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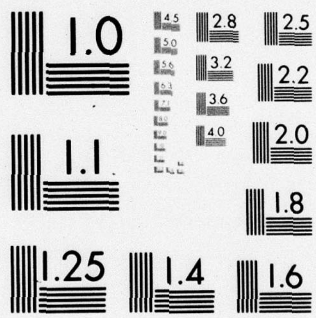

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

NECESSARY CONDITIONS FOR PROBLEMS

WITH HIGHER DERIVATIVE BOUNDED STATE VARIABLES

I. Bert Russak

June 1977

Final Report for periods Apr - Jun 75 and Jan - Mar 76

Approved for public release; distribution unlimited

Prepared for: Chief of Naval Research
Arlington, Virginia 22217

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The work reported herein was supported by the Foundation Research Program with funds provided by the Chief of Naval Research, Arlington, Virginia.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS-53Ru77061	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) NECESSARY CONDITIONS FOR PROBLEMS WITH HIGHER DERIVATIVE BOUNDED STATE VARIABLES.		5. TYPE OF REPORT & PERIOD COVERED Final rept. Apr-Jun 75 and Jan-Mar 76
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. Bert Russak		8. CONTRACT OR GRANT NUMBER(s) 61152N; RR 000-01-10 N0001477WR70044
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RR00001 RR0000110
11. CONTROLLING OFFICE NAME AND ADDRESS Chief of Naval Research Arlington, VA 22217		12. REPORT DATE June 1977
		13. NUMBER OF PAGES 45 48p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimization, Calculus of Variations, State Constraints

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A set of necessary conditions is obtained for the general control problem of Bolza with state inequality constraints in which the control enters in the second derivative of the constraint. Important features of this paper are in: i) relaxing the solvability assumption on the state constraints, ii) extending the maximum principle to a larger set than previously realized, iii) obtaining modified forms of the relation $\dot{H} = H_t$ and of the transversality relation usually obtained in problems of this type and iv) proving a condition concerning

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$\dot{\mu}_\alpha(t^1)$, the derivative of the multiplier functions at the final point.

The results obtained do not directly follow from the results for the case when the control enters in the first derivative of the constraint.

This work was supported by an NPS Foundation grant.

1. INTRODUCTION

A general problem in optimal control can be stated as follows: Let C be the class of arcs

$$a: \quad x^i(t) \quad \dot{x}^i(t) \quad u^k(t) \quad b^\sigma \quad t^0 \leq t \leq t^1$$

$$i = 1,\ldots,N \quad k = 1,\ldots,K \quad \sigma = 1,\ldots,r$$

whose points $(t,x(t),\dot{x}(t),u(t))$ lie in a region R in t-x-$\dot{x}$-u space with b in a region B in b space, $u(t)$ piecewise continuous, $x(t)$ of class C^1 and which in addition satisfy the constraints:

$$\ddot{x}^i = f^i(t,x,\dot{x},u) \qquad i = 1,\ldots,N$$

$$\psi^\alpha(t,x) \leq 0 \quad \alpha = 1,\ldots,m' \qquad \psi^\alpha(t,x) = 0 \quad \alpha = m'+1,\ldots,m$$

$$\theta^\eta(t,x,\dot{x},u) \leq 0 \quad \eta = 1,\ldots,L' \qquad \theta^\eta(t,x,\dot{x},u) = 0 \quad \eta = L'+1,\ldots,L$$

$$I_\gamma(a) \leq 0 \quad \gamma = 1,\ldots,p' \qquad I_\gamma(a) = 0 \quad \gamma = p'+1,\ldots,p$$

$$x^i(t^s) = \chi^{is}(b) \qquad \dot{x}^i(t^s) = \dot{\chi}^{is}(b) \qquad t^s = T^s(b) \qquad s = 0,1$$

where:[1]

$$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x,\dot{x},u)\,dt.$$

It is desired to minimize the integral

$$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x,\dot{x},u)\,dt$$

on the class C.

---

(1) With few exceptions, superscript $\cdot$ will denote differentiation with respect to t. The exceptions are used for notational convenience and will be explicitly noted. The current exception is $\dot{\chi}^{is}$ and refers to the constraints on $\dot{x}^i(t^s)$. Also, unless otherwise specified, the indices i,k,σ,α,η will have the respective ranges

$$1 \leq i \leq N, \quad 1 \leq k \leq K, \quad 1 \leq \sigma \leq r, \quad 1 \leq \alpha \leq m, \quad 1 \leq \eta \leq L.$$

A point $(t,x,\dot{x},u)$ such that $\psi^\alpha(t,x)=0$ is called a boundary point for the constraint ψ^α. An important feature of this paper is in relaxing the assumption upon the rank of the matrix $\begin{bmatrix}\psi^\alpha_{x^i} f^i_{u^k}\end{bmatrix}$ generally used in attacking problems of this type.(1) This is accomplished even though we admit solution arcs which have an infinite number of intervals on the boundary. Furthermore: i) the maximum principle is shown to apply to a larger set than previously obtained (see [1](2)); ii) modified forms of the relation $\dot{H}=H_t$ and of the transversality relation usually obtained in problems of this type are shown to hold and iii) a condition on $\dot{\mu}_\alpha(t^1)$, the derivative of the multiplier function at the final point, is obtained.

The results obtained do not directly follow from results for the case when the differential equations involve only first derivatives of the state.

With the dimensions of our terms allowed to adopt new values, the above problem is expressible as one in which the constraints have the simpler form

$$\ddot{x}^i = f^i(t,x,\dot{x},u)$$

$$\psi^\alpha(t,x) \leq 0 \qquad 1 \leq \alpha \leq m$$

$$I_\gamma(a) \leq 0 \qquad 1 \leq \gamma \leq p' \qquad I_\gamma(a) = 0 \qquad p' < \gamma \leq p$$

$$x^i(t^s) = \chi^{is}(b) \qquad \dot{x}^i(t^s) = \dot{\chi}^{is}(b) \qquad s = 0,1 \qquad 1 \leq i \leq N.$$

The methods of Hestenes are extended to obtain a set of first order necessary conditions which a solution arc to the problem stated above must satisfy. The extension of this result to the more general problem is the subject of a succeeding paper.

2. STATEMENT OF THE PROBLEM

We are concerned with arcs a

$$a: \quad x^i(t), \quad \dot{x}^i(t), \quad u^k(t), \quad b^\sigma \qquad t^o \leq t \leq t^1$$

with points $(t,x(t),\dot{x}(t),u(t))$ in a region R in t-x-$\dot{x}$-u space, b in a region B in b space, $u(t)$ piecewise continuous and $x(t)$ of class C^1.

(1) Unless otherwise specified, repeated indices are summed.

(2) Bracketed numbers refer to references.

A general problem can be stated as follows: It is desired to minimize the integral

(1) $$I_o(a) = g_o(b) + \int_{t^o}^{t^1} L_o(t,x,\dot{x},u)dt$$

on the class C of arcs which satisfy the conditions

(2-1) $$\ddot{x}^i = f^i(t,x,\dot{x},u)$$

(2-2) $$\psi^\alpha(t,x) \leq 0$$

(2-3) $$I_\gamma(a) \leq 0 \qquad 1 \leq \gamma \leq p' \qquad I_\gamma(a) = 0 \qquad p' < \gamma \leq p$$

(2-4) $$x^i(t^s) = \chi^{is}(b) \qquad \dot{x}^i(t^s) = \dot{\chi}^{is}(b) \qquad s = 0,1$$

where

(3) $$I_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} L_\gamma(t,x,\dot{x},u)dt \qquad \gamma = 1,\ldots,p\ .$$

We assume that the functions ψ^α are of class C^3 and that the functions f^i, χ^{is}, L_γ, g_γ, $\dot{\chi}^{is}$, are of class C^1 on R or B as the case may be.

Define the functions

(4-1) $$\tilde{\phi}^\alpha(t,x,\dot{x}) = \psi^\alpha_t + \psi^\alpha_{x^i}\dot{x}^i$$

(4-2) $$\phi^\alpha(t,x,\dot{x},u) = \psi^\alpha_{t^2} + 2\psi^\alpha_{tx^i}\dot{x}^i + \psi^\alpha_{x^ix^j}\dot{x}^i\dot{x}^j + \psi^\alpha_{x^i}f^i(t,x,\dot{x},u)\ , \quad i,j = 1,\ldots,N.$$

For arcs which satisfy (2-1), these functions act as $\frac{d\psi^\alpha}{dt}$ and $\frac{d^2\psi^\alpha}{dt^2}$ and are of class C^2 and C^1 respectively.

Let R_o be the set of points $(t,x,\dot{x},u)$ in R satisfying the conditions

(5-1) $$\psi^\alpha \leq 0$$

(5-2) $$\tilde{\phi}^\alpha = 0 \text{ and } \phi^\alpha \geq 0 \quad \forall\alpha \ni \psi^\alpha = 0 \quad \text{or} \quad \tilde{\phi}^\alpha = 0 \text{ and } \phi^\alpha \leq 0 \quad \forall\alpha \ni \psi^\alpha = 0$$

$$\alpha = 1,\ldots,m.$$

Let R_o^- and R_o^+ be those subsets of R_o whose points satisfy the first and second parts of (5-2) respectively.

We wish to test whether a given arc

a_o: $x_o(t)$, $\dot{x}_o(t)$, $u_o(t)$, b_o $t^o \le t \le t^1$

which is in the class C is a solution to the above problem.

The set R_o is seen to contain the collection of points $(t,x_o(t),\dot{x}_o(t),u_o(t))$. As an extension of this collection, we define R_1 as the set of points $(t,x_o(t),\dot{x}_o(t),u)$ which are in R_o.

We will assume that the matrix[1]

$$(6) \qquad \begin{bmatrix} \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \alpha,\beta = 1,\ldots,m$$

has rank m on R_1.

3. FIRST ORDER NECESSARY CONDITIONS FOR A MINIMUM

Define the functions

$$(7\text{-}1) \qquad H(t,x,\dot{x},u,p,\tilde{p},\mu) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha$$

and

$$G(b) = \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{io} + \lambda_{p+N+i} \dot{X}^{io}$$

$$\gamma = 1,\ldots,p\,.$$

Then we shall prove:

Theorem 3.1. Suppose that the arc a_o is a solution to the above problem. Then there are multipliers

$$(8) \qquad \lambda_\gamma,\ p_i(t),\ \tilde{p}_i(t),\ K^\tau,\ \mu_\alpha(t),$$

$$\gamma = 0,1,\ldots,p+2N\,, \qquad i = 1,\ldots,N\,; \qquad \alpha = 1,\ldots,m\,, \qquad \tau = 1,\ldots,2m$$

such that when these are used as arguments for the functions H and G defined above, then the following conditions are satisfied: The inequality

$$(9) \quad H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t)) \le H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t))$$

---

(1) $\delta_{\alpha\beta}$ denotes the Kronecker delta.

is valid for all u with $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. The multipliers $p_i(t)$, $\tilde{p}_i(t)$ are each continuous with a piecewise continuous derivative on $[t^o,t^1]$. Together with $\mu_\alpha(t)$ these multipliers satisfy the relations

$$\dot{\tilde{p}}_i = -H_{x^i}, \quad \dot{p}_i = -H_{\dot{x}^i}, \quad H_{u^k} = 0, \quad \ddot{x}^i = H_{p_i} \tag{10}$$

along a_o on intervals of continuity of $u_o(t)$.

The function $H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t))$ is continuous on $[t^o,t^1]$. On intervals of continuity of $u_o(t)$ this function has a continuous derivative satisfying[(1)]

$$\frac{dH}{dt} + \dot{\mu}_\alpha \phi^\alpha = H_t . \tag{11}$$

The transversality condition

$$dG + \Big[p_i(t^s)d\dot{X}^{is} + \tilde{p}_i(t^s)dX^{is} \Big]_{s=0}^{s=1} = 0 \tag{12}$$

is valid along a_o for all db.

The multipliers λ_ρ, k^τ are constants which satisfy[(2)]

$$\begin{aligned} &\lambda_o \geq 0 \quad \lambda_\gamma \geq 0 \quad \text{with} \quad \lambda_\gamma = 0 \quad \text{if} \quad I_\gamma(a_o) < 0 \quad \gamma = 1,\dots,p' \\ &K^\alpha \geq 0 \quad \lambda_{p+i} = K^\alpha q_i^\alpha(t^o) + K^{m+\alpha}\dot{q}_i^\alpha(t^o) \quad \lambda_{p+N+i} = K^{m+\alpha} q_i^\alpha(t^o) . \end{aligned} \tag{13}$$

Furthermore, together with $p(t),\tilde{p}(t),\mu(t)$ they are not of the form:[(1)]

$$\begin{aligned} &\lambda_\gamma = 0 \quad \gamma = 0,1,\dots,p \quad K^\tau = 0 \quad \tau = 1,\dots,2m \quad \dot{\mu}_\alpha(\bar{t}) = 0 \quad \text{if} \quad \psi^\alpha(\bar{t}) < 0 \\ &\mu_\alpha(\bar{t}) = 0 \quad \alpha = 1,\dots,m, \quad p_i(\bar{t}) = 0, \quad \tilde{p}_i(\bar{t}) = 0 \quad i = 1,\dots,N \end{aligned} \tag{14}$$

for any point $\bar{t}$ in $[t^o,t^1]$.

---

(1) In the proof of (11),(14), the terms involving $\dot{\mu}_\alpha$ will be shown to exist.

(2) The symbol $q_i^\alpha(t)$ denotes $\psi^\alpha_{x^i}(t,x_o(t))$. In addition, functions $M(t,x,\dot{x},u)$ when evaluated along a_o at points $(t,x_o(t),\dot{x}_o(t),u_o(t))$ will often be denoted as $M(t)$.

For each α, the multiplier $\mu_\alpha(t)$ is continuous on intervals of continuity of $u_o(t)$ and satisfies the following properties: i) there are constants a_α, b_α such that $\mu_\alpha(t) - (a_\alpha t + b_\alpha)$ is a nonincreasing function on $[t^o, t^1]$, ii) it is of the form $\bar{a}_\alpha t + \bar{b}_\alpha$ on intervals upon which $\psi^\alpha(t) < 0$, iii) $\mu_\alpha(t^1) = 0$ if $\psi^\alpha(t^1) < 0$, and iv) [(1)] $\dot{\mu}_\alpha(t^1) = 0$ if $\psi^\alpha(t^1) < 0$.

Next, let the arc $\mathcal{a}_o$ be a solution to our problem which together with the multipliers K^τ, λ_ρ, $\mu_\alpha(t)$, $p_i(t)$, $\tilde{p}_i(t)$ satisfy Theorem 3.1. Let the multipliers $\mu_\alpha(t)$, $p_i(t)$, $\tilde{p}_i(t)$ and the functions H, G be modified to the respective forms

$$\mu_\alpha(t)+(a_\alpha t+b_\alpha) \qquad p_i(t)+(a_\alpha t+b_\alpha)q_i^\alpha(t) \qquad \tilde{p}_i(t)+(a_\alpha t+b_\alpha)\dot{q}_i^\alpha(t)$$

$$(15) \qquad G(b) + \left[-(a_\alpha t^s + b_\alpha)(\dot{q}_i^\alpha(t^s)\chi^{is}(b) + q_i^\alpha(t^s)\dot{\chi}^{is}(b))\right]_{s=0}^{s=1}$$

$$H(t,x,\dot{x},u,p,\tilde{p},\mu) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha - a_\alpha \tilde{\phi}^\alpha, \qquad \gamma = 1,\ldots,p$$

for any constants a_α, b_α. Then by substitution, we see that the arc $\mathcal{a}_o$ together with the modified multipliers and modified H and G functions satisfy all of the statements of Theorem 3.1 except for properties iii) and iv) of the multipliers $\mu_\alpha(t)$ and the statement involving 14).

Then using the above definitions, set

$$H(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu}) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}_\alpha \tilde{\phi}^\alpha$$

$$(16) \quad G(b) = \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+i}\chi^{io} + \lambda_{p+N+i}\dot{\chi}^{io} - \lambda_{p+2N+\alpha}\Big(\dot{q}_i^\alpha(t^1)\chi^{i1}(b) + q_i^\alpha(t^1)\dot{\chi}^{i1}(b)\Big)$$
$$+ \lambda_{p+2N+m+\alpha}\Big(\dot{q}_i^\alpha(t^o)\chi^{io}(b) + q_i^\alpha(t^o)\dot{\chi}^{io}(b)\Big), \quad \gamma=1,\ldots,p.$$

As a consequence of these remarks, we have the following results:

<u>Theorem 3.2.</u> Suppose that the arc $\mathcal{a}_o$ is a solution to our problem. Then there are multipliers

$$K^\tau, \lambda_\rho, \mu_\alpha(t), \tilde{\mu}_\alpha, p_i(t), \tilde{p}_i(t)$$

$\tau = 1,\ldots,2m \qquad \rho = 0,1,\ldots,p+2N+2m \qquad \alpha = 1,\ldots,m \qquad i = 1,\ldots,N$

(1) See previous footnote concerning $\dot{\mu}_\alpha$ terms.

and functions H and G as described above such that with these terms as arguments, then the following conditions hold.

The inequality

(17)
$$H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu})$$
$$\leq H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu})$$

is valid for all u with $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. The multipliers $p_i(t)$, $\tilde{p}_i(t)$ are each continuous with a piecewise continuous derivative on $[t^o,t^1]$. Together with $\mu_\alpha(t)$, $\tilde{\mu}_\alpha$, these multipliers satisfy the relations

(18)
$$\dot{p}_i = -H_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -H_{x^i} \qquad \ddot{x}^i = H_{p_i} \qquad H_{u^k} = 0$$

along a_o on intervals of continuity of $u_o(t)$.

The function $H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu})$ is continuous on $[t^o,t^1]$. On intervals of continuity of $u_o(t)$ this function has a continuous derivative satisfying

(19)
$$\frac{dH}{dt} + \dot{\mu}_\alpha\phi^\alpha = H_t .$$

The transversality condition

(20)
$$dG + \left[p_i(t^s)d\dot{X}^{is} + \tilde{p}_i(t^s)dX^{is}\right]_{s=0}^{s=1} = 0$$

is valid along a_o for all db.

The multipliers λ_ρ, K^τ are constants which satisfy

(21)
$$\lambda_o \geq 0 \qquad \lambda_\gamma \geq 0 \quad \text{with} \quad \lambda_\gamma = 0 \quad \text{if} \quad I_\gamma(a_o) < 0 \qquad 1 \leq \gamma \leq p'$$
$$K^\alpha \geq 0 \qquad \lambda_{p+i} = K^\alpha q_i^\alpha(t^o) + K^{m+\alpha}\dot{q}_i^\alpha(t^o) \qquad \lambda_{p+N+i} = K^{m+\alpha} q_i^\alpha(t^o) .$$

Furthermore, together with $\mu(t)$, $\tilde{\mu}$, $p(t)$, $\tilde{p}(t)$ they are not of the form

$$
(22)\quad
\begin{aligned}
&\lambda_\gamma = 0 \qquad \gamma = 0,1,\dots,p \qquad K^\tau = 0 \qquad \tau = 1,\dots,2m \\
&\dot{\mu}_\alpha(\bar{t}) = a_\alpha \quad \text{if} \quad \psi^\alpha(\bar{t}) < 0 \qquad \mu_\alpha(\bar{t}) = a_\alpha \bar{t} + b_\alpha \\
&\tilde{\mu}_\alpha = a_\alpha \qquad \lambda_{p+2N+\alpha} = a_\alpha t^1 + b_\alpha \qquad \lambda_{p+2N+m+\alpha} = a_\alpha t^o + b_\alpha \\
&p_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha) q_i^\alpha(\bar{t}), \qquad \tilde{p}_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha)\dot{q}_i^\alpha(\bar{t})
\end{aligned}
$$

for any constants a_α, b_α at any point $\bar{t}$ in $[t^o,t^1]$.

For each α the multiplier $\mu_\alpha(t)$ is continuous on intervals of continuity of $u_o(t)$ and satisfies the following properties: i) there are constants a_α, b_α such that $\mu_\alpha(t) - (a_\alpha t + b_\alpha)$ is a nonincreasing function on $[t^o,t^1]$, ii) it is of the form $\bar{a}_\alpha t + \bar{b}_\alpha$ on intervals upon which $\psi_\alpha(t) < 0$ and iii) satisfies $\dot{\mu}_\alpha(t^1) = \tilde{\mu}_\alpha$ if $\psi^\alpha(t^1) < 0$.

<u>Lemma 3.1</u>. Theorem 3.2 follows from Theorem 3.1.

Set

$$
(23)\qquad \lambda_{p+2N+\alpha} = a_\alpha t^1 + b_\alpha \qquad \lambda_{p+2N+m+\alpha} = a_\alpha t^o + b_\alpha \qquad \tilde{\mu}_\alpha = a_\alpha
$$

where a_α, b_α are the constants of (15). By the remarks below (15), we see that it is only necessary to prove the statement involving (22) and property iii) of this multipliers $\mu_\alpha(t)$. The latter statement follows by substitution. In order to prove (22), we assume that the situation of (22) exists. Then by (23), we see that the situation of (14) exists for the original multipliers. Thus, Theorem 3.2 and also Lemma 3.1 are proven.

4. <u>REFORMULATION OF THE PROBLEM</u>

It is convenient to reformulate the problem stated above. In order to do this, introduce the additional variables

$$x^{N+\alpha}, \quad \dot{x}^{N+\alpha}, \quad u^{K+\alpha} \qquad \alpha = 1,\ldots,m, \qquad b^{r+s} \qquad s = 1,\ldots,4m$$

and the conditions

(24) $$\ddot{x}^{N+\alpha} = f^{N+\alpha} = u^{K+\alpha} \qquad x^{N+\alpha} < 0 \qquad \alpha = 1,\ldots,m .$$

Let $\bar{R},\bar{B}$ be the respective Cartesian product sets

$$R \times X^{N+1} \times \cdots X^{N+m} \times \dot{X}^{N+1} \times \cdots \dot{X}^{N+m} \times U^{K+1} \times \cdots U^{K+m}$$

and

$$B \times B^{1} \times \cdots B^{4m}$$

where for example $X^{N+\alpha}$ is the space of the variable $x^{N+\alpha}$.

Define the functions (without summing on α)

(25-1) $$\bar{\psi}^{\alpha}(t,x) = \psi^{\alpha}(t,x) + x^{N+\alpha}\psi^{\alpha}(t)$$

(25-2) $$\tilde{\bar{\phi}}^{\alpha}(t,x,\dot{x}) = \tilde{\phi}^{\alpha}(t,x,\dot{x}) + \dot{x}^{N+\alpha}\psi^{\alpha}(t) + x^{N+\alpha}\tilde{\phi}^{\alpha}(t)$$

(25-3) $$\bar{\phi}^{\alpha}(t,x,\dot{x},u) = \phi^{\alpha}(t,x,\dot{x},u) + u^{K+\alpha}\psi^{\alpha}(t) + 2\dot{x}^{N+\alpha}\tilde{\phi}^{\alpha}(t) + x^{N+\alpha}\phi^{\alpha}(t)$$

where $\psi^{\alpha}(t)$, $\phi^{\alpha}(t)$, $\tilde{\phi}^{\alpha}(t)$ refer to values along a_o.

We are now concerned with arcs in $\bar{R}$. Such an arc will be designated by

$$\bar{a}: \qquad \bar{x}^{j}(t), \qquad \dot{\bar{x}}^{j}(t), \qquad \bar{u}^{k}(t), \qquad \bar{b}^{\sigma}, \qquad t^{o} \leq t \leq t^{1}$$

$$j = 1,\ldots,N+m, \qquad k = 1,\ldots,K+m, \qquad \sigma = 1,\ldots,r+4m .$$

Consider the following problem. It is desired to minimize the integral

(26) $$I_o(\bar{a}) = g_o(\bar{b}) + \int_{t^o}^{t^1} L_o(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))dt$$

on the class $\bar{C}$ of arcs $\bar{a}$ which have; i) $(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))$ in $\bar{R}$, ii) $\bar{b}$ in $\bar{B}$, iii) $\bar{u}(t)$ piecewise continuous, iv) $\bar{x}(t)$ of class C^1 and which in addition, satisfy the conditions

$$(27\text{-}1) \qquad \ddot{\bar{x}}^j = f^j \qquad j = 1,\ldots,N+m$$

$$(27\text{-}2) \qquad \bar{\psi}^\alpha \leq 0 \qquad \bar{x}^{N+\alpha} < 0 \qquad \alpha = 1,\ldots,m$$

$$(27\text{-}3) \qquad I_\gamma(\bar{a}) \leq 0 \quad \gamma = 1,\ldots,p' \qquad I_\gamma(\bar{a}) = 0 \quad \gamma = p' + 1,\ldots,p$$

$$(27\text{-}4) \qquad \bar{x}^j(t^s) = \chi^{js}(b) \quad \dot{\bar{x}}^j(t^s) = \dot{\chi}^{js}(b) \qquad s = 0,1 \quad j = 1,\ldots,N+m$$

where $\chi^{is}(b)$ and $\dot{\chi}^{is}(b)$, $i=1,\ldots,N$, $s=0,1$ have the meaning already specified and

$$(27\text{-}5) \qquad \chi^{N+\alpha,0}(b)=b^{r+\alpha} \qquad \chi^{N+\alpha,1}(b)=b^{r+m+\alpha} \qquad \dot{\chi}^{N+\alpha,0}(b)=b^{r+2m+\alpha} \qquad \dot{\chi}^{N+\alpha,1}(b)=b^{r+3m+\alpha}$$

$$(27\text{-}6) \qquad I_\gamma(\bar{a}) = g_\gamma(\bar{b}) + \int_{t^o}^{t^1} L_\gamma(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))dt \qquad \gamma = 1,\ldots,p\,.$$

Let $\bar{a}$ be an arc in $\bar{C}$ and let a be the projection of this arc into R, B space and let S^α be the set $S^\alpha = \{t|\psi^\alpha(t) = 0\}$. Then by the definition of $\bar{\psi}^\alpha$, the arc a is in C for the original problem and furthermore there is a positive constant τ depending on a such that

$$(28) \qquad \psi^\alpha(t,x(t))/\psi^\alpha(t) > \tau > 0 \qquad t \in [t^o,t^1] - S^\alpha\,.$$

Define C_o as the subclass of arcs of C which satisfy the above inequality for some $\tau>0$. Then starting with an arc a in C_o we can construct an arc $\bar{a}$ in $\bar{C}$. The arc $\bar{a}$ is the projection of a into $\bar{R}$, $\bar{B}$ space and will in addition have

$$(29) \qquad \begin{array}{lll} \bar{x}^{N+\alpha}(t) \equiv -\bar{\tau} & \dot{\bar{x}}^{n+\alpha}(t) \equiv 0 & \bar{u}^{K+\alpha}(t) \equiv 0 \\ \bar{b}^{r+\alpha} = \bar{b}^{r+m+\alpha} = -\bar{\tau} & \bar{b}^{r+2m+\alpha} = \bar{b}^{r+3m+\alpha} = 0 & \alpha = 1,\ldots,m \end{array}$$

where $\bar{\tau}$ is a positive constant satisfying $\bar{\tau} \leq \tau$ for τ the constant of a. In so doing, we have shown that the reformulated problem is equivalent to the problem of minimizing I_o on C_o. Furthermore, since the arc a_o is in $C_o \subset C$ then a_o solves the latter problem and a correspondent $\bar{a}_o$ constructed as outlined above will be a solution to the reformulated problem. Thus, define the arc

$\bar{a}_o$: $\bar{x}^j_o(t)$, $\dot{\bar{x}}^j_o(t)$, $\bar{u}^k_o(t)$, $\bar{b}^\sigma_o$ $t^o \leq t \leq t^1$

with

$$\bar{x}^i_o = x^i_o \qquad i = 1,\ldots,N \qquad \bar{x}^{N+\alpha}_o \equiv -\theta$$

$$\dot{\bar{x}}^i_o = \dot{x}^i_o \qquad i = 1,\ldots,N \qquad \dot{\bar{x}}^{N+\alpha}_o \equiv 0$$

$$\bar{u}^h_o = u^h_o \qquad h = 1,\ldots,K \qquad \bar{u}^{K+\alpha}_o \equiv 0$$

$$\bar{b}^\sigma_o = b^\sigma_o \qquad \sigma = 1,\ldots,r \qquad \bar{b}^{r+\alpha}_o = \bar{b}^{r+m+\alpha}_o = -\theta$$

$$\bar{b}^{r+2m+\alpha}_o = \bar{b}^{r+3m+\alpha}_o = 0 \qquad \alpha = 1,\ldots,m$$

where θ is any constant satisfying $0 < \theta < 1$. By the preceding line of reasoning, the arc $\bar{a}_o$ is a solution to the present problem. Furthermore, by construction, we have[(1)]

(31) $$0 \geq \bar{\psi}^\alpha(t) \geq \psi^\alpha(t) \qquad \text{and} \qquad 0 = \bar{\psi}^\alpha(t) \quad \text{iff} \quad 0 = \psi^\alpha(t).$$

From our definitions, we see that the functions $\bar{\psi}^\alpha$ are of class C^3 with respect to x and of class C^2 with respect to t on sets

$$[t_1\ ,\ t_2] \qquad |x - x_o(t)| < \rho \qquad |\dot{x} - \dot{x}_o(t)| < \rho \qquad |u - u_o(t)| < \rho$$

where $[t_1\ ,\ t_2]$ is an interval of continuity of $u_o(t)$ and ρ is some positive constant. We shall henceforth call such sets as $\bar{C}$ sets. Furthermore, from the definition of $\tilde{\bar{\phi}}^\alpha$ and $\bar{\phi}^\alpha$ we see that the functions $\tilde{\bar{\phi}}^\alpha$ are C^2 in x, $\dot{x}$ and C^1 in t on $\bar{C}$ sets while the functions $\bar{\phi}^\alpha$ are C^1 in $u,x,\dot{x}$ on $\bar{C}$ sets and C^o in t on $\bar{C}$ sets. Finally $\tilde{\bar{\phi}}^\alpha$, $\bar{\phi}^\alpha$ act respectively as $d\bar{\psi}^\alpha/dt$ and $d^2\bar{\psi}^\alpha/dt^2$ along arcs satisfying (27-1).

---

(1) Functions with superscript "bars" and argument t will refer to evaluation along $\bar{a}_o$.

Next, define the set $\bar{R}_o$ as those points in $\bar{R}$ such that

(32-1) $\bar{\psi}^\alpha \leq 0 \qquad x^{N+\alpha} < 0$

and

(32-2) $\tilde{\phi}^\alpha = 0, \quad \bar{\phi}^\alpha \geq 0 \quad \forall\alpha \ni \bar{\psi}^\alpha = 0 \quad$ or $\quad \tilde{\phi}^\alpha = 0, \quad \bar{\phi}^\alpha \leq 0, \quad \forall\alpha \ni \bar{\psi}^\alpha = 0$

and define $\bar{R}_o^-, \bar{R}_o^+$ respectively as those points in $\bar{R}_o$ satisfying the former, latter conditions of (32-2). Lastly, define the set $\bar{D}$ as the points $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u)$ in $\bar{R}_o$ with either $u = \bar{u}_o(t)$ or t interior to an interval of continuity of $u_o(t)$. We next prove:

<u>Lemma 4.1.</u> The matrix $\begin{bmatrix} \bar{\phi}^\alpha_{u^k} \end{bmatrix}$ $\quad 1 \leq \alpha \leq m \quad 1 \leq k \leq K+m$ has rank m on $\bar{D}$.

<u>Proof</u>: By the definition of our functions, we see that if $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u)$ is a point in $\bar{D}$ then it's projection[1] $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u)_p$ in R is in R_o. Then by using these definitions again, together with the properties of the matrix of (6), the lemma is proven.

5. FIRST NECESSARY CONDITIONS FOR A MINIMUM IN THE REFORMULATED PROBLEM.

In order to state the main result to be proven for the present problem, we make the following definitions. The set $\bar{R}_1$ is the subset of $\bar{R}_o$ consisting of the points, $(t,\bar{x}_o(t), \dot{\bar{x}}_o(t),u)$ in $\bar{R}_o$. The set $\bar{R}_2$ is that subset of $\bar{R}_1$ with t interior to an interval of continuity of $\bar{u}_o(t)$ and the set R_2 is defined analogously with respect to R_2 and a_o.

We shall prove that given an arc

$\bar{a}_o: \qquad \bar{x}_o(t), \qquad \dot{\bar{x}}_o(t), \qquad \bar{u}_o(t), \qquad \bar{b}_o \qquad t^o \leq t \leq t^1$

which is a solution to our problem, then $\bar{a}_o$ satisfies the following theorem:

---

(1) Subscript p denotes projection.

<u>Theorem 5.1.</u> Suppose that $\bar{a}_o$ is a solution to our problem. Then there is a set of multipliers

$$\bar{\lambda}_\rho, \quad p_j(t), \quad \tilde{p}_j(t), \quad K^\tau, \quad \mu_\alpha(t),$$

$$\rho = 0,1,\ldots,p+2N+2m, \quad j = 1,\ldots,N+m, \quad \alpha = 1,\ldots,m, \quad \tau = 1,\ldots,2m$$

and functions

$$\bar{H}(t,x,\dot{x},u,p,\tilde{p},\mu) = p_j f^j + \tilde{p}_j \dot{x}^j - \bar{\lambda}_o L_o - \bar{\lambda}_\gamma L_\gamma - \mu_\alpha \bar{\phi}^\alpha \tag{33-1}$$

$$\bar{G}(b) = \bar{\lambda}_o g_o + \bar{\lambda}_\gamma g_\gamma + \bar{\lambda}_{p+i} \chi^{io} + \bar{\lambda}_{p+N+\alpha} b^{r+\alpha} + \bar{\lambda}_{p+N+m+i} \dot{\chi}^{io} + \bar{\lambda}_{p+2N+m+\alpha} b^{r+2m+\alpha} \tag{33-2}$$

$$j = 1,\ldots,N+m; \quad \gamma = 1,\ldots,p; \quad i = 1,\ldots,N; \quad \alpha = 1,\ldots m$$

such that with these multipliers as arguments, then the following conditions hold.

The inequality

$$\bar{H}(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,p(t),\tilde{p}(t),\mu(t)) \leq \bar{H}(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),\bar{u}_o(t),p(t),\tilde{p}(t),\mu(t)) \tag{34}$$

is valid for all u with $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u)$ in $\bar{R}_2$. The multipliers $p_j(t)$, $\tilde{p}_j(t)$, are each continuous with a piecewise continuous derivative on $[t^o,t^1]$. Together with $\mu_\alpha(t)$ they satisfy the relations

$$\ddot{x}^j = \bar{H}_{p_j}, \quad \dot{\tilde{p}}_j = -\bar{H}_{x^j}, \quad \dot{p}_j = -\bar{H}_{\dot{x}^j}, \quad \bar{H}_{u^k} = 0, \quad 1 \leq j \leq N+m, \quad 1 \leq k \leq K+m. \tag{35}$$

on intervals of continuity of $u_o(t)$.

The transversality condition

$$d\bar{G} + \Big[p_j(t^s) d\dot{X}^{js} + \tilde{p}_j(t^s) dX^{js} \Big]_{s=0}^{s=1} = 0 \tag{36}$$

is valid along $\bar{a}_o$ for all db.

The multipliers $\bar{\lambda}_\rho$, K^α are constants which satisfy(1)

---

(1) The symbols $\bar{q}_i^\alpha(t)$, $\dot{\bar{q}}_i^\alpha(t)$ denote quantities for $\bar{\psi}^\alpha$ along $\bar{a}_o$ which are analogous to $q_i^\alpha(t)$, $\dot{q}_i^\alpha(t)$ and ψ^α along a_o.

$$\bar{\lambda}_o \geq 0, \quad \bar{\lambda}_\gamma \geq 0 \quad \text{with} \quad \bar{\lambda}_\gamma = 0 \quad \text{if} \quad I_\gamma(\bar{a}_o) < 0, \quad 1 \leq \gamma \leq p', \quad K^\alpha \geq 0, \quad \alpha = 1,\ldots,m$$

(37)

$$\bar{\lambda}_{p+j} = K^\alpha \bar{q}_j^\alpha(t^o) + K^{m+\alpha} \dot{\bar{q}}_j^\alpha(t^o), \quad \bar{\lambda}_{p+N+m+j} = K^{m+\alpha} \bar{q}_j^\alpha(t^o), \quad j=1,\ldots,N+m \quad .$$

Furthermore together with $p(t)$, $\tilde{p}(t)$, they are not of the form:

$$\bar{\lambda}_\gamma = 0, \qquad 0 \leq \gamma \leq p \qquad K^\tau = 0, \qquad \tau = 1,\ldots,2m$$

(38)

$$p_j(\bar{t}) = 0, \qquad \tilde{p}_j(\bar{t}) = 0, \qquad j = 1,\ldots,N+m$$

for any point $\bar{t}$ in $[t^o,t^1]$.

For each α, the multiplier $\mu_\alpha(t)$ is continuous on intervals of continuity of $\bar{u}_o(t)$ and there are constants a_α, b_α such that $\mu_\alpha(t) - (a_\alpha t + b_\alpha)$ is a nonincreasing function on $[t^o,t^1]$.

6. RELATIONSHIP BETWEEN THEOREM 3.1 AND THEOREM 5.1.

The statements of Theorem 3.1, except for (11) and its preceding remark follow from Theorem 5.1. The excepted statement will be proven in a succeeding paper. In order now to prove the remainder of Theorem 3.1, set

$$\lambda_\rho = \bar{\lambda}_\rho \qquad \rho=0,1,\ldots,p+N$$

(39)

$$\lambda_{p+N+i} = \bar{\lambda}_{p+N+m+i} \qquad i=1,\ldots,N \quad .$$

By the definition of $\bar{\phi}^\alpha$, $\tilde{\bar{\phi}}^\alpha$ and $f^{N+\alpha}$ we see that $\bar{H}$ can be written as

(40) $$\bar{H} = H + p_{N+\alpha} u^{K+\alpha} + \tilde{p}_{N+\alpha} \dot{x}^{N+\alpha} - \mu_\alpha[u^{K+\alpha}\psi^\alpha(t) + 2\dot{x}^{N+\alpha}\tilde{\phi}^\alpha(t) + x^{N+\alpha}\phi^\alpha(t)]$$

where H is the function of Theorem 3.1. Thus,

$$\bar{H}_{x^i} = H_{x^i}, \quad \bar{H}_{p_i} = H_{p_i}, \quad \bar{H}_{\dot{x}^i} = H_{\dot{x}^i}, \quad \bar{H}_{u^k} = H_{u^k}$$

(41)

$$1 \leq i \leq N \qquad 1 \leq k \leq K$$

and statement (10) together with it's preceding remark about the multipliers $p(t)$, and $\tilde{p}(t)$ follow from Theorem 5.1 together with the fact that intervals of continuity of $u_o(t)$ and of $\bar{u}_o(t)$ agree.

Since the functions g_γ do not depend upon b^{r+s}, $1 \le s \le 4m$, then we may rewrite (36) as the separate statements

$$\bar{\lambda}_o dg_o + \bar{\lambda}_\gamma dg_\gamma + \bar{\lambda}_{p+i} dX^{io} + \bar{\lambda}_{p+N+m+i} d\dot{X}^{io} + \left[p_i(t^s) d\dot{X}^{is} + \tilde{p}_i(t^s) dX^{is} \right]_{s=0}^{s=1} = 0 \tag{42-1}$$

$$1 \le \gamma \le p, \qquad 1 \le i \le N.$$

$$\begin{aligned} & db^{r+\alpha}[\bar{\lambda}_{p+N+\alpha} - \tilde{p}_{N+\alpha}(t^o)] + \tilde{p}_{n+\alpha}(t^1) db^{r+m+\alpha} \\ & + db^{r+2m+\alpha}[\bar{\lambda}_{p+2N+m+\alpha} - p_{N+\alpha}(t^o)] + p_{N+\alpha}(t^1) db^{r+3m+\alpha} = 0 . \end{aligned} \tag{42-2}$$

Thus, (12) follows from (42-1).

Next since

$$I_\gamma(a_o) = I_\gamma(\bar{a}_o) \qquad \gamma = 0,1,\ldots,p \tag{43}$$

then (13) follows from (37) and the definition of $\bar{\psi}^\alpha$.

By (40) and (35) we get

$$0 = \bar{H}_{u^{k+\alpha}} = p_{N+\alpha} - \mu_\alpha(t)\psi^\alpha(t) \qquad (\alpha \text{ not summed}) \tag{44}$$

and

$$\dot{p}_{N+\alpha}(t) = -\bar{H}_{\dot{x}^{N+\alpha}} = -\tilde{p}_{N+\alpha}(t) + 2\mu_\alpha(t)\tilde{\phi}^\alpha(t) \qquad (\alpha \text{ not summed}). \tag{45}$$

Next, by differentiating (44)[1] there results

$$\dot{p}_{N+\alpha}(t) = \dot{\mu}_\alpha(t)\psi^\alpha(t) + \mu_\alpha(t)\tilde{\phi}^\alpha(t) \qquad (\alpha \text{ not summed}) . \tag{46}$$

---

[1] To be more precise, in (46) the term $\dot{\mu}_\alpha\psi^\alpha$ represents the difference $\frac{d}{dt}[\mu_\alpha\psi^\alpha] - \mu_\alpha\tilde{\phi}^\alpha$ which exists on $[t^o,t^1]$. At points t where $\psi^\alpha(t) < 0$, this difference equals $\dot{\mu}_\alpha\psi^\alpha$. At other points this difference is zero.

Equating (45) with (46) yields after solving for $\tilde{p}_{N+\alpha}(t)$

(47) $$\tilde{p}_{N+\alpha}(t) = \mu_\alpha(t)\tilde{\phi}^\alpha(t) - \dot{\mu}_\alpha(t)\psi^\alpha(t) \qquad (\alpha \text{ not summed}).$$

Differentiating (47) yields[1]

(48-1) $$\dot{\tilde{p}}_{N+\alpha}(t) = \mu_\alpha(t)\phi^\alpha(t) - \ddot{\mu}_\alpha(t)\psi^\alpha(t) \qquad (\alpha \text{ not summed})$$

while using (35) again gives

(48-2) $$\dot{\tilde{p}}_{N+\alpha}(t) = -\bar{H}_{x^{N+\alpha}} = \mu_\alpha(t)\phi^\alpha(t) \qquad (\alpha \text{ not summed}).$$

Comparing (48-1) with (48-2) yields

(49) $$\ddot{\mu}_\alpha(t)\psi^\alpha(t) = 0 \qquad (\alpha \text{ not summed})$$

proving the property ii) of the multipliers $\mu_\alpha(t)$.

Next, by (42-2) we note that

(50) $$\tilde{p}_{N+\alpha}(t^1) = p_{N+\alpha}(t^1) = 0$$

so that by (44) evaluated at t^1, we obtain

(51-1) $$\mu_\alpha(t^1)\psi^\alpha(t^1) = 0 \qquad (\alpha \text{ not summed})$$

proving property iii) of the multipliers $\mu_\alpha(t)$. Notice that by the properties of a_o then $\tilde{\phi}^\alpha(t)$ can be $\neq 0$ only if $\psi^\alpha(t) < 0$. Thus (51-1) could be rewritten

(51-2) $$\mu_\alpha(t^1)\tilde{\phi}^\alpha(t^1) = 0 \qquad (\alpha \text{ not summed}) .$$

Also by evaluating (47) at t^1 and using (50) together with (51-2), we see that

(52) $$0 = -\tilde{p}_{N+\alpha}(t^1) = -\mu_\alpha(t^1)\tilde{\phi}^\alpha(t^1) + \dot{\mu}_\alpha(t^1)\psi^\alpha(t^1) = \dot{\mu}_\alpha(t^1)\psi^\alpha(t^1), \qquad (\alpha \text{ not summed})$$

proving the property iv) of the multipliers $\mu_\alpha(t)$. The other properties of $\mu_\alpha(t)$ follow from Theorem 5.1.

---

[1] Remarks, similar to those referring to the term $\dot{\mu}_\alpha\psi^\alpha$ in (46) apply to the term $\ddot{\mu}_\alpha\psi^\alpha$ and $\dot{\mu}_\alpha\tilde{\phi}^\alpha$.

In order now to prove the statement involving (14), assume that the situation of (14) exists. Then by (47) and (44) we obtain

$$p_{N+\alpha}(\bar{t}) = \tilde{p}_{N+\alpha}(\bar{t}) = 0 \tag{53}$$

so that the situation of (38) exists. Thus, the statement involving (14) is proven.

It remains only to prove (9). In order to do this, we use (34) and (40) to get that

$$\begin{aligned} &H(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,p(t),\tilde{p}(t),\mu(t))+\left[u^{K+\alpha}-\bar{u}_o^{K+\alpha}(t)\right][p_{N+\alpha}-\mu_\alpha(t)\psi^\alpha(t)] \\ &\leq H(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),\bar{u}_o(t),p(t),\tilde{p}(t),\mu(t)) \end{aligned} \tag{54}$$

for all $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u)$ in $\bar{R}_2$.

Then by (44) we see that (54) is equivalent to (9) but for the projection of $\bar{R}_2$ into R_o . Now by reasoning similar to that used in proving Lemma 4.1, we see that given a point $p\epsilon R_2$, we can select values of $u^{K+\alpha}$ $\alpha=1,\ldots,m$ to get a corresponding point $\bar{p}$ in $\bar{R}_2$ with projection p in R_2 . Thus (9) is valid for points in R_2 . By continuity considerations, it is then sufficient to prove the denseness of R_2 in R_1 in order to completely prove (9). This denseness property is established by considerations similar to those used in the proof of Lemma 5.3 of [1] and will not be repeated. This thus completes the proof of Theorem 3.1 (except for the statement involving (11)) from Theorem 5.1.

7. ALTERNATE FORM OF THEOREM 5.1.

In the following pages, we will omit the superscript bar (-) from all terms and in addition will replace N+m, K+m, r+4m, by N, K, r respectively.

By Lemma 4.1, there are terms $\zeta^k_\beta(t)$ such that the matrix $[\zeta^k_\beta(t)]$ acts as a right inverse for the matrix $\left[\phi^\alpha_{u^k}(t)\right]$.

Next, set

(55-1) $$A^i_j(t) = f^i_{x^j}(t) - f^i_{u^k}(t)\,\zeta^k_\alpha(t)\phi^\alpha_{x^j}(t)$$

(55-2) $$\tilde{A}^i_j(t) = f^i_{\dot{x}^j}(t) - f^i_{u^k}(t)\zeta^k_\alpha(t)\phi^\alpha_{\dot{x}^j}(t)$$

(55-3) $$M_{\gamma j}(t) = L_{\gamma x^j}(t) - L_{\gamma u^k}(t)\zeta^k_\alpha(t)\phi^\alpha_{x^j}(t)$$

(55-4) $$\tilde{M}_{\gamma j}(t) = L_{\gamma \dot{x}^j}(t) - L_{\gamma u^k}(t)\zeta^k_\alpha(t)\phi^\alpha_{\dot{x}^j}(t), \quad \gamma=0,\ldots,p \; .$$

Let $S_{\gamma j}(t)$, $q_{ij}(t)$, $\tilde{q}_{ij}(t)$, $R_{\gamma j}(t)$, $Y_{ij}(t)$, and $Z_{ij}(t)$ be solutions on $[t^o,t^1]$ of

(56-1) $$\dot{S}_{\gamma j} + R_{\gamma h}A^h_j = M_{\gamma j}$$

(56-2) $$S_{\gamma j} + \dot{R}_{\gamma j} + R_{\gamma h}\tilde{A}^h_j = \tilde{M}_{\gamma j} \quad \text{with} \quad S_{\gamma j}(t^1) = 0 = R_{\gamma j}(t^1)$$
$$\gamma=0,\ldots,p$$

(56-3) $$\dot{q}_{ij} + Y_{ih}A^h_j = 0$$

(56-4) $$q_{ij} + \dot{Y}_{ij} + Y_{ih}\tilde{A}^h_j = 0 \quad \text{with} \quad q_{ij}(t^1) = 0 \quad Y_{ij}(t^1) = \delta_{ij}$$

(56-5) $$\dot{\tilde{q}}_{ij} + Z_{ih}A^h_j = 0$$

(56-6) $$\tilde{q}_{ij} + \dot{Z}_{ij} + Z_{ih}A^h_j = 0 \quad \text{with} \quad \tilde{q}_{ij}(t^1) = \delta_{ij} \quad Z_{ij}(t^1) = 0$$
$$h,i,j=1,\ldots,N \; .$$

Define

(57-1) $$F_\gamma(t,x,\dot{x},u) = L_\gamma - R_{\gamma j}f^j - (\dot{R}_{\gamma j} + S_{\gamma j})\dot{x}^j - \dot{S}_{\gamma j}x^j \qquad \gamma=0,\ldots,p$$

(57-2) $$F_{p+i}(t,x,\dot{x},u) = Y_{ij}f^j + (\dot{Y}_{ij} + q_{ij})\dot{x}^j + \dot{q}_{ij}x^j$$

(57-3) $$F_{p+N+i}(t,x,\dot{x},u) = Z_{ij}f^j + (\dot{Z}_{ij} + \tilde{q}_{ij})\dot{x}^j + \dot{\tilde{q}}_{ij}x^j \; .$$

Then by (56) and (57) we have along a_o

(58-1) $$F_{\rho x^i} - F_{\rho u^k}\zeta^k_\alpha \phi^\alpha_{x^i} = 0$$

(58-2) $$F_{\rho \dot{x}^i} - F_{\rho u^k}\zeta^k_\alpha \phi^\alpha_{\dot{x}^i} = 0 \qquad \rho=0,1,\ldots,p+2N \quad .$$

Define the $p+4N+1$ functionals on an arc a as

(59-1) $$J_\gamma(a) = \left[R_{\gamma j}(t^S)\dot{X}^{jS}(b) + S_{\gamma j}(t^S)X^{jS}(b)\right]_{S=0}^{S=1} + g_\gamma(b) + \int_{t^o}^{t^1} F_\gamma dt$$

$$\gamma=0,1,\ldots,p$$

(59-2) $$J_{p+i}(a) = -\left[Y_{ij}(t^S)\dot{X}^{jS}(b) + q_{ij}(t^S)X^{jS}(b)\right]_{S=0}^{S=1} + \int_{t^o}^{t^1} F_{p+i}dt$$

(59-3) $$J_{p+N+i}(a) = -\left[Z_{ij}(t^S)\dot{X}^{jS}(b) + q_{ij}(t^S)X^{jS}(b)\right]_{S=0}^{S=1} + \int_{t^o}^{t^1} F_{p+N+i}dt \ .$$

Next, augment the above functionals by defining $J_{p+2N+i}(a) = \dot{X}^{io}(b) - \dot{x}^i(t^o)$ and $J_{p+3N+i}(a) = X^{io}(b) - x^i(t^o)$. Now let a be an arc which satisfies (27-1). Then by (56) and (57) we have

(60-1) $$J_\gamma(a) = I_\gamma(a) - R_{\gamma j}(t^o)(\dot{X}^{jo}(b)-\dot{x}^j(t^o)) - S_{\gamma j}(t^o)(X^{jo}(b)-x^j(t^o)),$$

$$\gamma=0,1,\ldots,p$$

(60-2) $$J_{p+i}(a) = Y_{ij}(t^o)(\dot{X}^{jo}(b)-\dot{x}^j(t^o)) + q_{ij}(t^o)(X^{jo}(b)-x^j(t^o)) + \dot{x}^i(t^1) - \dot{X}^{i1}(b)$$

(60-3) $$J_{p+N+i}(a) = Z_{ij}(t^o)(\dot{X}^{jo}(b)-\dot{x}^j(t^o)) + \tilde{q}_{ij}(t^o)(X^{jo}(b)-x^j(t^o)) + x^i(t^1) - X^{i1}(b)$$

(60-4) $$J_{p+2N+i}(a) = \dot{X}^{io}(b) - \dot{x}^i(t^o)$$

(60-5) $$J_{p+3N+i}(a) = X^{io}(b) - x^i(t^o) \qquad i,j = 1,\ldots,N.$$

Notice that on arcs which satisfy (27-1) and also

(61) $$J_{p+2N+s}(a) = 0, \qquad s = 1,\dots,2N$$

then

(62) $$J_\gamma(a) = I_\gamma(a), \qquad \gamma = 0,1,\dots,p$$

$$J_{p+i}(a) = \dot{x}^i(t^1) - \dot{X}^{i1}(b), \qquad J_{p+N+i}(a) = x^i(t^1) - X^{i1}(b) .$$

Next, define A as the class of arcs satisfying (27-1) and (27-2). Then we have:

<u>Theorem 7.1</u>. The arc a_o minimizes J_o on A subject to the constraints

(63) $$J_\gamma(a) \le 0, \quad 1 \le \gamma \le p', \qquad J_\rho(a) = 0, \quad p' < \rho \le p + 4N .$$

Now, set

(64) $$\mu_{\rho\alpha}(t) = -F_{\rho u^k}(t)\zeta_\alpha^k(t)$$

(65) $$G_\rho(t,x,\dot{x},u) = F_\rho(t,x,\dot{x},u) + \mu_{\rho\alpha}(t)\phi^\alpha(t,x,\dot{x},u), \quad \rho=0,1,\dots,p+4n.$$

Then as a result of Theorem 7.1 we have:

<u>Theorem 7.2.</u> Let the arc a_o be a solution to our problem. Then there are multipliers K^τ, $\tilde{\lambda}_\rho$, $\tau=1,\dots,2m$, $\rho=0,1,\dots,p+4N$ with

$$K^\alpha \ge 0, \qquad 1 \le \alpha \le m$$

(66) $$\tilde{\lambda}_\gamma \ge 0, \quad 0 \le \gamma \le p' \quad \text{with} \quad \tilde{\lambda}_\gamma = 0, \quad \text{if} \quad J_\gamma(a_o) < 0, \quad 1 \le \gamma \le p'.$$

$$\tilde{\lambda}_{p+2N+i} = K^{m+\alpha}q_i^\alpha(t^o), \qquad \tilde{\lambda}_{p+3N+i} = K^\alpha q_i^\alpha(t^o) + K^{m+\alpha}\dot{q}_i^\alpha(t^o) .$$

Furthermore, if we set

$$\mu_\alpha(t) = \tilde{\lambda}_\rho \mu_{\rho\alpha}(t)$$

$$F = \tilde{\lambda}_\rho F_\rho$$

$$G = \tilde{\lambda}_\rho G_\rho$$

(67)
$$G(b) = \tilde{\lambda}_\gamma g_\gamma + \tilde{\lambda}_{p+2N+i}\dot{X}^{io} + \tilde{\lambda}_{p+3N+i}X^{io}$$

$$p_j(t) = \tilde{\lambda}_\gamma R_{\gamma j} - \tilde{\lambda}_{p+i}Y_{ij} - \tilde{\lambda}_{p+N+i}Z_{ij}$$

$$\tilde{p}_j(t) = \tilde{\lambda}_\gamma S_{\gamma j} - \tilde{\lambda}_{p+i}q_{ij} - \tilde{\lambda}_{p+N+i}\tilde{q}_{ij}$$

$$\alpha = 1,\ldots,m, \quad \rho = 0,1,\ldots,p + 4N, \quad \gamma = 0,1,\ldots p, \quad i,j = 1,\ldots,N$$

then we have

(68)
$$dG + \left[p_i(t^S)d\dot{X}^{iS} + \tilde{p}_i(t^S)dX^{iS} \right]_{S=0}^{S=1} = 0$$

for all db. The multipliers $\mu_\alpha(t)$ are given by the formulae

(69)
$$\mu_\alpha(t) = -F_{u^k}(t)\zeta^k_\alpha(t)$$

and are thus continuous on intervals of continuity of $u_o(t)$. Furthermore, for each α there are constants a_α, b_α such that $\mu_\alpha(t) - (a_\alpha t + b_\alpha)$ is a nonincreasing function of t on $t^o \leq t \leq t^1$.

The function G satisfies the inequality

(70)
$$G(t,x_o(t),\dot{x}_o(t),u) \geq G(t)$$

for all points $(t,x_o(t),\dot{x}_o(t),u)\varepsilon R_2$. Furthermore, along $\mathcal{a}_o$, this function satisfies

(71)
$$G_{u^k} = 0, \quad k=1,\ldots,K .$$

The multipliers $\tilde{\lambda}_\rho$, K^τ, $u(t)$, $p(t)$ are not of the form

$$\tilde{\lambda}_\gamma = 0, \quad \gamma=0,1,\ldots,p \qquad K^\tau = 0, \quad \tau = 1,\ldots,2m$$

$$p_i(\bar{t}) = 0 \qquad \tilde{p}_i(\bar{t}) = 0$$

for any $\bar{t}$ in $[t^o,t^1]$.

8. RELATIONSHIP BETWEEN THEOREM 7.2 AND THEOREM 5.1

Theorem 7.2 will be proven in the following sections. In the present section, it is shown that Therorem 5.1 follows from Theorem 7.2.

Firstly, we note that the relation (71) follows from (70) and (69) by directly analogous reasoning to that used in Corollary 1 of [1].

We now prove:

Theorem 8.1. Theorem 5.1 follows from Theorem 7.2.

Proof: Set

(73-1) $$\lambda_\gamma = \tilde{\lambda}_\gamma,\ 0 \le \gamma \le p, \quad \lambda_{p+j} = \tilde{\lambda}_{p+3N+j}, \quad \lambda_{p+N+j} = \tilde{\lambda}_{p+2N+j}, \quad 1 \le j \le N$$

and

(73-2) $$H(t,x,\dot{x},u,p,\tilde{p},\mu) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha \quad . \qquad \gamma = 0,\dots,p$$

By our definitions of p, $\tilde{p}$, G and F we see that

(74) $$H = -G - \dot{p}_j \dot{x}^j - \dot{\tilde{p}}_j x^j = -F - \mu_\alpha \phi^\alpha - \dot{p}_j \dot{x}^j - \dot{\tilde{p}}_j x^j \quad .$$

In addition, by the definition of μ_α together with (58), we obtain along a_o

(75-1) $$G_{x^i} = F_{x^i} - F_{u^k} \zeta^k_\alpha \phi^\alpha_{x^i} = 0$$

(75-2) $$G_{\dot{x}^i} = F_{\dot{x}^i} - F_{u^k} \zeta^k_\alpha \phi^\alpha_{\dot{x}^i} = 0, \qquad i = 1,\dots,N .$$

so that by (74) and (75), we have

(76-1) $$H_{x^i} = -G_{x^i} - \dot{\tilde{p}}_i = -\dot{\tilde{p}}_i$$

(76-2) $$H_{\dot{x}^i} = -G_{\dot{x}^i} - \dot{p}_i = -\dot{p}_i \,.$$

Finally (74) and (71) yield that along a_o

(77) $$H_{u^k} = -G_{u^k} = 0 \,.$$

Thus (76) and (77) and the form of H establish (35). The inequality (34) follows from the first part of (74) together with (70). Similarly (36), (37) and (38) follow from the definitions of G and λ_ρ together with (68), (66), (62) and (72). Finally, the statements concerning the properties of $p(t)$, $\tilde{p}(t)$, and $\mu_\alpha(t)$ follow from (67) and corresponding statements in Theorem 7.2

9. AUXILIARY LEMMAS.

Next define the set R_3 as that subset of R_2 with points satisfying the following condition. Given a point $(t,x,\dot{x},u)$ in R_2, let U be the collection of α indices such that $\psi^\alpha(t) = 0$, then either

$$\phi^\alpha(t,x,\dot{x},u) > 0, \quad \forall\alpha\varepsilon U, \quad \text{or} \quad \phi^\alpha(t,x,\dot{x},u) < 0, \quad \forall\alpha\varepsilon U .$$

We prove the following lemmas which are necessary for later results.

Lemma 9.1. Let

$$\ddot{\rho}_\alpha(t) \equiv \phi^\alpha(t,x_o(t),\dot{x}_o(t),u_o(t)). \tag{78}$$

Then the equations

$$\phi^\alpha(t,x,\dot{x},u) = z^\alpha, \qquad \alpha=1,\ldots,m \tag{79}$$

have solutions $u^k = U_o^k(t,x,\dot{x},z)$ on a set

$$t^o \leq t \leq t^1 , \quad |x-x_o(t)| < \delta' \quad |\dot{x}-\dot{x}_o(t)| < \delta' \quad |z-\phi(t)| < \delta' \tag{80}$$

such that $U_o^k(t,x_o(t),\dot{x}_o(t),\ddot{\rho}(t)) = u_o^k(t)$. The functions U_o^k together with their first partial derivatives with respect to x, $\dot{x}$, z are continuous on sets

$$t' \leq t \leq t'', \quad |x-x_o(t)| < \delta', \quad |\dot{x}-\dot{x}_o(t)| < \delta', \quad |z-\ddot{\rho}(t)| < \delta'$$

where $t' \leq t \leq t''$ is an interval of continuity of $u_o(t)$. Furthermore the functions

$$M_\rho(t,x,\dot{x},z) = F_\rho(t,x,\dot{x},U_o(t,x,\dot{x},z)) - F_\rho(t,x_o(t),\dot{x}_o(t),u_o(t)) \tag{81}$$

satisfy

$$M_{\rho x^i}(t,x_o(t),\dot{x}_o(t),\ddot{\rho}(t)) = 0$$

(82) $$M_{\rho \dot{x}^i}(t,x_o(t),\dot{x}_o(t),\ddot{\rho}(t)) = 0$$

$$M_{\rho z^\alpha}(t,x_o(t),\dot{x}_o(t),\ddot{\rho}(t)) = -\mu_{\rho\alpha}(t) = F_{\rho u^k}(t)\zeta^k_\alpha(t)$$

$$\rho = 0,1,\ldots,p+2N.$$

Proof: A point $(t,x_o(t),\dot{x}_o(t),u_o(t))$ in R_2 is also in $\bar{D}$. Then by using the result of Lemma 4.1, the continuity properties of the functions ϕ^α as listed below (30) and reasoning directly analogous to that of Lemma 5.1 of [1] (which will not be repeated here) the lemma is proven. The form of U^k_o is

(83) $$U^k_o(t,x,\dot{x},z) = u^k_o(t) + \zeta_\beta{}^k(t)V_\beta(t,x,\dot{x},z) . \qquad \beta = 1,\ldots,m$$

Lemma 9.2. Let $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ be a point in R_2 and set $\bar{z}_\alpha = \phi^\alpha(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$. The equations

(84) $$\phi^\alpha(t,x,\dot{x},u) = z^\alpha$$

have solutions $u^k = \bar{U}^k(t,x,\dot{x},z)$ on sets described by

(85) $$|t-\bar{t}| < \delta, \qquad |x-\bar{x}| < \delta, \qquad |\dot{x}-\bar{\dot{x}}| < \delta, \qquad |z-\bar{z}| < \delta$$

where δ is a certain positive constant depending on the point $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$. The solution satisfies $\bar{U}(\bar{t},\bar{x},\bar{\dot{x}},\bar{z}) = \bar{u}$ and together with its first derivatives with respect to $x,\dot{x},z$ is continuous on the above set.

Proof: The proof of this uses a similar procedure to that of Lemma 9.1.

Lemma 9.3: The set R_3 is dense in R_2.

Proof: The proof is entirely analogous to part of the proof used in Lemma 5.3 of [1] and will not be repeated here.

Next, according to the definition of ϕ^α we see that the following matrix relation holds

(86)
$$\left[\phi^{\alpha}_{u^k}\right] = \left[\psi^{\alpha}_{x^i}\right]\left[f^{i}_{u^k}\right]$$

so that by Lemma 4.1 at points along a_o the first matrix has rank m, and in particular at $t=t^o$, we have that there are indices $i_1,\ldots,i_m$ such that the matrix

(87)
$$\left[\psi^{\alpha}_{x^{i_\rho}}(t^o)\right] \qquad \rho,\alpha = 1,\ldots,m$$

has rank m.

We next prove Lemma 9.4.

<u>Lemma 9.4</u>:

Let $D_h = \left(d^1_h, d^2_h, \ldots, d^N_h\right)$ and $E_h = \left(e^1_h,\ldots,e^N_h\right)$, $h = 1,\ldots,\bar{h}$ be $\bar{h}$ arbitrary N dimensional vectors. The double system of equations

(88-1)
$$\psi^{\alpha}(t^o,x) - \left[\psi^{\alpha}(t^o) + \beta_h \psi^{\alpha}_{x^j}(t^o) d^j_h\right] = 0$$

(88-2)
$$\tilde{\phi}^{\alpha}(t^o,x,\dot{x}) - \left[\tilde{\phi}^{\alpha}(t^o) + \beta_h\left(\tilde{\phi}^{\alpha}_{x^j}(t^o)d^j_h + \tilde{\phi}^{\alpha}_{\dot{x}^j}(t^o)e^j_h\right)\right] = 0 \qquad \alpha = 1,\ldots,m;$$

has the solution $\qquad h = 1,\ldots,\bar{h};\ \ j = 1,\ldots,N$

(88-3)
$$x^j = X^j(\beta)$$

(88-4)
$$\dot{x}^j = \dot{X}^j(X(\beta),\beta) = \dot{X}(\beta), \qquad j=1,\ldots,N, \qquad |\beta| \le \beta'$$

where: i) $\beta=(\beta_1,\ldots,\beta_{\bar{h}})$, ii) β' is a certain positive constant and iii) (88-4) means that the solution (88-3) is used as the x part of the argument for the solution (88-4). This solution satisfies

$$\text{(a)}\ X(0) = x_o(t^o), \qquad \text{(b)}\ X\epsilon C^2, \qquad \text{(c)}\ \frac{\partial X^j(0)}{\partial\beta^h} = d^j_h$$

(89)
$$\text{(d)}\ \dot{X}(0) = \dot{x}_o(t^o), \qquad \text{(e)}\ \dot{X}\epsilon C^2 \qquad \text{(f)}\ \frac{\partial \dot{X}^j(0)}{\partial\beta^h} = e^j_h$$

$$j=1,\ldots,N, \quad h=1,\ldots,\bar{h}$$

Proof: Statements (a),(b), and (c) follow from the implicit function theorem together with (87) and the continuity properties of the functions ψ^α. Statements (d),(e), and (f) follow in an analogous way by noting that

$$\tilde{\phi}^\alpha_{\dot{x}^j}(t^o) = \psi^\alpha_{x^j}(t^o), \quad j=1,\ldots,N$$

and by using the functions of (88-3) as indicated in (88-4).

We next introduce a number of definitions. For each α, let S^α be the set introduced above (28). Let δa indicate the set of vectors

$$\delta a: \quad \begin{matrix} \delta x^j(t), & \delta\dot{x}^j(t), & \delta u^k(t), & \delta b^\sigma & t^o \leq t \leq t^1 \\ j=1,\ldots,N, & & k=1,\ldots,K, & \sigma=1,\ldots,r & \end{matrix}$$

which has $\delta x(t)$, $\delta\dot{x}(t)$ continuous, $\delta u(t)$ piecewise continuous. We refer to δa as a variation of a_o. For an arbitrary function $M(t,x,\dot{x},u,b)$ which possesses first partial derivatives with respect to x,u,b on a_o let

(90) $$\delta M(t) = M_{x^j}(t)\delta x^j(t) + M_{\dot{x}^j}(t)\delta\dot{x}^j(t) + M_{u^k}(t)\delta u^k(t) + M_{b^\sigma}(t)\delta b^\sigma, \quad t^o \leq t \leq t^1.$$

We call the right hand side of (90) the variation of M along a_o due to δa. We define an admissable variation of a_o if this variation satisfies

(91) $$\delta\dot{x}^j = \frac{d\delta x^j}{dt}, \quad \delta\ddot{x}^j = \delta f^j, \quad t^o \leq t \leq t^1, \quad \delta\psi^\alpha \leq 0 \text{ on } N^\alpha, \quad 1 \leq \alpha \leq m$$

where N^α depend upon δa and are neighborhoods of S^α. If δa satisfies the first two parts of (91) then we call δa a differentially admissible variation (henceforth D.A.V.) of a_o.

Next let W be the class of all m dimensional arcs

(92) $$w: \quad w^\alpha(t), \quad t^o \leq t \leq t^1, \quad \alpha = 1,\ldots,m$$

which is of class C^1 with piecewise continuous second derivative and in addition satisfies

(93) $$w^\alpha(t) \leq 0, \text{ on } N^\alpha \quad \alpha=1,\ldots,m.$$

Let δa be a D.A.V. of a_o with the induced variations $\delta\psi^\alpha$, $\delta\tilde{\phi}^\alpha$, $\delta\phi^\alpha$ in the functions ψ^α, $\tilde{\phi}^\alpha$, ϕ^α as introduced above. Define the arc g with components $g^\alpha(t) = \delta\psi^\alpha(t)$. According to the definition of an admissible variation and of the class W, we see that if δa is admissible, then g is in W. The converse of this is true as follows.

<u>Lemma 9.5:</u> Let w be in W. Then one can construct an admissible variation δa of a_o with δb, $\delta\dot{x}^{j_s}(t^o)$ and $\delta x^{j_s}(t^o)$, $s=1,\ldots,N-m$, $j_s \neq i_\rho$ (where i_ρ are the indices of (87)) arbitrary and such that

(94-1) $$\delta\psi^\alpha(t) = w^\alpha(t)$$

(94-2) $$\delta\tilde{\phi}^\alpha(t) = \dot{w}^\alpha(t)$$

(94-3) $$\delta\phi^\alpha(t) = \ddot{w}^\alpha(t) \qquad t^o \leq t \leq t^1$$

Proof: Consider the triple system of equations

(95-1) $$\psi^\alpha_{x^{i_\rho}} h^\rho - [w^\alpha(t^o) - \psi^\alpha_{x^{j_s}} \ell^s] = 0$$

(95-2) $$\tilde{\phi}^\alpha_{\dot{x}^{i_\rho}} c^\rho - [\dot{w}^\alpha(t^o) - \tilde{\phi}^\alpha_{\dot{x}^{j_s}} e^s - \tilde{\phi}^\alpha_{x^j} v^j] = 0, \quad \rho=1,\ldots,m, \; s=1,\ldots,N-m, \; i_\rho \neq j_s$$

(95-3) $$\phi^\alpha_{u^k}(t) d^k - [\ddot{w}^\alpha(t) - \phi^\alpha_{x^j}(t) z^j - \phi^\alpha_{\dot{x}^j} \dot{z}^j] = 0, \; t^o \leq t \leq t^1, \; \alpha=1,\ldots,m, \; j=1,\ldots,N, \; k=1,\ldots,K$$

where the arguments of (95-1) and (95-2) are $(t^o, x_o(t^o))$ and $(t^o, x_o(t^o), \dot{x}_o(t^o))$ respectively and where $h, \ell, c, e, v, d, z, \dot{z}$, are vectors of dimension m, N-m, m, N-m, N, K, N, N respectively. Then by the non-singularity of the matrix of (87); and the result of Lemma 4.1, the system (95) has the solution

(96) $$h^\rho = \bar{h}^\rho(\ell)$$
$$c^\rho = \bar{c}^\rho(e,v), \qquad \rho=1,\ldots,m$$
$$d^k = \bar{d}^k(t,z,\dot{z}) \qquad k=1,\ldots,K, \qquad t^o \leq t \leq t^1 .$$

Furthermore, $\bar{h}^\rho$, $\bar{c}^\rho$ are linear in their arguments while $\bar{d}^k$ is linear in $z, \dot{z}$ for each fixed t and continuously linear in these arguments for t on an interval of continuity of $u_o(t)$.

Next, consider the system of differential equations

(97-1)
$$\dot{z}^\tau = \frac{dz^\tau}{dt}$$
$$\frac{d\dot{z}^\tau}{dt} = f^\tau_{x^i}(t)z^i + f^\tau_{\dot{x}^i}(t)\dot{z}^i + f^\tau_{u^k}(t)\bar{d}^k(t,z,\dot{z}) \qquad \tau=1,\ldots,N \quad ,$$

with

(97-2)
$$z^{i_\rho}(t^o) = \bar{h}^\rho(\ell) = v^{i_\rho} , \qquad \dot{z}^{i_\rho}(t^o) = \bar{c}^\rho(e,v), \qquad \rho=1,\ldots,m$$
$$z^{j_s}(t^o) = \ell^s = v^{j_s} , \qquad \dot{z}^{j_s}(t^o) = e^s \qquad s=1,\ldots,N-m$$

when ℓ^s e^s are arbitrary. By the properties of $f^\tau, \bar{d}^k$, this system has the solution $z^\tau(t)$ yielding $\bar{d}(t) = \bar{d}(t,z(t),\dot{z}(t))$.

By construction, we obtain

(98-1)
$$\psi^\alpha_{x^j}(t^o)z^j(t^o) = w^\alpha(t^o)$$

(98-2)
$$\tilde{\phi}^\alpha_{x^j}(t^o)z^j(t^o) + \tilde{\phi}^\alpha_{\dot{x}^j}(t^o)\dot{z}^j(t^o) = \dot{w}^\alpha(t^o)$$

(98-3)
$$\phi^\alpha_{x^j}(t)z^j(t) + \phi^\alpha_{\dot{x}^j}(t)\dot{z}^j(t) + \phi^\alpha_{u^k}(t)\bar{d}^k(t) = \ddot{w}^\alpha(t) \qquad [t^o,t^1] .$$

Now define the variation

(99) δa: $\delta x(t)$, $\delta\dot{x}(t)$, $\delta u(t)$, δb, $t^o \leq t \leq t^1$

with $\delta x(t) = z(t)$, $\delta\dot{x}(t) = \dot{z}(t)$, $\delta u(t) = \bar{d}(t)$ and δb arbitrary. By our construction together with (98) this variation is admissible and satisfies (94) proving the lemma.

10. FIRST IMBEDDING THEOREM

Next, let δa be an admissible variation

(100) δa: $\delta x(t)$, $\delta\dot{x}(t)$, $\delta u(t)$, δb, $t^o \leq t \leq t^1$.

By Lemma 9.4 there are the functions

(100-1) $x = X(c)$, $\dot{x} = \dot{X}(c)$, $|c| \leq \beta'$

satisfying:

(101-1) $\psi^\alpha(t^o, X(c)) - [\rho^\alpha(t^o) + c\delta\psi^\alpha(t^o)] = 0$

(101-2) $\tilde{\phi}^\alpha(t^o, X(c), \dot{X}(c)) - [\dot{\rho}^\alpha(t^o) + c\delta\tilde{\phi}^\alpha(t^o)] = 0$

(101-3) $\dfrac{\partial X^j(0)}{\partial c} = \delta x^j(t^o)$, $\dfrac{\partial \dot{X}^j(0)}{\partial c} = \delta\dot{x}^j(t^o)$, $1 \leq j \leq N$

where $\delta\psi^\alpha(t)$, $\delta\tilde{\phi}^\alpha(t)$, $\delta\phi^\alpha(t)$ in (101) and (102) to follow, result from δa and $\rho^\alpha(t) \equiv \psi^\alpha(t)$. Set

(102) $\rho^\alpha(t,c) = \rho^\alpha(t) + c\delta\psi^\alpha(t)$

then

(103-1) $\dot{\rho}^\alpha(t,c) = \dot{\rho}^\alpha(t) + c\delta\tilde{\phi}^\alpha(t)$

(103-2) $\ddot{\rho}^\alpha(t,c) = \ddot{\rho}^\alpha(t) + c\delta\phi^\alpha(t)$, $\alpha=1,\ldots,m$, $t^o \leq t \leq t^1$.

Then by: i) suitably restricting c and solving the system of differential equations

(104)
$$\ddot{x} = f(t,x,\dot{x},U_o(t,x,\dot{x},\ddot{\rho}(t,c))) \qquad x(t^o,c) = X(c), \quad \dot{x}(t^o,c) = \dot{X}(c)$$

(with U_o from Lemma 9.1); ii) defining functions $b(c)$ such that

(105) $b(0) = b_0$, $b_c(0) = \delta b$

and performing analogous steps to those used in the proof of Theorem 6.1 of [1], we obtain the family of arcs

$a(c)$: $x(t,c)$, $\dot{x}(t,c)$, $u(t,c)$, $b(c)$, $t^o \leq t \leq t^1$, $0 \leq c \leq \bar{\delta}$

(for some $\bar{\delta} > 0$) in the class A introduced below (62). This family satisfies

(106) $$\psi^{\alpha}(t,x,(t,c)) = \rho^{\alpha}(t,c), \qquad 0 \le c \le \bar{\delta} \qquad t^{o} \le t \le t^{1}.$$

Proceeding as in [1], we evaluate the functionals (59) on this family, use the continuity properties of this family together with (101-3), (82), (106) and set

(107) $$k^{\rho} = \frac{\partial J_{\rho}(\mathcal{a}(0))}{\partial c} \qquad \rho = 0,1,\ldots,p+4N$$

to obtain

(108-1) $$k^{\gamma} = \left[R_{\gamma j}(t^{S})\delta\dot{\chi}^{jS} + S_{\gamma j}(t^{S})\delta\chi^{jS} \right]_{S=0}^{S=1} + \delta g_{\gamma} + \int_{t^{o}}^{t^{1}} F_{\gamma u^{k}}\zeta^{k}_{\alpha}\delta\phi^{\alpha}dt$$

and

(108-2) $$k^{p+i} = -\left[Y_{ij}(t^{S})\delta\dot{\chi}^{jS} + q_{ij}(t^{S})\delta\chi^{jS} \right]_{S=0}^{S=1} + \int_{t^{o}}^{t^{1}} F_{p+i,u^{k}}\zeta^{k}_{\alpha}\delta\phi^{\alpha}dt$$

(108-3) $$k^{p+N+i} = -\left[z_{ij}(t^{S})\delta\dot{\chi}^{jS} + \tilde{q}_{ij}(t^{S})\delta\chi^{jS} \right]_{S=0}^{S=1} + \int_{t^{o}}^{t^{1}} F_{p+N+i,u^{k}}\zeta^{k}_{\alpha}\delta\phi^{\alpha}dt$$

(108-4) $$k^{p+2N+i} = \delta\dot{\chi}^{i0} - \delta\dot{x}^{i}(t^{o})$$

(108-5) $$k^{p+3N+i} = \delta\chi^{i0} - \delta x^{i}(t^{o})$$

where e.g., $\delta\chi^{jS}$ means $\left(\frac{\partial\chi^{jS}(b(0))}{\partial b^{\sigma}}\right)\delta b^{\sigma}$. Thus we have proven:

Theorem 10.1: Let $\delta\mathcal{a}$ be an admissible variation and let $\mathcal{a}(c)$ be the one parameter family of arcs in A and related to $\mathcal{a}_{o}$ and $\delta\mathcal{a}$ as described above. Then the derivative k^{ρ} of $J_{\rho}(\mathcal{a}(c))$ with respect to c at c = 0 is given by (108)

11. SECOND IMBEDDING THEOREM

Let the point $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ be in R_3. Thus, $\bar{x} = x_o(\bar{t})$, $\bar{\dot{x}} = \dot{x}_o(\bar{t})$ and also

$$\bar{z}^\alpha = \phi^\alpha(\bar{t},\bar{x},\bar{\dot{x}},\bar{u}) \neq \ddot{\rho}^\alpha(\bar{t}) \qquad \alpha=1,\ldots,m \quad . \tag{109}$$

Let δ range on the interval $[0,\delta']$ and let $\varepsilon = \delta^2$. Next, let $S'(\varepsilon)$ be the interval $[\bar{t},\bar{t}+\varepsilon]$, while $S''(\varepsilon)$ and $S'''(\varepsilon)$ are respectively the intervals $[\bar{t}+\varepsilon,\bar{t}+\delta]$ and the null set or the interval $[\bar{t}+\varepsilon-\delta,\bar{t}]$ and $[\bar{t}+\varepsilon,\bar{t}+\delta]$ accordingly as $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ is or is not in R_o^+. Define a function $\rho^\alpha(t,\varepsilon)$ as follows

$$\rho^\alpha(t,\varepsilon) = \rho^\alpha(t) \quad \text{exterior to} \quad S'(\varepsilon)\cup S''(\varepsilon)\cup S'''(\varepsilon) \tag{110-1}$$

$$\ddot{\rho}^\alpha(t,\varepsilon) = \bar{z}^\alpha - \ddot{\rho}^\alpha(\bar{t}) + \ddot{\rho}^\alpha(t) \quad \text{on} \quad S'(\varepsilon) \tag{110-2}$$

and if $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ is in R_o^+

$$\ddot{\rho}^\alpha(t,\varepsilon)=\ddot{\rho}^\alpha(t) + \frac{\varepsilon}{(\delta-\varepsilon)^3}[(2\delta+\varepsilon)(\delta-\varepsilon)+6\delta(t-(\bar{t}+\delta))][\bar{z}^\alpha-\ddot{\rho}^\alpha(\bar{t})] \quad \text{on} \quad S'(\varepsilon) \tag{110-3}$$

while otherwise

$$\ddot{\rho}^\alpha(t,\varepsilon)=\ddot{\rho}^\alpha(t) + \frac{\varepsilon}{(\delta-\varepsilon)^3}[(2\delta+\varepsilon)(\delta-\varepsilon)+6\delta(t-\bar{t})][\bar{z}^\alpha-\ddot{\rho}^\alpha(\bar{t})] \quad \text{on} \quad S''(\varepsilon) \tag{110-4}$$

$$\ddot{\rho}^\alpha(t,\varepsilon)=\ddot{\rho}^\alpha(t) - \frac{12\varepsilon\delta}{(\delta-\varepsilon)^3}[t-\bar{t} - \frac{(\varepsilon+\delta)}{2}][\bar{z}^\alpha-\ddot{\rho}^\alpha(\bar{t})] \quad \text{on} \quad S'''(\varepsilon) \quad . \tag{110-5}$$

By this construction, we have that

$$\rho^\alpha(t,\varepsilon) = \rho^\alpha(t) + v^\alpha(t,\varepsilon) \tag{111}$$

where $v^\alpha(t,\varepsilon)$ is zero exterior to $S'(\varepsilon)US''(\varepsilon)US'''(\varepsilon)$ and with graph determined as follows:

Case: 1 $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ is in R_o^+

$$\ddot{v}^\alpha(t,\varepsilon) = \begin{cases} \bar{z}^\alpha - \ddot{\rho}^\alpha(\bar{t}) \quad \text{on } S'(\varepsilon) & (112\text{-}1) \\ \frac{\varepsilon}{(\delta-\varepsilon)^3}[(2\delta+\varepsilon)(\delta-\varepsilon) + 6\delta(t-\bar{t}-\delta)][\bar{z}^\alpha-\ddot{\rho}^\alpha(\bar{t})] \quad \text{on } S''(\varepsilon) & (112\text{-}2) \end{cases}$$

Case: 2 $(\bar{t},\bar{x},\dot{\bar{x}},\bar{u})$ is not in R_o^+

$$\ddot{v}^{\alpha}(t,\varepsilon) = \begin{cases} \dfrac{\varepsilon}{(\delta-\varepsilon)^3}[(2\delta+\varepsilon)(\delta-\varepsilon)+6\delta(t-\bar{t})][\bar{z}^{\alpha}-\ddot{\rho}^{\alpha}(\bar{t})] & \text{on } S''(\varepsilon) \quad (112\text{-}3) \\ \bar{z}^{\alpha}-\ddot{\rho}^{\alpha}(\bar{t}) & \text{on } S'(\varepsilon) \quad (112\text{-}4) \\ \dfrac{-12\varepsilon\delta}{(\delta-\varepsilon)^3}[t-\bar{t}-\dfrac{(\varepsilon+\delta)}{2}][\bar{z}^{\alpha}-\ddot{\rho}^{\alpha}(\bar{t})] & \text{on } S'''(\varepsilon)\ . \quad (112\text{-}5) \end{cases}$$

Choose δ' so small that: i) $[\bar{t}-\delta', \bar{t}+\delta']$ is contained in an interval of continuity of $u_o(t)$, ii) $\rho^{\alpha}(t,\varepsilon) \leq 0$ on $[t^o,t^1]$, and iii) $|\ddot{\rho}(t,\varepsilon)-\bar{z}| < \zeta$ on $S'(\varepsilon)$, $|\ddot{\rho}(t,\varepsilon)-\ddot{\rho}(t)| < \zeta$, otherwise (where ζ is the smaller of the constants of Lemmas 9.1 and 9.2) and define

(113) $$U(t,x,\dot{x},\varepsilon) = \begin{cases} \bar{U}(t,x,\dot{x},\ddot{\rho}(t,\varepsilon)) & \text{on } S'(\varepsilon) \\ U_o(t,x,\dot{x},\ddot{\rho}(t,\varepsilon)) & \text{otherwise} \end{cases}$$

where U_o, $\bar{U}$ are respectively the functions of Lemmas 9.1 and 9.2. By further reduction, if necessary, in the value of δ' and by using Theorem 1.1 in the appendix of [1], we can solve the system

(114-1) $$\ddot{x} = f(t,x,\dot{x},U(t,x,\dot{x},\varepsilon))$$

(114-2) $$x(t^o) = x_o(t^o)$$

(114-3) $$\dot{x}(t^o) = \dot{x}_o(t^o)\ .$$

By steps analogous to those used in the proof of Lemma 6.2 of [1], we obtain the family of arcs

$a(\varepsilon)$: $x(t,\varepsilon)$, $\dot{x}(t,\varepsilon)$, $u(t,\varepsilon)$, $b(\varepsilon)$, $t^o \leq t \leq t^1$, $0 \leq \varepsilon \leq (\delta')^2$

with $u(t,\varepsilon) = U(t,x(t,\varepsilon),\dot{x}(t,\varepsilon),\varepsilon)$, $b(\varepsilon) = b_o$, containing the arc a_o for $\varepsilon=0$ and satisfying $\dfrac{|x(t,\varepsilon)-x_o(t)|}{\varepsilon}$ and $\dfrac{|\dot{x}(t,\varepsilon)-\dot{x}_o(t)|}{\varepsilon}$ are bounded for $0\leq\varepsilon\leq(\delta')^2$ $t^o \leq t \leq t^1$.

Furthermore, we have that

(115) $$\psi^{\alpha}(t,x(t,\varepsilon)) = \rho^{\alpha}(t,\varepsilon) \leq 0, \quad 0 \leq \varepsilon \leq (\delta')^2 \qquad t^0 \leq t \leq t^1$$

so that by (115) and (114) the family $\mathcal{a}(\varepsilon)$ is in the class A introduced below (62).

Evaluating the functionals (59) on $\mathcal{a}(\varepsilon)$ yields

(116)
$$\begin{aligned} J_\gamma(\mathcal{a}(\varepsilon)) = & \left[R_{\gamma j}(t^S)\dot{X}^{jS}(b_o) + S_{\gamma j}(t^S)X^{jS}(b_o)\right]_{s=0}^{s=1} + g_\gamma(b_o) \\ & + \int_{S'(\varepsilon)} [F_\gamma(t,x(t,\varepsilon),\dot{x}(t,\varepsilon),u(t,\varepsilon)) - F_\gamma(t)]dt \\ & + \int_{S''(\varepsilon)} M_\gamma(t,x(t,\varepsilon),\dot{x}(t,\varepsilon),\ddot{\rho}(t,\varepsilon))dt. \\ & + \int_{S'''(\varepsilon)} M_\gamma(t,x(t,\varepsilon),\dot{x}(t,\varepsilon),\ddot{\rho}(t,\varepsilon))dt \\ & + \int_{t_R}^{t^1} M_\gamma(t,x(t,\varepsilon),\dot{x}(t,\varepsilon),\ddot{\rho}(t))dt + \int_{t^0}^{t^1} F_\gamma(t)dt \qquad \gamma = 0,1,\ldots,p \end{aligned}$$

with similar expressions for the other functionals and where: i) t_R means the right hand end point of $S'(\varepsilon)US''(\varepsilon)US'''(\varepsilon)$, ii) M_γ are the functions of Lemma 9.1 and iii) the facts that $x(t,\varepsilon) = x_o(t)$, $\dot{x}(t,\varepsilon) = \dot{x}_o(t)$, $u(t,\varepsilon) = u_o(t)$ to the left of $S'(\varepsilon)US''(\varepsilon)US'''(\varepsilon)$ have been used.

By using the continuity properties of the family $\mathcal{a}(\varepsilon)$ (as provided by Theorem 1.1 of the appendix of [1]) together with the remarks above (115) and proceeding in a manner analagous to that used in proving Theorem 6.2 of [1] we obtain:

<u>Theorem 11.1</u> Let $(\bar{t},\bar{x},\dot{\bar{x}},\bar{u})$ be an element in R_3 and let $\mathcal{a}(\varepsilon)$ be the family of arcs which is in the class A and related to $\mathcal{a}_o$, $(\bar{t},\bar{x},\dot{\bar{x}},\bar{u})$ and constructed as above. The derivative k^ρ of $J_\rho(\mathcal{a}(\varepsilon))$ with respect to ε at $\varepsilon=0$ is given by

(117-1) $$k^\rho = G_\rho(\bar{t},\bar{x},\dot{\bar{x}},\bar{u}) - G_\rho(\bar{t}), \qquad \rho = 0,1,\ldots,p+2N$$

(117-2) $$k^\rho = 0, \qquad \rho = p+2N+1,\ldots,p+4N \ .$$

12. PROOF OF THEOREM 7.2

At this point, we provide the following definition. We call a set K of $p+4N+1$ dimensional vectors k, a derived set for the functionals (59) at a_o on the class A, if for each finite collection of vectors $k_1,\ldots,k_s$ in K there is an s parameter family of arcs $a(E_1,\ldots,E_s)$ in A with $a(0) = a_o$ and such that the functions $\Pi_\rho(E) = J_\rho(a(E))$ have a differential at $E=0$ satisfying

(118) $$k_i^\rho = \left[\frac{\partial \Pi_\rho(E)}{\partial E_i}\right]_{E=0} \qquad \rho=0,1,\ldots,p+4N, \quad i=1,\ldots,s .$$

To apply this definition, let K be the set of all vectors of the following types:

Type I:

(119-1) $$k^\gamma = \left[R_{\gamma j}(t^S)\delta\dot{X}^{jS} + S_{\gamma j}(t^S)\delta X^{jS}\right]_{S=0}^{S=1} + \delta g_\gamma + \int_{t^o}^{t^1} F_{\gamma u^k}\zeta_\alpha^k\delta\phi^\alpha dt, \quad \gamma=0,1,\ldots,p$$

(119-2) $$k^{p+i} = -\left[Y_{ij}(t^S)\delta\dot{X}^{jS} + q_{ij}(t^S)\delta X^{jS}\right]_{S=0}^{S=1} + \int_{t^o}^{t^1} F_{p+i,u^k}\zeta_\alpha^k\delta\phi^\alpha dt$$

(119-3) $$k^{p+N+i} = -\left[z_{ij}(t^S)\delta\dot{X}^{jS} + \tilde{q}_{ij}(t^S)\delta X^{jS}\right]_{S=0}^{S=1} + \int_{t^o}^{t^1} F_{p+N+i,u^k}\zeta_\alpha^k\delta\phi^\alpha dt$$

(119-4) $$k^{p+2N+i} = \delta\dot{X}^{i0} - \delta\dot{x}^i(t^o)$$

(119-5) $$k^{p+3N+i} = \delta X^{i0} - \delta x^i(t^o), \qquad i=1,\ldots,N$$

where the terms $\delta\dot{X}^{jS}$, δX^{jS}, δg_γ, $\delta\phi^\alpha$, $\delta\dot{X}^{i0}$ $\delta\dot{x}^i(t^o)$, δX^{i0}, $\delta x^i(t^o)$ are related to an admissible variation δa as described in Theorem 10.1.

Type 2:

(119-6) $$k^\rho = G_\rho(\bar{t},\bar{x},\dot{\bar{x}},\bar{u}) - G_\rho(\bar{t}) \qquad \rho=0,1,\ldots,p+2N$$

(119-7) $$k^\rho = 0, \qquad \rho=p+2N+1,\ldots,p+4N$$

for $\bar{t},\bar{x},\dot{\bar{x}},\bar{u}$ in R_3. It will be proven in Section 13 that the set K is a derived set for J_ρ at a_o on A. For the moment, let us accept the truth of this statement. By Theorem 7.1 together with the generalized multiplier rule of Theorem 3.1 in Chapter 4 of [2], there are multipliers $\tilde{\lambda}_\rho$, $0 \le \rho \le p+4N$ not all zero, which satisfy

(120-1) $$\tilde{\lambda}_\gamma \ge 0, \quad 0 \le \gamma \le p' \quad \text{and} \quad \tilde{\lambda}_\gamma = 0 \ \text{if} \ J_\gamma(a_o) < 0, \quad 1 \le \gamma \le p'$$

(120-2) $$\tilde{\lambda}_\rho k^\rho \ge 0, \qquad 0 \le \rho \le p+4N$$

for all vectors in K.

Next, let $C = (c^1,\ldots,c^N)$ and $D = (d^1,\ldots,d^N)$ be vectors which satisfy

(121) $$\psi^\alpha_{x^j}(t^o)c^j \le 0, \qquad \tilde{\phi}^\alpha_{x^j}(t^o)c^j + \tilde{\phi}^\alpha_{\dot{x}^j}(t^o)d^j = 0 .$$

Select an m dimensional arc w which satisfies

(122) $$w^\alpha(t^o) = \psi^\alpha_{x^j}(t^o)c^j, \quad \dot{w}^\alpha(t^o) = 0, \quad \ddot{w}^\alpha(t) \equiv 0, \quad t^o \le t \le t^1 .$$

Then w is in the class W introduced in the remarks involving (92). By Lemma 9.5, there is an admissible variation

(123) $$\delta a: \quad \delta x(t), \quad \delta\dot{x}(t), \quad \delta u(t), \quad \delta b, \quad t^o \le t \le t^1$$

with δb arbitrary and with

(124) $$\delta x^{j_s}(t^o) = c^{j_s}, \quad \delta\dot{x}^{j_s}(t^o) = d^{j_s}, \quad j_s \ne i_\rho \qquad s=1,\ldots,N-m$$

(where i_ρ are the indices of (87)) such that

(125-1) $$\psi^{\alpha}_{x^j}(t)\delta x^j(t) = \delta\psi^{\alpha}(t) = w^{\alpha}(t)$$

(125-2) $$\tilde{\phi}^{\alpha}_{x^j}(t)\delta x^j(t) + \tilde{\phi}^{\alpha}_{\dot{x}^j}(t)\delta\dot{x}^j(t) = \delta\tilde{\phi}^{\alpha}(t) = \dot{w}^{\alpha}(t)$$

(125-3) $$\delta\phi^{\alpha}(t) = \ddot{w}^{\alpha}(t) = 0 , \qquad \underline{1<\alpha<m}, \ \underline{t^o<t<t^1} .$$

Furthermore, by (124), (121), (122), the definition of $\tilde{\phi}^{\alpha}$, the non-singularity of the matrix of (87) and (125) evaluated at t^o, then we have $\delta x^{i_\rho}(t^o) = c^{i_\rho}$, $\delta\dot{x}^{i_\rho}(t^o) = d^{i_\rho}$, $1 \leq \rho \leq m$ so that

(126) $$\delta x^j(t^o) = c^j, \qquad \delta\dot{x}^j(t^o) = d^j, \qquad 1 \leq j \leq N .$$

Then by (120-2), (125-3), (126) and the definition of k in (119-1) through (119-5), we see that the inequality

$$\tilde{\lambda}_\gamma\left[\delta g_\gamma + R_{\gamma j}(t^S)\delta\dot{X}^{jS} + S_{\gamma j}(t^S)\delta X^{jS}\right]_{S=0}^{S=1} - \tilde{\lambda}_{p+i}\left[Y_{ij}(t^S)\delta\dot{X}^{jS} + q_{ij}(t^S)\delta X^{jS}\right]_{S=0}^{S=1}$$

(127) $$-\tilde{\lambda}_{p+N+i}\left[z_{ij}(t^S)\delta\dot{X}^{jS} + \tilde{q}_{ij}(t^S)\delta X^{jS}\right]_{S=0}^{S=1} + \tilde{\lambda}_{p+2N+i}\delta\dot{X}^{i0} + \tilde{\lambda}_{p+3N+i}\,\delta X^{i0}$$

$$\geq \tilde{\lambda}_{p+2N+i}d^i + \tilde{\lambda}_{p+3N+i}c^i$$

holds for all δb and for all C,D vectors satisfying (121). Since the left hand side of (127) is linear in δb and holds for all δb we must have

(128) $$-\tilde{\lambda}_{p+2N+i}d^i - \tilde{\lambda}_{p+3N+i}c^i \geq 0$$

for all vectors D,C satisfying (121). Thus the expression (128) is minimized at D = C = 0 subject to the constraints (121). Then by the multiplier rule for linear functions of N variables, there are multipliers $K^1,\ldots,K^{2m}$ with $K^{\alpha} \geq 0$, $1 \leq \alpha \leq m$ such that

$$\tilde{\lambda}_{p+2N+i} = K^{m+\alpha} q_i^\alpha(t^o)$$

(129)

$$\tilde{\lambda}_{p+3N+i} = K^\alpha q_i^\alpha(t^o) + K^{m+\alpha} \dot{q}_i^\alpha(t^o)$$

where the definition of $\tilde{\phi}^\alpha$ has been used in (129). By these remarks, together with (120-1) we see that (66) is proven.

Now let δb be an arbitrary vector. Consider the variation δa with $\delta x(t) \equiv 0$, $\delta \dot{x}(t) \equiv 0$, $\delta u(t) \equiv 0$ and δb as the given vector. Then δa is an admissible variation. By (120-2) and the definition of K we have that the left hand side of (127) is non negative. However, since δb was arbitrary, then the left hand side of (127) must vanish. Then by the definition of $p(t)$, $\tilde{p}(t)$, we see that (68) is proven.

Next, let $(t,x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u})\varepsilon R_3$. According to (120-2), the definition of K in (119-6) and (119-7) and the definition of the function G, we have

(130) $$G(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u}) - G(\bar{t}) \geq 0, \qquad (\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u})\varepsilon R_3 .$$

Thus by Lemma 9.3, together with a continuity argument, we see that (130) holds also for $(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u})\varepsilon R_2$. Thus, (70) is proven.

Next, define the collection W^- of m dimensional arcs

(131) $$w: \quad w^\alpha(t), \qquad t^o \leq t \leq t^1 , \qquad 1 \leq \alpha \leq m$$

which is of class C^1 with piecewise continuous second derivative and in addition satisfies

(132) $$w^\alpha(t^S) = 0, \quad \dot{w}^\alpha(t^S) = 0, \quad S=0,1, \quad w^\alpha(t) \leq 0, \quad t^o \leq t \leq t^1 , \quad 1 \leq \alpha \leq m.$$

Then $W^- \subset W$. Now select β with $1 \leq \beta \leq m$ and let $w \varepsilon W^-$ be an arc with $w^\alpha(t) \equiv 0$, $1 \leq \alpha \leq m$, $\alpha \neq \beta$. According to Lemma 9.5, we can find an admissible variation δa with

(133-1) $\delta b = 0, \quad \delta x^{j_s}(t^o) = 0, \quad \delta \dot{x}^{j_s}(t^o) = 0, \quad s=1,\ldots,N-m, \quad j_s \neq i_\rho$

and

(133-2) $\delta\psi^\alpha(t) = w^\alpha(t), \quad \delta\tilde{\phi}^\alpha(t) = \dot{w}^\alpha(t), \quad \delta\phi^\alpha(t) = \ddot{w}^\alpha(t), \quad 1 \leq \alpha \leq m, \quad t^o \leq t \leq t^1$

[where i_ρ arc the indices of (87)]. We have

$$0 = w^\alpha(t^o) = \delta\psi^\alpha(t^o) = \psi^\alpha_{x^j}(t^o)\delta x^j(t^o)$$

(134)

$$0 = \dot{w}^\alpha(t^o) = \delta\tilde{\phi}^\alpha(t^o) = \tilde{\phi}^\alpha_{x^j}(t^o)\delta x^j(t^o) + \tilde{\phi}^\alpha_{\dot{x}^j}(t^o)\delta\dot{x}^j(t^o), \quad 1 \leq \alpha \leq m$$

and

(135) $$0 \equiv \ddot{w}^\alpha(t) = \delta\phi^\alpha(t) \qquad t^o \leq t \leq t^1, \quad \alpha \neq \beta .$$

Then by (134), and the non-singularity of the matrix of (87) together with (133-1) we see that $\delta x^{i_\rho}(t^o) = \delta \dot{x}^{i_\rho}(t^o) = 0, \quad \rho = 1,\ldots,m$. Thus,

(136) $$\delta x^j(t^o) = \delta\dot{x}^j(t^o) = 0, \quad j = 1,\ldots,N$$

By (120-2), (133), (135), (136), the definition of the class K, the definition of $\mu_\beta(t)$ and the selection of the arc w , we have that

(137) $$\int_{t^o}^{t^1} - \mu_\beta \ddot{w}^\beta dt \geq 0$$

where β has the value selected above. By the definition of the class W^- together with Lemma 2 in the appendix [1], then $\mu_\beta(t) - (a_\beta t + b_\beta)$ is a non-increasing function for certain constants a_β, b_β . This establishes the properties of $\mu_\beta(t)$ in Theorem 7.2.

In order to prove the statement involving (72) concerning the trivial form of the multipliers, we see that according to the definition of these multipliers and the selection of the initial points of the functions $R_{\gamma j}(t)$, $Y_{ij}(t)$, $Z_{ij}(t)$, $S_{\gamma j}(t)$, $q_{ij}(t)$, $\tilde{q}_{ij}(t)$ in (56), then

(138) $$p_i(t^1) = -\tilde{\lambda}_{p+i} , \quad \tilde{p}_i(t^1) = -\tilde{\lambda}_{p+N+i} .$$

Next, let $\tilde{\lambda}_\gamma = 0$, $\gamma = 0,1,\ldots,p$. By the definition of $p(t)$, $\tilde{p}(t)$, $S_{\gamma j}(t)$, $q_{ij}(t)$, $\tilde{q}_{ij}(t)$, $R_{\gamma j}(t)$, $Y_{ij}(t)$, $Z_{ij}(t)$, we see that for the present case the multipliers $p(t)$, $\tilde{p}(t)$ satisfy

$$\begin{aligned}
&\dot{p}_i + p_j \left[f^j_{\dot{x}^i} - f^j_{u^k} \zeta^k_\alpha \phi^\alpha_{\dot{x}^i} \right] + \tilde{p}_i = 0 \\
(139)\qquad &\dot{\tilde{p}}_i + p_j \left[f^j_{x^i} - f^j_{u^k} \zeta^k_\alpha \phi^\alpha_{x^i} \right] = 0 \qquad i,j=1,\ldots,N,\ k=1,\ldots,K,\ \alpha=1,\ldots,m
\end{aligned}$$

where the unlisted arguments are t, $t^o \le t \le t^1$. Thus, if $p_i(\bar{t}) = \tilde{p}_i(\bar{t}) = 0$ then

$$(140)\qquad p_i(t) \equiv \tilde{p}_i(t) \equiv 0, \qquad t^o \le t \le t^1, \qquad i=1,\ldots,N\,.$$

Now assume that the situation of (72) exists. Then by (140), (evaluated at $t=t^1$), (138) and (129) we see that all of the terms $\tilde{\lambda}_\rho$, $\rho=0,1,\ldots,p+4N$ vanish, contradicting the statement above (120). Thus, the statement involving (72) and also Theorem 7.2 are proven.

13. FINAL IMBEDDING LEMMA

Lemma 13.1 The set K is a derived set for the functionals J_ρ of (59) at a_o on the class A.

Proof: Let $h_1,\ldots,h_{r_1}$ be r_1 vectors of the form

$$\begin{aligned}
&h^\gamma_r = \left[R_{\gamma j}(t^S)\delta_r \dot{X}^{jS} + S_{\gamma j}(t^S)\delta_r X^{jS} \right]^{S=1}_{S=0} + \delta_r g_\gamma + \int_{t^o}^{t^1} F_{\gamma u^k} \zeta^k_\alpha \delta_r \phi^\alpha dt, \quad \gamma=0,1,\ldots,p \\
&h^{p+i}_r = -\left[Y_{ij}(t^S)\delta_r \dot{X}^{jS} + q_{ij}(t^S)\delta_r X^{jS} \right]^{S=1}_{S=0} + \int_{t^o}^{t^1} F_{p+i,u^k} \zeta^k_\alpha \delta_r \phi^\alpha dt \\
(141)\qquad &h^{p+N+i}_r = -\left[Z_{ij}(t^S)\delta_r \dot{X}^{jS} + \tilde{q}_{ij}(t^S)\delta_r X^{jS} \right]^{S=1}_{S=0} + \int_{t^o}^{t^1} F_{p+N+i,u^k} \zeta^k_\alpha \delta_r \phi^\alpha dt \\
&h^{p+2N+i}_r = \delta_r \dot{X}^{io} - \delta_r \dot{x}^i(t^o), \\
&h^{p+3N+i}_r = \delta_r X^{io} - \delta_r x^i(t^o), \quad i=1,\ldots,N \qquad r=1,\ldots,r_1
\end{aligned}$$

where for example $\delta_r\phi^\alpha$ denotes the variation in the function ϕ^α due to the admissible variation

(142) $\delta_r a$: $\quad \delta_r x^i(t), \quad \delta_r \dot{x}^i(t), \quad \delta_r u^k(t), \quad \delta_r b^\sigma \quad t^o \le t \le t^1, \quad 1 \le r \le r_1.$

Let $k_1,\ldots,k_{r_2}$ be r_2 vectors of the form

(143)
$$k_s^\rho = G_\rho(t_s,x_s,\dot{x}_s,u_s) - G_\rho(t_s) \qquad \rho = 0,1,\ldots,p+2N$$
$$k_s^\rho = 0 \qquad \rho = p+2N+1,\ldots,p+4N$$

where $(t_s,x_s,\dot{x}_s,u_s)$, $s=1,\ldots,r_2$, are points in R_3 with $t_i < t_j$ if $i < j$.

Using the above r_1 admissible variations, and setting $\rho^\alpha(t) = \psi^\alpha(t)$ define

(144-1) $$\rho^\alpha(t,c) = \rho^\alpha(t) + c_r\delta_r\psi^\alpha(t).$$

By differentiating and using the admissibility of $\delta_r a$, we have

(144-2) $$\dot{\rho}^\alpha(t,c) = \dot{\rho}^\alpha(t) + c_r\delta_r\tilde{\phi}^\alpha(t)$$

(144-3) $$\ddot{\rho}^\alpha(t,c) = \ddot{\rho}^\alpha(t) + c_r\delta_r\phi^\alpha(t), \quad \alpha=1,\ldots,m, \quad t^o\le t\le t^1, \quad r=1,\ldots,r_1.$$

According to the properties of $\delta_r a$, we can select δ' so small that

(145) $$\rho^\alpha(t,c) \le 0, \quad \alpha=1,\ldots,m \quad t^o \le t \le t^1 \quad 0 \le c_r \le \delta' .$$

We further reduce the value of δ' if necessary according to the following restrictions: i) $t_i + r_2\delta' < t_j - r_2\delta'$ if $t_i < t_j$; ii) $[t_s - r_2\delta', t_s + r_2\delta']$ is interior to an interval of continuity of $u_o(t)$, $s=1,\ldots,r_2$; iii) if t_s is not in S^α, then $[t_s - r_2\delta', t_s + r_2\delta']$ is disjoint from S^α .

Next, let δ_s , $s=1,\ldots,r_2$, range on the interval $[0,\delta']$ and let $\varepsilon_s = \delta_s^2$. Set

(146) $$z_s^\alpha = \phi^\alpha(t_s,x_s,\dot{x}_s,u_s) \qquad s=1,\ldots,r_2 \qquad \alpha=1,\ldots,m$$

and $$T_1 = t_1, \quad T_s = t_s + \varepsilon_1 + \varepsilon_2 + ,\ldots, + \varepsilon_{s-1} \qquad s > 1$$

and let $S_s'(\varepsilon)$ be the interval $[T_s, T_s + \varepsilon_s]$ while $S_s''(\varepsilon)$ and $S_s'''(\varepsilon)$ are respectively the interval $[T_s + \varepsilon_s, T_s + \delta_s]$ and the null set or the intervals $[T_s + \varepsilon_s - \delta_s, T_s]$ and $[T_s + \varepsilon_s, T_s + \delta_s]$ accordingly as the point $(t_s, x_s, \dot{x}_s, u_s)$ is or is not in $R_o^+ \cap R_3$.

Define functions $v_s^\alpha(t,\varepsilon)$, $s=1,\ldots,r_2$, $\alpha=1,\ldots,m$ as follows:

(147-1) $v_s^\alpha(t,\varepsilon) = 0$ exterior to $S_s'(\varepsilon) \cup S_s''(\varepsilon) \cup S_s'''(\varepsilon)$

(147-2) $\ddot{v}_s^\alpha(t,\varepsilon) = z_s^\alpha - \ddot{\rho}^\alpha(t_s)$ on $S_s'(\varepsilon)$

and if $(t_s, x_s, \dot{x}_s, u_s)$ is in R_o^+ then

(147-3) $$\ddot{v}_s^\alpha(t,\varepsilon) = \frac{\varepsilon_s}{(\delta_s-\varepsilon_s)^3}[(2\delta_s+\varepsilon_s)(\delta_s-\varepsilon_s) + 6\delta_s(t-T_s-\delta_s)][z_s^\alpha - \ddot{\rho}^\alpha(t_s)] \text{ on } S_s''(\varepsilon)$$

while otherwise

(147-4) $$\ddot{v}_s^\alpha(t,\varepsilon)= \begin{cases} \dfrac{\varepsilon_s}{(\delta_s-\varepsilon_s)^3}[(2\delta_s+\varepsilon_s)(\delta_s-\varepsilon_s) + 6\delta_s(t-T_s)][z_s^\alpha - \ddot{\rho}^\alpha(t_s)] & \text{on } S_s''(\varepsilon) \\ \dfrac{-12\varepsilon_s\delta_s}{(\delta_s-\varepsilon_s)^3}\left[t-T_s - \dfrac{(\varepsilon_s+\delta_s)}{2}\right][z_s^\alpha - \ddot{\rho}^\alpha(t_s)] & \text{on } S_s'''(\varepsilon) \end{cases}$$

Next, define the functions

(148) $$\rho^\alpha(t,c,\varepsilon) = \rho^\alpha(t,c) + \sum_{s=1}^{r_2} v_s^\alpha(t,\varepsilon) .$$

By using (145), (147), (148) and reducing δ' further if necessary, we can guarantee that

(149) $\rho^\alpha(t,c,\varepsilon) \leq 0$, $\alpha=1,\ldots,m$, $t^o \leq t \leq t^1$, $0 \leq c_r, \delta_s \leq \delta'$ $\varepsilon_s = \pi_s^2$, $r=1,\ldots,r_1, s=1,\ldots,r_2$

and that

(150) $|\ddot{\rho}(t,c,\varepsilon) - z_s| < \bar{\xi}$ on $S_s'(\varepsilon)$, $s=1,\ldots,r_2$

(151) $|\ddot{\rho}(t,c,\varepsilon) - \ddot{\rho}(t)| < \bar{\xi}$ otherwise on $[t^o, t^1]$ $\alpha=1,\ldots,m$, $0 \leq c_r, \delta_s \leq \delta'$, $\varepsilon_s = \delta_s^2$,

$s=1,\ldots,r_2$, $r=1,\ldots,r_1$

where $\bar{\xi}$ is the smaller of the constants of Lemmas 9.1 and 9.2.

Define

$$(152)\qquad U(t,x,\dot{x},c,\varepsilon) = \begin{cases} U_s(t,x,\dot{x},\ddot{\rho}(t,c,\varepsilon)) & \text{on } S_s'(\varepsilon) \\ U_o(t,x,\dot{x},\ddot{\rho}(t,c,\varepsilon)) & \text{otherwise} \end{cases}$$

where U_o, U_s are respectively the functions of Lemmas 9.1 and 9.2.

By further reduction if necessary in the value of δ' and by using Lemma 9.4, we know that there are functions $X(c)$, $\dot{X}(c)$ such that

$$(153\text{-}1)\qquad \psi^\alpha(t^o,X(c)) = [\rho^\alpha(t^o) + c_r\delta_r\psi^\alpha(t^o)] = \rho^\alpha(t^o,c,\varepsilon)$$

$$(153\text{-}2)\qquad \tilde{\phi}^\alpha(t^o,X(c),\dot{X}(c)) = [\dot{\rho}^\alpha(t^o) + c_r\delta_r\tilde{\phi}^\alpha(t^o)] = \dot{\rho}^\alpha(t^o,c,\varepsilon)$$

$$\alpha=1,\dots,m,\quad 0 \le c_r,\delta_s \le \delta',\quad \varepsilon_s = \delta_s^2,\quad s=1,\dots,r_2,\quad r=1,\dots,r_1.$$

Finally by further reducing the value of δ' if necessary and by using Theorem 1.1 in the appendix of [1], we can solve the initial value system

$$(154\text{-}1)\qquad \ddot{x} = f(t,x,\dot{x},U(t,x,\dot{x},c,\varepsilon))$$

$$(154\text{-}2)\qquad x(t^o,c,\varepsilon) = X(c)$$

$$(154\text{-}3)\qquad \dot{x}(t^o,c,\varepsilon) = \dot{X}(c)$$

within the $t,x,\dot{x},c,\varepsilon$ set about $t,x_o(t),\dot{x}_o(t),0,0$ specified by

$$|(t,x)-(t,x_o(t))| < \min(\delta',\theta),\ |(t,\dot{x})-(t,\dot{x}_o(t))| < \delta',\quad t^o \le t \le t^1$$

$$0 \le c_r,\varepsilon_s^{1/2} \le \delta' \qquad s=1,\dots,r_2, \qquad r=1,\dots,r_1$$

where θ is the constant introduced below (30). Using the solution to this system and defining the functions $\Theta^\sigma(c)$, $\sigma=1,\dots,r$, of class C^2 which also satisfy

$$(155)\qquad \Theta^\sigma(0) = b_o^\sigma,\quad \Theta^\sigma_{c_r}(0) = \delta_r b^\sigma,\qquad r = 1,\dots,r_1$$

we obtain the family of arcs

$$a(c,\varepsilon):\quad x(t,c,\varepsilon),\quad \dot{x}(t,c,\varepsilon),\quad u(t,c,\varepsilon),\ b(c,\varepsilon),\quad 0 \le c_r \le \delta',\quad 0 \le \varepsilon_s \le (\delta')^2$$

with $\quad u(t,c,\varepsilon) = U(t,x(t,c,\varepsilon),\dot{x}(t,c,\varepsilon),c,\varepsilon)\qquad b(c,\varepsilon) = \Theta(c)$

containing a_o for $c=\varepsilon=0$. Furthermore, by our construction, we have that

$$(156\text{-}1)\qquad \psi^\alpha(t^o,x(t^o,c,\varepsilon)) = \rho^\alpha(t^o,c,\varepsilon)$$

$$(156\text{-}2)\qquad \tilde{\phi}^\alpha(t^o,x(t^o,c,\varepsilon),\dot{x}(t^o,c,\varepsilon)) = \dot{\rho}^\alpha(t^o,c,\varepsilon)$$

$$(156\text{-}3)\qquad \phi^\alpha(t,x(t,c,\varepsilon),\dot{x}(t,c,\varepsilon),u(t,c,\varepsilon)) = \ddot{\rho}^\alpha(t,c,\varepsilon)$$

so that

(157) $\psi^{\alpha}(t,x(t,c,\varepsilon)) = \rho^{\alpha}(t,c,\varepsilon) \leq 0, \quad \alpha=1,\ldots,m, \quad t^{o}\leq t\leq t^{1}, \quad 0\leq c_r\leq\delta', \quad 0\leq\varepsilon_s\leq(\delta')^2$

$r=1,\ldots,r_1, \quad s=1,\ldots,r_2$

the last inequality following from (149).

By (157) and (154) the family $a(c,\varepsilon)$ is in the class A . With J_ρ as the functionals of (59), then by using the properties of this family, the functions $J_\rho(a(c,\varepsilon))$, $0 \leq \rho \leq p+4N$ are of class C^1 on the set $0 \leq c_r \leq \delta'$, $0 \leq \varepsilon_s \leq (\delta')^2$. For an arbitrary r , $1 \leq r \leq r_1$, set $c_i = 0$, $i \neq r$ and $\varepsilon = 0$. The family $a(c,\varepsilon)$ becomes the family $a(c_r)$ and by Lemma 10.1 we have

(158) $$h_r^{\rho} = \left.\frac{\partial J_\rho(a(c,\varepsilon))}{\partial c_r}\right|_{0=c=\varepsilon} \qquad \rho=0,1,\ldots,p+4N$$

where h_r^{ρ} are the terms of (141). In a similar manner but by using Lemma 11.1 and with k_s^{ρ} as the terms of (143) we find that

(159) $$k_s^{\rho} = \left.\frac{\partial J_\rho(a(c,\varepsilon))}{\partial \varepsilon_s}\right|_{0=c=\varepsilon} \qquad \rho=0,1,\ldots,p+4N \quad .$$

Since r,s were arbitrary then (158), (159) hold respectively for $r=1,\ldots,r_1$ and $s=1,\ldots,r_2$ and Lemma 13.1 is proven.

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