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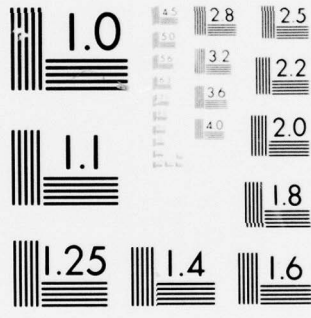
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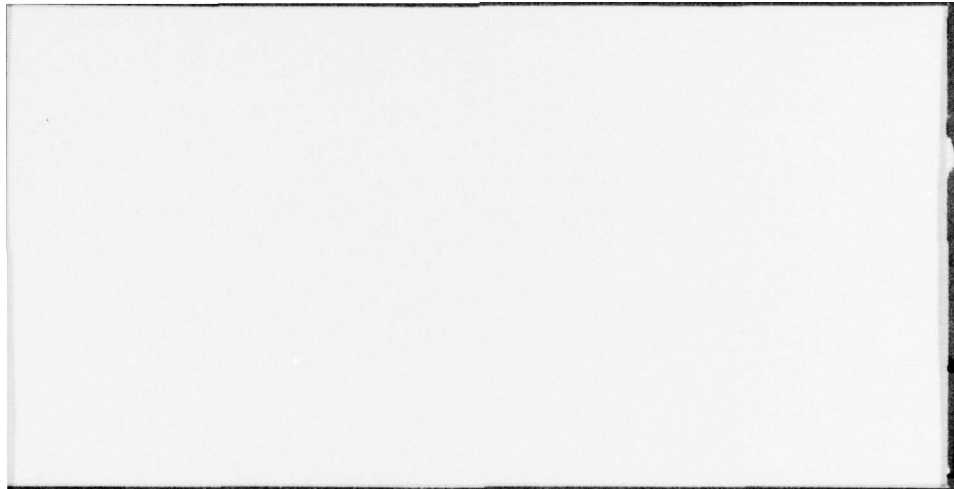


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Steady-State Wave Propagation in Simple and Compound  
ACOUSTIC WAVEGUIDES

J. C. Guillot and C. H. Wilcox

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ABSTRACT

A limiting absorption principle and corresponding existence theorem is presented for steady-state acoustic wave fields in simple and compound waveguides. The simple guides are semi-infinite cylinders, sectoral horns and conical horns. The compound guides are structures built by coupling one or more simple guides to a bounded cavity. The theory is developed under minimal regularity hypotheses about the waveguide walls. The class of allowable guides includes all the simple, but non-smooth, structures that arise in applications such as guides built from polyhedra and sections of cylinders, cones, spheres and other simple geometrical structures.

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## SECTION ONE

### INTRODUCTION

The acoustic waveguides studied in this paper are characterized geometrically by a class of unbounded domains  $\Omega$  in 3-dimensional Euclidean space  $E^3$  of the form

$$(1.1) \quad \Omega = \Omega_0 \cup \left( \bigcup_{i=1}^I S_i \right) \cup \left( \bigcup_{j=1}^{I'} S'_j \right) \cup \left( \bigcup_{k=1}^{I''} S''_k \right)$$

where  $\Omega_0$  is bounded, each  $S_i$  is a semi-infinite cylinder, each  $S'_j$  is a sectoral horn and each  $S''_k$  is a conical horn. Each set  $S_i, S'_j, S''_k$  is defined by a coordinate system in  $E^3$ . In particular

$$(1.2) \quad S_i = S_i(y_0^i) = \{q \in E^3: (x_1^i(q), x_2^i(q)) \in G_i \text{ and } y^i(q) > y_0^i\}$$

where  $(x_1^i, x_2^i, y^i) \in R^3$  are Cartesian coordinates,  $y_0^i \in R$  and  $G_i \subset R^2$  is a bounded domain. Similarly,

$$(1.3) \quad S'_j = S'_j(\rho_0^j) = \{q \in E^3: \rho^j(q) > \rho_0^j \text{ and } (\theta^j(q), z^j(q)) \in G'_j\}$$

where  $(\rho^j, \theta^j, z^j) \in R^3$  are cylindrical coordinates,  $\rho_0^j \geq 0$ ,  $G'_j = M'_j \times (0, z_0^j)$ ,  $M'_j$  is a domain in the unit circle  $S^1$  (which may coincide with  $S^1$ ) and  $z_0^j > 0$ . Finally,

$$(1.4) \quad S''_k = S''_k(r_0^k) = \{q \in E^3: r^k(q) > r_0^k \text{ and } \eta^k(q) \in G''_k\}$$

where  $(r^k, \eta^k) \in R \times S^2$  are spherical coordinates,  $S^2$  is the unit 2-sphere,  $r_0^k \geq 0$  and  $G''_k$  is a domain in  $S^2$ . The pair-wise intersections of the

$I + I' + I''$  sets  $S_i, S_j', S_k''$  are assumed to be bounded. For suitable choices of  $\Omega_0, y_0^i, \rho_0^j, r_0^k$  one then has

$$(1.5) \quad \text{the } I + I' + I'' \text{ sets } S_i, S_j', S_k'' \text{ are pair-wise disjoint}$$

By further specialization of  $\Omega_0, y_0^i, \rho_0^j, r_0^k$  it can be arranged that

$$(1.6) \quad \begin{aligned} &\text{the } I + I' + I'' + 1 \text{ sets } \Omega_0, S_i(y_0^i + 1), S_j'(\rho_0^j + 1), \\ &S_k''(r_0^k + 1) \text{ are pair-wise disjoint.} \end{aligned}$$

Throughout the paper  $\Omega$  is assumed to satisfy (1.1)-(1.6). The simple waveguides are the special cases where  $I + I' + I'' = 1$  and  $\Omega_0$  is empty. The remaining cases are called compound waveguides. The domains  $\Omega$  include a large number of geometries that arise in applications. The sectoral horns include parallel plate waveguides ( $M_j' = S^1$ ). The conical horns include the exteriors of spheres ( $G_k'' = S^2$ ), half-spaces, wedges and the complement of a plane with a bounded aperture.

Each waveguide is assumed to be filled with a homogeneous fluid whose equilibrium state is characterized by a sound speed  $c_0$ , density  $\rho_0$  and pressure  $p_0$ . It is assumed here that  $c_0 = 1$  and  $\rho_0 = 1$  since this can be achieved by a suitable choice of units. The acoustic field in  $\Omega$  is characterized by a real-valued acoustic potential  $u(t, q)$  which satisfies the d'Alembert equation

$$(1.7) \quad \frac{\partial^2 u}{\partial t^2} - \Delta u = f(t, q), \quad t \in \mathbb{R}, \quad q \in \Omega$$

where  $\Delta$  is the Laplacian in  $E^3$ . The function  $f$  will be called the source function. It has the structure [19, p. 280]

$$(1.8) \quad f(t, q) = \partial V(t, q) / \partial t + Q(t, q)$$

where  $V(t, q)$  is a potential for the (conservative) force field and  $Q(t, q)$  is the volume flow in  $\Omega$ . On the waveguide walls, described by the boundary  $\partial\Omega$ ,  $u(t, q)$  satisfies the Dirichlet condition

$$(1.9) \quad u \Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a soft boundary}$$

and the Neumann condition

$$(1.10) \quad \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a hard boundary}$$

where  $\partial u / \partial \nu = \nabla u \cdot \vec{\nu}$  and  $\vec{\nu}$  is a normal vector on  $\partial\Omega$ .

Steady-state acoustic fields are generated by source functions of the form

$$(1.11) \quad f(t, q) = g_1(q) \cos \omega t + g_2(q) \sin \omega t = \operatorname{Re} \{g(q) e^{-i\omega t}\}$$

where  $\omega \geq 0$  and  $g = g_1 + ig_2$ . The corresponding steady-state acoustic potentials have the same form

$$(1.12) \quad u(t, q) = v_1(q) \cos \omega t + v_2(q) \sin \omega t = \operatorname{Re} \{v(q) e^{-i\omega t}\}$$

where  $v = v_1 + iv_2$ .  $u(t, q)$  satisfies (1.7) and (1.9) or (1.10) for all  $t \in \mathbb{R}$  if and only if

$$(1.13) \quad \Delta v + \omega^2 v = -g(q), \quad q \in \Omega$$

and

$$(1.14) \quad v \Big|_{\partial\Omega} = 0 \text{ or } \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} = 0$$

These conditions do not determine  $v$  uniquely. Instead, solutions may differ by any solution of the reduced problem with  $g(q) \equiv 0$  in  $\Omega$ . Among these are the solutions in  $L_2(\Omega)$ , corresponding to eigenvalues  $\omega^2$  of the operator  $-\Delta$  acting in  $L_2(\Omega)$ . F. Rellich [18] and D. S. Jones [10] have shown that such eigenvalues may exist when  $I'' = 0$  (no conical horns in  $\Omega$ ). If  $\partial\Omega$  has sharp edges or vertices the reduced problem may have "edge wave" solutions which correspond physically to sources hidden in the edge. For example if  $\Omega = \{q \in E^3: 0 < \theta(q) < \pi/2\}$  then

$$(1.15) \quad v = H_0^{(1)}(\omega\rho)$$

satisfies (1.13) with  $g(q) \equiv 0$  in  $\Omega$  and the Neumann condition on  $\partial\Omega$ . Edge waves are eliminated by requiring that the time-averaged energy of the field be locally finite:

$$(1.16) \quad \int_{K \cap \Omega} \{|\nabla v|^2 + \omega^2 |v|^2\} dV < \infty \text{ for every compact } K \subset E^3$$

This condition is called the "edge condition" in the older literature [2, 14].

The physically important steady-state fields behave like outgoing waves in each unbounded component of  $\Omega$ . These solutions are characterized by a radiation condition of the Sommerfeld type which eliminates another type of non-uniqueness caused by "incoming wave" terms in  $v(q)$ .

The paper is organized as follows. In section 2 selfadjoint operators  $\Delta_{\Omega}^D$  and  $\Delta_{\Omega}^N$  in  $L_2(\Omega)$  are associated with the Laplacian and the Dirichlet and Neumann conditions, respectively. In section 3 the radiation condition is defined and used to formulate the uniqueness theorem. The existence of outgoing steady-state fields with prescribed source functions  $g$  is proved in section 4 as a corollary of a limiting absorption theorem. The latter,

which is the central analytical result of the paper, asserts that the resolvent  $R_{\Omega}^D(z) = -(\Delta_{\Omega}^D + z)^{-1}$  (resp.  $R_{\Omega}^N(z) = -(\Delta_{\Omega}^N + z)^{-1}$ ) has limits when  $z = \omega^2 \pm i\sigma \rightarrow \omega^2 \pm i0$  provided that  $\omega^2$  is not an eigenvalue of  $-\Delta_{\Omega}^D$  (resp.  $-\Delta_{\Omega}^N$ ) and  $\omega$  is not a cutoff frequency (see section 3) of one of the waveguides  $S_i$  or  $S'_j$ .

Section 5 contains a discussion of the proofs of the theorems formulated in sections 2-4. These theorems are generalizations to waveguide domains of results of Wilcox on scattering in exterior domains. The latter were presented in the monograph [23] to which reference is made for details of the proofs. The discussion in section 5 is limited to those points in which the proofs of [23] must be modified to treat waveguides.

Related Literature. The principle of limiting absorption has a long history. An early statement of it was given in 1905 by W. Ignatowsky [9] and it has often been used as a heuristic principle in the physical literature. It was proved for the waveguides formed by parallel plates and perfect cylinders by A. G. Svešnikov [20, 21]. The first general proof is due to D. M. Eidus [4] who studied elliptic operators with variable coefficients in exterior domains and perturbed cylinders. C. I. Goldstein has also proved the principle for perturbations of cylinders as part of his spectral and scattering theory of waveguides [5, 6, 7]. The case of finite perturbations of a cone in  $E^2$  was treated by Goldstein in [5, III]. Generalizations to higher-dimensional cones and more general differential operators and boundary conditions were announced but not proved in that paper. W. C. Lyford [12] has proved the limiting absorption principle for compound cylindrical waveguides in the same generality as here. The authors know of no treatment by the methods of this paper of compound waveguides containing sectoral and conical horns. Most of the results of this paper were announced by the authors in [8].

## SECTION TWO

## HILBERT SPACE THEORY OF THE LAPLACIAN

The basic Hilbert space of this section is the Lebesgue space  $L_2(\Omega)$  with scalar product

$$(2.1) \quad (u, v) = \int_{\Omega} \overline{u(q)} v(q) dV_q$$

where  $dV_q$  denotes integration with respect to Lebesgue measure in  $E^3$ . The definition of the selfadjoint realizations of  $\Delta$  in  $L_2(\Omega)$  associated with the Dirichlet and Neumann conditions also requires the Hilbert space  $L_2(\Omega, C^3)$  of Lebesgue square-integrable vector fields on  $\Omega$  with scalar product

$$(2.2) \quad (\vec{A}, \vec{B}) = \int_{\Omega} \overline{\vec{A}(q)} \cdot \vec{B}(q) dV_q$$

and the following linear subsets of  $L_2(\Omega)$  and  $L_2(\Omega, C^3)$ :

$$(2.3) \quad L_2^1(\Omega) = L_2(\Omega) \cap \{u: \nabla u \in L_2(\Omega, C^3)\}$$

$$(2.4) \quad L_2^1(\Delta, \Omega) = L_2^1(\Omega) \cap \{u: \Delta u \in L_2(\Omega)\}$$

$$(2.5) \quad L_2(\nabla \cdot, \Omega, C^3) = L_2(\Omega, C^3) \cap \{\vec{A}: \nabla \cdot \vec{A} \in L_2(\Omega)\}$$

In the definitions  $\nabla u$ ,  $\Delta u$  and  $\nabla \cdot \vec{A}$  are to be interpreted in the sense of distribution theory. The spaces (2.3)-(2.5) are also Hilbert spaces with respect to suitable scalar products [22, 23].

The classical Dirichlet and Neumann conditions are replaced here by the generalized Dirichlet condition

$$(2.6) \quad (u, \nabla \cdot \vec{V}) + (\nabla u, \vec{V}) = 0 \text{ for all } \vec{V} \in L_2(\nabla \cdot, \Omega, C^3)$$

and the generalized Neumann condition

$$(2.7) \quad (\Delta u, v) + (\nabla u, \nabla v) = 0 \text{ for all } v \in L_2^1(\Omega)$$

respectively. These conditions are applicable to arbitrary domains  $\Omega$  and are equivalent to the classical conditions when  $\partial\Omega$  and  $u$  are sufficiently regular [22, 23]. Operators  $\Delta_\Omega^D$  and  $\Delta_\Omega^N$  in  $L_2(\Omega)$  are defined by  $\Delta$  acting on the domains

$$(2.8) \quad D(\Delta_\Omega^D) = L_2^1(\Delta, \Omega) \cap \{u: u \text{ satisfies (2.6)}\}$$

$$(2.9) \quad D(\Delta_\Omega^N) = L_2^1(\Delta, \Omega) \cap \{u: u \text{ satisfies (2.7)}\}$$

They provide the starting point for the theory of acoustic waveguides developed below. Their utility is due to [12, 23]

Theorem 1. For every domain  $\Omega \subset E^3$  the operators  $\Delta_\Omega^D$  and  $\Delta_\Omega^N$  are selfadjoint non-positive operators in  $L_2(\Omega)$ .

The theorems in the remainder of the paper are based on a local compactness property of operators in  $L_2(\Omega)$  described by the

Definition. A linear operator  $A$  in  $L_2(\Omega)$  with domain  $D(A) \subset L_2^1(\Omega)$  is said to have the local compactness property (in symbols,  $A \in LC$ ) if and only if for every bounded measurable set  $K \subset E^3$  and subset  $S \subset D(A)$  which is bounded in  $L_2^1(K \cap \Omega)$  the set  $S$  is precompact in  $L_2(K \cap \Omega)$ .

The embedding of  $L_2^1(\Omega)$  in  $L_2^1(K \cap \Omega)$  and  $L_2(K \cap \Omega)$  implied by the definition is the natural embedding which assigns to any function on  $\Omega$  its restriction to  $K \cap \Omega$ .

A domain  $\Omega$  is said to have the Rellich property if, for each bounded measurable  $K \subset E^3$ , the natural embedding of  $L_2^1(K \cap \Omega)$  into  $L_2(K \cap \Omega)$  is

compact. If this property holds then every operator  $A$  with  $D(A) \subset L_2^1(\Omega)$  satisfies  $A \in LC$ . Not all domains have the Rellich property [3]. The property was proved by Rellich [17] for domains with piecewise smooth boundaries and by S. Agmon for domains having the "segment property" [1, Theorem 3.8]. Wilcox [23] extended Agmon's proof to domains with the "finite tiling property." Thus  $\Delta_\Omega^N \in LC$  for these classes. Of course  $\Delta_\Omega^D \in LC$  for the same class. However, in the case of the Dirichlet condition restrictions on  $\partial\Omega$  are not needed. Instead, one has

$$(2.10) \quad \Delta_\Omega^D \in LC \text{ for every domain } \Omega \subset E^3$$

A proof is indicated in section 5.

In the remainder of the paper the class of domains  $\Omega$  defined by (1.1)-(1.6) is denoted by  $W^D$  and the subset consisting of domains  $\Omega \in W^D$  such that  $\Delta_\Omega^N \in LC$  is denoted by  $W^N$ . Moreover, in discussions of  $\Delta_\Omega^D$  (resp.  $\Delta_\Omega^N$ ) it is always assumed that  $\Omega \in W^D$  (resp.  $W^N$ ). With this understanding the theories of  $\Delta_\Omega^D$  and  $\Delta_\Omega^N$  are identical. For conciseness the symbol  $\Delta_\Omega$  is used below to denote either  $\Delta_\Omega^D$  or  $\Delta_\Omega^N$  in statements that are valid for both.

The study of the point spectrum of  $\Delta_\Omega$  was begun by F. Rellich [18] and extended by D. S. Jones [10]. They showed that domains  $\Omega$  having a conical horn ( $I'' \geq 1$ ) have no point spectrum, while domains with no conical horn ( $I'' = 0$ ) may have point spectrum. Geometric criteria for its existence were given and Jones showed that it must be discrete; i.e., each interval can have only finitely many eigenvalues and each has finite multiplicity. The work of Rellich and Jones involved stronger local regularity hypotheses on  $\partial\Omega$  than are made here. However, their methods of proof are based on the quadratic forms associated with  $\Delta_\Omega$  and can be extended to all domains for which  $\Delta_\Omega \in LC$ .

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## SECTION THREE

## THE UNIQUENESS OF STEADY-STATE WAVE FIELDS

The asymptotic behavior of steady-state fields in the unbounded cylinder and/or horn portions of  $\Omega$  is determined in this section and used to formulate a radiation condition and corresponding uniqueness theorem.

The basic space of this section is defined by

$$(3.1) \quad L_2^{\text{loc}}(\bar{\Omega}) = \mathcal{D}'(\Omega) \cap \{u: u \in L_2(K \cap \Omega) \text{ for every compact } K \subset E^3\}$$

where  $\mathcal{D}'(\Omega)$  is L. Schwartz's space of distributions in  $\Omega$ .  $L_2^{\text{loc}}(\bar{\Omega})$  must be distinguished from the more familiar space

$$(3.2) \quad L_2^{\text{loc}}(\Omega) = \mathcal{D}'(\Omega) \cap \{u: u \in L_2(K) \text{ for every compact } K \subset \Omega\}$$

Note that the condition  $u \in L_2^{\text{loc}}(\bar{\Omega})$  restricts  $u$  near  $\partial\Omega$  while  $u \in L_2^{\text{loc}}(\Omega)$  does not. The local analogues of the remaining spaces of section 2 are also needed. They are

$$(3.3) \quad L_2^{\text{loc}}(\bar{\Omega}, C^3) = \mathcal{D}'(\Omega, C^3) \cap \{\vec{A}: \vec{A} \in L_2(K \cap \Omega, C^3) \text{ for every compact } K \subset E^3\}$$

$$(3.4) \quad L_2^{1, \text{loc}}(\bar{\Omega}) = L_2^{\text{loc}}(\bar{\Omega}) \cap \{u: \nabla u \in L_2^{\text{loc}}(\bar{\Omega}, C^3)\}$$

$$(3.5) \quad L_2^{1, \text{loc}}(\Delta, \bar{\Omega}) = L_2^{1, \text{loc}}(\bar{\Omega}) \cap \{u: \Delta u \in L_2^{\text{loc}}(\bar{\Omega})\}$$

$$(3.6) \quad L_2^{\text{loc}}(\nabla \cdot, \bar{\Omega}, C^3) = L_2^{\text{loc}}(\bar{\Omega}, C^3) \cap \{\vec{A}: \nabla \cdot \vec{A} \in L_2^{\text{loc}}(\bar{\Omega})\}$$

The following spaces are also needed

$$(3.7) \quad L_2^{\text{com}}(\bar{\Omega}) = E'(E^3) \cap L_2(\Omega)$$

$$(3.8) \quad L_2^{1, \text{com}}(\bar{\Omega}) = L_2^{\text{com}}(\bar{\Omega}) \cap L_2^1(\Omega)$$

$$(3.9) \quad L_2^{\text{com}}(\bar{\Omega}, C^3) = E'(E^3, C^3) \cap L_2(\Omega, C^3)$$

$$(3.10) \quad L_2^{\text{com}}(\nabla \cdot, \bar{\Omega}, C^3) = L_2^{\text{com}}(\bar{\Omega}, C^3) \cap L_2(\nabla \cdot, \Omega, C^3)$$

where  $E'(E^3)$  is L. Schwartz's space of distributions in  $E^3$  with compact support.

Operators  $\Delta_{\Omega}^{D, \text{loc}}$  and  $\Delta_{\Omega}^{N, \text{loc}}$  in  $L_2^{\text{loc}}(\bar{\Omega})$  are defined by  $\Delta$  acting on the domains

$$(3.11) \quad D(\Delta_{\Omega}^{D, \text{loc}}) = L_2^{1, \text{loc}}(\Delta, \bar{\Omega}) \cap \{u: (2.6) \text{ holds for all } \vec{v} \in L_2^{\text{com}}(\nabla \cdot, \bar{\Omega}, C^3)\}$$

$$(3.12) \quad D(\Delta_{\Omega}^{N, \text{loc}}) = L_2^{1, \text{loc}}(\Delta, \bar{\Omega}) \cap \{u: (2.7) \text{ holds for all } v \in L_2^{1, \text{com}}(\bar{\Omega})\}$$

In section 4 it is shown that steady-state fields with source functions  $g \in L_2^{\text{com}}(\bar{\Omega})$  exist in the domains (3.11), (3.12). Such fields satisfy the edge condition (1.15) and the generalized Dirichlet or Neumann condition. For brevity, the notation  $\Delta_{\Omega}^{\text{loc}}$  will be used to denote either  $\Delta_{\Omega}^{D, \text{loc}}$  or  $\Delta_{\Omega}^{N, \text{loc}}$  in statements which are valid for both.

Let  $v \in D(\Delta_{\Omega}^{\text{loc}})$  satisfy

$$(3.13) \quad \Delta v + \omega^2 v = -g(q) \in L_2^{\text{com}}(\bar{\Omega})$$

where  $\omega \geq 0$ . To study the asymptotic behavior of  $v(q)$  it will be convenient to restrict it to the sets  $S_i, S_j', S_k''$  by means of a smooth cutoff function. To do this choose a function  $j \in C^{\infty}(\mathbb{R})$  such that

$$(3.14) \quad j(\tau) \equiv 0 \text{ for } \tau \leq 0, \quad j(\tau) \equiv 1 \text{ for } \tau \geq 1, \quad 0 \leq j(\tau) \leq 1$$

and associate with each function  $u$  on  $\bar{\Omega}$  the functions  $u_i$  on  $S_i$ ,  $u_j'$  on  $S_j'$ ,  $u_k''$  on  $S_k''$  defined by

$$(3.15) \quad u_i(q) = j(y^i(q) - y_0^i - 1)u(q) \text{ for all } q \in S_i$$

$$(3.16) \quad u'_j(q) = j(\rho^j(q) - \rho_0^j - 1)u(q) \text{ for all } q \in S'_j$$

$$(3.17) \quad u''_k(q) = j(r^k(q) - r_0^k - 1)u(q) \text{ for all } q \in S''_k$$

The corresponding linear operators, defined by

$$(3.18) \quad J_i u = u_i, \quad J'_j u = u'_j, \quad J''_k u = u''_k$$

have the properties described by (cf. [12])

Proposition 3.1. The restrictions of  $J_i$ ,  $J'_j$ ,  $J''_k$  to the spaces  $L_2^{\text{loc}}(\bar{\Omega})$ ,  $D(\Delta_\Omega^{\text{loc}})$ ,  $L_2(\Omega)$  and  $D(\Delta_\Omega)$  are linear continuous mappings of these spaces into the corresponding spaces for  $S_i$ ,  $S'_j$  and  $S''_k$  respectively.

The topologies of the spaces  $L_2^{\text{loc}}(\bar{\Omega})$ ,  $D(\Delta_\Omega^{\text{loc}})$  referred to in Proposition 3.1 are the Fréchet topologies defined in [12, 23]. The structure of the functions  $v_i$ ,  $v'_j$ ,  $v''_k$  corresponding to (3.13) will be analyzed by means of the special coordinate systems that define  $S_i$ ,  $S'_j$ ,  $S''_k$  and spectral analysis with respect to the Laplace-Beltrami operators for  $G_i \subset \mathbb{R}^2$ ,  $G'_j \subset S^1 \times \mathbb{R}$ ,  $G''_k \subset S^2$ . The corresponding selfadjoint non-positive operators in  $L_2(G_i)$ ,  $L_2(G'_j)$ ,  $L_2(G''_k)$  will be denoted by  $\Delta_i^D$  and  $\Delta_i^N$ ,  $\Delta'_j^D$  and  $\Delta'_j^N$ ,  $\Delta''_k^D$  and  $\Delta''_k^N$ , respectively. Their definitions are entirely analogous to those of  $\Delta_\Omega^D$  and  $\Delta_\Omega^N$  in section 2. In fact, the definitions given there are applicable to the Laplace-Beltrami operators in subdomains of any Riemannian manifold. The notations  $\Delta_i$ ,  $\Delta'_j$ ,  $\Delta''_k$  will be used to denote either the Dirichlet or the Neumann operators in statements that are valid for both. The discreteness of the spectra of these operators follows from the following proposition whose proof is indicated in section 5.

**Proposition 3.2.** The hypothesis  $\Omega \in W^D$  (resp.,  $\Omega \in W^N$ ) implies that  $\Delta_i^D$  and  $\Delta_k^{D,N}$  (resp.,  $\Delta_i^N$  and  $\Delta_k^{D,N}$ ) have the local compactness property.

Note that  $\Delta_j^D \in LC$  and  $\Delta_j^N \in LC$  because  $G_j^D$  is just a rectangle in  $S^1 \times \mathbb{R}$ , a domain with the Rellich property. These observations and the boundedness of  $G_i, G_j^D, G_k^N$  imply that  $-\Delta_i, -\Delta_j^D, -\Delta_k^N$  have discrete spectra lying in  $[0, \infty)$ . The eigenvalues, enumerated in increasing order and repeated according to multiplicity, will be denoted by  $\{\omega_{i\ell}^2\}, \{\mu_{j\ell}^2\}, \{\nu_{k\ell}^2\}$  respectively where  $\ell = 1, 2, 3, \dots$ . Corresponding complete orthonormal sets of eigenfunctions in  $L_2(G_i), L_2(G_j^D), L_2(G_k^N)$  will be denoted by  $\{\phi_{i\ell}\}, \{\phi_{j\ell}^D\}, \{\phi_{k\ell}^N\}$ . The sequences  $\{\mu_{j\ell}^2\}, \{\phi_{j\ell}^D\}$  will be constructed from those for the operators  $-\Delta_{M_j^D} = -d^2/d\theta^2$  in  $L_2(M_j^D)$  and  $-\Delta_{(0, z_0^j)} = -d^2/dz^2$  in  $L_2(0, z_0^j)$  by separation of variables; i.e.,

$$(3.19) \quad \mu_{j\ell}^2 = \nu_{jm}^2 + \omega_{jn}^2$$

$$(3.20) \quad \phi_{j\ell}^D(\theta, z) = a_{jm}(\ell)(\theta) b_{jn}(\ell)(z)$$

where  $(d^2/d\theta^2 + \nu_{jm}^2)a_{jm} = 0$ ,  $(d^2/dz^2 + \omega_{jn}^2)b_{jn} = 0$  and  $\ell \Leftrightarrow (m(\ell), n(\ell))$  is a suitable enumeration of the pairs  $(m, n)$  of positive integers.

It will be convenient to represent  $u_i, u_j^D, u_k^N$  as functions of the special coordinates in  $S_i, S_j^D, S_k^N$ . The following notation will be used

$$(3.21) \quad u_i(x^i, y^i) = u_i(q) \text{ where } (x^i, y^i) \Leftrightarrow q \in S_i$$

$$(3.22) \quad u_j^D(\rho^j, \theta^j, z^j) = u_j^D(q) \text{ where } (\rho^j, \theta^j, z^j) \Leftrightarrow q \in S_j^D$$

$$(3.23) \quad u_k^N(r^k, \eta^k) = u_k^N(q) \text{ where } (r^k, \eta^k) \Leftrightarrow q \in S_k^N$$

The expandability of these functions in the eigenfunctions of  $\Delta_i, \Delta_j^D, \Delta_k^N$  whenever  $u \in D(\Delta_\Omega^{\text{loc}})$  follows from [22, Theorem 2.2]

Proposition 3.3. The correspondences (3.21)-(3.23) define continuous embeddings

$$(3.24) \quad D(\Delta_{S_i}^{\text{loc}}) \subset C(\{y_0^i, \infty\}, L_2(G_i))$$

$$(3.25) \quad D(\Delta_{S_j'}^{\text{loc}}) \subset C(\{\rho_0^j, \infty\}, L_2(G_j'))$$

$$(3.26) \quad D(\Delta_{S_k''}^{\text{loc}}) \subset C(\{r_0^k, \infty\}, L_2(G_k''))$$

Propositions 3.1 and 3.3 imply that for every  $u \in D(\Delta_{\Omega}^{\text{loc}})$  the coefficient functions

$$(3.27) \quad u_{i\ell}(y) = \int_{G_i} \overline{\phi_{i\ell}(x)} u_i(x, y) dx$$

$$(3.28) \quad u_{j\ell}'(\rho) = \int_{G_j'} \overline{\phi_{j\ell}'(\theta, z)} u_j'(\rho, \theta, z) d\theta dz$$

$$(3.29) \quad u_{k\ell}''(r) = \int_{G_k''} \overline{\phi_{k\ell}''(\eta)} u_k''(r, \eta) d\eta$$

are continuous and one has the following two propositions.

Proposition 3.4. For every  $u \in D(\Delta_{\Omega}^{\text{loc}})$  the following expansions are valid.

$$(3.30) \quad u_i(x, y) = \sum_{\ell=1}^{\infty} u_{i\ell}(y) \phi_{i\ell}(x), \text{ convergent in } D(\Delta_{S_i}^{\text{loc}})$$

$$(3.31) \quad u_j'(\rho, \theta, z) = \sum_{\ell=1}^{\infty} u_{j\ell}'(\rho) \phi_{j\ell}'(\theta, z), \text{ convergent in } D(\Delta_{S_j'}^{\text{loc}})$$

$$(3.32) \quad u_k''(r, \eta) = \sum_{\ell=1}^{\infty} u_{k\ell}''(r) \phi_{k\ell}''(\eta), \text{ convergent in } D(\Delta_{S_k''}^{\text{loc}})$$

Proposition 3.5. Let  $v \in D(\Delta_{\Omega}^{\text{loc}})$  satisfy (3.13) and let  $g_i(x, y) = 0$  for  $y > y^i(g)$ ,  $g_j^i(\rho, \theta, z) = 0$  for  $\rho > \rho^j(g)$  and  $g_k^i(r, \eta) = 0$  for  $r > r^k(g)$ . Then the coefficients  $v_{i\ell}$ ,  $v'_{j\ell}$ ,  $v''_{k\ell}$  satisfy the differential equations

$$(3.33) \quad L \left[ \frac{d}{dy}, \omega^2 - \omega_{i\ell}^2, 1/2 \right] v_{i\ell}(y) = 0 \text{ for } y > y^i(g)$$

$$(3.34) \quad L \left[ \frac{d}{d\rho}, \omega^2 - \omega_{jn(\ell)}^2, v'_{jm(\ell)} \right] \rho^{1/2} v'_{j\ell}(\rho) = 0 \text{ for } \rho > \rho^j(g)$$

$$(3.35) \quad L \left[ \frac{d}{dr}, \omega^2, (v_{k\ell}^2 + 1/4)^{1/2} \right] r v''_{k\ell}(r) = 0 \text{ for } r > r^k(g)$$

where

$$(3.36) \quad L \left[ \frac{d}{dx}, a, b \right] = \frac{d^2}{dx^2} + (a - (b^2 - 1/4)/x^2)$$

A solution basis for (3.33) is defined by

$$(3.37) \quad \exp \{i(\omega^2 - \omega_{i\ell}^2)^{1/2} y\}, \exp \{-i(\omega^2 - \omega_{i\ell}^2)^{1/2} y\}, \omega \neq \omega_{i\ell}$$

$$(3.38) \quad 1, y, \omega = \omega_{i\ell}$$

Solution bases for (3.35), (3.36) may be based on the usual Hankel functions  $H_{\nu}^{(1)}(z)$ ,  $H_{\nu}^{(2)}(z)$  which have the asymptotic forms [13]

$$(3.39) \quad H_{\nu}^{(1,2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)}, z \rightarrow \infty$$

where  $1 \Leftrightarrow +$  and  $2 \Leftrightarrow -$ . With this notation a solution basis for (3.34) is

$$(3.40) \quad H_{\nu}^{(1,2)}((\omega^2 - \omega_{jn(\ell)}^2)^{1/2} \rho), \nu = v'_{jm(\ell)}, \omega \neq \omega'_{jn(\ell)}$$

$$(3.41) \quad \rho^{-v'_{jm(\ell)}}, \rho^{v'_{jm(\ell)}}, \omega = \omega'_{jn(\ell)}, v'_{jm(\ell)} \neq 0$$

$$(3.42) \quad 1, \quad \ell n \rho \quad , \quad \omega = \omega'_{jn}(\ell), \quad v'_{jm}(\ell) = 0$$

Similarly, a solution basis for (3.35) is

$$(3.43) \quad r^{-1/2} H_v^{(1,2)}(\omega r), \quad v = (v_{k\ell}^2 + \frac{1}{4})^{1/2}, \quad \omega \neq 0$$

$$(3.44) \quad r^{-\frac{1}{2} - (v_{k\ell}^2 + \frac{1}{4})^{1/2}}, \quad r^{-\frac{1}{2} + (v_{k\ell}^2 + \frac{1}{4})^{1/2}}, \quad \omega = 0$$

When multiplied by the time-factor  $e^{-i\omega t}$  of (1.11) the solutions (3.37) describe propagating waves in  $S_i$  if and only if  $\omega > \omega_{i\ell}$ . Similarly, (3.40) and (3.43) define propagating waves in  $S'_j$  and  $S''_k$  if and only if  $\omega > \omega'_{jn}(\ell)$  and  $\omega > 0$ , respectively. The numbers  $\{\omega_{i\ell}\}$ ,  $\{\omega'_{jn}\}$ ,  $0$  are called the "cutoff frequencies" for the waveguides  $S_i$ ,  $S'_j$ ,  $S''_k$ , respectively. The notation

$$(3.45) \quad \sigma_{\text{cutoff}}(\Omega) = \left\{ \bigcup_{i=1}^I \bigcup_{\ell=1}^{\infty} \{\omega_{i\ell}\} \right\} \cup \left\{ \bigcup_{j=1}^{I'} \bigcup_{n=1}^{\infty} \{\omega'_{jn}\} \right\} \cup \{0\}$$

will be used.

Concerning the point spectrum of  $\Delta_\Omega$ , it is clear from Propositions 3.4 and 3.5 that every eigenfunction must have coefficients  $v_i$ ,  $v'_j$ ,  $v''_k$  which are zero in all propagating modes and decrease exponentially in all cutoff modes. In particular, if  $\Omega$  contains a conical horn ( $I'' \geq 1$ ) then  $\Delta_\Omega$  can have no point spectrum, as was proved by Rellich.

The goal of this paper is to show that for all frequencies  $\omega \geq 0$  except cutoff frequencies and the square roots of eigenvalues of  $-\Delta_\Omega$  there exists in  $D(\Delta_\Omega^{\text{loc}})$  a unique solution of (3.13) which is a pure outgoing wave in each propagating mode and is exponentially decreasing in each cutoff mode. The description of these solutions is facilitated by the use of the functions  $(\lambda \pm i0)^{1/2}$  defined by

$$(3.46) \quad \text{Im } z^{1/2} > 0 \text{ for all } z \in \mathbb{C} - [0, \infty)$$

$$(3.47) \quad (\lambda \pm i0)^{1/2} = \lim_{\epsilon \rightarrow +0} (\lambda \pm i\epsilon)^{1/2}$$

Note that

$$(3.48) \quad \pm(\lambda \pm i0)^{1/2} = \lambda^{1/2} \geq 0 \text{ for all } \lambda \geq 0$$

$$(3.49) \quad -i(\lambda \pm i0)^{1/2} = (-\lambda)^{1/2} > 0 \text{ for all } \lambda < 0$$

Definition. Let  $\omega \in [0, \infty) - \sigma_{\text{cutoff}}(\Omega)$ . Then a solution  $v \in D(\Delta_{\Omega}^{\text{loc}})$  of (3.13) is said to satisfy the outgoing radiation condition if and only if there exist constants  $c_{i\ell}(\omega)$ ,  $c'_{j\ell}(\omega)$ ,  $c''_{k\ell}(\omega)$  such that

$$(3.50) \quad v_{i\ell}(y) = c_{i\ell}(\omega) \exp \{i(\omega^2 - \omega_{i\ell}^2 + i0)^{1/2} y\} \text{ for } y > y^i(g)$$

$$(3.51) \quad v'_{j\ell}(\rho) = c'_{j\ell}(\omega) H_v^{(1)}((\omega^2 - \omega_{jn}^2 + i0)^{1/2} \rho),$$

$$v = v'_{jm}(\ell) \text{ for } \rho > \rho^j(g)$$

$$(3.52) \quad v''_{k\ell}(r) = c''_{k\ell}(\omega) r^{-1/2} H_v^{(1)}((\omega^2 + i0)^{1/2} r),$$

$$v = (v_{k\ell}^2 + \frac{1}{4})^{1/2} \text{ for } r > r^k(g)$$

$v$  is said to satisfy the incoming radiation condition if (3.50)-(3.52) hold with  $(\lambda + i0)^{1/2}$  replaced by  $(\lambda - i0)^{1/2}$ .

The uniqueness theorem can now be formulated as follows.

Theorem 2. Let  $\omega \in [0, \infty) - \sigma_{\text{cutoff}}(\Omega)$  and let  $v \in D(\Delta_{\Omega}^{\text{loc}})$  satisfy

$$(3.53) \quad \Delta v + \omega^2 v = 0 \text{ in } \Omega$$

$$(3.54) \quad v \text{ satisfies the outgoing or incoming radiation condition}$$

(3.55)  $v$  is orthogonal in  $L_2(\Omega)$  to the eigenspace of  $-\Delta_\Omega$  for the eigenvalue  $\omega^2$  (which contains only the zero vector if  $\omega^2$  is not an eigenvalue)

Then  $v(q) = 0$  in  $\Omega$ . Moreover, if  $\Omega$  has a conical horn component ( $I'' \geq 1$ ) the same result holds for all  $\omega \geq 0$ .

Note that condition (3.55) is meaningful, even though  $v$  is not in  $L_2(\Omega)$ , because the eigenfunctions of  $\Delta_\Omega$  are exponentially decreasing in each waveguide and (3.54) implies that the coefficients  $v_j, v_j', v_k''$  are bounded.

Theorem 2 clearly implies the uniqueness of solutions in  $D(\Delta_\Omega^{\text{loc}})$  of (3.13) which satisfy (3.54), (3.55). Of course, this result is of interest only if such solutions exist. In section 4 Theorem 2 is shown to imply the validity of a limiting absorption principle and, as a consequence, the existence of outgoing and incoming solutions in  $D(\Delta_\Omega^{\text{loc}})$  for every  $g \in L_2^{\text{com}}(\bar{\Omega})$ .

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## SECTION FOUR

## THE LIMITING ABSORPTION PRINCIPLE

The spectrum, the point spectrum and the continuous spectrum of the operator  $-\Delta_\Omega$  will be denoted by  $\sigma(\Omega)$ ,  $\sigma_p(\Omega)$  and  $\sigma_c(\Omega)$ , respectively. Thus  $\sigma(\Omega) \subset [0, \infty)$  and  $\sigma(\Omega) = \sigma_p(\Omega) \cup \sigma_c(\Omega)$  [11]. The resolvent operator of  $-\Delta_\Omega$ ,

$$(4.1) \quad R_\Omega(z) = -(\Delta_\Omega + z)^{-1}, \quad z \in \mathbb{C} - \sigma(\Omega)$$

is bounded in  $L_2(\Omega)$  and  $z \rightarrow R_\Omega(z)$  defines a holomorphic  $L_2(\Omega)$  operator-valued function in  $\mathbb{C} - \sigma(\Omega)$  [11]. This function cannot be extended to points  $z \in \sigma(\Omega)$ . However, it will be shown that if  $g \in L_2^{\text{com}}(\bar{\Omega})$  then the mapping  $z \rightarrow R_\Omega(z)g \in D(\Delta_\Omega^{\text{loc}})$  has one-sided limits at the points of the set

$$(4.2) \quad \Lambda = \sigma_c(\Omega) - (\sigma_{\text{cutoff}}(\Omega))^2$$

This result is a corollary of

Theorem 3. For each  $g \in L_2^{\text{com}}(\bar{\Omega})$  the mappings  $T_+$  and  $T_-$  from  $\{(\lambda, \sigma): \lambda \geq 0 \text{ and } \sigma > 0\}$  to  $D(\Delta_\Omega^{\text{loc}})$  defined by

$$(4.3) \quad T_\pm(\lambda, \sigma) = R_\Omega(\lambda \pm i\sigma)g \in D(\Delta_\Omega^{\text{loc}})$$

are continuous. Moreover, for each interval  $I = [a, b] \subset \Lambda$  and each  $\sigma_0 > 0$ ,  $T_+$  and  $T_-$  are uniformly continuous on  $I \times (0, \sigma_0]$ .

This result was proved in [23] for the special case of exterior domains ( $G_k'' = S^2$ ). The extension of the proof to the class of waveguide domains  $W^D$  and  $W^N$  is given in section 5. An immediate corollary of the uniform continuity statement of Theorem 3 is the

Limiting Absorption Principle. The mappings  $T_+$  and  $T_-$  have continuous extensions to the set

$$(4.4) \quad \{(\lambda, \sigma): \lambda \geq 0, \sigma > 0\} \cup \{\Lambda \times [0, \infty)\}$$

Moreover, the limits  $T_{\pm}(\lambda, 0+) = R_{\Omega}(\lambda \pm i0)g$ ,  $\lambda \in \Lambda$ , define operators  $R_{\Omega}(\lambda \pm i0)$  from  $L_2^{\text{com}}(\bar{\Omega})$  to  $D(\Delta_{\Omega}^{\text{loc}})$  which are bounded uniformly on compact subsets of  $\Lambda$ .

A second corollary is the following existence theorem for steady-state fields in waveguides.

Theorem 4. For each  $\omega \geq 0$  such that  $\omega^2 \in \Lambda$  and each  $g \in L_2^{\text{com}}(\bar{\Omega})$  let

$$(4.5) \quad v^{\pm}(\cdot, \omega) = R_{\Omega}(\omega^2 \pm i0)g \in D(\Delta_{\Omega}^{\text{loc}})$$

Then  $v^+(\cdot, \omega)$  (resp.  $v^-(\cdot, \omega)$ ) is the (unique) outgoing (resp. incoming) solution in  $D(\Delta_{\Omega}^{\text{loc}})$  of (3.13).

## SECTION FIVE

## PROOFS OF THE THEOREMS

The validity of  $\Delta_{\Omega}^D \in LC$  for every  $\Omega \subset E^3$  follows from the characterization

$$(5.1) \quad D(\Delta_{\Omega}^D) = L_2^{1,0}(\Omega) \cap L_2^1(\Delta, \Omega),$$

where  $L_2^{1,0}(\Omega)$  is the closure of  $C_0^{\infty}(\Omega)$  in  $L_2^1(\Omega)$ , and from the fact that for bounded domains  $\Omega'$  the natural embedding of  $L_2^{1,0}(\Omega')$  into  $L_2(\Omega')$  is compact [1, Theorem 8.3]. To prove (5.1) note that  $L_2^{1,0}(\Omega) \cap L_2^1(\Delta, \Omega) = D(A)$  where  $A$  is the selfadjoint operator associated with the sesquilinear form  $A(u, v) = (\nabla u, \nabla v)$  with domain  $D(A) = L_2^{1,0}(\Omega)$  [11, p. 322]. It is obvious that  $A \subset -\Delta_{\Omega}^D$ . Hence  $A = -\Delta_{\Omega}^D$  by the maximality of selfadjoint operators which proves (5.1).

Proof of Proposition 3.2. A proof will be indicated for the operators  $\Delta_i$  corresponding to the cylinders  $S_i$ . The other cases can be proved similarly. The property  $\Delta_i^D \in LC$  follows by the method of the preceding paragraph. To prove  $\Delta_i^N \in LC$  let  $S = \{v(x)\} \subset D(\Delta_i^N)$  be bounded in  $L_2^1(G_i)$ . Let  $\chi_i(q)$  be the characteristic function of  $S_i$ ,  $\phi \in C_0^{\infty}(R)$ ,  $\phi(y) \equiv 1$  for  $y_0^i + 1 \leq y \leq y_0^i + 2$ ,  $\phi(y) \equiv 0$  outside  $y_0^i + \frac{1}{2} \leq y \leq y_0^i + \frac{5}{2}$  and define  $S' = \{u: u(q) = v(x^i(q)) \phi(y^i(q)) \chi_i(q), v \in S\}$ . Also let  $K_i = \{q \in E^3: x^i(q) \in G_i \text{ and } y_0^i + 1 < y^i(q) < y_0^i + 2\}$ . Then  $K_i$  is bounded and measurable and  $K_i \cap \Omega = K_i$ . Moreover,  $\|u\|_{L_2^1(K_i)} = \|v\|_{L_2^1(G_i)} \leq C$  for all  $v \in S$ . Since  $\|u\|_{L_2(K_i)} = \|v\|_{L_2(G_i)}$  for all  $v \in S$  the hypothesis  $\Omega \in \mathcal{L}^N$  implies that  $S$  is compact in  $L_2(G_i)$ .

Proof of Theorem 2. Note that (3.53) implies that (3.50)-(3.52) hold with  $y^i(g) = y_0^i$ ,  $\rho^j(g) = \rho_0^j$ ,  $r^k(g) = r_0^k$ . Choose a number  $R > \text{Max}(y_0^i, \rho_0^j, r_0^k)$  and define (cf. (1.2)-(1.4))

$$(5.2) \quad \Omega_R = \Omega - \bigcup_{i=1}^I S_i(R) - \bigcup_{j=1}^{I'} S_j'(R) - \bigcup_{k=1}^{I''} S_k''(R)$$

Now apply Green's theorem to  $v$  and  $\bar{v}$  in  $\Omega_R$ . This must be done by applying the generalized boundary condition (2.6) (resp. (2.7)) with  $u \rightarrow v$  and  $\vec{V} \rightarrow \chi \nabla \bar{v}$  (resp.  $u \rightarrow v$  and  $v \rightarrow \chi \bar{v}$ ) where  $\chi$  is a smooth cutoff function. The technique is explained in [23, p. 57ff]. The result is, by (3.53),

$$(5.3) \quad 0 = \int_{\Omega_R} \{v \Delta \bar{v} - \bar{v} \Delta v\} dV$$

$$= 2 \text{ i Im} \left[ \sum_{i=1}^I \int_{G_i} \left\langle v \frac{\partial \bar{v}}{\partial y} \right\rangle_{y=R} dx + \sum_{j=1}^{I'} \int_{G_j'} \left\langle v \frac{\partial \bar{v}}{\partial \rho} \right\rangle_{\rho=R} R d\theta dz \right. \\ \left. + \sum_{k=1}^{I''} \int_{G_k''} \left\langle v \frac{\partial \bar{v}}{\partial r} \right\rangle_{r=R} R^2 d\eta \right]$$

Applying Proposition 3.4 to  $v$  and using Parseval's relation in  $L_2(G_i)$ ,  $L_2(G_j')$  and  $L_2(G_k'')$  gives

$$(5.4) \quad 0 = \text{Im} \left[ \sum_{i=1}^I \sum_{\ell=1}^{\infty} v_{i\ell}(R) \frac{d\bar{v}_{i\ell}(R)}{dR} + \sum_{j=1}^{I'} \sum_{\ell=1}^{\infty} v'_{j\ell}(R) \frac{d\bar{v}'_{j\ell}(R)}{dR} \right. \\ \left. + \sum_{k=1}^{I''} \sum_{\ell=1}^{\infty} v''_{k\ell}(R) \frac{d\bar{v}''_{k\ell}(R)}{dR} \right]$$

Finally, using (3.50)-(3.52) and the Wronskians for equations (3.33)-(3.35) gives, after simplification,

$$(5.5) \quad \frac{\pi}{2} \sum_{i=1}^I \sum_{\omega > \omega_{i\ell}} (\omega^2 - \omega_{i\ell}^2 + i0)^{1/2} |c_{i\ell}(\omega)|^2 + \sum_{j=1}^{I'} \sum_{\omega > \omega'_{jn(\ell)}} |c'_{j\ell}(\omega)|^2 + \sum_{k=1}^{I''} \sum_{\ell=1}^{\infty} |c''_{k\ell}(\omega)|^2 = 0$$

All the terms in these sums are non-negative; see (3.48). Hence, the sums (3.30)-(3.32) for  $v$  are finite and have only exponentially decreasing terms since  $\omega \in [0, \infty) - \sigma_{\text{cutoff}}(\Omega)$ . Thus  $v \in L_2(\Omega)$  and therefore  $v \in D(\Delta_\Omega)$  and  $-\Delta_\Omega v = \omega^2 v$  which, with (3.55), implies  $v = 0$ . Note that if  $\omega \in \sigma_{\text{cutoff}}(\Omega)$  then one cannot conclude that  $v \in L_2(\Omega)$  because of the exceptional solutions (3.38), (3.41), (3.42), (3.44). However, if  $I'' \geq 1$  then all  $c''_{k\ell}(\omega) = 0$ , as before, and hence  $v(q) = 0$  in  $S''_k$ . The analyticity of solutions of (3.53) then implies that  $v(q) = 0$  in  $\Omega$ .

Proof of Theorem 3. The proof is the same, except for one step, as the proof given in [23, pp. 65-75]. Briefly, the convergence of a sequence  $\{u_m\}$  in  $L_2^{1, \text{loc}}(\Delta, \bar{\Omega})$  is to be proved where  $F_m = -(\Delta + z_m)u_m$  has support in  $\Omega_r$  (defined here by (5.2)). Step 1 is to prove the convergence of  $\{u_m\}$  in  $L_2^1(\Delta, \Omega_{r''})$  where  $r < r'' < r'$ . This can be done exactly as in [23]. Step 2 is to prove the convergence of  $\{u_m\}$  in  $L_2^1(\Delta, \Omega_R - \Omega_{r''})$  for any  $R > r''$ . The proof in [23, p. 69] makes use of the Green's function for  $-\Delta$  in  $E^3$  which is an elementary function. An analogous proof for the waveguide domains considered here would require the Green's functions for  $-\Delta$  in each  $S_i, S'_j$  and  $S''_k$  and these are not known explicitly. As an alternative to this method it will be shown here that there exists a constant  $C = C(R)$  such that

for all  $u_m$

$$(5.6) \quad \|u_m\|_{L_2(\Omega_R - \Omega_{r''})} \leq C \|u_m\|_{L_2(\Omega_{r''})}$$

Convergence of  $\{u_m\}$  in  $L_2(\Omega_{r''})$  therefore implies its convergence in  $L_2^{\text{loc}}(\bar{\Omega})$ .

The remainder of the proof follows that of [23].

To prove (5.6) recall that  $u_m = R_\Omega(z_m)F_m$  where  $\text{Im } z_m \neq 0$ ,  $\text{supp } F_m \subset \Omega_r$  and  $r$  is fixed. Applying Propositions 3.4 and 3.5 to  $u_m \in L_2(\Omega)$  gives (cf. (3.46))

$$(5.7) \quad (u_m)_{i\ell}(y) = c_{i\ell}(z_m) e^{i(z_m - \omega_{i\ell}^2)^{1/2} y}$$

$$(5.8) \quad (u_m)'_{j\ell}(\rho) = c'_{j\ell}(z_m) H_\nu^{(1)}((z_m - \omega_{jn}^2)^{1/2} \rho), \quad \nu = \nu'_{jm}(\ell)$$

$$(5.9) \quad (u_m)''_{k\ell}(r) = c''_{k\ell}(z_m) r^{-1/2} H_\nu^{(1)}(z^{1/2} r), \quad \nu = (\nu_{k\ell}^2 + \frac{1}{4})^{1/2}$$

Inequality (5.6) follows from (5.7)-(5.9) and the following

**Proposition 5.1.** Let  $[a, b] \subset (0, \infty)$  and  $\sigma_0 > 0$ . Then there exist constants  $C = C(a, b, \sigma_0)$  and  $R = R(a, b, \sigma_0)$  such that

$$(5.10) \quad r'^{1/2} |H_\nu^{(1)}(z^{1/2} r')| \leq C r^{1/2} |H_\nu^{(1)}(z^{1/2} r)|$$

for all  $r$  and  $r'$  such that  $R \leq r \leq r'$ , all  $\nu \geq 1/2$  and all  $z \in \mathbb{C}$  such that  $a \leq \text{re } z \leq b$  and  $0 \leq \text{Im } z \leq \sigma_0$ .

This estimate follows from F. Olver's uniform estimates for Hankel functions [15, 16].

**Proof of Theorem 4.** The existence of the limits  $v^\pm(x, \omega)$  in  $D(\Delta_\Omega^{\text{loc}})$  follows from Theorem 3. Hence, only the radiation condition remains to be verified. This follows from (5.7)-(5.9) because  $v^\pm(\cdot, \omega) = \lim_{\sigma \rightarrow 0^+} R_\Omega(\omega^2 \pm i\sigma)g$ .

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A limiting absorption principle and corresponding existence theorem is presented for steady-state acoustic wave fields in simple and compound waveguides. The simple guides are semi-infinite cylinders, sectoral horns and conical horns. The compound guides are structures built by coupling one or more simple guides to a bounded cavity. The theory is developed under minimal regularity hypotheses about the waveguide walls. The class of allowable guides includes all the simple, but non-smooth, structures that arise in applications such as guides built from polyhedra and sections of cylinders, cones, spheres and other simple geometrical structures.			

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