal stress behavior and the Weissenberg effect. The results were identical. The prediction of identical results for these three constitutive equations has been shown to be a consequence of two-dimensionality; it has been demonstrated that the two-dimensional vorticity equation for the entire (disturbed) flow, which is obtained after elimination of pressure, is identical for all the three fluids.

The same problem was also tried with more realistic type of disturbances, namely those in the form of longitudinal rolls invariant in the flow direction. Only the second-order fluid was investigated for this type of disturbances. The resulting equations, however, involved a large number of nonlinear terms, and the problem was linearized by assuming that the disturbances were small. The validity of this process of linearization in an application of the energy method is questionable; however, it should be noted that the linearization was necessary only in the elastic terms, and the solution is probably not meaningless. The results (Fig. 3.5) indicate that the critical Reynolds number may increase or decrease due to the presence of elasticity in the fluid, depending on the sign of the nondimensional quantity $H = (2\beta+3\gamma)/4\rho\delta^2$. It may be noted that the case $H > 0$ which gives a destabilization in this flow also gives a positive Weissenberg effect (observed by the free surface in a circular Couette flow).

7.2 Plane Poiseuille Flow

The linear stability theory was applied to a plane Poiseuille flow of centerline velocity U_m and gap width 2δ. The fluid was assumed to obey the Oldroyd constitutive equation

$$S^{ij} + \lambda_1 \frac{\delta S^{ij}}{\delta t} = 2\mu(E^{ij} + \lambda_2 \frac{\delta E^{ij}}{\delta t}) \qquad (7.2)$$

where E^{ij} is the strain rate tensor, λ_1 is the stress relaxation time, $\lambda_2 (< \lambda_1)$ is the strain rate relaxation time, and $\delta/\delta t$ is the Oldroyd convective derivative. The stability equations were numerically integrated by modifying the existing computer technique for a Newtonian fluid. The resulting neutral stability curves (see Fig. 4.8, in which $A = \lambda_1 U_m/\delta$ and $B = \lambda_2 U_m/\delta$ are two elastic parameters) indicate that

(i) The presence of viscoelasticity destabilizes the flow.

(ii) At a fixed λ_1, an increase of $\lambda_2 (< \lambda_1)$ stabilizes the flow. This of course is expected since the Newtonian results should be approached as $\lambda_2 \to \lambda_1$.

(iii) For small λ_1 and λ_2 it is only $(\lambda_1 - \lambda_2)$ which is important, not λ_1 and λ_2 separately.

(iv) The critical wave number increases (that is, the most dangerous waves become shorter) as the fluid elasticity is increased.

(v) Our results do not agree with the results of Tlapa & Bernstein (1970) for the Maxwell fluid ($\lambda_2 = 0$); an examination of their computer program revealed at least two errors.

It has been noted that the results of <u>all</u> the investigators, who assumed different constitutive equations with different normal stress behavior and Weissenberg effect, lead to an identical conclusion for the

plane Poiseuille flow. This fact has been shown to be a result of two-dimensionality of the problem.

7.3 Circular Couette Flow

Here again the linear stability theory was applied to the flow in the annular region between two circular cylinders, with R_1 and R_2 as the radii of the inner and outer cylinders respectively, Ω_1 and Ω_2 as their angular velocities, and $\delta = R_2 - R_1$. The small gap approximation was used, and the fluid was assumed to obey the Oldroyd constitutive equation (7.2). The stability equations were solved by means of an expansion in terms of a suitable set of orthonormal functions. Numerical values were computed for two cases, namely (i) the cylinders rotating in the same direction with the same angular velocity, that is, $a = \Omega_2/\Omega_1 - 1 = 0$, and (ii) outer cylinder stationary, that is, $a = -1$. The critical values of the nondimensional Taylor number, $T = -2a\Omega_1^2\delta^3 R_1\rho^2/\mu^2$, were found by drawing the neutral stability curves. The final results are given as curves of the critical Taylor number T_c vs. A, for various values of B (Fig. 5.4-5.5), and the critical wavenumber k_c vs. A, for various values of B (Fig. 5.6-5.7).

The results indicate that the conclusions (i) through (iv) of the last section are also valid for this flow.

The analytical results cannot be compared with the existing experimental results on the stability of circular Couette flow of polymer solutions, since the experimental results of different observers are sharply contradictory.

7.4 Comparison of the Three Stability Problems Studied

For many different constitutive equations the plane Couette flow has predicted a stabilization due to elasticity, while the plane Poiseuille flow has predicted a destabilization. The insensitivity to the constitutive equation has been shown to be a result of two-dimensionality. However, it is not easy to explain why the plane Couette flow should be stabilized and the plane Poiseuille flow should be destabilized. The obscurity of the stability behavior of these two flows are well-known for the Newtonian fluid, for which even the mechanisms responsible for causing the instability are little understood. It is quite possible that these mechanisms, whatever they are, are modified by elasticity differently for the two flows. And of course there is the experimental evidence of the Toms phenomenon that the plane Couette flow should be stabilized, and the conclusion of destabilization of the plane Poiseuille flow agrees with all the other theoretical investigations of the same flow.

In presence of three-dimensional disturbances there is no proof of the insensitivity to the constitutive equations, and the results of the circular Couette flow show that the conclusions are indeed dependent on the constitutive equations used. All the investigations (including the present one) which used a constitutive equation predicting a positive free surface Weissenberg effect predict destabilization, while the one investigation of Fong (1965) which used a constitutive equation having a negative free surface Weissenberg effect predicts stabilization. Since a real viscoelastic fluid generally displays the positive effect, the conclusion of destabilization seems to be valid.

7.5 Normal Stresses and Weissenberg Effect

It has been found that the sign, and the magnitude, of the Weissenberg effect may depend on the method of observation, namely through the shape of the free surface or through the height difference of radial tappings.

The Oldroyd A fluid and the Walters A' fluid (both long and short memories) display the negative Weissenberg effect as far as the free surface is concerned, but they would display the opposite effect when observed through radial tappings. The Oldroyd B as well as the Walters B' fluid (both long and short memories) display identical positive effect in both types of observations. The second-order fluid would display the positive Weissenberg effect when observed through radial tappings, but it may display either effect when the free surface is observed. A simple experiment, namely that of observing the Weissenberg effect by the two methods, is suggested for measuring the normal stress functions of a viscoelastic liquid.

7.6 Energy Consideration of Constitutive Equations

It has been mentioned in Section 7.1 that in order for the energy method of Orr to become applicable to a second-order fluid, it was necessary to bring in the concept of the elastic potential energy, and to discover an expression for it.

For a purely viscous liquid all the work spent in deforming a material particle is dissipated into internal thermal energy. It is expected that in a viscoelastic liquid a part of the shearing deformation work must be stored as elastic energy. Consequently the rate of

deformation work in a viscoelastic liquid has been written as a sum of dissipation and a rate of increase of elastic potential energy:

$$T_{ij}U_{i,j} = \Psi + \frac{D(PE)}{Dt} \tag{7.3}$$

where T_{ij} is the stress tensor, U_i is the velocity, Ψ is the dissipation, and PE is the elastic potential energy. Expressions have been found for Ψ and PE of the various commonly used constitutive equations, namely the second-order fluid, Oldroyd fluid, Maxwell fluid, and Walters fluid with short memory.

It has been found that the dissipation Ψ for the Oldroyd and Maxwell fluids is always positive, but the dissipation for the second-order and the Walters fluids is not necessarily so. However, it has also been found that Ψ is positive for the latter two fluids for all two-dimensional motions, and also for all slow flows; this conclusion is consistent with the fact that these two constitutive equations were derived as slow flow approximations to more general constitutive equations.

7.7 Some Unsteady Parallel Flows

The following three parallel flow problems of the Oldroyd fluid have been solved:

(i) Flow due to a single flat plate oscillating in an infinite fluid.

(ii) Flow in the space between two parallel plates, one of which is oscillating.

(iii) A shear wave motion given by $U_1 = Ae^{i(kx_2-\omega t)}$, with k real and $\omega = \omega_r + i\omega_i$.

No startling facts have been found in the first two problems. However, an interesting fact was found for the third problem for which it was found that

$$\omega = \frac{i(1+\nu k^2\lambda_2) + \sqrt{4\lambda_1\nu^2 - (1+k^2\nu\lambda_2)^2}}{2\lambda_1} \tag{7.4}$$

where ν is the kinematic viscosity. Two cases are therefore possible, according as $4\lambda_1\nu k^2$ is more or less than $(1 + \lambda_2\nu k^2)^2$. In the first case (large elasticity) $\omega_r > 0$ and $\omega_i < 0$, so that the waves are propagated as well as damped in time. In the second case (small elasticity) $\omega_r = 0$ and $\omega_i < 0$, so that the waves do not propagate but are "standing" and damped, as in Newtonian fluids.

For a Maxwell fluid, the above phenomenon is explained as follows: in one case, the propagation speed of elastic waves is greater than that of the viscous diffusion, so that there is a net propagation; in the other case, the viscous diffusion overcomes the elastic wave speed, so that there is no net propagation.

7.8 Final Remarks

The soundness of the basis of the entire field of viscoelastic flows seems questionable to us. For example, the conclusions sometimes depend on the constitutive equations used. Leaving aside subtle questions like the hydrodynamic stability, they display different types of normal stresses and Weissenberg effects in the simplest of all flows—a steady

viscometric flow. It is highly desirable that the normal stress functions of a real viscoelastic liquid be thoroughly investigated experimentally. The various existing studies on this subject have failed to agree even on the sign of these functions. It is suggested that many more studies should be undertaken to settle this question.

It may be possible that different materials have different behavior. Some constitutive equations may then be appropriate for one fluid and others for another. From a thorough experimental study of the normal stress functions of different materials, a rational ground regarding the choice of the appropriate constitutive equation for that particular material would emerge.

Lastly, too much confidence should not be placed in our explanation of the thickening of the viscous sublayer during the Toms phenomenon, as given in Sections 3.7 and 7.1. Firstly, the disturbances in the sublayer are observed to be of the type of streamwise vortices, for which we have demonstrated a destabilization of the plane Couette flow (and therefore a thinning of the viscous sublayer), if the (second-order) fluid is to display the positive Weissenberg effect. Secondly, there are some deficiencies in Taylor's argument, as pointed out by Lumley & McMahon (1963). The author therefore feels that there may not be any connection between the Toms phenomenon and the stability of viscoelastic flows; the explanation of the phenomenon may lie elsewhere.

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13. ABSTRACT

Three problems in the stability of viscoelastic flows have been theoretically investigated.

The case of plane Couette flow has been solved by the classical energy method of Orr. For the method to be applicable it has been found that the concept of elastic potential energy of a viscoelastic liquid has to be introduced, and an expression has been found for it. For two-dimensional disturbances of any magnitude the presence of elasticity has been found to stabilize the flow. This result suggests that the viscous sublayer thickness must increase during the so-called Toms phenomenon. It has been found that the results are identical for three fluids, namely the second-order fluid and the Walters fluids A' and B', although the normal stress behavior of these fluids are quite different. This insensitivity to the constitutive equation has been shown to be a result of the two-dimensionality of the problem.

Linear stability theory has been used to investigate the stability of two flows of the Oldroyd fluid, namely the plane Poiseuille flow and the circular Couette flow. For both flows the presence of elasticity has been found to destabilize the flow and to increase the critical wavenumber.

As an aside, it has also been shown that the sign, and magnitude, of the Weissenberg effect may depend on the method of observation, namely through the shape of the free surface or through the height difference of radial tappings.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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Poiseuille Flow	8					
Toms Phenomenon	8					
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Weissenberg Effect	8					

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