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On Nonlocal Continuum Mechanics

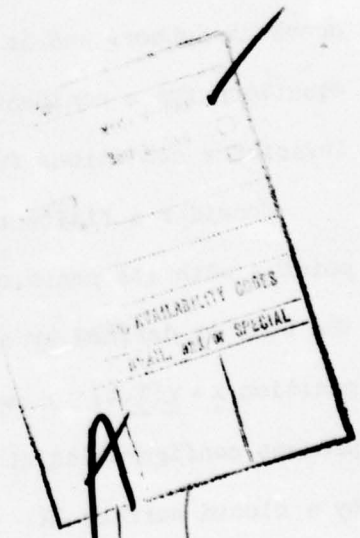
by

A. E. Green[†] and P. M. Naghdi^{*}

Abstract. It is shown in this paper that the conservation laws of nonlocal continuum mechanics are equivalent to, and can be derived from, the conservation of energy and the invariance conditions under superposed rigid body motions. Also, the theory of nonlocal thermoelasticity is reconsidered in the light of recent developments in thermodynamics, taking invariance conditions fully into account.

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1. Introduction

In the mechanics of single phase continua in which nonlocal effects are excluded, it is known that the equations of conservation of mass, momentum and moment of momentum may be derived from the energy equation with the help of invariance conditions under superposed rigid body motions. In the first part of the paper, we prove that the same is true for nonlocal continuum mechanics. From another point of view, this shows that there is consistency between the equations of conservation of mass, momentum and moment of momentum, and the equation of conservation of energy if one accepts the invariance criteria.

Constitutive equations for a nonlocal elastic and thermoelastic solid have been studied by a number of writers (see, e.g. [4,6,7] and references contained therein). There appears to be a number of differences between these authors over invariance conditions under superposed rigid body motions and subsidiary conditions to be satisfied by force, couple and energy residuals. Recently an entirely new approach to continuum thermodynamics has been proposed by Green and Naghdi [8]. In view of the differences between previous authors and in the light of the developments in [8], constitutive equations for a nonlocal thermoelastic solid are reconsidered here taking the invariance conditions fully into account.

Consider a finite body $\mathcal{B}$ with material points X and identify the material point X with its position $\underline{X}$ in a fixed reference configuration. A motion of the body is defined by a sufficiently smooth vector function $\underline{x}$ which assigns position $\underline{x} = \underline{x}(\underline{X}, t)$ to each material point $\underline{X}$ at each instant of time t . In the present configuration at time t , the body $\mathcal{B}$ occupies a region of space $\mathcal{R}$ bounded by a closed surface $\partial\mathcal{R}$. Similarly, in the present configuration, an arbitrary material volume of $\mathcal{B}$ occupies a portion of the region of space $\mathcal{R}$, which we denote by a part $\mathcal{P} (\subseteq \mathcal{R})$ bounded by a closed surface $\partial\mathcal{P}$. Let $\rho = \rho(\underline{X}, t)$ be the mass density in the present configuration and designate the velocity vector

at time t by $\underline{\dot{x}} = \underline{\dot{x}}$, where a superposed dot stands for the material time derivative. Throughout the paper, we assume that mass is conserved locally.

We assume that in its present configuration, the body $\mathcal{B}$ is acted upon by an external body force $\underline{b} = \underline{b}(\underline{X}, t)$ per unit mass and an external surface force $\underline{\bar{t}}$ per unit area acting on the boundary $\partial\mathcal{R}$. The rate of work of these external forces are, respectively, $\underline{b} \cdot \underline{v}$ per unit mass and $\underline{\bar{t}} \cdot \underline{v}$ per unit area. We also admit the existence of an internal body force $\underline{f} = \underline{f}(\underline{X}, t)$ and an internal body couple represented by an axial vector $\underline{\lambda} = \underline{\lambda}(\underline{X}, t)$, each per unit mass, with corresponding rates of work per unit mass equal to $\underline{f} \cdot \underline{v}$ and $\frac{1}{2} \underline{\lambda} \cdot \underline{w}$, respectively, where $\underline{w} = \text{curl } \underline{v}$ is the (axial) vorticity vector. In addition, there is an internal surface force $\underline{t} = \underline{t}(\underline{X}, t; \underline{n})$, acting over each surface $\partial\mathcal{P}$ with outward unit normal $\underline{n}$, such that $\underline{t} \cdot \underline{v}$ represents rate of work per unit area; the field $\underline{t}$, called the stress vector, is measured per unit area of $\partial\mathcal{P}$ and assumes the value $\underline{\bar{t}}$ on $\partial\mathcal{R}$.

With reference to the thermal properties of the body, we first introduce the absolute temperature at each material point by a scalar field $\theta = \theta(\underline{X}, t) > 0$, as well as an external rate of supply of heat $r = r(\underline{X}, t)$ per unit mass and an external rate of surface supply of heat $-\bar{h}$ per unit area acting across $\partial\mathcal{R}$. Also, we admit the existence of an internal surface flux of heat $-h = -h(\underline{X}, t; \underline{n})$ per unit area across the surface $\partial\mathcal{P}$; the field h , called the heat flux and measured per unit area per unit time, assumes the value $\bar{h}$ on $\partial\mathcal{R}$. We define the ratios of r and h to temperature by

$$s = s(\underline{X}, t) = \frac{r}{\theta}, \quad k = k(\underline{X}, t; \underline{n}) = \frac{h}{\theta} \quad (1.1)$$

and refer to $(1.1)_{1,2}$ as the external rate of supply of entropy per unit mass and the internal surface flux of entropy per unit area of $\partial\mathcal{P}$, respectively; the latter assumes the value $\bar{k} = \bar{h}/\theta$ on the external boundary of $\mathcal{B}$, where $\bar{k}$ is

the external rate of surface supply of entropy per unit area of ∂R . Further, we assume the existence of the specific entropy specified by a scalar field $\eta = \eta(\underline{X}, t)$ and an internal rate of production of entropy $\xi = \xi(\underline{X}, t)$ per unit mass, with a contribution to the internal rate of production of heat equal to $\theta \xi$ per unit mass. In addition, we admit the existence of a potential $\epsilon = \epsilon(\underline{X}, t)$ called the specific internal energy and a residual rate of production of energy $\phi = \phi(\underline{X}, t)$ per unit mass.

2. The energy equation. Invariance conditions.

The balance of energy for an arbitrary material volume occupying a part P in the present configuration can be stated as

$$\begin{aligned} \frac{d}{dt} \int_P \rho(\epsilon + \frac{1}{2} \underline{\underline{v}} \cdot \underline{\underline{v}}) dv \\ = \int_P \rho(r + \underline{\underline{b}} \cdot \underline{\underline{v}} + \underline{\underline{f}} \cdot \underline{\underline{v}} + \frac{1}{2} \underline{\underline{\lambda}} \cdot \underline{\underline{w}} + \phi) dv + \int_{\partial P} (\underline{\underline{t}} \cdot \underline{\underline{v}} - h) da . \end{aligned} \quad (2.1)$$

Under superposed rigid body motions represented by

$$\underline{\underline{x}}^+ = \underline{\underline{c}} + Q \underline{\underline{x}} , \quad (2.2)$$

where $\underline{\underline{c}}(t)$ is a vector function and $Q(t)$ a proper orthogonal tensor function of time, we assume that the mass density ρ , the heat supply r , and the fields

$$\eta, \theta, \epsilon, \phi, h, \underline{\underline{f}}, \underline{\underline{\lambda}}, \underline{\underline{t}} , \quad (2.3)$$

all are unaltered apart from orientation in the case of vectors. Thus, if $\eta, \theta, \rho, r, \epsilon, \dots, \underline{\underline{t}}$ become $\eta^+, \theta^+, \dots, \underline{\underline{t}}^+$ under superposed rigid body motions, we have

$$\begin{aligned} \eta^+ &= \eta , \quad \theta^+ = \theta , \\ \rho^+ &= \rho , \quad r^+ = r , \quad \epsilon^+ = \epsilon , \quad \phi^+ = \phi , \quad h^+ = h , \end{aligned} \quad (2.4)$$

$$\underline{\underline{f}}^+ = Q \underline{\underline{f}} , \quad \underline{\underline{\lambda}}^+ = Q \underline{\underline{\lambda}} , \quad \underline{\underline{t}}^+ = Q \underline{\underline{t}} .$$

In addition to (2.1), which holds for every part $P (\subseteq \mathcal{R})$, we assume that the energy equation for the whole body is the same as that for local continuum mechanics so that

$$\frac{d}{dt} \int_R (\epsilon + \frac{1}{2} \underline{\underline{v}} \cdot \underline{\underline{v}}) dv = \int_R \rho(r + \underline{\underline{b}} \cdot \underline{\underline{v}}) dv + \int_{\partial R} (\underline{\underline{t}} \cdot \underline{\underline{v}} - \bar{h}) da . \quad (2.5)$$

It follows from (2.1) and (2.5) that

$$\int_R \rho(\underline{\underline{f}} \cdot \underline{\underline{v}} + \frac{1}{2} \underline{\underline{\lambda}} \cdot \underline{\underline{w}} + \phi) dv = 0 . \quad (2.6)$$

For local continuum mechanics, a derivation of field equations and jump conditions from the energy equation has been given by Green and Rivlin [1] and by Green and Naghdi [2]. Here we employ a method of derivation which is modelled on that used by Naghdi [3, p. 487] in a different context. Let $\underline{\underline{\Omega}}$ be a second order tensor-valued function of time defined by $\underline{\underline{\Omega}}(t) = \dot{\underline{\underline{Q}}}(t) \underline{\underline{Q}}(t)^T$, where the notation $\underline{\underline{Q}}^T$ stands for the transpose of $\underline{\underline{Q}}$. Then, since $\underline{\underline{Q}}$ is a proper orthogonal tensor, we have

$$\dot{\underline{\underline{Q}}} = \underline{\underline{\Omega}} \underline{\underline{Q}} , \quad \underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T . \quad (2.7)$$

Hence, there exists an axial vector $\underline{\underline{w}}$ corresponding to the skew tensor $\underline{\underline{\Omega}}$ such that

$$\underline{\underline{\Omega}} \underline{\underline{a}} = \underline{\underline{w}} \times \underline{\underline{a}} \quad (2.8)$$

for all vectors $\underline{\underline{a}}$. Naghdi [3, p. 453] has shown that $\underline{\underline{Q}}$ can be chosen so that

$$\underline{\underline{\Omega}} = \underline{\underline{\Omega}}_0 , \quad \underline{\underline{w}} = \underline{\underline{w}}_0 , \quad (2.9a)$$

$$\underline{\underline{Q}}(t) = \underline{\underline{I}} , \quad \dot{\underline{\underline{Q}}} = \underline{\underline{\Omega}}_0 , \quad (2.9b)$$

where $\underline{\underline{\Omega}}_0$ is a constant skew-symmetric tensor, $\underline{\underline{w}}_0$ is a constant vector and $\underline{\underline{I}}$ is the identity tensor. With this choice, in all subsequent work we may write

$$\underline{\underline{\Omega}} \underline{\underline{w}} = \underline{\underline{0}} , \quad \frac{d}{dt} \underline{\underline{Q}}^T \underline{\underline{w}} = \underline{\underline{0}} , \quad \underline{\underline{\Omega}}_0 \underline{\underline{a}} \cdot \underline{\underline{a}} = 0 \quad (2.10)$$

for all vectors $\underline{\underline{a}}$.

Consider first a motion of the continuum which differs from the given motion only by a uniform rigid body translational velocity specified by

$$\begin{aligned}\tilde{x}^+ &= \tilde{x} + \tilde{c} \quad , \quad \ddot{\tilde{c}} = \underline{0} \quad , \\ \tilde{v}^+ &= \tilde{v} + \dot{\tilde{c}} \quad , \quad \tilde{w}^+ = \tilde{w} \quad .\end{aligned}\tag{2.11}$$

From (2.4) and (2.11) it follows that $\rho, r, \epsilon, \phi, h, f, \lambda$ and $\tilde{t}$ are all unchanged under this superposed rigid body motion. We assume that the new motion can be maintained by the same body force $\tilde{b}$. The energy equation associated with this motion is obtained from (2.1) by replacing $\tilde{v}$ by $\tilde{v} + \dot{\tilde{c}}$, where $\dot{\tilde{c}}$ is a constant vector. Then, remembering that the volume element dv and surface element da are also unchanged, by subtraction we obtain

$$\frac{1}{2} \dot{\tilde{c}}^2 \frac{d}{dt} \int_{\mathcal{P}} \rho \, dv + \dot{\tilde{c}} \cdot \left[\frac{d}{dt} \int_{\mathcal{P}} \rho \tilde{v} \, dv - \int_{\mathcal{P}} \rho (\tilde{b} + \tilde{f}) \, dv - \int_{\partial \mathcal{P}} \tilde{t} \, da \right] = 0$$

for all arbitrary $\dot{\tilde{c}}$, the quantities in the brackets being independent of $\dot{\tilde{c}}$. Hence,

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dv = 0 \quad ,\tag{2.12}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \tilde{v} \, dv = \int_{\mathcal{P}} \rho (\tilde{b} + \tilde{f}) \, dv + \int_{\partial \mathcal{P}} \tilde{t} \, da \quad .\tag{2.13}$$

These are the equations of conservation of mass and linear momentum. From (2.12) and (2.13), under suitable smoothness assumptions, follow the field equations

$$\dot{\rho} + \rho \operatorname{div} \tilde{v} = 0 \quad ,\tag{2.14}$$

$$\tilde{t} = \tilde{T} \tilde{n} \quad , \quad \operatorname{div} \tilde{T} + \rho (\tilde{b} + \tilde{f}) = \rho \dot{\tilde{v}} \quad ,$$

where $\tilde{T}$ is the stress tensor.

By applying the same invariance conditions to the energy equation (2.5) for the whole body, we obtain

$$\frac{d}{dt} \int_R \rho \underline{v} \, dv = \int_R \rho \underline{b} \, dv + \int_{\partial R} \underline{t} \, da . \quad (2.15)$$

From comparison of (2.15) with (2.13), we conclude that

$$\int_R \rho \underline{f} \, dv = 0 . \quad (2.16)$$

The condition (2.16) also follows from (2.6) by a similar consideration of invariance. It states that the total internal force acting on the whole body is zero.

Next, we consider a superposed rigid body motion specified by

$$\underline{x}^+ = Q \underline{x} \quad (2.17)$$

and restrict our attention to the case in which the conditions (2.9) hold.

Then,[§]

$$\underline{v}^+ = Q \underline{v} + \dot{Q} Q^T \underline{x} , \quad (2.18)$$

$$\dot{\underline{v}}^+ = \dot{\underline{v}} + 2\dot{Q} Q^T \underline{v} + \dot{Q} Q^T \underline{x} , \quad \underline{w}^+ = \underline{w} + 2\omega_0 ,$$

and

$$\underline{f}^+ \cdot \underline{v}^+ = \underline{f} \cdot \underline{v} + \omega_0 \cdot \underline{x} \times \underline{f} , \quad \underline{\lambda}^+ \cdot \underline{w}^+ = \underline{\lambda} \cdot \underline{w} + 2\omega_0 \cdot \underline{\lambda} ,$$

$$\underline{t}^+ \cdot \underline{v}^+ = \underline{t} \cdot \underline{v} + \omega_0 \cdot \underline{x} \times \underline{t} , \quad (2.19)$$

$$\underline{v}^+ \cdot \underline{v}^+ = \underline{v} \cdot \underline{v} + 2Q^T \omega_0 \cdot \underline{x} \times \underline{v} - \omega_0^2 \cdot Q \underline{x} Q^T \underline{x} ,$$

where in obtaining (2.19) use has been made of the requirements (2.4). We now need to consider the invariance condition to be satisfied by the body force $\underline{b}$

[§]The result for $\underline{v}^+$ is recorded without the use of (2.9b) since $\underline{v}^+ \cdot \underline{v}^+$ occurs later in an integral which is differentiated with respect to time.

under superposed rigid body motion (2.17), i.e., we must find the new body force $\tilde{b}^+$ which will sustain the motion (2.17). From (2.14) and (2.4), we obtain the relation⁺

$$\tilde{b}^+ - \dot{\tilde{v}}^+ = Q(\tilde{b} - \dot{\tilde{v}}) \quad (2.20)$$

and the rate of work of body force $\tilde{b}^+$ per unit mass due to $\tilde{v}^+$ is

$$\tilde{b}^+ \cdot \tilde{v}^+ = \tilde{b} \cdot \tilde{v} + \omega_0 \cdot \tilde{x} \times \tilde{b} - \frac{1}{2} \Omega_0^2 \cdot (\tilde{v} \tilde{x} + \tilde{x} \tilde{v}) \quad , \quad (2.21)$$

where (2.9), (2.18) and (2.20) have been used.

Consider now an energy equation of the form (2.1) associated with the motion (2.17) and subtract from it the energy equation (2.1). Then, with the help of (2.10), (2.19) and (2.21), it follows that

$$\begin{aligned} \omega_0 \cdot \left[\frac{d}{dt} \int_P \rho \tilde{x} \times \tilde{v} \, dv - \int_P \rho (\tilde{x} \times (\tilde{b} + \tilde{f}) + \tilde{\lambda}) \, dv - \int_{\partial P} \tilde{x} \times \tilde{t} \, da \right] \\ - \frac{1}{2} \Omega_0^2 \cdot \left[\frac{d}{dt} \int_P \rho (\tilde{Q} \tilde{x}) (\tilde{Q} \tilde{x}) \, dv - \int_P \rho (\tilde{v} \tilde{x} + \tilde{x} \tilde{v}) \, dv \right] = 0 \end{aligned}$$

for all arbitrary ω_0 . From the last result follow the two equations

$$\frac{d}{dt} \int_P \rho \tilde{x} \times \tilde{v} \, dv = \int_P \rho [\tilde{x} \times (\tilde{b} + \tilde{f}) + \tilde{\lambda}] \, dv + \int_{\partial P} \tilde{x} \times \tilde{t} \, da \quad , \quad (2.22)$$

$$\frac{d}{dt} \int_P \rho (\tilde{Q} \tilde{x}) (\tilde{Q} \tilde{x}) \, dv = \int_P \rho (\tilde{v} \tilde{x} + \tilde{x} \tilde{v}) \, dv \quad , \quad (2.23)$$

the first of which represent the conservation of moment of momentum. With the use of (2.14), the field equation resulting from (2.22) can be reduced to

$$\rho \tilde{\Gamma} + \tilde{T} - \tilde{T}^T = 0 \quad , \quad (2.24)$$

⁺Alternatively, the result (2.20) can be obtained directly from (2.13) and (2.14); in this connection, see [3, p. 486].

where $\underline{\Gamma}$ is the skew tensor corresponding to the axial vector $\underline{\lambda}$ so that

$$\underline{\Gamma} \underline{a} = \underline{\lambda} \times \underline{a} \quad (2.25)$$

for all arbitrary vectors $\underline{a}$.

Suppose now that there are no surfaces of discontinuity in ρ . Then, in view of (2.9) and (2.14)₁, the local equation resulting from (2.23) is satisfied identically and yields no new information. The same conclusion holds even if the mass density ρ has a jump discontinuity across a surface $\mathcal{P}$, assuming that $\underline{x}$ is continuous.

If we consider the superposed rigid body motion (2.17) and use the energy equation (2.5), by a procedure similar to that which led to (2.22) we find

$$\frac{d}{dt} \int_{\mathcal{R}} \rho \underline{x} \times \underline{v} \, dv = \int_{\mathcal{R}} \rho \underline{x} \times \underline{b} \, dv + \int_{\partial \mathcal{R}} \underline{x} \times \underline{t} \, da \quad (2.26)$$

Comparison of (2.26) and (2.22) implies that

$$\int_{\mathcal{R}} \rho (\underline{x} \times \underline{f} + \underline{\lambda}) \, dv = 0 \quad (2.27)$$

The condition (2.27) may also be deduced from a similar consideration of invariance of (2.6) under the superposed rigid body motion (2.17). It states that the total internal couple acting on the whole body is zero. We observe that (2.16) can also be derived from (2.27) with the help of invariance conditions. If we make use of (2.24) and (2.25) the condition (2.27) can be put in the alternative form

$$\int_{\mathcal{R}} [\rho \{ \underline{x} \otimes \underline{f} - \underline{f} \otimes \underline{x} \} + \underline{T} - \underline{T}^T] \, dv = 0 \quad (2.28)$$

For later convenience we set

$$\mu = \phi + \underline{f} \cdot \underline{v} + \frac{1}{2} \underline{\lambda} \cdot \underline{w} \quad (2.29)$$

so that, from (2.6), we have

$$\int_R \rho \mu \, dv = 0 \quad . \quad (2.30)$$

In view of the invariance conditions (2.4), (2.11) and (2.18), it follows that μ is not unaltered by superposed rigid body motions. The energy equation (2.1) in which we use the substitution (2.29) for μ , together with (2.5) and the derived equations (2.16), (2.12), (2.13), (2.15), (2.16), (2.22), (2.26), (2.27), (2.28) and (2.30) are equivalent to those postulated by Edelen, Green and Laws [4] and Green and Naghdi [5], apart from notational differences. They are also similar to those employed by Eringen and Edelen [6] and Eringen [7] except that these authors allowed for nonlocal mass changes. However, in [6], a different view from that given here is adopted for the invariance condition to be satisfied by the function which corresponds to μ .

3. Balance of entropy

Following the recent approach of Green and Naghdi [8] to thermodynamics, we assume a balance of entropy in the form

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \eta \, dv = \int_{\mathcal{P}} \rho (s + \xi) \, dv - \int_{\partial \mathcal{P}} k \, da \quad (3.1)$$

for every material volume occupying a part $\mathcal{P}$ in the present configuration, where s and k are defined by (1.1). The field equation resulting from (3.1) is

$$\rho \dot{\eta} = \rho (s + \xi) - \operatorname{div} \underline{p} \quad , \quad (3.2)$$

where the entropy flux and the heat flux vectors $\underline{p}$ and $\underline{q}$ are related by

$$\underline{q} = \theta \underline{p} \quad , \quad k = \underline{p} \cdot \underline{n} \quad , \quad h = \underline{q} \cdot \underline{n} \quad . \quad (3.3)$$

Invariance conditions additional to those in (2.4) are

$$k^+ = k \quad , \quad \underline{p}^+ = \underline{Q} \underline{p} \quad , \quad \underline{q}^+ = \underline{Q} \underline{q} \quad , \quad \xi^+ = \xi \quad , \quad s^+ = s \quad . \quad (3.4)$$

With the help of (2.14), (2.29), (3.2) and (3.3), the field equation resulting from the energy equation (2.1) can be reduced to

$$\begin{aligned} & - \rho (\dot{\psi} + \eta \dot{\theta}) + \underline{T} \cdot \operatorname{grad} \underline{v} - \rho \underline{f} \cdot \underline{v} + \rho \mu \\ & - \rho \xi \theta - \underline{p} \cdot \operatorname{grad} \theta = 0 \quad , \end{aligned} \quad (3.5)$$

where the specific Helmholtz free energy $\psi = \psi(X, t)$ is defined by $\psi = \epsilon - \eta \theta$.

In view of (2.4) and (3.4) it is clear that μ cannot, in general, be invariant under superposed rigid body motions if the energy equation is to remain invariant. The behavior of μ under such motions, which is determined by (2.4), (2.24) and (2.29), is precisely that which will render (3.5) invariant.

Once constitutive equations have been specified for

$$\psi, \eta, \mu, \xi, \underline{f}, \underline{\lambda}, \underline{p}, \underline{T} \quad (3.6)$$

the energy equation (3.5) is regarded as an identity for all processes. We also regard (2.24) and the subsidiary conditions (2.16), (2.28) and (2.30) as identities to be satisfied for all thermo-mechanical processes, i.e., for all motions and all temperature distributions. As far as the last four of these subsidiary conditions are concerned, the view taken here appears to differ from that adopted in [6,7].

When the body is in equilibrium with $\underline{v} = \underline{0}$ and all constitutive variables independent of the time (but dependent on $\underline{\chi}$), we assume that ϕ , and hence μ , vanishes. Then, it follows from (3.5) and (3.2) that

$$\rho \xi \theta + \underline{p} \cdot \text{grad } \theta = 0, \quad (3.7)$$

$$\rho(s + \xi) - \text{div } \underline{p} = 0,$$

or

$$\rho r - \text{div } \underline{q} = 0, \quad (3.8)$$

It was observed in [8] that the quantities ξ, η, ψ may be arbitrary to the extent of additive functions $\dot{f}, f, -\theta f$, respectively, where f is an arbitrary function of $\underline{\chi}, \theta$, their space and time derivatives and a functional of their past histories throughout $\mathcal{R}$. The additive functions have the property that they make no contribution to the differential equations for $\underline{\chi}, \theta$ or the energy identity (3.5), boundary and initial conditions. The arbitrariness was removed by setting

$$\dot{f} = 0. \quad (3.9)$$

The same situation holds for the present nonlocal theory except that now f may

also be a functional of χ, θ over the whole region $\mathcal{R}$. The arbitrariness is again removed by the specification (3.9).

A mathematical interpretation of the second law of thermodynamics was shown in [8] to lead to an inequality which reflects the fact that, for every process associated with a dissipative material, a part of the external mechanical work supplied to a material volume $\mathcal{P}$ is always converted into heat and cannot be withdrawn from $\mathcal{P}$ as mechanical work. We assume that the constitutive response functions for ϵ, η include dependence on the set of variables $\underline{v}', \dot{\theta}', \underline{g}'$ and their higher space and time derivatives, where a superposed prime denotes the values of the variables at any point in the present configuration $\mathcal{R}$ [see Eq. (4.21)]. We refer to this set of variables collectively as $\mathcal{W}$. Further, let ϵ^*, η^* denote the respective values of ϵ, η when the variables $\mathcal{W}$ are set equal to zero in the response functions. Thus, for example,

$$\epsilon = \epsilon(\underline{F}', \underline{x}', \theta', \mathcal{W}) \quad , \quad \epsilon^* = \epsilon(\underline{F}', \underline{x}', \theta', 0) \quad , \quad \mathcal{W} = (\underline{v}', \dot{\theta}', \underline{g}', \dots) \quad , \quad (3.10)$$

where the dots in $(3.10)_3$ refer to the higher space and time derivatives of $\underline{v}', \dot{\theta}', \underline{g}'$. Then, the inequality for nonlocal continuum mechanics corresponding to that obtained previously for local continuum mechanics (see (4.3) of [8]) is

$$\int_{\mathcal{R}} \rho w \, dv \geq 0 \quad , \quad (3.11)$$

where

$$\begin{aligned} \rho w = & -\rho(\dot{\psi}^* + \eta^* \dot{\theta}) + \underline{T} \cdot \text{grad } \underline{v} - \rho \underline{f} \cdot \underline{v} = \rho[\dot{\psi} - \dot{\psi}^* + (\eta - \eta^*) \dot{\theta}] + \rho \underline{\xi} \theta + \underline{p} \cdot \underline{g} - \rho \mu \quad , \\ & \psi^* = \epsilon^* - \theta \eta^* \quad . \end{aligned} \quad (3.12)$$

In addition to a thermodynamic restriction corresponding to (3.11), two further inequalities which place restrictions on the heat conduction vector and internal energy are proposed in [8, Sec. 4]. But the corresponding additional inequalities will not be needed in the present paper.

4. Nonlocal thermoelasticity

As noted in section 1, a number of writers have studied nonlocal elasticity and thermoelasticity in the context of nonlinear continuum mechanics. Without giving an extensive bibliography, we refer to [4,6,7] and references cited therein. An examination of these papers shows that there is no clear agreement on the constitutive theory of thermoelasticity. By way of additional background, we recall that a fairly general class of thermoelastic solids was considered in [4]; and, although this work was not the most general possible within the framework of the nonlocal theory, the results given satisfy all the invariance conditions and subsidiary equations in a manner which is in line with those of the present paper. More general constitutive assumptions were made initially in [6,7], but the results in [6,7] in the case of zero heat conduction appear to be different from each other, partly because of the difference of interpretation of invariance conditions.

We introduce further notations by

$$\begin{aligned} \underline{g} &= \text{grad } \theta, \quad \underline{F} = \text{Grad } \underline{\chi}, \quad \dot{\underline{F}} = \underline{L} \underline{F}, \quad \underline{L} = \text{grad } \underline{v}, \\ \underline{T} &= (\rho/\rho_0) \underline{P} \underline{F}^T, \quad \underline{T} \cdot \underline{L} = (\rho/\rho_0) \underline{P} \cdot \dot{\underline{F}}, \quad \dot{\underline{g}} = \text{grad } \dot{\theta} - \underline{L}^T \underline{g}, \end{aligned} \quad (4.1)$$

where the operator *grad* is with respect to $\underline{x}$, the operator *Grad* with respect to $\underline{X}$ and ρ_0 is the density of the solid in a reference configuration. Also,

$$\begin{aligned} \rho &= \rho(\underline{X}, t), \quad \theta = \theta(\underline{X}, t), \quad \underline{g} = \underline{g}(\underline{X}, t), \quad \underline{x} = \underline{\chi}(\underline{X}, t), \quad \underline{F} = \underline{F}(\underline{X}, t), \quad \underline{v} = \dot{\underline{\chi}}(\underline{X}, t), \\ \rho' &= \rho(\underline{X}', t), \quad \theta' = \theta(\underline{X}', t), \quad \underline{g}' = \underline{g}(\underline{X}', t), \quad \underline{x}' = \underline{\chi}(\underline{X}', t), \quad \underline{F}' = \underline{F}(\underline{X}', t), \quad \underline{v}' = \dot{\underline{\chi}}(\underline{X}', t), \\ \rho_0 &= \rho_0(\underline{X}), \quad \rho'_0 = \rho_0(\underline{X}'), \quad dV = dV(\underline{X}), \quad dV' = dV(\underline{X}'), \\ dv &= dv(\underline{X}, t), \quad dv' = dv(\underline{X}', t). \end{aligned} \quad (4.2)$$

In conformity with (4.2), henceforth we use a superposed prime to designate the operation of replacing $\underline{x}$ by $\underline{x}'$ and $\underline{x}'$ by $\underline{x}$. For example, corresponding to

$$f = f(\theta, \underline{g}, \underline{x}, \underline{F}, \theta', \underline{g}', \underline{x}', \underline{F}', \underline{x}, \underline{x}') ,$$

we write

$$f' = f(\theta', \underline{g}', \underline{x}', \underline{F}', \theta, \underline{g}, \underline{x}, \underline{F}, \underline{x}', \underline{x}) .$$

For a nonlocal thermoelastic solid, we adopt the constitutive assumptions that

$$\psi, \eta, \underline{T}, \underline{f}, \underline{\xi}, \underline{p}, \underline{\mu}, \underline{\lambda} \quad (4.3)$$

are functions of

$$\theta, \underline{g}, \underline{x}, \underline{F}, \underline{x} \quad (4.4)$$

and functionals of

$$\theta', \underline{g}', \underline{x}', \underline{F}', \underline{x}' \quad (4.5)$$

over the whole region $\mathcal{R}$. In equilibrium, the deformation function $\underline{x}$ and the temperature θ on the right-hand sides of $(4.2)_{2,4}$ and $(4.2)_{8,10}$ are independent of t but can be chosen arbitrarily. The function $\underline{\mu}$ vanishes in equilibrium and hence vanishes when the functions in (4.4) and (4.5) depend also on t . Subsequent investigations shows that $\underline{\mu} = 0$ yields an inadequate description for nonlocal thermoelasticity. Inspection of the energy equation (3.5) as an identity, particularly the term $\dot{\psi}$, suggests that we allow $\underline{\mu}$ to be a linear function of degree one in

$$\dot{\theta}, \dot{\underline{g}}, \dot{\underline{F}}, \underline{v} \quad (4.6)$$

and a linear functional of degree one in

$$\dot{\theta}', \dot{g}', \dot{F}', \dot{v}' \quad (4.7)$$

over $\mathcal{R}$, as well as depending nonlinearly on (4.4) and (4.5).

Before proceeding further and in order to simplify the rest of our discussion, it is convenient to dispose of some mathematical preliminaries. For this purpose, we first introduce the vector space $\mathcal{V}$ of all quadruplets $(a, \underline{a}, \underline{b}, \underline{A})$ consisting of the scalar a , the two vectors $\underline{a}, \underline{b}$ and the tensor $\underline{A}$. The space $\mathcal{V}$ is a 16-dimensional vector space with the inner product of the two quadruplets $(a, \underline{a}, \underline{b}, \underline{A})$ and $(c, \underline{c}, \underline{d}, \underline{B})$ as

$$(a, \underline{a}, \underline{b}, \underline{A}) \cdot (c, \underline{c}, \underline{d}, \underline{B}) = ac + \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{d} + \underline{A} \cdot \underline{B} \quad (4.8)$$

and with the norm

$$|(a, \underline{a}, \underline{b}, \underline{A})| = [(a, \underline{a}, \underline{b}, \underline{A}) \cdot (a, \underline{a}, \underline{b}, \underline{A})]^{\frac{1}{2}} \quad (4.9)$$

Let $\mathcal{U}$ be an open and connected subset of $\mathcal{V}$ consisting of those quadruplets $(a, \underline{a}, \underline{b}, \underline{A})$ with $a > 0$ and $\det \underline{A} > 0$. Then, $(\theta, \underline{g}, \underline{X}, \underline{F})$ certainly lies in $\mathcal{U}$. We assume all variables in (4.4) to (4.7) are continuous functions of $\underline{X}$ in $\mathcal{R}$ for all times and introduce the abbreviation

$$\underline{\Lambda}(\cdot) = (\theta(\cdot), \underline{g}(\cdot), \underline{X}(\cdot), \underline{F}(\cdot)) \quad (4.10)$$

As will become apparent shortly, the constitutive relations to be introduced below lead us to the study of the functional relations of the form

$$f(\underline{X}) = f(\underline{\Lambda}(\underline{X}'), \underline{\Lambda}(\underline{X})) = \bar{F}(\underline{\Lambda}(\cdot), \underline{\Lambda}(\underline{X})) \quad , \quad (4.11)$$

$$\underline{X}' \in \mathcal{R}_0$$

where dependence on t is understood and where $\mathcal{R}_0$ is the region occupied by the finite body $\mathcal{B}$ in its reference configuration.

Let $\gamma(\cdot)$ be a given influence function which is positive, monotone decreasing and continuous in $\mathcal{R}_0$ and introduce the norm

$$\|\underline{\Gamma}(\cdot)\| = \left[\int_{\mathcal{R}_0} \gamma(|\underline{x}-\underline{x}'|) \underline{\Gamma}(\underline{x}') \cdot \underline{\Gamma}(\underline{x}') dV' \right]^{\frac{1}{2}} \quad (4.12)$$

for all continuous functions $\underline{\Gamma}(\cdot)$ defined for every $\underline{x}' \in \mathcal{R}_0$ whose values lie in $\mathcal{U}$ and for definiteness also specify $\gamma(0)=1$. The collection of all such functions $\underline{\Gamma}(\cdot)$ forms a pre-Hilbert space $\mathcal{H}$ in which the inner product for the two functions $\underline{\Gamma}_1(\cdot)$ and $\underline{\Gamma}_2(\cdot)$ is

$$\langle \underline{\Gamma}_1(\cdot), \underline{\Gamma}_2(\cdot) \rangle = \int_{\mathcal{R}_0} \gamma(|\underline{x}-\underline{x}'|) \underline{\Gamma}_1(\underline{x}') \cdot \underline{\Gamma}_2(\underline{x}') dV' \quad (4.13)$$

We suppose that the functional $\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda})$ is defined for every $\underline{\Lambda}$ in $\mathcal{U}$ and for every function $\underline{\Gamma}(\cdot)$ in $\mathcal{H}$. Then, $\bar{f}$ is said to be continuous if for each fixed $\underline{\Gamma}(\cdot)$ and for every $\underline{v}(\cdot)$ in $\mathcal{H}$,

$$\bar{f}(\underline{\Gamma}(\cdot) + \underline{v}(\cdot), \underline{\Lambda} + \underline{\pi}) - \bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda}) \rightarrow 0 \quad (4.14)$$

as $|\underline{\pi}| + \|\underline{v}(\cdot)\| \rightarrow 0$. The functional $\bar{f}$ is continuously differentiable if the relation (4.14) can be replaced by

$$\frac{1}{|\underline{\pi}| + \|\underline{v}(\cdot)\|} \left| \bar{f}(\underline{\Gamma}(\cdot) + \underline{v}(\cdot), \underline{\Lambda} + \underline{\pi}) - \bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda}) - D\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda}) \cdot \underline{\pi} - \delta\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda})[\underline{v}(\cdot)] \right| \rightarrow 0 \quad (4.15)$$

as $|\underline{\pi}| + \|\underline{v}(\cdot)\| \rightarrow 0$. Here $D\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda})$ is a continuous functional with values in the space $\mathcal{U}$ and is simply the partial derivative of $\bar{f}$ with respect to $\underline{\Lambda}$ holding $\underline{\Gamma}(\cdot)$ fixed, i.e.,

$$D\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda}) = \frac{\partial \bar{f}}{\partial \underline{\Lambda}}(\underline{\Gamma}(\cdot), \underline{\Lambda}) \quad (4.16)$$

Also, for each fixed $\underline{\Gamma}(\cdot)$ and $\underline{\Lambda}$, $\delta\bar{f}(\underline{\Gamma}(\cdot), \underline{\Lambda})[\underline{v}(\cdot)]$ is a continuous scalar-valued functional defined in $\mathcal{H}$ which is linearly dependent on $\underline{v}(\cdot)$ and is continuously

dependent on the pairs $(\underline{\Gamma}(\cdot), \underline{\Lambda})$. If the functional $\bar{f}$ is continuously differentiable in the sense just described, then the derivative of the functional f in (4.11) with respect to t is given by

$$\dot{f} = \frac{\partial \bar{f}}{\partial \underline{\Lambda}} (\underline{\Lambda}(\cdot), \underline{\Lambda}) \cdot \dot{\underline{\Lambda}} + \delta \bar{f}(\underline{\Lambda}(\cdot), \underline{\Lambda}) \dot{\underline{\Lambda}}(\underline{X}') \quad (4.17)$$

Moreover, the functional $\delta \bar{f}(\underline{\Lambda}(\cdot), \underline{\Lambda}) \dot{\underline{\Lambda}}(\underline{X}')$, which is linear in $\dot{\underline{\Lambda}}(\underline{X}')$ admits the representation

$$\begin{aligned} \delta \bar{f} &= \int_{R_0} \underline{\Phi}(\underline{X}, \underline{X}') \cdot \dot{\underline{\Lambda}}(\underline{X}') (\rho'_0 / \rho') dV' \\ &= \int_R \underline{\Phi}(\underline{X}, \underline{X}') \cdot \dot{\underline{\Lambda}}(\underline{X}') dv' \quad , \end{aligned} \quad (4.18)$$

where $\underline{\Phi}(\underline{X}, \underline{X}')$ is specified by

$$\underline{\Phi}(\underline{X}, \underline{X}') = \hat{\underline{\Phi}} \left(\underline{\Lambda}(\underline{X}''), \underline{\Lambda}(\underline{X}'), \underline{\Lambda}(\underline{X}), \underline{X}'', \underline{X}', \underline{X} \right) \quad (4.19)$$

$$\underline{X}'' \in R_0$$

and $\hat{\underline{\Phi}}$ is a functional of $\underline{X}''$ over R_0 .

Returning to our main objective concerning the constitutive equations in nonlocal elasticity, we recall that in obtaining the local field equations we have already tacitly introduced the usual continuity and differentiability assumptions, which must be satisfied by the various fields and functions. Moreover, as noted above, it has been also assumed that all quantities in (4.4) to (4.7) are continuous functions of $\underline{X}$ throughout R for all times. We now introduce the constitutive assumptions that $\eta, \xi, \underline{f}, \underline{T}, \underline{p}, \underline{\mu}, \underline{\lambda}$ are continuous functionals of the type (4.11), and that ψ , in addition, is a continuously differentiable functional in the sense defined above. We also assume that μ is a continuous functional of $\dot{\underline{\Lambda}}(\cdot)$ and a continuous function of $\dot{\underline{\Lambda}}(\underline{X})$. Then, if $\underline{\Phi}$ is the quadruplet

$$\underline{\Phi} = (\underline{m}, \underline{m}, \underline{n}, \underline{K}) \quad , \quad (4.20)$$

we may write

$$\begin{aligned} \dot{\Psi} = & \frac{\partial \Psi}{\partial \theta} \dot{\theta} + \frac{\partial \Psi}{\partial \underline{g}} \cdot \dot{\underline{g}} + \frac{\partial \Psi}{\partial \underline{x}} \cdot \dot{\underline{x}} + \frac{\partial \Psi}{\partial \underline{F}} \cdot \dot{\underline{F}} \\ & + \int_R (\underline{m} \dot{\theta}' + \underline{m} \cdot \dot{\underline{g}}' + \underline{n} \cdot \dot{\underline{x}}' + \underline{K} \cdot \dot{\underline{F}}') dv' \quad . \end{aligned} \quad (4.21)$$

The last expression can be rewritten in the following form:

$$\begin{aligned} \rho \dot{\Psi} = & - \rho \hat{\eta} \dot{\theta} + \underline{\ell} \cdot \text{grad } \dot{\theta} + (\rho/\rho_0) \hat{\underline{P}} \cdot \dot{\underline{F}} \\ & - \underline{g} \otimes \underline{\ell} \cdot \underline{L} - \rho \hat{\underline{f}} \cdot \underline{v} + \rho \hat{\underline{\mu}} \quad , \end{aligned} \quad (4.22)$$

where

$$\begin{aligned} - \rho \hat{\eta} = & \rho \frac{\partial \Psi}{\partial \theta} + \int_R \rho' \underline{m}' dv' \quad , \quad \underline{\ell} = \rho \frac{\partial \Psi}{\partial \underline{g}} + \int_R \rho' \underline{m}' dv' \quad , \\ (\rho/\rho_0) \hat{\underline{P}} = & \rho \frac{\partial \Psi}{\partial \underline{F}} + \int_R \rho' \underline{K}' dv' \quad , \quad - \rho \hat{\underline{f}} = \rho \frac{\partial \Psi}{\partial \underline{x}} + \int_R \rho' \underline{n}' dv' \quad , \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} \rho \hat{\underline{\mu}} = & \int_R [\rho \underline{m} \dot{\theta}' - \rho' \underline{m}' \dot{\theta} + \rho \underline{m} \cdot (\text{grad } \dot{\theta})' - \rho' \underline{m}' \cdot \text{grad } \dot{\theta} \\ & - \rho \underline{m} \cdot \underline{L}'^T \underline{g}' + \rho' \underline{m}' \cdot \underline{L}^T \underline{g} \\ & + \rho \underline{n} \cdot \underline{v}' - \rho' \underline{n}' \cdot \underline{v} \\ & + \rho \underline{K} \cdot \dot{\underline{F}}' - \rho' \underline{K}' \cdot \dot{\underline{F}}] dv' \quad . \end{aligned} \quad (4.24)$$

We observe that $\hat{\underline{\mu}}$ satisfies the condition

$$\int \rho \hat{\underline{\mu}} dv = 0 \quad (4.25)$$

identically.

Since the expression (4.22) for $\rho \dot{\Psi}$ must be invariant under superposed rigid body motions, it follows from (4.22) and (4.24) that $\rho \hat{\underline{f}}$ has the alternative form

$$\hat{\rho} \hat{\mathbf{f}} = \int_R (\rho \mathbf{n} - \rho' \mathbf{n}') dv' \quad (4.26)$$

and

$$\begin{aligned} & \rho(\mathbf{x} \otimes \hat{\mathbf{f}} - \hat{\mathbf{f}} \otimes \mathbf{x}) + \hat{\mathbf{T}} - \hat{\mathbf{T}}^T \\ & + \int_R [\rho \mathbf{n} \otimes \mathbf{x}' - \rho' \mathbf{n}' \otimes \mathbf{x} + \rho \mathbf{K} \mathbf{F}'^T - \rho' \mathbf{K}' \mathbf{F}^T - \rho \mathbf{g}' \otimes \mathbf{m} + \rho' \mathbf{g} \otimes \mathbf{m}'] dv' \\ & - \int_R [\rho \mathbf{n} \otimes \mathbf{x}' - \rho' \mathbf{n}' \otimes \mathbf{x} + \rho \mathbf{K} \mathbf{F}'^T - \rho' \mathbf{K}' \mathbf{F}^T - \rho \mathbf{g}' \otimes \mathbf{m} + \rho' \mathbf{g} \otimes \mathbf{m}']^T dv' = \mathbf{0}, \quad (4.27) \end{aligned}$$

where

$$\hat{\mathbf{T}} = (\rho/\rho_0) \hat{\mathbf{P}} \mathbf{F}^T - \mathbf{g} \otimes \mathbf{l}. \quad (4.28)$$

From (4.26) and (4.27) we see that

$$\int_R \hat{\rho} \hat{\mathbf{f}} dv = \mathbf{0} \quad (4.29)$$

and

$$\int_R [\rho(\mathbf{x} \otimes \hat{\mathbf{f}} - \hat{\mathbf{f}} \otimes \mathbf{x}) + \hat{\mathbf{T}} - \hat{\mathbf{T}}^T] dv = \mathbf{0}. \quad (4.30)$$

The results (4.26) and (4.27) can also be obtained by making explicit the condition that ψ is an invariant function of (4.4) and a functional of (4.5) under superposed rigid body motions, and by using the results (4.23)_{2,3,4} and (4.28) (see, e.g. [4]).

For convenience, we now define $\bar{\eta}, \bar{\mathbf{f}}, \bar{\mu}, \bar{\mathbf{T}}$ by the equations

$$\begin{aligned} \eta &= \hat{\eta} + \bar{\eta}, \quad \mathbf{f} = \hat{\mathbf{f}} + \bar{\mathbf{f}}, \\ \mu &= \hat{\mu} + \bar{\mu}, \quad \mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}}. \end{aligned} \quad (4.31)$$

The functions $\bar{\eta}, \bar{\mathbf{f}}, \bar{\mu}$ satisfy the same type of constitutive equation as $\eta, \mathbf{f}, \mu$ and, in view of (2.16), (2.28), (4.29) and (4.30),

$$\int_R \bar{\rho} \bar{\mathbf{f}} dv = \mathbf{0}, \quad (4.32)$$

$$\int_R [\bar{\rho}(\mathbf{x} \otimes \bar{\mathbf{f}} - \bar{\mathbf{f}} \otimes \mathbf{x}) + \bar{\mathbf{T}} - \bar{\mathbf{T}}^T] dv = \mathbf{0}. \quad (4.33)$$

The function $\bar{\mu}$ has the same type of constitutive equation as μ and because of

(2.30) and (4.25),

$$\int_R \rho \bar{\mu} dv = 0 \quad (4.34)$$

Since the energy equation (3.5) is regarded as an identity for all processes and since ξ and $\underline{p}$ are independent of the rates (4.6) and (4.7) and μ is linear of degree one in these variables, with the help of (4.22) and (4.31), it follows from (3.5) that

$$\rho \xi \dot{\theta} + \underline{p} \cdot \text{grad } \dot{\theta} = 0 \quad (4.35)$$

and

$$-\rho \bar{\eta} \dot{\theta} - \underline{\ell} \cdot \text{grad } \dot{\theta} - \rho \bar{f} \cdot \underline{v} + \bar{T} \cdot \text{grad } \underline{v} + \rho \bar{\mu} = 0 \quad (4.36)$$

From (4.34) and (4.36) we have

$$\int_R [(\rho \bar{\eta} - \text{div } \underline{\ell}) \dot{\theta} + (\rho \bar{f} + \text{div } \bar{T}) \cdot \underline{v}] dv - \int_{\partial R} (\bar{T} \underline{n} \cdot \underline{v} - \dot{\theta} \underline{\ell} \cdot \underline{n}) da = 0 \quad (4.37)$$

for all processes. For given values of the variables (4.4) and (4.5) we may choose $\dot{\theta}$ and $\underline{v}$ arbitrarily and independently throughout $R \cup \partial R$, so that

$$\rho \bar{\eta} - \text{div } \underline{\ell} = 0, \quad \rho \bar{f} + \text{div } \bar{T} = 0 \quad \text{in } R, \quad (4.38)$$

$$\bar{T} \underline{n} = 0, \quad \underline{\ell} \cdot \underline{n} = 0 \quad \text{on } \partial R. \quad (4.39)$$

These are restrictions on constitutive equations for $\bar{\eta}, \underline{\ell}, \bar{T}$ to be satisfied for all processes. Also, from (4.36) and (4.38) we have

$$-\text{div}(\underline{\ell} \dot{\theta} - \bar{T}^T \underline{v}) + \rho \bar{\mu} = 0 \quad (4.40)$$

Again, this is a restriction on the constitutive equations for $\underline{\ell}, \bar{T}, \bar{\mu}$ for all processes.

For the class of constitutive equations considered here it is difficult to make explicit the restrictions implied by (4.38) to (4.40). Given these

restrictions then the subsidiary conditions (4.32) to (4.34) are satisfied.

If the Helmholtz function ψ does not depend explicitly on $\underline{g}, \underline{g}'$, then from (4.23)₂, (4.38) and (4.40) we see that

$$\bar{\eta} = 0, \quad \bar{\ell} = 0, \quad \rho \bar{\mu} + \operatorname{div}(\bar{T}^T \underline{v}) = 0, \quad (4.41)$$

while (4.28) becomes

$$\hat{T} = (\rho/\rho_0) \hat{P} \hat{F}^T. \quad (4.42)$$

In view of (4.38) and (4.39) we see that the internal body force $\bar{f}$ and stress $\bar{T}$ make no contribution to field equations and yield no applied stress on ∂R . It may happen that from the identities (4.38), (4.42) and (4.41) it follows that

$$\bar{T} = 0, \quad \bar{f} = 0, \quad \bar{\mu} = 0, \quad (4.43)$$

but we have not been able to prove this. In view of the properties associated with $\bar{f}, \bar{T}$ it seems natural to place special emphasis on the situation in which (4.43) hold. Given (4.41) to (4.43), it follows from (4.31) that

$$\eta = \hat{\eta}, \quad \underline{f} = \hat{\underline{f}}, \quad \mu = \hat{\mu}, \quad \underline{T} = \hat{\underline{T}} = (\rho/\rho_0) \hat{P} \hat{F}^T, \quad (4.44)$$

where $\hat{\eta}, \dots, \hat{P}$ are given by (4.23)_{1,3,4}, (4.24) and (4.26). The field equation (2.14) reduces to

$$\operatorname{div} \hat{\underline{T}} + \rho(\hat{\underline{b}} + \hat{\underline{f}}) = \rho \dot{\underline{v}} \quad (4.45)$$

while, with the help of (4.35), equation (3.2) becomes

$$\rho \theta \dot{\eta} = \rho r - \operatorname{div} \underline{q}. \quad (4.46)$$

Constitutive equations for $\underline{T}, \lambda$ or equivalently $\underline{T}, \Gamma$, must satisfy the identity (2.24).

The above discussion of constitutive equations has been carried out without the help of any form of second law of thermodynamics. When the free energy response function does not depend explicitly on $\underline{g}$ and $\underline{g}'$, then $\psi = \psi^*$ and, by (4.23)₁, (4.31)₁ and (4.41)₁, $\eta = \eta^*$ also. Moreover, from an examination of (3.12), (4.35) and (2.30), it can be seen that the inequality (3.11) is identically satisfied and yields no new information. We do not consider here the case in which ψ does depend on $\underline{g}$ and $\underline{g}'$ and set aside a discussion of the conditions to be imposed on the heat flux vector.

The constitutive equation for ψ and other results given in [4] may be regarded as special cases of those obtained here. The present results, however, differ in some respects from those given in [6,7].

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References

1. Green, A. E., and Rivlin, R. S., Z. Angew. Math. Phys. 15, 290-292 (1964).
2. Green, A. E., and Naghdi, P. M., Int. J. Engng. Sci. 2, 621-624 (1965).
3. Naghdi, P. M., The theory of shells and plates. In S. Flügge's Handbuch der Physik, Vol. VIa/2 (edited by C. Truesdell), pp. 425-640, Springer-Verlag, 1972.
4. Edelen, D. G. B., Green, A. E., and Laws, N., Arch. Rational Mech. Anal. 43, 36-44 (1971).
5. Green, A. E., and Naghdi, P. M., Arch. Rational Mech. Anal. 48, 352-378 (1972).
6. Eringen, A. C., and Edelen, D. G. B., Int. J. Engng. Sci. 10, 233-248 (1972).
7. Eringen, A. C., Int. J. Engng. Sci. 12, 1063-1077 (1974).
8. Green, A. E., and Naghdi, P. M., to appear.

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