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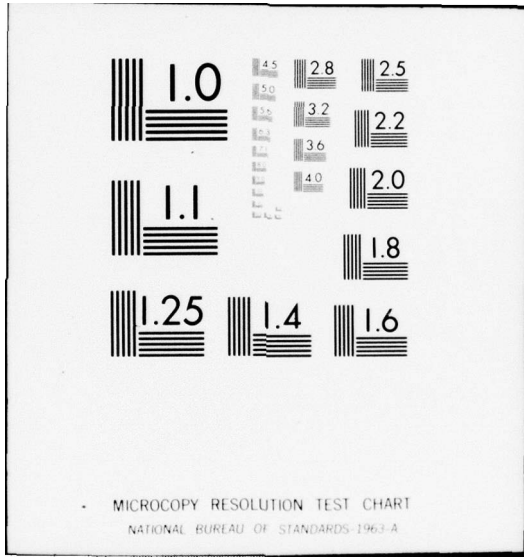
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# NSWC TECHNICAL REPORT C

WHITE OAK LABORATORY

THE DIFFERENTIAL SCATTERING CROSS SECTION FOR A PLASMA COLUMN IN RELATIVISTIC MOTION

BY  
R.L. Kligman

1 JANUARY 1977

NAVAL SURFACE WEAPONS CENTER  
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 14 NSWC/WOL/TR-76-101	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 The Differential Scattering Cross Section for a Plasma Column in Relativistic Motion.		5. TYPE OF REPORT & PERIOD COVERED Final rept.
7. AUTHOR(s) 10 Ronald L. Kligman		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Surface Weapons Center White Oak Laboratory White Oak, Silver Spring, Maryland 20910		8. CONTRACT OR GRANT NUMBER(s) 16 F32382
11. CONTROLLING OFFICE NAME AND ADDRESS		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62332N; SF32-382; 17 SF32-382-101 WR40BA;
		12. REPORT DATE 11 1 January 1977
		13. NUMBER OF PAGES 38 1241p
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) electromagnetic scattering relativistic plasma column differential cross section		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Expressions are found for the differential scattering cross section of an infinitely long plasma column which has been irradiated by a plane electromagnetic wave. The propagation direction of the wave makes an arbitrary angle with the cylinder axis, and the plasma medium is allowed to move with an arbitrary velocity.		

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Preface

The question of scattering an electromagnetic signal from a relativistic electron beam which is propagating through a gaseous medium is addressed. The dependence of the scattering cross section on the angle of incidence is studied and presented. This work has been supported by the Naval Sea Systems Command.

*D. L. Love*  
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I. INTRODUCTION

The scattering of electromagnetic waves from plasmas has been studied by numerous researchers in recent years. Wait<sup>1</sup> considered the case of a plane wave obliquely incident on a dielectric cylinder of infinite length. The incident magnetic field is transverse to the cylinder axis. Results for the scattered fields are cited and limiting cases such as the far field approximation, the highly conducting cylinder and the highly permeable cylinder are discussed. Scattering of electromagnetic waves from a plasma column at rest may be studied through an extension of Wait's work if one simply characterizes the plasma medium by a frequency dependent dielectric constant  $\epsilon(\omega)$ . Many researchers have approached plasma scattering problems in this fashion. Shapiro<sup>2</sup> has studied the problem of scattering from a homogeneous plasma. Others<sup>3,4</sup> have considered inhomogeneous and anisotropic plasma columns. Yeh and Rusch<sup>5</sup> have looked at the plasma sheath that is both anisotropic and radially inhomogeneous. Far field patterns as well as the back scattering cross sections were numerically obtained. Messian and Vandenplas<sup>6</sup> added complexity to the plasma scattering problem by allowing the column to have an axial drift velocity,  $v \ll c$ . Such motion couples an incident plane wave, whose magnetic vector is parallel to the column axis (an H wave) to a scattered wave which has a component parallel to the cylinder axis (an E wave). This phenomena naturally follows from the Lorentz covariant form of the Maxwell equations and is the plasma high frequency analogue of the

- 
1. J. R. Wait, "Scattering of a Plane Wave from a Circular Dielectric Cylinder at Oblique Incidence," Can. J. Phys. 33, 189, 1955.
  2. H. Shapiro, Antenna Lab. Tech. Rept. No. 11, California Institute of Technology, 1957.
  3. F. A. Albin and R. G. Jahn, "Reflection and Transmission of Electromagnetic Waves at Electron Density Gradients," J. Appl. Phys. 32, 75, 1961.
  4. P. M. Platzman and H. T. Ozaki, "Scattering of Electromagnetic Waves from an Infinitely Long Magnetized Cylindrical Plasma," J. Appl. Phys. 31, 1597, 1960.
  5. W. V. T. Rusch and C. Yeh, "Scattering by an Infinite Cylinder Coated with an Inhomogeneous and Anisotropic Plasma Sheath, IEEE Transactions on Antennas and Propagation, Vol. AP-15, 3, 452, 1967.
  6. A. M. Messian and P. E. Vandenplas, "High Frequency Effect Due to the Axial Drift Velocity of a Plasma Column," Phys. Rev. 149, 131, 1966.

Roentgen-Eichenwald current.<sup>7-9</sup> This is the current generated by pulling a dielectric slab between the parallel plates of a capacitor, which establishes a magneto-static field perpendicular to both the velocity of the slab and the electric field of the capacitor. Yeh<sup>10</sup> calculated the scattered fields emanating from a plasma column moving with an arbitrary velocity where the incident plane wave was obliquely incident. As previous investigators found, cross-polarized scattered fields are induced and for a particular angle of incidence, namely  $\theta_0 = \sin^{-1}(v/c)$  (where  $v$  is the velocity of the plasma medium,  $c$  is the speed of light<sup>2</sup> in vacuum). In particular, both E and H scattered waves will be excited when E or H waves are incident on a plasma cylinder, either obliquely or normally.

Hodjat<sup>11</sup> considered the problem of a plane wave incident at an angle upon an electron beam confined by a glass envelope, i.e. a dielectric material. He also considered the separate cases of a wave incident on a dielectric cylinder, a hollow dielectric cylinder and a moving electron beam, respectively. The coefficients of the scattered waves were found. An experiment was also conducted to measure the z component scattered electric field with a dipole antenna, and the cross-polarized scattered magnetic field with a loop antenna. The experiments were restricted to the cases of the dielectric cylinder, the hollow cylinder, and the electron beam surrounded by the dielectric cylinder.

Kriegsmann<sup>12</sup> has studied the scattering of plane waves from a cylindrically confined cold plasma in the geometric optics limit, for the E and B polarization cases. Solutions to the Helmholtz equation for an inhomogeneous medium are found using the geometric theory of

7. W. C. Roentgen, "Über die durch Bewegung eines in homogenen elektrische Felde befindlichen Dielectricums hervorgerufene elektrodynamische Kraft", Ann. Phys. 35, 264, 1888 and "Beschreibung des Apparates mit welchem die Versuche über die elektrodynamische Wirkung bewegter Dielectricia ausgeführt wurden," Ann. Phys. 40, 93, 1890.
8. A. Eichenwald, "Über die Magnestische Wirkungen bewegter Körper in elektrostatischen Felde," Ann Phys. 11, 421, 1903.
9. A. Sommerfeld, "Electrodynamics," (Academic Press Inc., New York, New York, 1952), pp. 283-285.
10. C. Yeh, "Scattering Obliquely Incident Microwaves by a Moving Plasma Column," J. Appl. Phys. 40, 5066, 1969.
11. F. Hodjat, "Scattering of Electromagnetic Waves from a Moving Cylindrical Electron Beam," UCLA doctoral thesis, University Microfilms No. 72-2829, 1971.
12. G. Kriegsmann, "An Application of the Method of Geometric Optics to the Scattering of Plane Electromagnetic Waves Off Cylindrically Confined Cold Plasmas," UCLA doctoral thesis, University Microfilms No. 74-29-265, 1974.

diffraction developed by Keller and his co-workers. The results of this method compare well with those of exactly soluble problems, giving the same results asymptotically. Such notions as rays, amplitudes and caustics are all fundamental features of such an approach.

In this paper we follow the general approach of Yeh,<sup>10</sup> however the physical parameter of interest is the differential scattering cross section per unit length.

## II. THE PLASMA CHARACTERIZATION

We consider a plasma that consists of electrons only. These electrons are free to move in response to an applied field of frequency  $\omega$ . If the plasma is rarefied enough, then the collision frequency  $\omega_c$  between electrons may be negligible compared to the frequency of the applied field. According to the Drude theory<sup>13</sup> the conductivity may be given by

$$(II.1) \quad \sigma = in_0e^2/m\omega$$

where  $n_0$  is the number of electrons per unit volume,  $e$  is the electron charge, and  $m$  is its mass. Thus the plasma we are considering may be regarded as collisionless.

Since no longitudinal fields can exist in a conducting medium in the absence of an applied current density,<sup>13</sup> we shall consider the transverse wave field that varies as  $\exp(i\vec{k}\cdot\vec{r}-i\omega t)$  where  $\vec{k}$  is the propagation wave vector and may be found from the Maxwell equations to satisfy the equation

$$(II.2) \quad k^2 = \left(\frac{\omega}{c}\right)^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right)$$

where  $n$  is the index of refraction of the plasma, and  $\epsilon$  is its dielectric constant.

If some non-equilibrium charge configuration exists in the plasma, some of the electrons will rearrange themselves in an attempt to screen out the resultant electric field. For example, there may exist an excess of electrons in some planar y-z region, a shortage in another planar region and a uniform electric field  $E_x$  between the charged planes. Any electron in the interplanar region will experience a force

$$(II.3) \quad m\ddot{x} = -eE_x$$

---

10. C. Yeh, Op. Cit.

13. J. D. Jackson, "Classical Electrodynamics," (John Wiley and Sons, Inc., New York, 1963), pp. 223-226.

where the electric field is given by

$$(II.4) \quad E_x = \int / \epsilon_0 = n_0 e x / \epsilon_0$$

where  $\int$  is the surface charge density on the positively charged plane, and  $\epsilon_0$  is the permittivity of free space. Substitution of eq (II.4) into eq (II.3) yields

$$(II.5) \quad \ddot{x} + \left( \frac{n_0 e^2}{m \epsilon_0} \right) x = 0$$

This is just a harmonic oscillator equation and we may define the plasma frequency  $\omega_p$  as

$$(II.6) \quad \omega_p^2 = n_0 e^2 / m \epsilon_0$$

Physically, the electron experiencing a force will move until it reaches the positively charged plane, however its kinetic energy will carry it beyond, continuing until its energy is ultimately reconverted to potential form. It will then repeat the motion in a periodic fashion. This is known as a plasma oscillation where the restoring force is the Coulomb interaction and the electron's mass is the inertia.

If we substitute eq (II.1) into eq (II.2) making use of eq (II.6) we obtain

$$(II.8) \quad k^2 \approx \frac{\omega^2}{c^2} (1 - \omega_p^2 / \omega^2)$$

Since  $k = n\omega/c$ , we find an expression for the index of refraction of a plasma given by

$$(II.9) \quad n^2 \approx 1 - \omega_p^2 / \omega^2$$

For the case  $\omega > \omega_p$  (sometimes referred to as the underdense case) the index of refraction is real and the electromagnetic waves simply propagate through the medium. The other situation, namely  $\omega < \omega_p$  (the overdense case) gives rise to an imaginary index and electromagnetic waves will scatter and be reflected from this plasma medium. In the general situation, where the charged particle density is a function of position, the waves propagate through the underdense region until reaching some point where the particle density has increased sufficiently to become overdense. At this critical surface the waves are reflected. There will be some penetration of this surface by the fields but they will quite rapidly decay exponentially in the interior of the critical surface.

It should be noted that the expression for the plasma frequency given, in eq (II.6) is valid for a cold plasma, i.e. when the electrons thermal kinetic energy has a small effect on the plasma density. When this is not the case one must consider fluctuations in

the particle density and a consequent modification of the expression for the plasma frequency. For our purposes the cold plasma approximation will suffice. Jackson<sup>13</sup> has considered the problem of the warm plasma oscillations and derives appropriate expressions for the frequencies of these oscillations in terms of the already defined cold plasma frequency and the density fluctuations. These expressions are appropriately called "dispersion relations" since they relate the oscillation frequency  $\omega$  to its wave number  $k$ .

### III. THE INCIDENT AND SCATTERED FIELDS

We shall consider a plane wave propagating in such a direction as to be obliquely incident on a plasma column that is infinite in extent, aligned parallel to the  $z$  axis. Fig. 1 clearly illustrates the geometry of the initial conditions. One may characterize the plasma by specifying either its conductivity  $\sigma(\omega)$  or its dielectric constant  $\epsilon(\omega)$ . Because the plasma conductivity will be finite, in general, it is more convenient to use the dielectric constant as the characterization parameter. One may then treat the problem of the scattering by a plasma column in the same fashion as the scattering from a dielectric cylinder.

We write the expression for the incident plane wave with respect to the laboratory  $S$  frame

$$(III.1) \quad \begin{cases} E_z^{(i)} = E_0 \cos \theta_0 \exp(-ik_0 y \cos \theta_0 + ik_0 z \sin \theta_0) e^{-i\omega t} \\ H_z^{(i)} = 0 \end{cases}$$

where  $E_0$  and  $\omega$  are the amplitude and frequency of the incoming, incident wave and

$$k_0 = \omega/c = \omega(\mu_0 \epsilon_0)^{1/2}$$

where we have chosen to work in the MKSA system of units. In a reference frame  $S'$  that is embedded in the column, moving with velocity  $\vec{v} = v_z \hat{z}$  relative to the  $S$  frame the incident wave is written:

$$(III.2) \quad \begin{aligned} E_z^{(i)'} &= E_0' \cos \theta' \exp(-ik_0' \cos \theta' y' + ik_0' \sin \theta' z') e^{-i\omega' t'} \\ &= F_E' \sum_{n=-\infty}^{\infty} (-1)^n J_n(k_0' r' \cos \theta') \exp(in\theta') \end{aligned}$$

13. J. D. Jackson,  
Op. Cite, pp. 223-226.

$$(III.3) \quad H_z^{i'} = 0$$

The right hand side of eq (III.2) is derived in Appendix A. The wave parameters measured in the  $S'$  frame are related to those measured in the  $S$  frame. The relationships are

$$(III.4) \quad \begin{aligned} \omega' &= \gamma\omega(1-\beta\sin\theta_0) \\ \gamma &= (1-\beta^2)^{-\frac{1}{2}} \\ \beta &= v_z/c \end{aligned}$$

$$(III.5) \quad k'_0 \cos\theta' = k_0 \cos\theta_0$$

$$(III.6) \quad \sin\theta' = (\sin\theta_0 - \beta)/(1 - \beta\sin\theta_0)$$

$$(III.7) \quad k'_0 = \omega'(\mu_0 \epsilon_0)^{\frac{1}{2}} = \gamma k_0(1 - \beta\sin\theta_0)$$

$$(III.8) \quad E'_0 = \gamma E_0(1 - \beta\sin\theta_0)$$

$$(III.9) \quad \begin{aligned} F'_E &= E'_0 \cos\theta' \exp(ik'_0 \sin\theta' z' - i\omega' t') \\ &= E_0 \cos\theta_0 \exp(ik_0 \sin\theta_0 z - i\omega t) \end{aligned}$$

Equations (III.4) through (III.9) are derived in appendix B.

The  $z$ -component scattered fields may be written as

$$(III.10) \quad E_{z'}^{(s)}(r', \theta', \phi') = F'_E \sum_{n=-\infty}^{\infty} (-1)^n A'_n H_n^{(1)}(k'_0 r' \cos\theta') e^{in\phi'}$$

$$(III.11) \quad H_{z'}^{(s)}(r', \theta', \phi') = F'_E \sum_{n=-\infty}^{\infty} (-1)^n B'_n i \cdot \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} H_n^{(1)}(k'_0 r' \cos\theta') e^{in\phi'}$$

where the expansion coefficients  $A'_n$  and  $B'_n$  will be determined from boundary conditions. The Hankel function of the first kind is used to represent outgoing radiation and is given by

$$(III.12) \quad H_n^{(1)}(x) = J_n(x) + iN_n(x)$$

where  $J_n(x)$  and  $N_n(x)$  are the ordinary Bessel and Neumann functions

respectively.

We may represent the z-component fields that penetrate into the plasma column by

$$(III-13) \quad E_z^{(p)}(r', \theta', \phi') = F_E' \sum_{n=-\infty}^{\infty} (-1)^n C_n' J_n(\lambda' r') e^{in\phi'}$$

$$(III-14) \quad H_z^{(p)}(r', \theta', \phi') = F_E' \sum_{n=-\infty}^{\infty} (-1)^n D_n' i \left( \frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}} J_n(\lambda' r') e^{in\phi'}$$

and

$$(III-15) \quad \lambda' = k_0' \left( \frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0} - \sin^2 \theta' \right)^{\frac{1}{2}}$$

with  $C_n'$  and  $D_n'$  determined from the boundary conditions and eq

(III.15) derived in appendix C.

#### IV. BOUNDARY CONDITIONS

The total tangential magnetic and electric field intensities are continuous across the plasma - vacuum interface, i.e. the cylinder surface.

$$(IV.1) \quad \begin{cases} \vec{n} \times (\vec{E}^{(i)'} + \vec{E}^{(s)'}) = \vec{n} \times \vec{E}^{(p)'} \\ \vec{n} \times (\vec{H}^{(i)'} + \vec{H}^{(s)'}) = \vec{n} \times \vec{H}^{(p)'} \end{cases}$$

where  $\vec{n}$  is the unit normal vector associated with the plasma surface.

If we use eqs (III.2), (III.10) and (III.13) in the first of eq (IV.1) we get

$$(IV.2) \quad J_n(k_0' a \cos \theta') + A_n' H_n^{(1)}(k_0' a \cos \theta') - C_n' J_n(\lambda' a) = 0$$

Making use of eqs (III.3), (III.11) and (III.14) in the second of eq (IV.1) yields

$$(IV.3) \quad B_n' H_n^{(1)}(k_0' a \cos \theta') - D_n' J_n(\lambda' a) = 0$$

So far we have two equations for the four undetermined coefficients  $A'_n$ ,  $B'_n$ ,  $C'_n$  and  $D'_n$ . We shall find two more equations, from the Maxwell equations which will give us a complete set of equations from which these constants may be determined. The Maxwell equations are:

$$(IV.4) \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$(IV.5) \quad \vec{\nabla} \cdot \vec{D} = 0 \quad , \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

suppressing the superscripts "s" and "p" and prime "l" for convenience. We shall utilize the constitutive relations as well, namely

$$(IV.6) \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P} = K \epsilon_0 \vec{E}$$

$$(IV.7) \quad \vec{B} = \mu_0 (\vec{H} + \vec{M}) = K_m \mu_0 \vec{H}$$

The quantities  $K$  and  $K_m$  are the relative permittivity and permeability respectively. For example, in vacuum  $K$  and  $K_m$  are equal to unity, in the plasma medium we may write

$$K \epsilon_0 = \epsilon_1 \quad , \quad K_m \mu_0 = \mu_1$$

We shall now find expressions for the azimuthal component of the fields in terms of the  $z$  components in the  $s'$  frame, i.e.,

$$(IV.8) \quad E'_\phi = E'_\phi(E'_z, H'_z)$$

$$(IV.9) \quad H'_\phi = H'_\phi(E'_z, H'_z)$$

Assume the harmonic time dependence of the fields to have the form  $\exp(-i\omega t')$ . Eqs (IV.4) and (IV.5) are written in cylindrical coordinates, and after some substitutions among the component equations of eqs (IV.5) and (IV.5) we arrive at

$$(IV.10) \quad E'_\phi = \frac{\mu_0 \epsilon_0}{K^2 \mu_0 \epsilon} \left( \frac{1}{r} \frac{\partial^2 E'_z}{\partial z \partial \phi} - \frac{\partial^2 E'_\phi}{\partial z^2} \right) + \frac{1}{i\omega \epsilon} \frac{\partial H'_z}{\partial r}$$

$$(VII.11) \quad H'_{\phi} = \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon} \left( \frac{1}{r'} \frac{\partial^2 H'_{z'}}{\partial z' \partial \phi'} - \frac{\partial^2 H'_{\phi'}}{\partial z'^2} \right) - \frac{1}{i\omega' \mu} \frac{\partial E'_{z'}}{\partial r'}$$

The results of eqs (IV.10) and (IV.11) are derived explicitly in appendix D. We may further approximate these equations if we make some assumptions about the plasma medium itself. First we assume

$$(IV.12) \quad E'_{\phi} \sim K \exp(ik'_0 \sin \theta' z) f(\phi', r')$$

where K is some constant that must be determined. Substituting eq (IV.12) into the right hand side of eq (IV.10) yields a term that is just  $E_{\phi}$  times the factor  $\mu_0 \epsilon_0 \sin^2 \theta' / \mu_1 \epsilon_1$ . If we approach the cylinder surface from the inside  $r \rightarrow a_-$ , this factor may be typically of the order of  $10^{-1}$ , for plasmas that are of interest to us. However an approach from just outside the cylinder  $r \rightarrow a_+$  produces a value of  $\sin^2 \theta'$  for the factor, and eq. (IV.10) becomes

$$(IV.13) \quad E'_{\phi} = \sec^2 \theta' \left( \frac{1}{k_0'^2 r} \frac{\partial^2 E'_{z'}}{\partial z' \partial \phi'} + \frac{1}{i\omega' \epsilon_0} \frac{\partial H'_{z'}}{\partial r'} \right), \quad r' \rightarrow a_+$$

and

$$(IV.14) \quad E'_{\phi} \approx \frac{1}{10k_0'^2 r} \frac{\partial^2 E'_{z'}}{\partial z' \partial \phi'} + \frac{1}{i\omega' \epsilon_1} \frac{\partial H'_{z'}}{\partial r'}, \quad r' \rightarrow a_-$$

Since the azimuthal field components must be continuous across the plasma surface we see eq (IV.13) and (IV.14) will be consistent if we choose the proper coefficients  $A'_n$ ,  $B'_n$ ,  $C'_n$  and  $D'_n$ .

Consequently Eqs (IV.10) and (IV.11) may then be written

$$(IV.15) \quad E'_{\phi} = \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon r'} \frac{\partial^2 E'_{z'}}{\partial z' \partial \phi'} + \frac{1}{i\omega' \epsilon} \frac{\partial H'_{z'}}{\partial r'}$$

$$(IV.16) \quad H'_{\phi} = \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon r'} \frac{\partial^2 H'_{z'}}{\partial z' \partial \phi'} + \frac{1}{i\omega' \epsilon} \frac{\partial E'_{z'}}{\partial r'}$$

using the boundary conditions, eq (IV.1) and eqs (III.2), (III.3), and (III.10) and (III.11) substituted into eqs (IV.15) and (IV.16) we get

$$\begin{aligned}
 \text{(IV.17)} \quad n \sin \theta' J_n(k'_0 a \cos \theta') &= -n \sin \theta' H_n^{(1)}(k'_0 a \cos \theta') A'_n + \\
 &+ k'_0 a \cos \theta' H_n^{(1)}(k'_0 a \cos \theta') B'_n + n \frac{k'_0}{\lambda'} \cos^2 \theta' \sin \theta' J_n(\lambda' a) C'_n - \\
 &- \left(\frac{k'_0}{\lambda'}\right)^2 \cos^2 \theta' \left(\frac{\mu_1}{\mu_0}\right) \lambda' a J_n(\lambda' a) D'_n
 \end{aligned}$$

$$\begin{aligned}
 \text{(IV.18)} \quad -k'_0 a \cos \theta' J'_n(k'_0 a \cos \theta') &= k'_0 a \cos \theta' H_n^{(1)}(k'_0 a \cos \theta') A'_n - \\
 &- n \sin \theta' H_n^{(1)}(k'_0 a \cos \theta') B'_n - \left(\frac{k'_0}{\lambda'}\right)^2 \frac{\epsilon_1}{\epsilon_0} \cos^2 \theta' (\lambda' a) J'_n(\lambda' a) C'_n + \\
 &+ n \sin \theta' \left(\frac{k'_0}{\lambda'}\right)^2 \cos^2 \theta' J'_n(\lambda' a) D'_n
 \end{aligned}$$

where we have used the approximation,

$$\text{(IV.19)} \quad \frac{\mu_0 \epsilon_0}{\mu_1 \epsilon_1} \approx \left(\frac{k'_0}{\lambda'}\right)^2 \cos^2 \theta'$$

Eqs (IV.17) and (IV.18) together with eqs (IV.2) and (IV.3) form a system of equations that completely specify the undetermined coefficients. The system of equations may be written as a matrix equation

$$\text{(IV.20)} \quad A \Lambda = \mathfrak{B}$$

where

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \dots & a_{14} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{41} & \dots & \dots & a_{44} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} A'_n \\ B'_n \\ C'_n \\ D'_n \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

We may solve eq (IV.20) for the column vector  $\Lambda$  by left multiplying eq (IV.20) by  $a^{-1}$ , the inverse of  $a$  which is defined

(IV.21)

$$a^{-1} = \frac{1}{\det|a|} \begin{pmatrix} a'_{11} & a'_{21} & \dots & a'_{41} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a'_{14} & \dots & \dots & a'_{44} \end{pmatrix}$$

The factor  $\det|a|$  is just the determinant value of the matrix  $a$ , the primed elements  $a'_{ij}$  are just the transpose of the cofactors of the elements  $a_{ij}$  associated with  $a$ . Specifically the elements are

$$a_{11} = H_n^{(1)}(k'_0 a \cos \theta')$$

$$a_{12} = 0$$

$$a_{13} = -J_n(\lambda' a)$$

$$a_{14} = 0$$

$$a_{21} = 0$$

$$a_{22} = H_n^{(1)}(k'_0 a \cos \theta')$$

$$a_{23} = 0$$

$$a_{24} = -J_n(\lambda' a)$$

$$a_{31} = -n \sin \theta' H_n^{(1)}(k'_0 a \cos \theta')$$

$$a_{32} = k'_0 a \cos \theta' H_n^{(1)'}(k'_0 a \cos \theta')$$

$$a_{33} = (k'_0/\lambda')^2 \cos^2 \theta' n J_n(\lambda' a) \sin \theta'$$

$$a_{34} = - (k'_0/\lambda')^2 \cos^2 \theta' (\lambda' a) J'_n(\lambda' a) (\mu_1/\mu_0)$$

$$a_{41} = k'_0 a \cos \theta' H_n^{(1)'}(k'_0 a \cos \theta')$$

$$a_{42} = - \sin \theta' n H_n^{(1)'}(k'_0 a \cos \theta')$$

$$a_{43} = - (k'_0/\lambda')^2 \cos^2 \theta' (\epsilon_1/\epsilon_0) (\lambda' a) J'_n(\lambda' a)$$

$$a_{44} = (k'_0/\lambda')^2 \cos^2 \theta' \sin \theta' n J_n(\lambda' a)$$

and for the elements of the column vector  $B$ ,

$$b_1 = - J_n(k'_0 a \cos \theta')$$

$$b_2 = 0$$

$$b_3 = \sin \theta' n J_n(k'_0 a \cos \theta')$$

$$b_4 = - k'_0 a \cos \theta' J'_n(k'_0 a \cos \theta')$$

The formal solution to the scattering problem of a plane wave incident on a stationary plasma column is given by

$$(IV.22) \quad \Lambda = A^{-1} B$$

It is necessary to make a Lorentz transformation to the observer's frame  $S$  which sees the plasma column translating with velocity  $v \vec{e}_z$ . One then obtains the scattered fields from a moving plasma column. The resultant fields are

$$(IV.23) \quad E_z^{(s)} = E_z^{(s)'}$$

$$(IV.24) \quad H_z^{(s)} = H_z^{(s)'}$$

$$(IV.25) \quad E_\phi^{(s)} = \gamma (E_\phi^{(s)'} - v_z \mu_0 H_r^{(s)'})$$

$$(IV.26) \quad H_\phi^{(s)} = \gamma (H_\phi^{(s)'} + v_z \epsilon_0 E_r^{(s)'})$$

$$(IV.27) \quad E_r^{(s)} = \gamma(E_r^{(s)})' + v_z \mu_0 H_\phi^{(s)}{}'$$

$$(IV.28) \quad H_r^{(s)} = \gamma(H_r^{(s)})' - v_z \epsilon_0 E_\phi^{(s)}{}'$$

These results are derived in Appendix E.

## V. THE DIFFERENTIAL AND TOTAL SCATTERING CROSS SECTIONS

The differential scattering cross section is defined as

$$(V.1) \quad \frac{d\sigma}{d\Omega} \equiv \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy flux/unit time/unit area}}$$

$$(V.2) \quad d\Omega = dA/r^2 = d\phi dz/r$$

is the element of solid angle in a cylindrical coordinate system. The incident energy flux is the magnitude of the time-averaged Poynting vector for a plane wave

$$(V.3) \quad |\vec{S}^{(i)}| = \frac{1}{2} |R_e \{ \vec{E}^{(i)} \times \vec{H}^{*(i)} \}|$$

where the symbol 'Re' means the real part of the quantity that follows and the horizontal bars denote the magnitude of the quantity between them. For the incident plane wave in our problem it is

$$(V.4) \quad |\vec{S}^{(i)}| = \frac{1}{2} \left( \frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}} |F_E|^2 \gamma$$

More specifically, for the case of the infinite plasma column

$$(V.5) \quad \frac{d^2\sigma}{d\phi dz} = \lim_{r \rightarrow \infty} r \left\{ \frac{\vec{S}_s \cdot \vec{e}_r}{\frac{1}{2} \left( \frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}} |F_E|^2 \gamma} \right\}$$

where

$$(V.6) \quad |\vec{S}_s| = \frac{1}{2} |R_e \{ \vec{E}^{(s)} \times \vec{H}^{*(s)} \}|$$

represents the energy radiated in the radial direction out from the cylinder. The component of the Poynting vector in the azimuthal

direction,  $\vec{S}_s \cdot \vec{e}_\phi$ , will not contribute to any real energy flow, but instead represents some type of "inductive" energy that is not radiative. The z component of the Poynting vector has no meaning for the problem since the plasma is of infinite extent in the z direction. The remainder of this chapter will be spent deriving an explicit expression for the scattered energy flux as given by eq (V.6) using eqs (IV.23) through (IV. 28) we find

$$(V.7) \quad (\vec{E}^{(s)} \times \vec{H}^{*(s)})_r = \gamma (H_z^{*(s)} \hat{r} \cdot \vec{E}^{(s)} - E_z^{(s)} \hat{r} \cdot H_\phi^{*(s)}) - \\ - \gamma v_z (\mu_0 H_z^{*(s)} \hat{r} \cdot H_r^{(s)} + \epsilon_0 E_z^{(s)} \hat{r} \cdot E_r^{*(s)})$$

$$(V.8) \quad (\vec{E}^{(s)} \times \vec{H}^{*(s)})_\phi = \gamma (E_z^{(s)} \hat{r} \cdot H_r^{*(s)} - H_z^{*(s)} \hat{r} \cdot E_r^{(s)}) - \\ - \gamma v_z (\epsilon_0 E_z^{(s)} \hat{r} \cdot E_\phi^{*(s)} + \mu_0 H_z^{*(s)} \hat{r} \cdot H_\phi^{(s)})$$

$$(V.9) \quad (\vec{E}^{(s)} \times \vec{H}^{*(s)})_z = \gamma^2 (E_r^{(s)} \hat{r} \cdot H_\phi^{*(s)} - E_\phi^{(s)} \hat{r} \cdot H_r^{*(s)}) \\ + \gamma^2 v_z^2 (\mu_0 [|H_\phi^{(s)} \hat{r}|^2 + |H_r^{(s)} \hat{r}|^2] + \epsilon_0 [|E_\phi^{(s)} \hat{r}|^2 + |E_r^{(s)} \hat{r}|^2]) \\ + \gamma^2 v_z^2 \mu_0 \epsilon_0 (H_\phi^{(s)} \hat{r} \cdot E_r^{*(s)} - H_r^{(s)} \hat{r} \cdot E_\phi^{*(s)})$$

If we use eqs (IV.15), (IV.16), (D.15) and (D.16) and substitute eqs (III.10) and (III.11) into them we find

$$(V.10) \quad (\vec{E}^{(s)} \times \vec{H}^{*(s)})_r = \gamma |F_E'|^2 \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} \left(\frac{2}{\pi k_0 r'}\right) \times \\ \times \left[ \left(1 + \frac{v_z \sin \phi'}{c}\right) \sum_{n, n' = -\infty}^{\infty} (-1)^{n+n'} B_n' B_n^{*'} e^{i(n-n')(\phi' - \pi/2)} + \right. \\ \left. + \left(1 - \frac{v_z \sin \phi'}{c}\right) \sum_{n, n' = -\infty}^{\infty} (-1)^{n+n'} A_n' A_n^{*'} e^{i(n-n')(\phi' - \pi/2)} \right]$$

$$(V.11) \quad (\vec{E}(s) \times \vec{H}^*(s))_{\phi} = \gamma |F'_E|^2 \left(\frac{\epsilon_C}{\mu_0}\right)^{\frac{1}{2}} \left(\frac{4i \sin \phi'}{\pi k_0 r}\right) \times$$

$$\times \sum_{n, n'=-\infty}^{\infty} (-1)^{n+n'} A'_n B_{n'}^{*'} e^{i(n-n')(\phi' - \pi/2)}$$

$$(V.12) \quad (\vec{E}(s) \times \vec{H}^*(s))_z = \gamma^2 |F'_E|^2 \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} \left(\frac{2 \cos \phi'}{\pi k_0 r}\right) \times$$

$$\times \left( \sin \theta' \left[ 1 + \left(\frac{v_z}{c}\right)^2 \right] + \frac{v_z}{c} [1 + \sin^2 \theta'] \right) \times$$

$$\times \sum_{n, n'=-\infty}^{\infty} (-1)^{n+n'} (A'_n A_{n'}^{*'} + B'_n B_{n'}^{*'}) e^{i(n-n')(\phi' - \pi/2)}$$

where we have used the asymptotic approximation to the Hankel function in eqs (V.10) through (V.12), i.e.

$$(V.13) \quad H_n^{(1)}(x) \longrightarrow \sqrt{\frac{2}{\pi x}} \exp\{i[z - \frac{\pi}{2}(n + \frac{1}{2})]\}$$

for large  $x$ , valid for the far field scattering zone.

## VI. THE SHORT WAVELENGTH LIMIT WITH THE PLASMA COLUMN STATIONARY

The differential scattering cross section as expressed in eq (V.5) may be written

$$(VI.1) \quad \frac{d^2 \sigma}{d\phi dz} = \frac{2}{\pi k_0} \operatorname{Re} \sum_{n, n'=-\infty}^{\infty} (-1)^{n+n'} (A'_n A_{n'}^{*'} + B'_n B_{n'}^{*'}) \times$$

$$e^{i(n-n')(\phi' - \pi/2)}$$

The backscattering differential cross section corresponds to  $\phi = \pi/2$ , for normal incidence, i.e.  $\theta_0 = 0^\circ$

$$(VI.2) \quad \sigma_{bs} \equiv \left( \frac{d^2 \sigma}{d\phi dz} \right)_{\substack{\phi = \pi/2 \\ \theta_0 = 0}} = \frac{2}{\pi k_0} \operatorname{Re} \sum_{n, n' = -\infty}^{\infty} (-1)^{n+n'} (A_n' A_n^{*\prime} + B_n' B_n^{*\prime})$$

which is of concern to us. The results of eqs (VI.1) and (VI.2) are found from eqs (V.5) and (V.10). For an incident plane wave with a frequency

$$\omega_0 = 163 \text{ GHz}$$

a plasma column radius

$$a = .3 \text{ cm}$$

with plasma index of refraction

$$n = i 3.45$$

and a plasma frequency

$$\omega_p = 563 \text{ GHz}$$

we find that the series in the right hand side of eq (VI.2) converges quite rapidly after two terms, namely

$$(VI.3) \quad A_n' \approx A_n^{*\prime} \approx B_n' \approx B_n^{*\prime} \approx 0, \text{ for } |n|, |n'| > 1$$

We also note the relations

$$(VI.4) \quad \begin{cases} A_{-n}' = A_n' \\ B_{-n}' = -B_n' \end{cases}$$

We find for normal incidence  $\theta_0 = 0^\circ$

$$(VI.5) \quad (\sigma_{bs})_{\theta_0 = 0^\circ} = 5.03 \text{ cm}$$

A general expression may be found for the differential scattering cross section by using eq (VI.1) along with the approximation of eq (VI.3) giving

$$\begin{aligned}
 \text{(VI.7)} \quad \frac{d^2\sigma}{d\phi dz} = & \frac{2}{\pi k_0} \left\{ |A_0|^2 + |B_0|^2 + 2|A_1|^2 \left[ 1 + \cos 2(\phi - \pi/2) \right] \right. \\
 & + 2|B_1|^2 \cdot \left[ 1 - \cos 2(\phi - \pi/2) \right] - 2\text{Re} \left\{ A_0 A_1^* + A_1 A_0^* \right\} \cos(\phi - \pi/2) \\
 & \left. - 2\text{Re} \left\{ B_0 B_1^* + B_1 B_0^* \right\} \cos(\phi - \pi/2) \right\}
 \end{aligned}$$

For the grazing angle incidence the magnitudes of the coefficients  $A_1$  and  $B_1$  dominate over all others and they are approximately equal, with the angular dependence just that of  $\cos(\phi - \pi/2)$ . The scattering is maximum in the forward transmitted direction, i.e. at  $\phi = 3\pi/2$ , and a minimum in the forward, reflected direction at  $\phi = \pi/2$ . For broadside incidence the magnitude of the coefficient  $A_0$  is much greater than all others and as before, the angular dependence is  $\cos(\phi - \pi/2)$  and qualitatively the scattering angular distribution is identical to the case of grazing incidence. Table 1 lists the real and imaginary parts of the complex coefficients  $A_0$ ,  $A_1$ ,  $B_0$  and  $B_1$  for grazing and broadside incidence.

The terms "forward" scattered and "back" scattered for non-normal incidence should be physically interpreted carefully. From Snell's law, and conservation of momentum it is clear that the forward scattered energy is just that energy that is reflected from the backside of the cylinder, at an angle that is identically equal to the angle of incidence. The back scattered energy is that which is reflected from the front side of the cylinder, at an angle identically equal to the angle of incidence. Thus in both cases, for non-normal incidence, no energy would be reflected back to the source point of the incident radiation. Of course for normal incidence the back scattered energy would reach the initial source point of the incident radiation.

## VII. CONCLUSIONS

The velocity-dependent differential scattering cross section has been found. For the special case of the plasma column at rest, and the incident plane wave frequency sufficiently high, we obtained explicit expressions for the scattered or radiated energy flux. The angular distribution of the scattered energy is transmitted forward for both cases of incidence, i.e. broadside and grazing. This is not surprising for the situation of grazing incidence, for the effective " $k_{0a}$ " is large enough that geometric optics effects begin to show, namely the treatment of the waves as rays. In such a case one would expect the radiated energy to be forward scattered. This is sometimes referred to as specular scattering. For broadside incidence the effective  $k_{0a}$  is zero. One would expect, as is the

case for the scattering from the perfectly conducting cylinder<sup>14</sup> most of scattered energy is in the backscattered direction. However, since the dielectric constant for the plasma column is  $O(1)$  we should not expect to be able to extrapolate that well from the qualitative behaviour of a medium whose conductivity approaches infinity. The effect of the strength of the dielectric constant on the angular distribution of the scattered energy is isotropic to zeroth order in the expansion coefficients and approximately so to the first order. These results compare favorably with those of Yeh<sup>10</sup> except he presents the angular dependence of the square of the z component of the E and H scattered fields to the incident field instead of the scattering cross section explicitly. His values for  $k_p a$  and  $k_o a$  are comparable, however.

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10. C. Yeh, Op. Cit. p. 5066

14. W. K. H. Panofsky and M. Phillips, "Classical Electricity and Magnetism," (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962), p.330.

## APPENDIX A. THE PLANE WAVE EXPANSION

A Fourier-Bessel expansion for a plane wave is developed from the expansion<sup>15</sup>

$$(A.1) \quad \exp\{z(t^2-1)/2t\} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

With the substitutions

$$t = -\exp(i\theta')$$

$$z = k'_0 r' \cos\theta'$$

we establish the identity

$$(A.2) \quad \exp(-ik'_0 \cos\theta' y') = \sum_{n=-\infty}^{\infty} (-1)^n \exp(in\theta') J_n(k'_0 r' \cos\theta')$$

which is just the result we presented in eq (III.2)

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15. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics" Vol. 1, (McGraw-Hill Book Company Inc., New York, 1953), pp 619-620.

## APPENDIX B. THE PLANE WAVE TRANSFORMATIONS

The phase of a plane wave is a Lorentz invariant quantity since one associates phase with the number of wave peaks or valleys (or any other easily identifiable reference points) that pass a given reference point in some given time. Imagine an observer in the S frame possesses a phase meter, located at point P(z), as shown in

Fig. 2. At time  $z = 0$ , the meter reads  $\vec{k} \cdot \vec{z}$  worth of phase. At time  $\tau = t$  later, the wave has propagated to the left and the meter reads a phase of  $\vec{k} \cdot \vec{z} - \omega t$ . The number of crests the meter has counted is given by

$$(B.1) \quad \begin{array}{l} \text{The} \\ \text{Number} \\ \text{of} \\ \text{Crests} \\ \text{Counted} \end{array} = \frac{\text{phase}}{\text{phase per crest}} = (\vec{k} \cdot \vec{z} - \omega t) / 2\pi$$

An observer in another reference frame  $S'$  moving with velocity  $\vec{V} = \vec{V}e_z$  is situated at point  $p'(z')$ . At  $z = 0$ , the two reference frames  $S$  and  $S'$  coincide. At time  $\tau = t'$  the points P(z) and P'(z') are coincident and the number of crests counted by the observers in either frame should be identical, i.e.

$$(B.2) \quad \vec{k} \cdot \vec{z} - \omega t = \vec{k}' \cdot \vec{z}' - \omega' t'$$

This is shown in Fig. 3.

Recall the Lorentz transformations between two frames that are in relative motion along the z axis,

$$(B.3) \quad \begin{array}{l} x' = x \\ y' = y \\ z' = \gamma(z - vt) \\ t' = \gamma(t - vz/c^2) \end{array}$$

Substitution of eq (B.3) into eq (B.2), collecting the coefficients of x, y, z and t and then equating them to zero (since x, y, z, t are linearly independent variables) yield the following results

$$(B.4) \quad \begin{cases} k_x = k'_x \\ k_y = k'_y \\ k_z = \gamma(k'_z + \omega v_z/c^2) \\ \omega' = \gamma(\omega + k'_z v_z) \end{cases}$$

The inverse relations are

$$(B.5) \quad \begin{cases} k'_z = \gamma(k_z - \omega v_z/c^2) \\ \omega' = \gamma(\omega - k_z v_z) \\ k'_x = k_x \\ k'_y = k_y \end{cases}$$

If  $\theta$  and  $\theta'$  are the angles  $\vec{k}$  and  $\vec{k}'$  make with the  $z$  axis, and for light waves  $|\vec{k}| = \omega/c$ ,  $|\vec{k}'| = \omega'/c$ , we may then recast the first two equations of (B.5) into the form

$$(B.6) \quad \begin{cases} \omega' = \gamma\omega(1 - v_z \cos\theta/c) \\ \tan\theta' = \sin\theta/\gamma(\cos\theta - v_z/c) \end{cases}$$

where

$$(B.7) \quad \begin{cases} k_z = |\vec{k}| \cos\theta \\ k'_z = |\vec{k}'| \cos\theta' \end{cases}$$

The first equation of (B.6) represents the Doppler shift with the  $\gamma$  factor an additional effect. Even at right angles,  $\theta = \pi/2$ , there is a shift in frequency known as the transverse Doppler shift. Relating these results to those of (III.4) through (III.9) we must first identify

$$\theta_o = \pi/2 - \theta$$

The first equation in (B.6) becomes

$$(B.8) \quad \omega' = \gamma\omega(1 - \beta \sin\theta_0)$$

Now

$$k_x x + k_y y = k'_x x' + k'_y y'$$

This implies

$$k \sin\theta = k' \sin\theta'$$

and for the case of the cylinder

$$(B.9) \quad k'_0 \cos\theta' = k_0 \cos\theta_0$$

The second equation in (B.6) becomes

$$\left(\frac{k}{k'}\right) \gamma \cos\theta' = (\cos\theta - v_z/c)^{-1}$$

With some manipulation and the use of the first equation in (B.6) we arrive at

$$(B.10) \quad \sin\theta' = (\sin\theta_0 - \beta) / (1 - \beta \sin\theta_0)$$

We have already verified eq (III.7) since it is no more than the Doppler shift equation of (B.6).

We must now examine the field transformation equations<sup>13</sup>

$$(B.11) \quad \begin{cases} \vec{E}'_{||} = \vec{E}_{||} \\ \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \end{cases}$$

More specifically

$$(B.12) \quad \vec{E}'_{||} = \vec{E}'_{||} = E_0 \cos\theta_0 \vec{e}_z$$

$$(B.13) \quad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) = \gamma(-E_0 \sin\theta_0 \vec{e}_y + \vec{v} \times \vec{B})$$

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13. J. D. Jackson, Op. Cit. P223-226.

Recall, for a plane wave

$$(B.14) \quad \vec{B} = \sqrt{\mu_0 \epsilon_0} \vec{e}_k \times \vec{E} = -\sqrt{\mu_0 \epsilon_0} E_0 \vec{e}_x$$

and consequently

$$(B.15) \quad \vec{v} \times \vec{B} = \frac{v}{c} E_0 \vec{e}_y$$

Substitution of eq (B.15) into eq (B.13) yields

$$(B.16) \quad \vec{E}'_{\perp} = \gamma E_0 (\beta - \sin \theta_0) \vec{e}_y$$

and

$$(B.17) \quad E_0'^2 = E_0^2 (\cos^2 \theta_0 + \gamma^2 \sin^2 \theta_0 + \gamma^2 \beta^2) - 2E_0^2 \gamma^2 \sin \theta_0$$

With some rearrangement and explicit use of the defining equation for  $\gamma$  we obtain

$$(B.18) \quad E_0' = E_0 \gamma (1 - \beta \sin \theta_0)$$

Thus eq (III.8) is confirmed.

The equality of phases, eq (B.2) combined with the first equation in (B.11) gives rise to eq (III.9). We have now established eqs (III.4) through (III.9) of Chapter III.

## APPENDIX C. THE WAVE NUMBER IN THE PLASMA

If we consider the incident plane wave striking a tangent plane of the plasma's cylindrical surface we immediately obtain Snell's law<sup>13</sup>

$$(C.1) \quad \sin\theta' / \sin\theta'' = \sqrt{\mu_1 \epsilon_1 / \mu_0 \epsilon_0}$$

where  $\theta'$  and  $\theta''$  are the angles the propagation vectors  $\vec{k}'_0$  and  $\vec{k}''_0$  make with the normal to the tangent plane outside and inside the cylinder respectively. Define

$$(C.2) \quad k'_0 = (k''_0 \cos\theta'') / \cos\theta' \equiv \lambda' / \cos\theta''$$

Since the  $S'$  frame is identically the same inside or outside the cylinder

$$(C.3) \quad \omega' t' = \omega'' t''$$

where

$$(C.4) \quad \omega' = k'_0 c$$

$$(C.5) \quad \omega'' = k''_0 c = \lambda c / \cos\theta''$$

Since  $t'$  is identically equal to  $t''$  we obtain

$$(C.6) \quad \omega' = k'_0 / (\mu_0 \epsilon_0)^{1/2} = \lambda' / (\mu_1 \epsilon_1)^{1/2} \cos\theta''$$

or

$$(C.7) \quad \left( \frac{\lambda'}{k'_0} \right)^2 = \left( \frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0} \right) \cos^2 \theta''$$

We may rearrange eq (C.1) into a more convenient form by squaring it and using a trigonometric identity to obtain

$$(C.8) \quad \left( \frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0} \right) \cos^2 \theta'' = \frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0} - \sin^2 \theta'$$

Substituting the left hand side of eq (C.7) into eq (C.8) gives

$$(C.9) \quad \lambda' = k'_0 \left( \frac{\mu_1 \epsilon_1}{\mu_0 \epsilon_0} - \sin^2 \theta' \right)^{\frac{1}{2}}$$

which is the desired expression.

APPENDIX D. THE TRANSVERSE FIELD COMPONENTS AS  
FUNCTIONS OF THE LONGITUDINAL FIELDS

Assuming harmonic time dependence of the form  $\exp(-i\omega t)$  the Maxwell equations become

$$(D.1) \quad \vec{\nabla} \times \vec{E}' = i\omega \vec{B}' = i\omega \kappa \mu_0 \vec{H}' \quad , \quad \vec{\nabla} \cdot \vec{H}' = 0$$

$$(D.2) \quad \vec{\nabla} \times \vec{H}' = -i\omega \kappa \epsilon_0 \vec{E}' \quad , \quad \vec{\nabla} \cdot \vec{E}' = 0$$

For any vector field  $\vec{A}(\vec{r})$  we may write its curl in cylindrical coordinates as

$$(D.3) \quad (\vec{\nabla} \times \vec{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial r}$$

$$(D.4) \quad (\vec{\nabla} \times \vec{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(D.5) \quad (\vec{\nabla} \times \vec{A})_z = \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi}$$

then the first equations in (D.1) and (D.2) become

$$(D.6) \quad i\omega \kappa \mu_0 H'_r = \frac{1}{r} \frac{\partial E'_z}{\partial \phi} - \frac{\partial E'_\phi}{\partial z}$$

$$(D.7) \quad i\omega \kappa \mu_0 H'_\phi = \frac{\partial E'_r}{\partial z} - \frac{\partial E'_z}{\partial r}$$

$$(D.8) \quad i\omega \kappa \mu_0 H'_z = \frac{1}{r} \frac{\partial}{\partial r} (r E'_\phi) - \frac{1}{r} \frac{\partial E'_r}{\partial \phi}$$

and

$$(D.9) \quad -i\omega \kappa \epsilon_0 E'_r = \frac{1}{r} \frac{\partial H'_z}{\partial \phi} - \frac{\partial H'_\phi}{\partial z}$$

$$(D.10) \quad -i\omega \kappa \epsilon_0 E'_\phi = \frac{\partial H'_r}{\partial z} - \frac{\partial H'_z}{\partial r}$$

$$(D.11) \quad -i\omega' \kappa \epsilon_0 E_z' = \frac{1}{r'} \frac{\partial}{\partial r'} (r' H_\phi') - \frac{1}{r'} \frac{\partial H_r'}{\partial \phi'}$$

Since we are first attempting to establish the relations

$$E_\phi = E_\phi (E_z, H_z), \quad H_\phi = H_\phi (E_z, H_z)$$

We shall differentiate eq (D.6), then substitute this result into eq (D.10). We find

$$(D.12) \quad E_\phi' = \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon} \left( \frac{1}{r'} \frac{\partial^2 E_z'}{\partial z' \partial \phi'} - \frac{\partial^2 E_\phi'}{\partial z'^2} \right) + \frac{1}{i\omega' \epsilon} \frac{\partial H_z'}{\partial r'}$$

where we have used the relation

$$(D.13) \quad \omega'^2 = k_0'^2 / \mu_0 \epsilon_0$$

Let us differentiate eq (D.9) and substitute the result into eq (D.7) getting

$$(D.14) \quad H_\phi' = \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon} \left( \frac{1}{r'} \frac{\partial^2 H_z'}{\partial z' \partial \phi'} - \frac{\partial^2 H_\phi'}{\partial z'^2} \right) - \frac{1}{i\omega' \mu} \frac{\partial E_z'}{\partial r'}$$

In a similar fashion one may find the radial components of the fields in terms of the longitudinal components, i.e.

$$E_r = E_r (E_z, H_z), \quad H_r = H_r (E_z, H_z)$$

Differentiate eq (D.7) with respect to  $z'$  and substitute the result into eq (D.9). We obtain

$$(D.15) \quad E_r' = \frac{i}{\omega' \epsilon} \left( \frac{1}{r'} \frac{\partial H_z'}{\partial \phi'} \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon r'} \frac{\partial^3 H_z'}{\partial z'^2 \partial \phi'} - \frac{i}{\omega' \mu} \frac{\partial^2 E_z'}{\partial z' \partial r'} \right)$$

and we have established the first relation.

Differentiate eq (D.10) and substitute this result into eq (D.6). The result is

$$(D.16) \quad H_r' = -\frac{i}{\omega \mu} \left( \frac{1}{r} \frac{\partial E_z'}{\partial \phi} - \frac{\mu_0 \epsilon_0}{k_0'^2 \mu \epsilon r} \frac{\partial^3 E_z'}{\partial z'^2 \partial \phi} - \frac{i}{\omega \epsilon} \frac{\partial^2 H_z'}{\partial z' \partial r'} \right)$$

In both eqs (D.15) and (D.16) the terms involving the second partials of the radial fields with respect to  $z'$  have been included implicitly in the expansion coefficients  $A_n'$ ,  $B_n'$ ,  $C_n'$  and  $D_n'$ , using the arguments of Chapter IV.

APPENDIX E. THE SCATTERED FIELD COMPONENTS  
IN THE OBSERVERS FRAME

If we use the Lorentz transformation equations appropriate to the electromagnetic field<sup>15</sup> we see

$$(E.1) \quad \begin{array}{l} E_{||} = E'_{||} \\ B_{||} = B'_{||} \end{array} \quad \Rightarrow \quad \begin{array}{l} E_z = E'_{z'} \\ B_z = B'_{z'} \end{array}$$

and

$$(E.2) \quad \vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \vec{v} \times \vec{B}'_{\perp})$$

For the case

$$\begin{aligned} \vec{E}'_{\perp} &= E'_{\phi'} \vec{e}_{\phi'} \\ (\vec{v} \times \vec{B}'_{\perp})_{\phi'} &= v_z B'_{r'} \vec{e}_{\phi'} \end{aligned}$$

and we have

$$(E.3) \quad E_{\phi} = \gamma(E'_{\phi'} - v_z \mu_0 H'_{r'})$$

For the other transverse component

$$\begin{aligned} \vec{E}'_{\perp} &= E'_{r'} \vec{e}_{r'} \\ (\vec{v} \times \vec{B}'_{\perp})_{r'} &= -v_z B'_{\phi'} \vec{e}_{r'} \end{aligned}$$

which gives

$$(E.4) \quad E_{r'} = \gamma(E'_{r'} + v_z \mu_0 H'_{\phi'})$$

---

15. W. K. H. Panofsky and M. Phillips, "Classical Electricity and Magnetism," (Addison-Wesley Publishing Company, Inc. Reading, Mass., 1962), p. 330.

The final transformation equation

$$(E.5) \quad \vec{B}_{\perp} = \gamma(\vec{B}'_{\perp} + \vec{v} \times \vec{E}'_{\perp}/c^2)$$

has two cases as well. For the first case

$$\begin{aligned} \vec{B}'_{\perp} &= B'_{\phi} \vec{e}'_{\phi} \\ (\vec{v} \times \vec{E}')_{\phi} &= v_z E'_{r'} \vec{e}'_{\phi} \end{aligned}$$

and we find

$$(E.6) \quad H_{\phi} = \gamma(H'_{\phi} + v_z \epsilon_0 E'_{r'})$$

where we have used the relation

$$c^2 = (\mu_0 \epsilon_0)^{-1}$$

For the second case

$$\begin{aligned} \vec{B}'_{\perp} &= B'_{r'} \vec{e}'_{r'} \\ (\vec{v} \times \vec{E}'_{\perp})_{r'} &= -v_z E'_{\phi} \vec{e}'_{r'} \end{aligned}$$

and

$$(E.7) \quad H_{r'} = \gamma(H'_{r'} - v_z \epsilon_0 E'_{\phi})$$

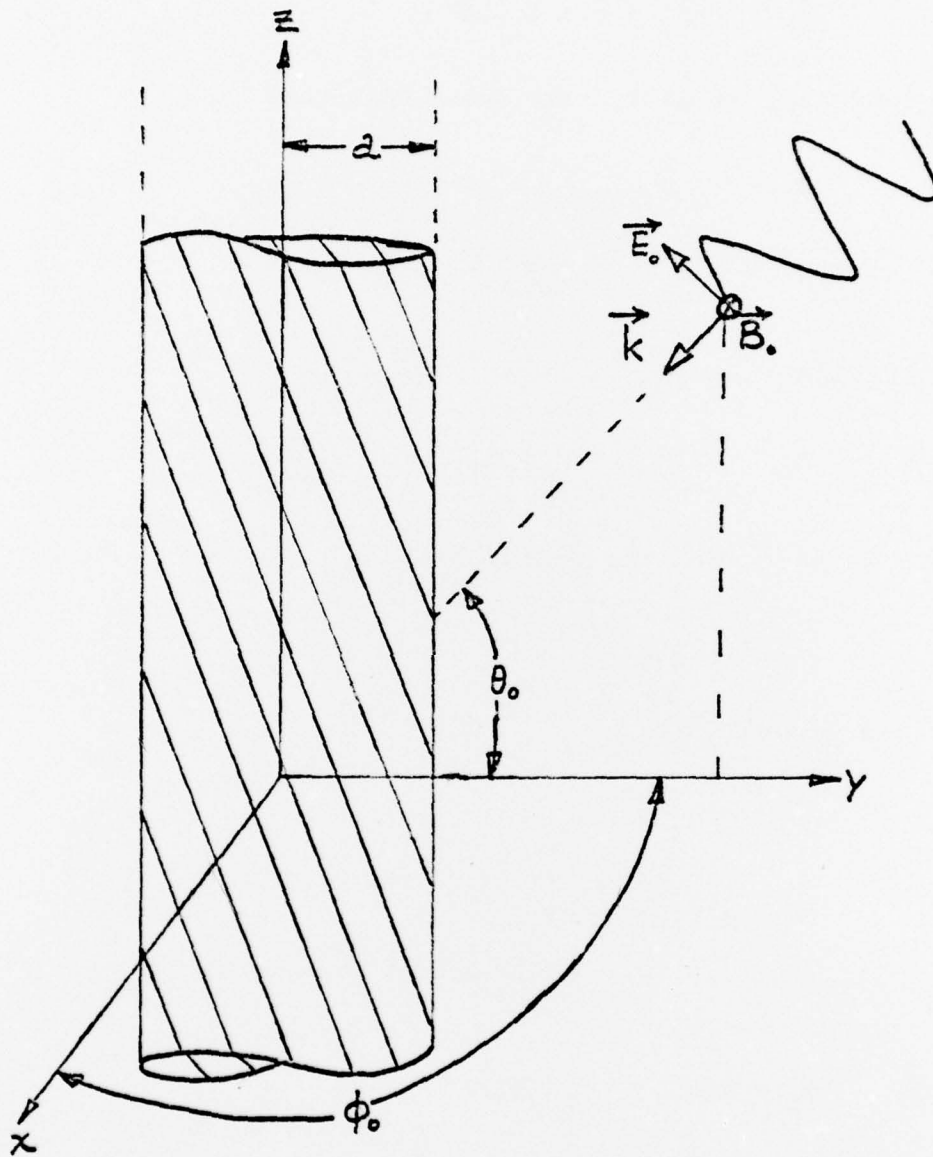
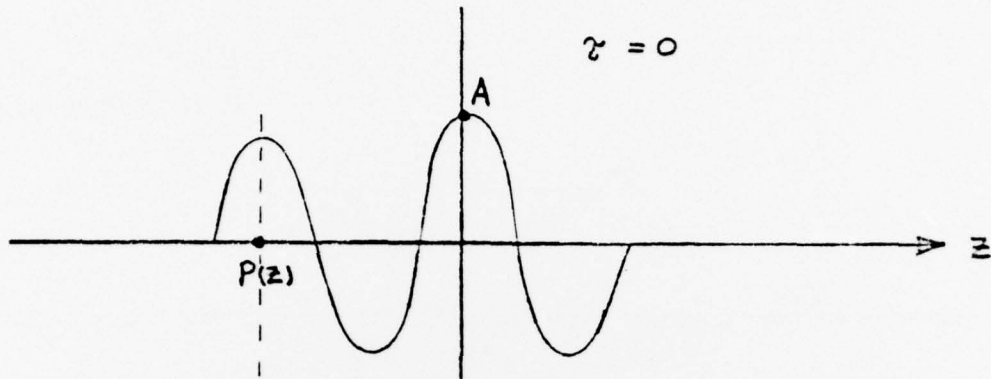
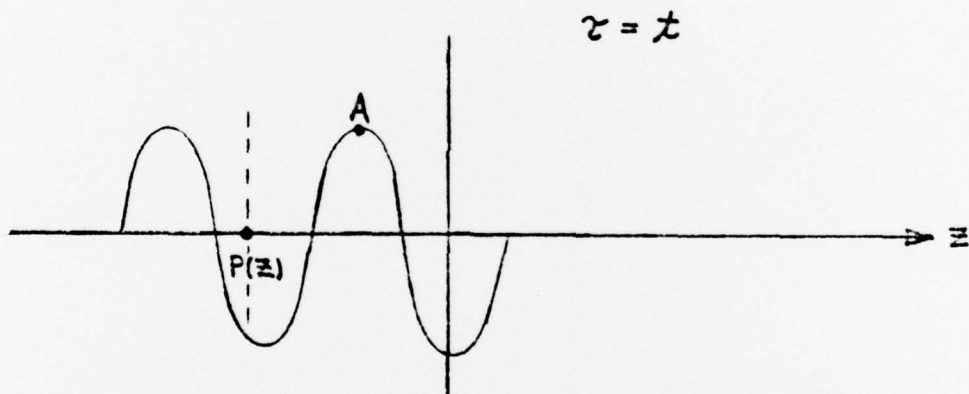


Fig. 1. Plane Wave Obliquely Incident on Plasma Column



Phase meter reads  $\vec{k} \cdot \vec{z}$



Phase meter reads  $\vec{k} \cdot \vec{z} - \omega t$

Fig.2 The Measurement of Phase by Counting Wave Crests

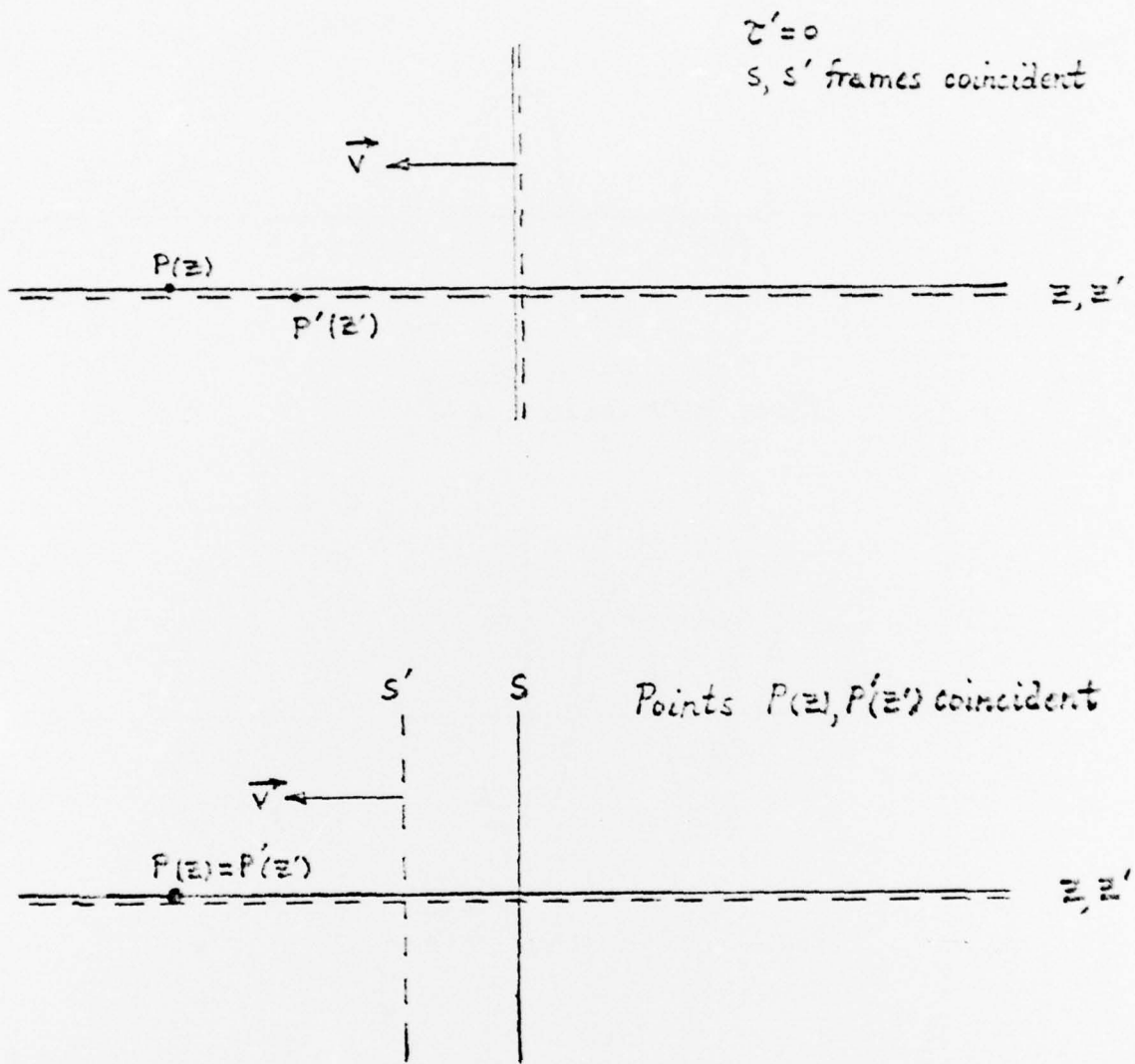


Fig. 3 The invariance of phase

$\theta_o$ , Angle of Incidence	$\text{Re}A_o$	$\text{Im}A_o$	$\text{Re}A_1$	$\text{Im}A_1$
$0^\circ$	8.50	-2.11	.858	-2.23
$80^\circ$	.247	-.0608	.845	-.0672
$\theta_o$ , Angle of Incidence	$\text{Re}B_o$	$\text{Im}B_o$	$\text{Re}B_1$	$\text{Im}B_1$
$0^\circ$	0	0	0	0
$80^\circ$	0	0	.833	-.0662

Table 1. The Dominant Expansion Coefficients for the Scattering Cross Section

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