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AN AUTOMATIC METHOD FOR THE NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATIONS

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AN INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATION MAY BE REPRESENTED BY A POWER SERIES-GIVEN AN INITIAL VALUE. ADDITIONAL COEFFICIENTS OF THE POWER SERIES CAN BE OBTAINED THROUGH NUMERIC CONTINUATION AND A POWER SERIES REPRESENTATION OF THE SOLUTION CAN BE OBTAINED FOR ANY FINITE RANGE ABOUT THE CENTER OF EXPANSION BY THE USE OF RECURRENCE RELATIONS DEVELOPED THROUGH THIS STUDY. THE USE OF POWER SERIES AS THE SOLUTION METHOD ENABLES THE MEMORY OR HISTORY OF THE EQUATION TO BE RETAINED.

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AN AUTOMATIC METHOD FOR THE NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATIONS

RESEARCH REPORT

Presented in Partial Fulfillment of the
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by

Arthur E. Goldberger, Jr.

Texas A&M University

1977

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

An automatic method for the determination of the numerical solution of integro-differential-difference equations is presented. The basic form of an integro-differential-difference equation is assumed to be;

$$\dot{y}(t) = Ly(t) + f(t) + \int_{o}^{t} G(t-\lambda)\, y(\lambda)\, d\lambda \qquad (1\text{-}1)$$

with initial values $y(o) = y^{(o)}$, where y and f are real vector functions of the independent variable, t. L and G are real matrix functions of t.

The numerical method is an extension of the method of Frobenius, Lapidus and Sienfeld (1971), Kreyszig (1967). The basic theory of existence and uniqueness for this equation is known, and some basic references are [Baker, Miller, Cushing, and Bellman and Cooke].

Background

Mathematical models in applied science are used to represent and simplify complex problems. Mathematical models are deterministic or stochastic. Deterministic models include linear programs, non-linear programs, differential equations, and other more specialized forms, such as integro-differential-difference equations. C.T.H. Baker (1974), states

> "There has been little or no investigation in the published literature of the efficient automatic implementation of methods for Volterra integro-differential equations."

Baker further described the potential that exists for boundary value problems in integro-differential-difference equations. His apparent theme is the programming difficulty of numerical integration of initial value problems; and that seems to be reason enough to avoid the more difficult boundary value problem.

Some integral equations and integro-differential equations are easily transformed into differential equations. However, this is usually difficult for integro-difference equations and integro-differential-difference equations, or impossible.

We will present a method for the automatic numerical integration of initial value problems in integro-differential-difference equations, and integro-difference equations. This method can then be used with the codes of Doiron (1967), Hunter and Childs (1977), and Schuetz (1976) to develop an automatic solution of boundary value problems. These codes are based upon the power series integration method, as is the code presented here.

Hunter reported success, using the power series method, in regression analysis with differential equation models. Dr. Bart Childs suggested that power series integration could be an effective tool for the solution of integro-differential-difference equations, and integro-difference equations. A primary advantage of the power series method is that it provides a convenient method for inclusion of the "history" or "memory" in calculations, see Nachlinger and Wheeler, 1973.

Integro-differential-difference equations arise in a number of applications. The most frequent are related to Problems of Heat Conduction - Nachlinger and Wheeler (1973), Nuclear Transfer and Transport - Delves and Walsh (1974), Population Models - Cushing (1976), Renewal Theory - Feller, (1941), and Viscoelasticity - Malone (1971).

CHAPTER II

A POWER SERIES REPRESENTATION OF AN INTEGRO-DIFFERENCE EQUATION

A power series is taken to be an infinite series of the form;

$$\sum_{i=o}^{\infty} c^{(i)} (t-a)^i = c^{(o)} + c^{(1)}(t-a) + c^{(2)}(t-a)^2 + \dots \tag{2-1}$$

where $c_o, c_1, c_2, \dots$ are constants called coefficients of the series, a is a constant called the center of expansion, and t is the independent variable. Notice: indexes are parenthesized, exponents are not.

If in particular $a=0$, we obtain a power series:

$$\sum_{i=o}^{\infty} c^{(i)} t^i = c^{(o)} + c^{(1)} t + c^{(2)} t^2 + \dots \tag{2-2}$$

The basic idea of the power series representation of integro-difference equations is straightforward.

Given: A scalar integro-difference equation:

$$y(t) = \int_o^t g(t-\lambda)\, y(\lambda)\, d\lambda \tag{2-3}$$

We can represent the result of the integral as a power series in t.

$$\int_o^t g(t-\lambda)\, y(\lambda)\, d\lambda = \sum_{i=o}^{\infty} h^{(i)} t^i = h^{(o)} + h^{(1)} t + h^{(2)} t^2 + \dots \tag{2-4}$$

where the $h^{(o)}, h^{(1)}, \dots$ are functions of the coefficients of the original series for g and y. The limits of integration yield $h^{(o)} \equiv 0$. To illustrate, we represent the given functions $y(\lambda)$ and $g(t-\lambda)$ in power series:

$$y(\lambda) = \sum_{k=o}^{\infty} y^{(k)} \lambda^k \tag{2-5}$$

$$g(t-\lambda) = \sum_{m=o}^{\infty} g^{(m)} (t-\lambda)^m \tag{2-6}$$

Substituting these series into equation (2-4) yields:

$$\int_o^t \left(\sum_{k=o}^{\infty} y^{(k)} \lambda^k \right) \left(\sum_{m=o}^{\infty} g^{(m)} (t-\lambda)^m \, d\lambda \right) = \sum_{i=o}^{\infty} h^{(i)} t^i \tag{2-7}$$

The results of the multiplication of the two power series may be rewritten as expressions giving coefficients of like powers of t:

$$\sum_{i=o}^{\infty} h^{(i)} t^i = \sum_{i=o}^{\infty} \left[\int_o^t \sum_{j=o}^{i-1} y^{(j)} g^{(i-1-j)} \lambda^j (t-\lambda)^{i-1-j} \, d\lambda \right] \tag{2-8}$$

Interchanging the order of independent operators of summation and integration gives:

$$\sum_{i=1}^{\infty} h^{(i)} t^i = \sum_{i=1}^{\infty} \sum_{j=o}^{i-1} y^{(j)} g^{(i-1-j)} \int_o^t \lambda^j (t-\lambda)^{i-1-j} \, d\lambda \tag{2-9}$$

The limits of integration yield a particularly compact result:

$$\int_o^t \lambda^j (t-\lambda)^k \, d\lambda = \frac{j!k!}{(k+j+1)!} t^{k+j+1} \tag{2-10}$$

Substituting the results (2-10) into equation (2-9) and collecting coefficients of like powers of t yields:

$$\int_o^t y(\lambda) g(t-\lambda) \, d\lambda$$

$$= \sum_{i=o}^{\infty} \left[\sum_{j=o}^{i-1} (y^{(j)} g^{(i-1-j)}) \left(\frac{1}{i}\right) \left(\frac{j}{i-j}\right) \left(\frac{j-1}{i-j+1}\right) \cdots \left(\frac{j-(i-2)}{i-j+(i-2)}\right) \right] t^i \tag{2-11}$$

or

$$h^{(i)} = \sum_{j=o}^{i-1} \left(y^{(j)}\, g^{(i-1-j)}\right) \left[\left(\frac{1}{i}\right)\left(\frac{j}{i-j}\right)\left(\frac{j-1}{i-j+1}\right) \cdots \left(\frac{j-(i-2)}{i-j+(i-2)}\right)\right] \tag{2-12}$$

We rewrite equation (2-12) in a form, in which the values of $h^{(i)}$ are:

$$h^{(i)} = \sum_{j=o}^{i-1} g^{(j)}\, y^{(i-1-j)} \left[\frac{(i-1)!\ (j-1)!}{(i+j-1)!}\right], \ for\ i > 0 \tag{2-13}$$

Notice that the limits of integration give a *null* $h^{(o)}$.

We again call the reader's attention to the use of parenthesized superscripts to denote coefficient indexes. This index may apply to coefficients that are scalars, vectors, or matrices.

CHAPTER III

A SCALAR INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATION

When a state of the system is arbitrarily specified at the initial time, $y = y(o)$ at $t = 0$, then a solution exists for $t > 0$ and is uniquely determined by these equations Richtmyer and Morton (1967). The basic form of the scalar integro-differential-difference equation is assumed to be;

$$\dot{y}(t) = f(y(t)) + \int_{o}^{t} g(t-\lambda)\ y(\lambda)\ d\lambda \tag{3-1}$$

with initial values $y(o)$. The functions $y, f,$ and g are real valued functions of the independent variable, t.

We denote the power series for y as

$$y(t) = \sum_{i=o}^{\infty} y^{(i)} t^{i} \tag{3-2}$$

From term by term differentiation of equation (3-2) it is easily determined that:

$$\dot{y}(t) = \sum_{i=o}^{\infty} (i+1)\ y^{(i+1)} t^{i} \tag{3-3}$$

Furthermore let $f(y(t))$ and $g(\lambda)$ be represented by the following power series:

$$f(y(t)) = \sum_{i=o}^{\infty} f^{(i)} t^{i}$$

$$g(t-\lambda) = \sum_{i=o}^{\infty} g^{(i)}\ (t-\lambda)^{i} \tag{3-4}$$

We are assuming that given $y^{(o)}$ we can calculate $f^{(o)}$ and as we develop additional y coefficients, we can develop additional f coefficients.

The integral in equation (3-1) may be represented by a power series:

$$\int_{o}^{t} g(t-\lambda)\ y(\lambda)\ d\lambda = \sum_{i=o}^{\infty} h^{(i)} t^{i} \tag{3-5}$$

The h coefficients are calculated by the equation (2-12) or (2-13). Again, the limits of integration give $h^{(o)}=0$. We can obtain a recurrence relation for the higher coefficients in equation (3-1). By the proper power series substitutions equation (3-1) becomes:

$$\sum_{i=o}^{\infty} (i+1)\ y^{(i+1)} t^{i} = \sum_{i=o}^{\infty} f^{(i)} t^{i} + \sum_{i=o}^{\infty} h^{(i)} t^{i} \tag{3-6}$$

Coefficients of like powers of t yields:

$$(i+1)\ y^{(i+1)} t^{i} = f^{(i)} t^{i} + h^{(i)} t^{i} \tag{3-7}$$

or

$$y^{(i+1)} = \frac{1}{i+1}\ (f^{(i)} + h^{(i)}) \tag{3-8}$$

with initial values $y(o) = y^{(o)}$. Through this relation, additional coefficients of the power series solution can be obtained.

We assume the $f(y)$ in equations (3-1), (3-4), (3-7) and (3-8) to be of the classes of power series operators discussed in Appendix A. That is, given $y^{(o)} \cdots y^{(i)}$, we can calculate $f^{(o)} \cdots f^{(i)}$. In other words, given $y^{(o)}$ as an initial value we can calculate $f^{(o)}$, then, using the recurrence relation (3-8) we can calculate $y^{(1)}$, from which we now can calculate $f^{(1)}$. This recurrence procedure can then be continued for as many terms as desired.

A power series solution of these types of models is usually "absolutely convergent" in the complex variables sense. "Numeric convergence" of a power series with finite arithmetic is not absolute. For this reason the

recurrence relation will be "numerically good" only within a given neighborhood of the center of expansion. A *numeric continuation* procedure will be necessary for an acceptable numerical integration method.

NUMERIC CONTINUATION

The power series representation of the solution of the scalar integro-differential-difference equation:

$$\dot{y}(t) = f(y(t)) + \int_{o}^{t} g(t-\lambda)\ y(\lambda)\ d\lambda \tag{4-1}$$

has been shown to be amenable to calculation with a reasonable effort. The power series has a finite range or neighborhood about the center of expansion, within which the solution values (as contrasted with the power series coefficients) can be calculated with sufficient accuracy.

We will denote by T the range of the independent variable $t \in [o,T]$ which yields $y(t)$ with sufficient accuracy. We will call the process of obtaining new coefficients, about $t = T$, *numeric continuation*. It is quite similar to *analytic continuation* from complex variable theory. In this process, we use the acceptably accurate values $y(T)$ as *initial values* and generate the additional power series coefficients by use of recurrence relations. This is well known in ordinary differential equations, but, a slight problem must be handled for the integro-difference term. This problem arises from the *memory* or *history* contributions of the integro-difference term.

We introduce τ, a shifted independent variable, $\tau = t - T$. We rewrite (4-1) as:

$$\dot{y}(T+\tau) = f(y(T+\tau)) + \int_{o}^{T} g(T+\tau-\lambda)\ y(\lambda)\ d\lambda + \int_{T}^{T+\tau} g(T+\tau-\lambda)\ y(\lambda)\ d\lambda \tag{4-2}$$

The first integral term is represented as

$$\int_0^T g(T+\tau-\lambda)\ y(\lambda)\ d\lambda = \sum_{i=0}^{\infty} q^{(i)}\tau^i \tag{4-3}$$

The given functions $y(\lambda)$ and $g(T+\tau-\lambda)$ are expressed as the power series:

$$y(\lambda) = \sum_{j=0}^{\infty} y^{(j)}\lambda^j \tag{4-4}$$

$$g(T+\tau-\lambda) = \sum_{k=0}^{\infty} g^{(k)}\ (T+\tau-\lambda)^k \tag{4-5}$$

These are the same coefficients as in (3-2), (3-4) and (3-8). Substituting these series into equation (4-3)

$$\int_0^T \left(\sum_{j=0}^{\infty} y^{(j)}\lambda^j \right) \left(\sum_{k=0}^{\infty} g^{(k)}(T+\tau-\lambda)^k \right) d\lambda = \sum_{i=0}^{\infty} q^{(i)}\tau^i \tag{4-6}$$

and rearranging:

$$\sum_{i=0}^{\infty} q^{(i)}\tau^i = \left(\sum_{j=0}^{\infty} y^{(j)} \sum_{k=0}^{\infty} g^{(k)} \int_0^T \lambda^j\ (T+\tau-\lambda)^k\ d\lambda \right. \tag{4-7}$$

The limits of integration yield:

$$\int_0^T \lambda^j\ (T+\tau-\lambda)^k\ d\lambda = \frac{j!k!T^{k+j+1}}{(k+j+1)!} \sum_{i=0}^{k} \binom{j+k+1}{i} \left(\frac{\tau}{T}\right)^i \tag{4-8}$$

The $\binom{\cdot}{\cdot}$ is the usual binomial factor.

Substituting the results (4-8) into equation (4-6)

$$\int_0^T g(T+\tau-\lambda)\ y(\lambda)\ d\lambda$$

$$= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} y^{(j)}T^j \sum_{k=i}^{\infty} g^{(k)}T^k \left(\frac{j!\ k!}{(k+j+1)!}\right) \binom{j+k+1}{i} \frac{T}{T^i} \right) \tau^i \tag{4-9}$$

or

$$q^{(i)} = \sum_{j=0}^{\infty} y^{(j)}T^j \sum_{k=i}^{\infty} g^{(k)}T^k \left(\frac{j!\ k!}{(k+j+1)!}\right) \binom{j+k+1}{i} \frac{T}{T^i} \tag{4-10}$$

By inspection we find all values in the above equation are known, and it is a simple process to calculate each $q^{(i)}$.

We can also represent the second integral term in (4-2) as a power series:

$$\int_T^{T+\tau} g(T+\tau-\lambda)\ y(\lambda)\ d\lambda = \sum_{i=0}^{\infty} h_T^{(i)} \tau^i \tag{4-11}$$

Shifting the limits of integration gives:

$$\int_T^{T+\tau} g(T+\tau-\lambda)\ y(\lambda)\ d\lambda = \int_0^{\tau} g_T(\tau-\lambda)\ y_T(\lambda)\ d\lambda \tag{4-12}$$

The subscript T indicates that those power series coefficients are about the center of expansion, $t=T$. Comparing equation (4-12) with (2-4) we see that the $h_T^{(i)}$ is calculated like (2-13)

$$h_T^{(i)} = \sum_{j=0}^{i-1} g_T^{(j)}\ y_T^{(i-1-j)} \quad \left[\frac{(i-1)!\ (j-1)!}{(i+j-1)!} \right] \tag{4-13}$$

and $h^{(0)} = 0$. These coefficients are for the shifted independent variable $\tau = t-T$.

The power series in terms of the shifted independent variable will also be numerically valid over a finite range. Further continuation procedures would change significantly because of the expense of storing power series over all previous ranges.

We have presented methods to calculate "exactly":

$$\int_0^{t} g(t-\lambda)\ y(\lambda)\ d\lambda = \int_0^{T} g(T+\tau-\lambda)\ y(\lambda)\ d\lambda + \int_0^{\tau} g_T(\tau-\lambda)\ y_T(\lambda)\ d\lambda \tag{4-14}$$

$$= \sum_{i=0}^{\infty} q_T^{(i)} \tau^i + \sum_{i=0}^{\infty} h_T^{(i)} \tau^i \tag{4-15}$$

where $t = T + \tau$.

The usual form of the kernel $g(x)$ is the negative exponential, $g(x) = exp(-x)$. See Nachlinger and Wheeler (1973), Nunziato (1971), Cushing (1976). Assuming this form, we rewrite the first integral term on the right side of (4-14):

$$\int_0^T exp(-(T+\tau-\lambda))\ y(\lambda)\ d\lambda = \int_0^T exp(-T)\ exp(-(\tau-\lambda))\ y(\lambda)\ d\lambda$$

$$= exp(-T) \int_0^T exp(-(\tau-\lambda))\ y(\lambda)\ d\lambda \qquad (4\text{-}16)$$

We note that the contribution of $q_T^{(i)}$ (from equations (4-3) through (4-10)) is like a "damped" $h_o^{(i)}$. The $q_T^{(i)}$ contribution should be small compared to $h_T^{(i)}$ if the value of T is large enough to cause significant damping $exp(-T)$. Previous experience in ordinary differential equations indicate that large ranges are probable, using say twelve to twenty terms of the power series Doiron (1970), Hunter and Childs (1977). We can shift a power series to an approximate series at the next center:

$$\sum_{i=o}^{\infty} q_{T_n}^{(i)}\ \tau_n^{\ i} = \sum_{i=o}^{\infty} q_{T_{n+1}}^{(i)}\ \tau_{n+1}^{\ i} \qquad (4\text{-}17)$$

where

$$\tau_{n+1} = \tau_n - (T_{n+1} - T_n) = \tau_n - \Delta_{n+1} \qquad (4\text{-}18)$$

$$q_{T_{n+1}}^{(i)} = \sum_{k=i}^{\infty} q_{T_n}^{(k)} \binom{k}{i} \Delta_{n+1}^{\ k} \qquad (4\text{-}19)$$

By the proper power series substitutions equation (4-2) becomes:

$$\sum_{i=o}^{\infty} (i+1)\ y_{T_n}^{(i+1)}\ \tau^i = \sum_{i=o}^{\infty} f_{T_n}^{(i)}\ \tau^i + \sum_{i=o}^{\infty} q_{T_n}^{(i)}\ \tau^i + \sum_{i=o}^{\infty} h_{T_n}^{(i)}\ \tau^i \qquad (4\text{-}20)$$

which yields the recurrence relation:

$$(i+1)\, y_{T_n}^{(i+1)}\, \tau^i = f_{T_n}^{(i)}\, \tau^i + q_{T_n}^{(i)}\, \tau^i + h_{T_n}^{(i)}\, \tau^i$$

or

$$y_{T_n}^{(i+1)} = \frac{1}{i+1}\, [\, f_{T_n}^{(i)} + q_{T_n}^{(i)} + h_{T_n}^{(i)}\,] \tag{4-21}$$

Our computational strategy is:

(1) Input the initial value $y^{(o)}$, and initialize the q coefficients to *zero*. Note that the first center of expansion is *zero*.

(2) We now calculate the q coefficients about the current center of expansion. If the current center of expansion is T_1, then use equation (4-10) to calculate the q coefficients. If the current center of expansion is greater than T_1, then use the shift algorithm (4-18) and (4-19) to calculate the q coefficients about the center.

(3) Calculate the coefficients $y^{(1)} \ldots y^{(\cdot)}$ by the recurrence relation (4-21).

(4) Determine the range of the power series which will yield a specific accuracy over the interval desired. Then we may now evaluate the $y(\tau)$ at any point within the range calculated.

(5) If the range does not extend over the interval desired, then we evaluate the function y at the extreme point of the range which becomes the next center of expansion. Repeat the computational process beginning with step (2).

CHAPTER V

THE BOUNDARY-VALUE PROBLEM IN INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATIONS

A general form of linear integro-differential-difference equations could be

$$\dot{y} = Ly + f + \int_0^t G(t-\lambda)\, y(\lambda)\, d\lambda \tag{5-1}$$

where L and G are known matrix functions of t. In addition y and $\dot{y}$ are vector functions of t, and f is a known vector function of t. A second order scalar linear integro-differential-difference equation like

$$\ddot{x} + \mu\, \dot{x} + \xi x = \sin(t) + \int_0^t \exp\,(t-\lambda)\, x(\lambda)\, d\lambda \tag{5-2}$$

is a well defined problem if $x(o)$ and $\dot{x}(o)$ are known. We can convert this to the form (5-1) by defining

$$y^{(1)} = x$$

$$y^{(2)} = \dot{x}$$

with

$$L = \begin{bmatrix} 0 & 1 \\ -\xi & -\mu \end{bmatrix} \qquad G(\alpha) = \begin{bmatrix} 0 & 0 \\ 0 & \exp(-\alpha) \end{bmatrix} \qquad f(t) = \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} \tag{5-3}$$

Most higher order problems in integro-differential-difference equations will likely be of the same type as this example. That is, one equation in a set of ordinary differential equations will have an integro difference term.

The solution of boundary value problems in integro-differential-difference equations of a general nature are possible with the use of

the codes of Doiron (1971) and Hunter and Childs (1977). The codes must be supplemented with subprograms implementations of equations (2-12) or (2-13), equations (4-9) or (4-10), equation (4-13), and equation (4-19).

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APPENDIX

POWER SERIES ALGORITHMS

A Summary

Power series operations are straightforward with the use of auxiliary series and repetitive applications of known algorithms for series addition, subtraction, multiplication, division, and elementary functions. The following power series algorithms (Doiron, 1970) are important to this study:

Addition: $f(t) = g(t) + h(t)$

$$f^{(i)} = g^{(i)} + h^{(i)} \tag{A-1}$$

Multiplication: $f(t) = f(g(t), h(t)) = g(t)*h(t)$

$$f^{(k)} = \sum_{i=o}^{k} g^{(i)} h^{(k-i)} \tag{A-2}$$

Division: $f(t) = f(g(t), h(t)) = g(t)/h(t)$

$$f^{(o)} = g^{(o)}/h^{(o)} \tag{A-3}$$

$$f^{(k)} = \frac{g^{(k)} - \sum_{i=o}^{k} h^{(i)} g^{(k-i)}}{h^{(o)}} \quad k > 1 \tag{A-4}$$

Sine and cosine:

$$h(t) = \cos(y(t))$$

$$g(t) = \sin(y(t))$$

$$h^{(o)} = \cos(y^{(o)}) \tag{A-5}$$

$$g^{(o)} = \sin(y^{(o)}) \tag{A-6}$$

$$h^{(k+1)} = -\frac{1}{k} \sum_{i=o}^{k} i\, y^{(i)} g^{(k-i)} \quad k > 0 \tag{A-7}$$

$$g^{(k+1)} = \frac{1}{k} \sum_{i=o}^{k} i\, y^{(i)}\, h^{(k-i)} \quad k > 0 \tag{A-8}$$

Exponential:

$$h(t) = exp(y(t))$$

$$h^{(o)} = 1 \tag{A-9}$$

$$h^{(k+1)} = \frac{1}{k+1} \sum_{i=o}^{k} (k-i)\, h^{(i)}\, y^{(k-i)} \quad k > 0 \tag{A-10}$$

These algorithms are supplemented by the integro-differential-difference equations and integro-difference equations presented in this study:

$$h(t) = \int_{o}^{t} y(\lambda)\, g(t-\lambda)\, d\lambda$$

$$h^{(o)} = 0 \tag{A-11}$$

$$h^{(k)} = \sum_{i=o}^{k-1} y^{(i)}\, g^{(k-1-i)} \left[\frac{1}{k} \left(\frac{i}{k-i}\right) \left(\frac{i-1}{k-i+1}\right) \cdots \left(\frac{i-(k-2)}{k-i+(k-2)}\right) \right] \quad k > 0 \tag{A-12.a}$$

or

$$h^{(k)} = \sum_{i=o}^{k-1} y^{(i)}\, g^{(k-1-i)} \quad \frac{(i-1)!\,(k-1)!}{(i+k-1)!} \quad k > 0 \tag{A-12.b}$$

$$q(\tau) = \int_{o}^{T} y(\lambda)\, g(T+\tau-\lambda)\, d\lambda$$

$$q^{(k)} = \sum_{j=o}^{\infty} y^{(j)}\, T^{j} \sum_{i=k}^{\infty} g^{(i)}\, T^{(i)} \quad \frac{j!\, i!}{(i+j+1)!} \quad \binom{i+j+1}{k} \frac{T}{T^{k}} \tag{A-13}$$

The shifting algorithm to transform a power series about one center T_n to an *approximate* power series about the next center T_{n+1} is:

$$f_{T_{n+1}}{}^{(k)} = \sum_{i=k}^{\infty} f_{T_n}{}^{(i)} \binom{i}{k} \Delta_{n+1}{}^{k} \tag{A-14}$$

where

$$\tau_{n+1} = \tau_n - (T_{n+1} - T_n) = \tau_n - \Delta_{n+1}$$