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SOME RESULTS ON SYMMETRIC STABLE
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For symmetric stable random variables, a condition equivalent to linear regression is obtained and moments of certain monomials are computed. For stochastic processes of random variables with p-th moments, $p>1$, a function space approach is developed and applied to the system identification problem. Conditions for measurable modifications and for integrable or absolutely continuous sample paths are given.

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MILLER, GRADY. Some Results on Symmetric Stable Distributions and Processes. (Under the direction of STAMATIS CAMBANIS.)

This work investigates properties of symmetric stable distributions and stochastic processes. A necessary and sufficient condition is presented for a regression involving symmetric stable random variables to be linear. We introduce the notion of n-fold dependence for symmetric stable random variables and under this condition characterize all monomials in such random variables for which moments exist. A function space approach to symmetric stable stochastic processes is developed and applied to the problem of system identification. Necessary and sufficient conditions are given for the existence of measurable modifications of such processes and for the almost sure integrability and absolute continuity of sample paths.

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INTRODUCTION AND SUMMARY

It is well known that the stable laws arise naturally as the limiting distributions of normed sums of independent identically distributed random variables (or vectors), and this result has been extended to Banach space valued random variables ([Kumar and Mandrekar 1972]) as well as to random variables with values in certain topological vector spaces ([Rajput 1975]). The limiting distributions that have infinite variance can be typed by a parameter α, $0 < \alpha < 2$, and only absolute moments of order strictly less than α are finite, whereas in the finite variance case ($\alpha = 2$) the limiting distribution is always normal and all moments exist. Even though stable laws on the real line are absolutely continuous, closed form expressions for their density functions are known in only a few cases. In contrast, the characteristic functions of stable measures on finite or infinite-dimensional spaces are quite simple ([Kuelbs 1973]) and therefore constitute a primary tool in our research.

Many easily formulated problems involving stable distributions on Euclidean n-space remain unsolved, and the study of multivariate stable distributions is continually being renewed (as in [Hosoya 1976]). Since the normal distributions have been extensively investigated, our efforts are directed mainly toward stable distributions that have $0 < \alpha < 2$ and (for simplicity) that are symmetric. If the symmetric stable random variables are defined on a probability space (Ω,F,P), then they belong to the complete metric space $L_p(\Omega,F,P)$ where $0 < p < \alpha < 2$ in the infinite variance case and to the Hilbert space $L_2(\Omega,F,P)$ in the normal case. Our basic approach is to extend results known for normal distributions to symmetric stable distributions, and many of the difficulties which arise are due to the more complicated structure of L_p spaces and (for $p \geq 1$) the lack of a simple representation for the dual elements such as exists for an inner product space. Consequently our development often originates with p-th order random variables and then is narrowed to include only symmetric stable random variables. One of the most notable advantages of specializing to the stable case is that here the notion of independence provides a satisfactory and useful analogy to the concept of orthogonality for random variables with finite second moments.

Summary

The first chapter begins with basic definitions and characterizations on infinite-dimensional spaces, but deals mostly with problems in an n-dimensional Euclidean space setting. The principal results give necessary and sufficient conditions for independence of random vectors, linear regression, and finite absolute moments of monomials.

In the second chapter we study the structure of the linear space of a symmetric stable process ($\alpha > 1$) and use this structure to represent elements in the linear space (under certain conditions) as stochastic integrals of elements of a function space Λ_α (or λ_α). Conditions for independence, best linear approximations, and expressions for dual elements are obtained in terms of these representations. These results are applied to the problem of linear system identification when the input is a symmetric stable process.

The final chapter contains necessary and sufficient conditions for the measurability of p-th order or symmetric stable processes and for the integrability or absolute continuity of sample paths of symmetric stable processes, as well as sufficient conditions for absolute continuity of sample paths of p-th order processes.

I. INDEPENDENCE, REGRESSION, AND MOMENTS

Multivariate stable distributions and their characteristic functions have been known and studied for many years, but the subject continues to attract the attention of researchers (*e.g.*, [Press 1972, Paulauskas 1976]). Still unanswered are some quite natural questions surrounding such topics as the properties of conditional distributions or the effects of nonlinear transformations on stable distributions. In the first two sections of this chapter we discuss definitions and results that will be of use to us later, and in the latter two we present some developments on regression analysis and moments for jointly symmetric stable random variables.

1. Fundamental definitions and characterizations.

In this section we define a stable measure on a Banach space and state some characterizations of the characteristic function (c.f.) of a symmetric stable measure on a Hilbert space. This material is well-known for stable measures on n-dimensional Euclidean space R^n, but has only recently been extended to infinite-dimensional spaces. Even though we shall rarely consider stable measures on spaces other than R^n, the additional generality provided here will occasionally be needed.

Let E be a real separable Banach space and for every $a \in R$ define the continuous map $T_a: E \to E$ by $T_a(x) = ax$. A probability measure μ on the Borel subsets of E is said to be *stable* if for any $a > 0$ and $b > 0$ there exists $c > 0$ and $x \in E$ such that

$$(\mu T_a^{-1}) \circledast (\mu T_b^{-1}) = (\mu T_c^{-1}) \circledast \delta_x ,$$

where δ_x is the Borel probability measure satisfying $\delta_x(\{x\}) = 1$ and $\circledast$ denotes the convolution operation.

Let E^* be the dual space of E and C be the space of complex numbers. The c.f. of a Borel probability measure μ on E is a map $\hat{\mu}: E^* \to C$ defined by

$$\hat{\mu}(w) = \int_E e^{iw(x)} \mu(dx)$$

for all $w \in E^*$. It has been shown ([Itô and Nisio 1968]) that a Borel probability measure on a real separable Banach space is uniquely determined by its c.f. The following characterization of a stable measure is given by [Kumar and Mandrekar 1972] and [Rajput 1975].

1.1.1 *A Borel probability measure* μ *on a real separable Banach space* E *is stable if and only if for every integer* $n \geq 1$ *there exists* $x_n \in E$ *such that*

$$[\hat{\mu}(w)]^n = \hat{\mu}(n^{1/\alpha} w) e^{iw(x_n)}$$

for every $w \in E^*$, *where* α *is uniquely determined by* μ *and satisfies* $0 < \alpha \leq 2$.

It is customary to say that the measure μ is α-stable whenever the condition in 1.1.1 holds. A measure μ on E is said to be symmetric if $\mu(B) = \mu(-B)$ for every Borel set B. For a symmetric α-stable (SαS) measure μ we have $x_n = 0$ in 1.1.1 for all n. It is straightforward to check that μ is a SαS measure on E if and only if μw^{-1} is a SαS measure on R for every $w \in E^*$.

Let us restrict attention to SαS measures on a real separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and unit sphere $S = \{x \in H: \langle x,x\rangle = 1\}$. Then the SαS c.f. is characterized in [Kuelbs 1973, Corollary 2.1].

1.1.2 *A map* ϕ: H → R *is the c.f. of an* SαS *measure on* H *if and only if it can be written in the form*

$$\phi(y) = \exp\{-\int_S |\langle x,y\rangle|^\alpha \Gamma(dx)\}$$

for every $y \in H$, *where* Γ *is a finite symmetric Borel measure on* S.

If $H = R^n$ and $0 < \alpha < 2$, then the symmetric measure Γ on S is uniquely determined by the SαS measure ([Kanter 1973, Lemma 1]), and we shall call Γ the *spectral measure* of the SαS distribution (or c.f.) as is done in [Paulauskas 1976, p. 357]. If $\alpha = 2$, then ϕ is the c.f. of a Gaussian measure (or distribution). Whenever the distribution of a random vector $(\xi_1,\ldots,\xi_n)$ is an SαS measure on R^n, we shall refer to $\xi_1,\ldots,\xi_n$ as jointly SαS random variables.

We now present another characterization (also due to Kuelbs) of the SαS c.f. on H after introducing some additional terminology. Let τ be the topology induced on H by the seminorms of the form $\langle Ty,y\rangle^{1/2}$, where T is a symmetric, positive, trace class operator on H. An even, real-valued function f on H satisfying $f(0) = 0$ is said to be *of negative type* if

$$\sum_{i,j=1}^{n} f(y_i - y_j)c_i c_j \le 0$$

for all n, all $y_1,\ldots,y_n \in H$, and all real numbers $c_1,\ldots,c_n$ such that $\sum_{j=1}^{n} c_j = 0$. If f^α is of negative type and if $f(\lambda y) = |\lambda| f(y)$ for all real λ and $y \in H$, then f is called a *homogeneous negative-definite function of order* α.

1.1.3 [Kuelbs 1973, Theorem 3.1] *A map* $\phi: H \to R$ *is the c.f. of a* SαS *measure on* H *if and only if it has the form*

$$\phi(y) = \exp\{-f^{\alpha}(y)\} ,$$

where f *is a homogeneous negative-definite function of order* α *which is* τ*-continuous on* H.

A stochastic process $\xi = \{\xi_t, t \in T\}$ is called SαS if its finite-dimensional distributions are SαS. When $\alpha = 2$, ξ is a zero mean Gaussian process and its statistical properties can be expressed in terms of a single function, the covariance function. However, when $0 < \alpha < 2$, there is in general no simple parametric description of the finite-dimensional distributions of the process.

A special class of SαS stochastic processes which are closely related to Gaussian processes and which have an equally simple parametric description are the so-called sub-Gaussian processes. Nevertheless the sub-Gaussian processes have quite different properties from the $\alpha = 2$ case, some of which are mentioned in [Bretagnolle, *et al.* 1966, p. 251].

To introduce the sub-Gaussian process we begin with a zero mean Gaussian process $\{\xi_t, t \in T\}$ with finite-dimensional c.f.'s of the form given in result 1.1.3: for every n and every $(t_1,\ldots,t_n) \in T^n$,

$$\phi_{\xi_{t_1},\ldots,\xi_{t_n}}(r_1,\ldots,r_n) = \exp\{-f^2_{t_1,\ldots,t_n}(r_1,\ldots,r_n)\},$$

where $f_{t_1,\ldots,t_n}$ is a homogeneous negative-definite function of order 2.

It is well known that if a function ψ is of negative type, then ψ^p is of negative type for all p such that $0 < p \le 1$. (See [Parthasarathy and Schmidt 1972] for a general discussion.) Thus it follows from 1.1.3 that

$$\exp\{-f^{\alpha}_{t_1,\ldots,t_n}(r_1,\ldots,r_n)\}$$

is an SαS c.f. on R^n for any α such that $0 < \alpha < 2$. The family of all such SαS c.f.'s, for $n = 1,2,\ldots$ and $(t_1,\ldots,t_n)\in T^n$, clearly specifies a consistent family of finite-dimensional distributions and hence a stochastic process. We shall use the term α-sub-Gaussian to refer to finite-dimensional distributions having c.f.'s of this form as well as to such SαS stochastic processes.

Note that the distribution of an α-sub-Gaussian vector is determined by α and a positive-definite matrix Σ and that the distribution of an α-sub-Gaussian process is determined by α and a positive-definite function $R(s,t)$. Hence sub-Gaussian distributions have a particularly simple parametric description, unlike the general SαS distribution. However, it is not known how the spectral measure of a sub-Gaussian vector is expressed in terms of α and Σ.

While stable measures on a separable Banach space suffice for our purposes, we may mention that [Dudley and Kanter 1974] and [DeAcosta 1975] treat stable measures on more general "measurable vector spaces," and [Rajput 1975] defines certain stable measures on topological vector spaces.

2. Independence.

The question of how to characterize the independence of jointly SαS random variables is a natural one and has been answered in [Schilder 1970, Theorem 5.1] and in [Paulauskas 1976, Proposition 4]. Although these results by Schilder and Paulauskas are stated correctly, the proofs as they appear are not convincing and a more detailed treatment seems justified. The implications of independence are important for us in the following chapter when we consider SαS processes having independent increments; so in this section we prove a characterization of independence for jointly SαS random variables or vectors in terms of the support of their spectral measure.

1.2.1 *THEOREM. Let* $\xi_1,\dots,\xi_n$ *be jointly* SαS *random variables with* $0 < \alpha < 2$ *and spectral measure* Γ. *For fixed* k *and* m *satisfying* $1 \le k < m \le n$, ξ_k *and* ξ_m *are independent if and only if* $\Gamma(\{(x_1,\dots,x_n)\in S: x_k x_m \neq 0\}) = 0$.

This result is essentially due to Schilder, but its proof here is based on Lemma 1.2.2 which we state and prove first. The technique used in the lemma was motivated by the proof of Lemma 1 in [Kanter 1973].

Define the σ-finite measure ρ on $(0,\infty)$ by $\rho(ds) = \dfrac{ds}{s^{1+\alpha}}$, define θ: $(0,\infty) \times R^n \to R^n$ by $\theta(s,x) = sx$, and define T: $R^n \to R^n$ by $T(x_1,\dots,x_n) = (y_1,\dots,y_n)$ where $y_k = x_k$, $y_m = x_m$, and $y_i = 0$ if $i \neq k$ and $i \neq m$. Let $\nu = \Gamma T^{-1}$ and $G = (\rho\times\nu)\theta^{-1}$. Choose four real numbers h_k, h_m, h_k', h_m' such that

$$f(v) = \sum_{j=k,m} (2-\cos h_j v_j - \cos h_j' v_j) > 0$$

whenever $v_k \neq 0$ or $v_m \neq 0$.

1.2.2 *LEMMA. The function* ψ: $R^n \to R$ *defined by*

$$\psi(r) = \int_S |r_k x_k + r_m x_m|^\alpha \Gamma(dx)$$

uniquely determines the measure $f(v)G(dv)$ *on* R^n .

Proof: Notice that

$$\psi(r) = \int_S |\langle r,Tx\rangle|^\alpha \Gamma(dx) = \int_{R^n} |\langle r,x\rangle|^\alpha \nu(dx)$$

for all $r \in R^n$ and that for all $z \in R$

$$|z|^\alpha \int_0^\infty (1-\cos s)\rho(ds) = \int_0^\infty (1-\cos zs)\rho(ds) .$$

Thus

$$\psi(r) \int_0^\infty (1-\cos s)\rho(ds) = \int_{R^n} \int_0^\infty |\langle r,x\rangle|^\alpha (1-\cos s)\rho(ds)\nu(dx)$$

$$= \int_{(0,\infty)\times R^n} (1-\cos\langle r,sx\rangle)(\rho\times\nu)(ds\times dx) = \int_{R^n} (1-\cos\langle r,v\rangle)G(dv)$$

for every $r \in R^n$. Let $\delta_k = (\ldots 0\ldots,h_k,\ldots 0\ldots)$ and $\delta_m = (\ldots 0\ldots,h_m,\ldots 0\ldots)$, where the coordinates h_k and h_m are in the k-th and m-th positions, respectively. Then for every $r \in R^n$ the function ψ determines

$$\frac{1}{2} \sum_{j=k,m} \int_{R^n} [(1-\cos\langle r+\delta_j,v\rangle) + (1-\cos\langle r-\delta_j,v\rangle) - 2(1-\cos\langle r,v\rangle)]G(dv)$$

$$= \int_{R^n} \cos\langle r,v\rangle \sum_{j=k,m} (1-\cos h_j v_j)G(dv)$$

$$= \int_{R^n} e^{i\langle r,v\rangle} \sum_{j=k,m} (1-\cos h_j v_j)G(dv) ,$$

since G is a symmetric measure. Thus ψ determines the value of

$\int_{R^n} e^{i\langle r,v\rangle} f(v)G(dv)$ for every $r \in R^n$. Since $f(v)G(dv)$ is a finite measure on R^n, the result follows from the uniqueness of the Fourier transform. □

Proof of Theorem 1.2.1: If ξ_k and ξ_m are independent, then their joint c.f. factors. Thus for every real r_k and r_m

$$\int_S |r_k x_k + r_m x_m|^\alpha \Gamma(dx) = |r_k|^\alpha \int_S |x_k|^\alpha \Gamma(dx) + |r_m|^\alpha \int_S |x_m|^\alpha \Gamma(dx) .$$

Consider the measure Γ_0 on S placing mass $\frac{1}{2} \int |x_k|^\alpha \Gamma(dx)$ on $(\ldots 0 \ldots, 1, \ldots 0 \ldots)$ and on $(\ldots 0 \ldots, -1, \ldots 0 \ldots)$, where the 1 and -1 are the k-th coordinates; placing mass $\frac{1}{2} \int |x_m|^\alpha \Gamma(dx)$ on $(\ldots 0 \ldots, 1, \ldots 0 \ldots)$ and on $(\ldots 0 \ldots, -1, \ldots 0 \ldots)$, where the 1 and -1 are the m-th coordinates; and placing mass zero on the remainder of S. Then clearly

$$(*) \qquad \int_S |r_k x_k + r_m x_m|^\alpha \Gamma(dx) = \int_S |r_k x_k + r_m x_m|^\alpha \Gamma_0(dx)$$

for all r_k, r_m. Let $\nu_0 = \Gamma_0 T^{-1}$ and $G_0 = (\rho \times \nu_0)\theta^{-1}$. Define $B = \{v \in R^n: v_k v_m \neq 0\}$ and observe that

$$\int_{R^n} \chi_B(v) f(v) G_0(dv) = \int_0^\infty \int_{R^n} \chi_B(sx) f(sx) (\rho \times \nu_0)(ds \times dx)$$

$$= \int_0^\infty \int_{R^n} \chi_{\{x_k x_m \neq 0\}}(x) f(sx) \nu_0(dx) \rho(ds) = 0 ,$$

since $\nu_0(\{x \in R^n: x_k x_m \neq 0\}) = \Gamma_0(\{x \in S: x_k x_m \neq 0\}) = 0$. By (*) and Lemma 1.2.2, $f(v)G(dv)$ and $f(v)G_0(dv)$ must agree on B. Hence

$$0 = \int_{R^n} \chi_B(v) f(v) G(dv)$$

$$= \int_0^\infty \int_{R^n} \chi_{\{x_k x_m \neq 0\}}(x) f(sx) \nu(dx) \rho(ds) ,$$

so that $\int_{R^n} \chi_{\{x_k x_m \neq 0\}}(x) f(s_0 x)\nu(dx) = 0$ for some $s_0 > 0$. Because $f(s_0x) > 0$ on $\{x_k x_m \neq 0\}$, we get that

$$0 = \nu(\{x \in R^n\colon x_k x_m \neq 0\}) = \Gamma(\{x \in S\colon x_k x_m \neq 0\}) .$$

Conversely, $\Gamma(\{x\in S\colon x_k x_m \neq 0\}) = 0$ implies that

$$\int_S |r_k x_k + r_m x_m|^\alpha \Gamma(dx) = \left(\int_{\substack{x_k\neq 0\\ x_m=0}} + \int_{\substack{x_k=0\\ x_m\neq 0}} + \int_{\substack{x_k=0\\ x_m=0}} + \int_{\substack{x_k\neq 0\\ x_m\neq 0}} \right) |r_k x_k + r_m x_m|^\alpha \Gamma(dx)$$

$$= |r_k|^\alpha \int_S |x_k|^\alpha \Gamma(dx) + |r_m|^\alpha \int_S |x_m|^\alpha \Gamma(dx) .$$

Thus ξ_k and ξ_m are independent since their joint c.f. factors. □

1.2.3 *COROLLARY. A subset* $\{\xi_{k_1},\ldots,\xi_{k_i}\}$ *of* $\{\xi_1,\ldots,\xi_n\}$ *is independent if and only if the random variables are pairwise independent.*

Proof: Necessity is clear. For the sufficiency, Γ is concentrated on the set $\{x_{k_p} x_{k_q} = 0,\ p \neq q \text{ in } 1,2,\ldots,i\}$ and therefore we have

$$\int_S |r_{k_1} x_{k_1} + \ldots + r_{k_i} x_{k_i}|^\alpha \Gamma(dx)$$

$$= \left(\int_{\{x_{k_2}=\ldots=x_{k_i}=0\}} + \ldots + \int_{\{x_{k_1}=\ldots=x_{k_{i-1}}=0\}} \right) |r_{k_1} x_{k_1} + \ldots + r_{k_i} x_{k_i}|^\alpha \Gamma(dx)$$

$$= \int_S |r_{k_1} x_{k_1}|^\alpha \Gamma(dx) + \ldots + \int_S |r_{k_i} x_{k_i}|^\alpha \Gamma(dx) .$$ □

1.2.4 *COROLLARY. Let* $\{k_1,\ldots,k_i\}$ *and* $\{m_1,\ldots,m_j\}$ *be disjoint subsets of* $\{1,\ldots,n\}$. *Then the random vectors* $(\xi_{k_1},\ldots,\xi_{k_i})$ *and* $(\xi_{m_1},\ldots,\xi_{m_j})$

are independent if and only if any two random variables, one selected from each vector, are independent.

Proof: Necessity is clear. For the sufficiency, observe that

$$\int |r_{k_1}x_{k_1}+\ldots+r_{k_i}x_{k_i}+r_{m_1}x_{m_1}+\ldots+r_{m_j}x_{m_j}|^{\alpha}\Gamma(dx)$$

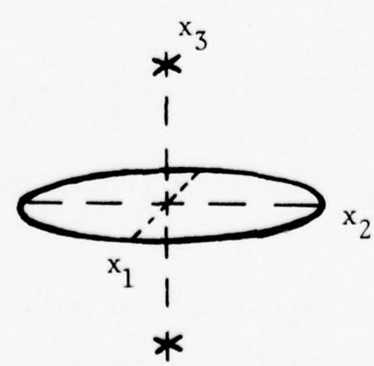
$$= \left(\int\limits_{\left\{\substack{x_{k_1}^2+\ldots+x_{k_i}^2\neq 0 \\ \text{and} \\ x_{m_1}^2+\ldots+x_{m_j}^2\neq 0}\right\}} \int\limits_{\left\{\substack{x_{k_1}^2+\ldots+x_{k_i}^2\neq 0 \\ \text{and} \\ x_{m_1}^2+\ldots+x_{m_j}^2= 0}\right\}} \int\limits_{\left\{\substack{x_{k_1}^2+\ldots+x_{k_i}^2= 0 \\ \text{and} \\ x_{m_1}^2+\ldots+x_{m_j}^2\neq 0}\right\}} \int\limits_{\left\{\substack{x_{k_1}^2+\ldots+x_{k_i}^2= 0 \\ \text{and} \\ x_{m_1}^2+\ldots+x_{m_j}^2= 0}\right\}} \right) |\cdots|^{\alpha}\Gamma(dx)$$

$$= \int\limits_S |r_{k_1}x_{k_1}+\ldots+r_{k_i}x_{k_i}|^{\alpha}\Gamma(dx) + \int\limits_S |r_{m_1}x_{m_1}+\ldots+r_{m_j}x_{m_j}|^{\alpha}\Gamma(dx) ,$$

since $\Gamma\{x_{k_1}^2+\ldots+x_{k_i}^2 \neq 0 \text{ and } x_{m_1}^2+\ldots+x_{m_j}^2 \neq 0\} = 0$ by Theorem 1.2.1. □

1.2.5 Example. If ξ_1,ξ_2, and ξ_3 are jointly SαS random variables with $0 < \alpha < 2$ and spectral measure Γ, then (ξ_1,ξ_2) and ξ_3 are independent if and only if Γ is concentrated on

$$\{x_1x_3=0\} \cup \{x_2x_3=0\} = \{x_3=0\} \cup \{x_1=0 \text{ and } x_2=0\}$$
$$= \{x_1^2 + x_2^2=1\} \cup \{x_3=\pm 1\} .$$



1.2.6 Example. It is easy to check from their c.f. that two nondegenerate jointly sub-Gaussian random variables $(0 < \alpha < 2)$ cannot be independent. Indeed, consider the bivariate normal c.f.

$$\exp\{-2^{-1}(\sigma_1^2 r_1^2 + 2\sigma_{12} r_1 r_2 + \sigma_2^2 r_2^2)\} ,$$

where $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$ are the marginal variances and σ_{12} is the covariance. Then

$$\phi(r_1,r_2) = \exp\{-2^{-\frac{\alpha}{2}}(\sigma_1^2 r_1^2 + 2\sigma_{12} r_1 r_2 + \sigma_2^2 r_2^2)^{\frac{\alpha}{2}}\}$$

is the joint c.f. of two nondegenerate sub-Gaussian random variables that are independent if and only if

$$(*) \qquad (\sigma_1^2 r_1^2 + 2\sigma_{12} r_1 r_2 + \sigma_2^2 r_2^2)^{\frac{\alpha}{2}} = \sigma_1^\alpha |r_1|^\alpha + \sigma_2^\alpha |r_2|^\alpha$$

for all r_1,r_2. The left-hand side of (*) is never zero when $r_2 \neq 0$; so we hold $r_2 \neq 0$ and differentiate both sides with respect to r_1 to get

$$\frac{\sigma_1^2 r_1 + \sigma_{12} r_2}{(\sigma_1^2 r_1^2 + 2\sigma_{12} r_1 r_2 + \sigma_2^2 r_2^2)^{\frac{2-\alpha}{2}}} = \sigma_1^\alpha (r_1)^{\alpha-1}$$

whenever $r_2 \neq 0$, which becomes

$$\sigma_1^2 \sigma_2^\alpha r_1 (r_2)^{\alpha-1} + \sigma_2^\alpha \sigma_{12} |r_2|^\alpha = \sigma_1^\alpha \sigma_{12} |r_1|^\alpha + \sigma_1^\alpha \sigma_2^2 (r_1)^{\alpha-1} r_2$$

after substituting (*) and simplifying. Taking $r_1 = 0$ and $r_2 = 1$ we see that $\sigma_{12} = 0$; so we now have

$$\sigma_1^2 \sigma_2^\alpha r_1 (r_2)^{\alpha-1} = \sigma_1^\alpha \sigma_2^2 (r_1)^{\alpha-1} r_2$$

whenever $r_2 \neq 0$, which implies that $\sigma_1 \sigma_2 = 0$, a contradiction. (Note: when raising a number u to a power p we shall use the convention

$$(u)^p = |u|^p \operatorname{sign}(u) \text{ .)}$$

3. Regression.

In this section we obtain a necessary and sufficient condition, expressed in terms of their spectral measure, for a regression involving SαS random variables to be linear (Theorem 1.3.7). This is a consequence of a result relating the form of the linear regression function to partial derivatives of the joint c.f. (Theorem 1.3.1). We also obtain a sufficient condition for linear regression (Proposition 1.3.8) which is simpler than the necessary and sufficient condition in Theorem 1.3.7, but nevertheless has some interesting applications.

Let $\xi_0, \xi_1, \ldots, \xi_n$ be jointly SαS random variables with $1 < \alpha < 2$. For an SαS distribution on R it is well-known that the moments of order $p < \alpha$ exist, and it is therefore meaningful to consider $E(\xi_0 | \xi_1, \ldots, \xi_n)$ and to study the form of f for which

$$E(\xi_0 | \xi_1, \ldots, \xi_n) = f(\xi_1, \ldots, \xi_n) \text{ a.s.}$$

Kanter has obtained several results which show that f is a linear function in certain cases. The regression $E(\xi_0 | \xi_1)$ is always linear (Corollary 1.3.4), as is the regression $E(\xi_0 | \xi_1, \ldots, \xi_n)$ provided $\xi_1, \ldots, \xi_n$ are independent (Corollary 1.3.6) ([Kanter 1972a]). In case $E(\xi_0 | \xi_1, \ldots, \xi_n)$ and $\xi_1, \ldots, \xi_n$ are jointly SαS (a condition for which criteria are not known), then once again the regression is linear ([Kanter 1972b]).

For our investigations we shall use a general result giving a necessary and sufficient condition for linear regression in terms of the joint c.f. of the random variables (not necessarily stable). The method of proof comes from a related result found in [Lukacs and Laha 1964, Theorem 6.1.1].

1.3.1 *THEOREM. Let* $\xi_0,\xi_1,\ldots,\xi_n$ *be random variables having first moments and with joint c.f.* ϕ. *Then*

$$(*) \qquad E(\xi_0|\xi_1,\ldots,\xi_n) = a_1\xi_1+\ldots+a_n\xi_n \quad \text{a.s.}$$

if and only if

$$\left[\frac{\partial}{\partial r_0}\phi(r_0,r_1,\ldots,r_n)\right]_{r_0=0}$$

$$= a_1\frac{\partial}{\partial r_1}\phi(0,r_1,\ldots,r_n)+\ldots+a_n\frac{\partial}{\partial r_n}\phi(0,r_1,\ldots,r_n)$$

for all $r_1,\ldots,r_n$.

Proof: Observe first that the condition may be written as

$$(**) \qquad E[\xi_0 e^{i(r_1\xi_1+\ldots+r_n\xi_n)}] = a_1E[\xi_1 e^{i(r_1\xi_1+\ldots+r_n\xi_n)}]+\ldots+a_nE[\xi_n e^{i(r_1\xi_1+\ldots+r_n\xi_n)}]$$

for all $r_1,\ldots,r_n$.

Necessity. (*) implies

$$E(\xi_0|\xi_1,\ldots,\xi_n)e^{i(r_1\xi_1+\ldots+r_n\xi_n)} = (a_1\xi_1+\ldots+a_n\xi_n)e^{i(r_1\xi_1+\ldots+r_n\xi_n)} \quad \text{a.s.,}$$

and (**) follows by taking expectations.

Sufficiency. Let $E(\xi_0-a_1\xi_1-\ldots-a_n\xi_n|\xi_1,\ldots,\xi_n) = f(\xi_1,\ldots,\xi_n)$, where f is a Borel-measurable function on R^n. Then for all $r_1,\ldots,r_n$

we have that

$$\int_{R^n} f(x_1,\ldots,x_n)e^{i(r_1x_1+\ldots+r_nx_n)}\,dP(\xi_1,\ldots,\xi_n)^{-1}(x_1,\ldots,x_n)$$

$$= E[f(\xi_1,\ldots,\xi_n)e^{i(r_1\xi_1+\ldots+r_n\xi_n)}]$$

$$= E[(\xi_0-a_1\xi_1-\ldots-a_n\xi_n)e^{i(r_1\xi_1+\ldots+r_n\xi_n)}] = 0$$

by (**). Now

$$\nu(B) = \int_B f(x_1,\ldots,x_n)dP(\xi_1,\ldots,\xi_n)^{-1}(x_1,\ldots,x_n)$$

defines a finite signed measure (f.s.m.) on R^n which is therefore uniquely determined by its Fourier transform. Thus

$$\int_{(\xi_1,\ldots,\xi_n)^{-1}(B)} f(\xi_1,\ldots,\xi_n)dP$$

$$= \int_B f(x_1,\ldots,x_n)dP(\xi_1,\ldots,\xi_n)^{-1}(x_1,\ldots,x_n) = 0$$

for all Borel subsets B of R^n, and since $f(\xi_1,\ldots,\xi_n)$ is a $\sigma(\xi_1,\ldots,\xi_n)$-measurable random variable, we get that $f(\xi_1,\ldots,\xi_n) = 0$ a.s. [P] and (*) follows. □

If the regression is linear, then it is clear that the coefficients $a_1,\ldots,a_n$ are uniquely determined by ϕ if and only if $\xi_1,\ldots,\xi_n$ are linearly independent elements of $L_1(\Omega,F,P)$. For each choice of $r\in R^n$ the condition of Theorem 1.3.1 provides a linear equation involving the a_j's, but it is not clear in general what n choices of $r\in R^n$ will provide n linearly independent equations which can be solved for the a_j's.

1.3.2 <u>Example</u>. If $\xi_0, \xi_1, \ldots, \xi_n$ are jointly α-sub-Gaussian random variables, then the regression is linear and the coefficients are the same as in the Gaussian case. For, let

$$\phi(r_0, \ldots, r_n) = \exp\left\{-2^{-\frac{\alpha}{2}}\left(\sum_{i,j=0}^{n} \sigma_{ij} r_i r_j\right)^{\frac{\alpha}{2}}\right\},$$

where $\Sigma = (\sigma_{ij})$ is a covariance matrix. Then for $r_1, \ldots, r_n$ not all zero,

$$\left.\frac{\partial\phi(r_0, r_1, \ldots, r_n)}{\partial r_0}\right|_{r_0=0} = \frac{-\alpha 2^{-\frac{\alpha}{2}}\phi(0, r_1, \ldots, r_n)\sum_{j=1}^{n}\sigma_{0j} r_j}{\left(\sum_{i,j=1}^{n}\sigma_{ij} r_i r_j\right)^{\frac{2-\alpha}{2}}}$$

and for $1 \le k \le n$

$$\frac{\partial\phi(0, r_1, \ldots, r_n)}{\partial r_k} = \frac{-\alpha 2^{-\frac{\alpha}{2}}\phi(0, r_1, \ldots, r_n)\sum_{j=1}^{n}\sigma_{jk} r_j}{\left(\sum_{i,j=1}^{n}\sigma_{ij} r_i r_j\right)^{\frac{2-\alpha}{2}}}.$$

Therefore the condition of Theorem 1.3.1 is written as

$$\sum_{j=1}^{n}\sigma_{0j} r_j = \sum_{k=1}^{n} a_k \sum_{j=1}^{n}\sigma_{jk} r_j$$

or

$$\sum_{j=1}^{n}\left[\sigma_{0j} - \sum_{k=1}^{n}\sigma_{jk} a_k\right] r_j = 0$$

for all $r_1, \ldots, r_n$, and thus it is satisfied by the a_k's which are the solutions of the system of equations

$$\sum_{k=1}^{n}\sigma_{jk} a_k = \sigma_{0j}, \quad j = 1, \ldots, n .$$

Hence the regression is linear and the regression coefficients satisfy the same equations $\Sigma\underline{a} = \underline{\sigma}_0$, $\underline{a}^T = (a_1,\ldots,a_n)$, $\underline{\sigma}_0^T = (\sigma_{01},\ldots,\sigma_{0n})$, as when $\xi_0,\xi_1,\ldots,\xi_n$ are jointly Gaussian with mean zero and covariance matrix Σ.

1.3.3 *COROLLARY. If* $\xi_0,\xi_1,\ldots,\xi_n$ *are jointly* SαS *with spectral measure* Γ *on the unit sphere* S *in* R^{n+1}, *then*

$$E(\xi_0|\xi_1,\ldots,\xi_n) = a_1\xi_1+\ldots+a_n\xi_n \quad a.s.$$

if and only if

$$\int_S (x_0-a_1x_1-\ldots-a_nx_n)(r_1x_1+\ldots+r_nx_n)^{\alpha-1}\Gamma(dx) = 0$$

for all $r_1,\ldots,r_n$.

Before illustrating the use of Corollary 1.3.3, we define the *covariation* $C_{\eta\zeta}$ *of* η *with* ζ as

$$C_{\eta\zeta} = \int_S x_1(x_2)^{\alpha-1}\Gamma_{\eta,\zeta}(dx) ,$$

where η and ζ are jointly SαS with spectral measure $\Gamma_{\eta,\zeta}$. (Note the lack of symmetry in η and ζ here.) The next result provides $C_{\eta\zeta}$ with an interesting interpretation.

1.3.4 *COROLLARY.* [Kanter 1972a, Theorem 1.4]. *If* η *and* ζ *are jointly* SαS *random variables, then*

$$E(\eta|\zeta) = \frac{C_{\eta\zeta}}{C_{\zeta\zeta}}\zeta \quad a.s.$$

Proof: By Corollary 1.3.3, $E(\eta|\zeta) = a\zeta$ a.s. if and only if

$$\int_S (x_1 - ax_2)(rx_2)^{\alpha-1}\Gamma_{\eta,\zeta}(dx) = 0$$

for all $r \in R$. Solving for a yields

$$a = \frac{\int_S x_1(x_2)^{\alpha-1}\Gamma_{\eta,\zeta}(dx)}{\int_S |x_2|^{\alpha}\Gamma_{\eta,\zeta}(dx)} = \frac{C_{\eta\zeta}}{C_{\zeta\zeta}} . \qquad \square$$

For jointly Gaussian random variables η and ζ with zero mean (the case $\alpha = 2$) it is well-known that a result analogous to Corollary 1.3.4 holds with $C_{\eta\zeta}$ replaced by the covariance of η and ζ.

By appropriate choice of Γ it is easy to see from Corollary 1.3.3 that the regression can be nonlinear. For example, take $n = 2$ and suppose that

$$\Gamma(3^{-1/2}, 3^{-1/2}, 3^{-1/2}) = \Gamma(0,1,0) = \Gamma(0,0,1) = 1$$

and that Γ places zero mass on the remainder of S. (Note that Γ need not be symmetric unless we are concerned about uniqueness.) Then $E(\xi_0|\xi_1,\xi_2)$ is not a linear function of ξ_1 and ξ_2; however, even in this simple case we do not know the form of the regression.

1.3.5 *COROLLARY. If* ξ_0,ξ_1,ξ_2 *are jointly* SαS *and if*

$$E(\xi_0|\xi_1,\xi_2) = a_1\xi_1 + a_2\xi_2 \quad a.s.,$$

then a_1 *and* a_2 *satisfy*

$$(*) \qquad \begin{aligned} a_1c_{11} + a_2c_{21} &= c_{01} \\ a_1c_{12} + a_2c_{22} &= c_{02} , \end{aligned}$$

where c_{ij} *is the covariation of* ξ_i *with* ξ_j. *Moreover, equations* (*) *uniquely determine* a_1 *and* a_2 *if and only if* ξ_1 *and* ξ_2 *are linearly independent elements of* $L_1(\Omega)$.

Proof: If the regression is linear, then equations (*) follow immediately from the condition of Corollary 1.3.3 by taking $r_1 = 1$, $r_2 = 0$ and $r_1 = 0$, $r_2 = 1$. These equations have a unique solution unless

$$c_{11}c_{22} = c_{12}c_{21} ,$$

i.e.,

$$\int_S |x_1|^\alpha \Gamma(dx) \int_S |x_2|^\alpha \Gamma(dx) = \int_S x_1 (x_2)^{\alpha-1} \Gamma(dx) \int_S x_2 (x_1)^{\alpha-1} \Gamma(dx) ,$$

which implies that $x_1 = \lambda x_2$ a.e. $[\Gamma]$ for some $\lambda \epsilon R$ by Hölder's inequality, hence $\xi_1 = \lambda \xi_2$ a.s. □

If $n > 2$ and the regression is linear, then the regression coefficients a_j again satisfy linear equations given by the condition in Corollary 1.3.3. Unfortunately, just as in the non-stable case (Theorem 1.3.1), we do not know in general how to choose n linearly independent equations that can be solved for the a_j's.

The following corollary shows that the regression is always linear and the regression coefficients are easily obtained whenever $\xi_1, \ldots, \xi_n$ are independent.

1.3.6 *COROLLARY*. [Kanter 1972a, Theorem 3.4]. *If* $\xi_0, \xi_1, \ldots, \xi_n$ *are jointly* SαS *random variables and if* $\xi_1, \ldots, \xi_n$ *are independent and nondegenerate, then*

$$E(\xi_0 | \xi_1, \ldots, \xi_n) = a_1 \xi_1 + \ldots + a_n \xi_n \quad a.s.,$$

and the coefficients a_k *are given by*

$$a_k = \frac{C_{0k}}{C_{kk}},$$

where C_{0k} *is the covariation of* ξ_0 *with* ξ_k *and* C_{kk} *is the covariation of* ξ_k *with itself.*

The proof follows easily from Corollaries 1.2.3 and 1.3.3.

We now obtain a condition for linear regression by applying to Corollary 1.3.3 the methods used in Lemma 1.2.2. Although the resulting condition appears surprisingly involved, it is not clear that further simplification is possible.

Define $T: R^{n+1} \to R^{n+1}$ by $T(y_0, y_1, \ldots, y_n) = (0, y_1, \ldots, y_n)$, define $\theta: (0,\infty) \times S \to R^{n+1}$ by $\theta(s,x) = sx$, and define the σ-finite measure ρ on $(0,\infty)$ by $\rho(ds) = ds/s^\alpha$. Let $f(x) = x_0 - a_1x_1 - \cdots - a_nx_n$, and define a f.s.m. ν on S by $\nu(dx) = f(x)\Gamma(dx)$. Let G be the measure on R^{n+1} defined by $G = (\rho\times\nu)\theta^{-1}$, and let $\mathcal{G}$ be the σ-field of subsets

$$\{R\times B: B \text{ is a Borel subset of } R^n - (0,\ldots,0)\}$$

of $R\times[R^n - (0,\ldots,0)]$.

1.3.7 *THEOREM. Let* $\xi_0, \xi_1, \ldots, \xi_n$ *be jointly* SαS *variables,* $1 < \alpha < 2$, *with corresponding measure* Γ *on* S. *Then* $E(\xi_0|\xi_1,\ldots,\xi_n) = a_1\xi_1 + \cdots + a_n\xi_n$ *a.s. if and only if* G *is the zero measure on* $\mathcal{G}$.

<u>Proof</u>: Assume that $E(\xi_0|\xi_1,\ldots,\xi_n) = a_1\xi_1 + \cdots + a_n\xi_n$ a.s. Then by Corollary 1.3.3,

$$\int_S f(x)(\langle r, Tx\rangle)^{\alpha-1}\Gamma(dx) = 0$$

for all $r \in R^{n+1}$. Note that for any $z \in R$,

$$(z)^{\alpha-1}\int_0^\infty \sin s\,\rho(ds) = \int_0^\infty \sin(zs)\rho(ds) \;.$$

Thus

$$0 = \int_S f(x)(\langle r,Tx\rangle)^{\alpha-1}\Gamma(dx) \int_0^\infty \sin s\,\rho(ds)$$

$$= \int_0^\infty \int_S \sin\langle r,T(sx)\rangle \nu(dx)\rho(ds)$$

$$= \int_{R^{n+1}} \sin\langle r,Tv\rangle G(dv)$$

$$(*) \qquad = \int_{R^{n+1}} \sin\langle r,v\rangle GT^{-1}(dv)$$

for all $r \in R^{n+1}$.

Let $h_1,\ldots,h_n$ be any real numbers, and let $\delta_1 = (0,h_1,0,\ldots,0)$, $\delta_2 = (0,0,h_2,0,\ldots,0),\ldots,\ \delta_n = (0,\ldots,0,h_n)$. Then for every $r \in R^{n+1}$

$$0 = \frac{1}{2}\sum_{j=1}^{n} \int_{R^{n+1}} [2\sin\langle r,v\rangle - \sin\langle r+\delta_j,v\rangle - \sin\langle r-\delta_j,v\rangle] GT^{-1}(dv)$$

$$= \int_{R^{n+1}} \sin\langle r,v\rangle \sum_{j=1}^{n} (1-\cos h_jv_j) GT^{-1}(dv) \;.$$

We can choose $h_1,\ldots,h_n$ and $h_1',\ldots,h_n'$ such that either $\sum_{j=1}^n(1-\cos h_jv_j)$ or $\sum_{j=1}^n(1-\cos h_j'v_j)$ is nonzero for every $v \in \{(v_0,v_1,\ldots,v_n) \in R^{n+1}: v_1^2+\ldots+v_n^2 > 0\}$. Let $g(v) = \sum_{j=1}^n(2-\cos h_jv_j - \cos h_j'v_j)$. Then for every $r \in R^{n+1}$,

$$\int_{R^{n+1}} \sin\langle r,v\rangle g(v) GT^{-1}(dv) = 0 \;.$$

Now it is easy to see that $g(v)GT^{-1}(dv)$ defines a f.s.m. on R^{n+1} which is antisymmetric in the sense that

$$\int_{-B} g(v)GT^{-1}(dv) = -\int_{B} g(v)GT^{-1}(dv)$$

for every Borel subset B of R^{n+1}. Thus for every $r \in R^{n+1}$,

$\int_{R^{n+1}} e^{i<r,v>} g(v)GT^{-1}(dv) = 0$. It follows by the uniqueness of the Fourier transform that $g(v)GT^{-1}(dv)$ is the zero measure on R^{n+1}. Hence GT^{-1} is the zero measure on $\{(v_0,v_1,\ldots,v_n) \in R^{n+1}: v_1^2+\ldots+v_n^2 > 0\}$. Thus for every Borel set $B \subset R^n - (0,\ldots,0)$,

$$G(R\times B) = GT^{-1}(R\times B) = 0 \ .$$

Therefore G is the zero measure on $\mathcal{G}$.

For the converse, note that $\sin<r,Tv>$ is a $\mathcal{G}$-measurable function on $R\times[R^n - (0,\ldots,0)]$ for every $r \in R^{n+1}$. Thus, if G is the zero measure on $\mathcal{G}$, then

$$\int_{R^{n+1}} \sin<r,Tv>G(dv) = 0$$

for every $r \in R^{n+1}$. The linearity of the regression now follows from equation (*) and Corollary 1.3.3. □

The condition in Theorem 1.3.7 is too complicated for easy verification; so we shall present in the following proposition a simpler sufficient condition that provides nontrivial examples in which the regression is linear. To simplify the presentation and to make geometric visualization possible, we shall treat the case $n = 2$, *i.e.*, the regression $E(\xi_0|\xi_1,\xi_2)$.

Therefore, consider jointly SαS random variables ξ_0,ξ_1,ξ_2 with $1 < \alpha < 2$ and spectral measure Γ on the unit sphere $S = \{(x_0,x_1,x_2) \in R^3: x_0^2+x_1^2+x_2^2 = 1\}$. Geometrically, we regard S as being separated into two hemispheres S^+ and S^- by the plane $\{x_0=0\}$ and obtain from the spectral measure Γ two new measures Γ_1 and Γ_2 on the plane $\{x_0 = 0\}$ by "collapsing" these two hemispheres. Specifically, let $S^+ = S\cap\{x_0\geq 0,\ x_0\neq 1\}$, $S^- = S\cap\{x_0<0, x_0\neq -1\}$, and $U = \{x\in R^2:\ x_1^2+x_2^2\leq 1\}$, and define $T\colon S \to U$ by $T(x_0,x_1,x_2) = (x_1,x_2)$. Then, for all Borel subsets B of U,

$$\Gamma_1(B) = \Gamma(S^+\cap T^{-1}B)$$

and

$$\Gamma_2(B) = \Gamma(S^-\cap T^{-1}B)$$

define two measures on U which we notice place zero mass on the point (0,0). Finally, we introduce two functions f_1 and f_2 on U representing the function $x_0 - a_1x_1 - a_2x_2$ on S^+ and S^-, respectively:

$$f_1(x_1,x_2) = \sqrt{1 - x_1^2-x_2^2} - a_1x_1 - a_2x_2$$

$$f_2(x_1,x_2) = -\sqrt{1 - x_1^2-x_2^2} - a_1x_1 - a_2x_2 ,$$

and we define a f.s.m. on U by

$$\nu(dx) = f_1(x)\Gamma_1(dx) + f_2(x)\Gamma_2(dx) .$$

1.3.8 *PROPOSITION. If ν is the zero measure on U, then $E(\xi_0|\xi_1,\xi_2) = a_1\xi_1 + a_2\xi_2$ a.s.*

Proof: The result follows from Corollary 1.3.3 and the following equality:

$$\int_S (r_1x_1+r_2x_2)^{\alpha-1}(x_0-a_1x_1-a_2x_2)\Gamma(dx)$$

$$= \int_{S^+} (r_1x_1+r_2x_2)^{\alpha-1}f_1(Tx)\Gamma(dx) + \int_{S^-} (r_1x_1+r_2x_2)^{\alpha-1}f_2(Tx)\Gamma(dx)$$

$$= \int_U (r_1x_1+r_2x_2)^{\alpha-1} f_1(x)\Gamma_1(dx) + \int_U (r_1x_1+r_2x_2)^{\alpha-1} f_2(x)\Gamma_2(dx)$$

$$= \int_U (r_1x_1+r_2x_2)^{\alpha-1} \nu(dx) . \qquad \square$$

If Γ_1 and Γ_2 are absolutely continuous with respect to a measure μ (*e.g.*, $\mu = \Gamma_1 + \Gamma_2$) with Radon-Nikodym derivatives g_1 and g_2, respectively, then the condition of Proposition 1.3.6 can be expressed as

$$0 = \nu(B) = \int_B [f_1(x)g_1(x) + f_2(x)g_2(x)]\mu(dx)$$

for all Borel subsets B of U, or equivalently,

$$(*) \qquad \int_U |f_1(x)g_1(x) + f_2(x)g_2(x)|\mu(dx) = 0 .$$

1.3.9 <u>Example</u>. Given real numbers a_1 and a_2, define a Borel subset B of U by

$$B = \{(x_1,x_2)\epsilon U: \sqrt{1-x_1^2-x_2^2} - a_1x_1-a_2x_2 > 0 \quad \text{and}$$
$$-\sqrt{1-x_1^2-x_2^2} - a_1x_1 - a_2x_2 > 0\}$$

and two functions g_1 and g_2 on U by

$$g_1(x) = \begin{cases} \dfrac{1}{\sqrt{1-x_1^2-x_2^2} - a_1x_1 - a_2x_2} & \text{if } x\epsilon B , \\ 0 & \text{otherwise} , \end{cases}$$

$$g_2(x) = \begin{cases} \dfrac{1}{\sqrt{1-x_1^2-x_2^2} + a_1x_1 + a_2x_2} & \text{if } x\epsilon B , \\ 0 & \text{otherwise} , \end{cases}$$

and let μ be any σ-finite measure on U with respect to which g_1 and g_2 are integrable. Then (*) is satisfied, and consequently $E(\xi_0|\xi_1,\xi_2) = a_1\xi_1 + a_2\xi_2$ a.s.

There appears to be little more that can be said about the form of the regression function using the techniques of this section, primarily because higher moments of SαS variables do not exist (see the next section). One quite restricted kind of problem which can be solved simply is to obtain necessary and sufficient conditions for $E(\xi_1\xi_2|\xi_3,\xi_4) = \xi_3\xi_4$ a.s. when the appropriate moments exist.

1.3.10 *PROPOSITION. Let* ξ_1,ξ_2,ξ_3,ξ_4 *be jointly* SαS *random variables with* $1 < \alpha < 2$ *and spectral measure* Γ *such that* ξ_1 *and* ξ_2 *are independent and* ξ_3 *and* ξ_4 *are independent. Assume that* $\xi_3 \neq 0$ *and* $\xi_4 \neq 0$, *and for each* $i,j \in \{1,2,3,4\}$ *let* c_{ij} *be the covariation of* ξ_i *with* ξ_j. *Then* $E(\xi_1\xi_2|\xi_3,\xi_4) = \xi_3\xi_4$ *a.s. if and only if either* $c_{13}c_{24} = c_{33}c_{44}$ *and* $c_{14} = c_{23} = 0$ *or else* $c_{14}c_{23} = c_{33}c_{44}$ *and* $c_{13}=c_{24} = 0$.

We remark that our analysis shows it impossible to have a first-order term such as $c\xi_3$ in the above regression.

<u>Proof</u>: In a manner similar to the argument in Theorem 1.3.1, it follows that $(\xi_1\xi_2|\xi_3,\xi_4) = \xi_3\xi_4$ a.s. if and only if

$$\left.\frac{\partial^2\phi(r_1,r_2,r_3,r_4)}{\partial r_1\partial r_2}\right|_{r_1=r_2=0} = \frac{\partial^2\phi(0,0,r_3,r_4)}{\partial r_3\partial r_4}$$

for all (r_3,r_4), where ϕ is the joint c.f. of ξ_1,ξ_2,ξ_3,ξ_4. Obtaining the partial derivatives, this condition becomes

$$(r_3r_4)^{\alpha-1}c_{33}c_{44}$$

$$= (r_3r_4)^{\alpha-1}c_{13}c_{24} + (r_3r_4)^{\alpha-1}c_{14}c_{23} + |r_3|^{2(\alpha-1)}c_{13}c_{23} + |r_4|^{2(\alpha-1)}c_{14}c_{24}$$

for all (r_3,r_4). Taking alternately $(r_3,r_4) = (1,0)$ and $(r_3,r_4) = (0,1)$ we get $c_{13}c_{23} = 0 = c_{14}c_{24}$ and our condition then becomes

$$c_{33}c_{44} = c_{13}c_{24} + c_{14}c_{23} .$$

But one of the terms on the right-hand side must be zero since $c_{13}c_{23} = 0 = c_{14}c_{24}$, and both terms cannot be zero since $\xi_3 \neq 0$ and $\xi_4 \neq 0$ imply $c_{33} \neq 0$ and $c_{44} \neq 0$. □

1.3.11. Example. To illustrate Proposition 1.3.10 consider a measure Γ which concentrates its mass as follows: mass $2^{(\alpha/2)-1}$ is placed on the four points $(0,2^{-1/2},2^{-1/2},0)$, $(0,2^{-1/2},-2^{-1/2},0)$, $(2^{-1/2},0,0,2^{-1/2})$, $(2^{-1/2},0,0,-2^{-1/2})$; and mass $5^{\alpha/2}$ is placed on the two points $((\frac{4}{5})^{1/2},0,(\frac{1}{5})^{1/2},0)$, $(0,(\frac{4}{5})^{1/2},0,(\frac{1}{5})^{1/2})$. Then $c_{13} = c_{24} = c_{33} = c_{44} = 2$ and $c_{14} = c_{23} = 0$, and therefore $E(\xi_1\xi_2|\xi_3,\xi_4) = \xi_3\xi_4$ a.s. (Notice that neither ξ_1 and ξ_4 nor ξ_2 and ξ_3 are independent.)

We conclude this section with a discussion of regression in the infinite-dimensional case. The reading of these remarks might be deferred since some of the concepts which arise here are dealt with extensively in Chapter II.

Suppose that $\{\eta,\xi_t,t\epsilon T\}$ is a SαS family of random variables with $1 < \alpha < 2$, and consider the regression of η on $\{\xi_t,t\epsilon T\}$. There exists a countable subset T_∞ of T such that

$$E(\eta|\xi_t, t\in T) = E(\eta|\xi_t, t\in T_\infty) .$$

If we order the points in T_∞ and let T_n be the set containing the first n points, then

$$E(\eta|\xi_t, t\in T_\infty) = \lim_{n\to\infty} E(\eta|\xi_t, t\in T_n)$$

by [Doob 1953, p. 319], where the convergence is in $L_p(\Omega)$, $1 < p < \alpha$. If $E(\eta|\xi_t, t\in T_n) \in L_n$ for every n, where L_n is the linear space of $\{\xi_t, t\in T_n\}$, then $E(\eta|\xi_t, t\in T_\infty)$ will belong to the linear space of $\{\xi_t, t\in T_\infty\}$ and consequently the regression $E(\eta|\xi_t, t\in T)$ will be linear.

We have seen two cases where the regressions $E(\eta|\xi_t, t\in T_n)$ are always linear: when the process is (1) α-sub-Gaussian (Example 1.3.2) and (2) independent (Corollary 1.3.6). Case (2) is interesting when $\{\xi_t, t\in T\}$ is a SαS process with independent increments, $T = [0,\infty)$, and $\xi_0 = 0$. For then $\sigma(\xi_{t_1},\ldots,\xi_{t_n}) = \sigma(\xi_{t_1},\xi_{t_2}-\xi_{t_1},\ldots,\xi_{t_n}-\xi_{t_{n-1}})$, $0 \le t_1\le\ldots\le t_n$, and therefore

$$E(\eta|\xi_{t_1},\ldots,\xi_{t_n}) = a_1\xi_{t_1} + a_2(\xi_{t_2}-\xi_{t_1})+\ldots+a_n(\xi_{t_n}-\xi_{t_{n-1}}) \text{ a.s.}$$

In both cases (1) and (2) we have expressions for the regression coefficients a_j and can therefore write $E(\eta|\xi_t, t\in T)$ as the limit of a sequence of finite linear combinations of elements in $\{\xi_t, t\in T\}$. We shall further investigate case (2) in Section 3 of the following chapter.

4. Moments.

It is known that if ξ is an SαS random variable and $p > 0$, then $E|\xi|^p < \infty$ if and only if $0 < p < \alpha$. We consider here the question of determining the values of positive constants $p_1,\ldots,p_n$ such that

$$(*) \qquad E(|\xi_1|^{p_1}\ldots|\xi_n|^{p_n}) < \infty$$

when $\xi_1,\ldots,\xi_n$ are jointly SαS random variables. Clearly if they are independent the necessary and sufficient condition is $0 < p_i < \alpha$, $i = 1,\ldots,n$. If $n = 2$ and ξ_1 and ξ_2 are dependent, we show that (*) is equivalent to $0 < p_1+p_2 < \alpha$ (Corollary 1.4.5). For general n the necessary and sufficient condition on the p_i's is the same,

$$0 < p_1+\ldots+p_n < \alpha$$

provided the SαS distribution in R^n satisfies a condition which we shall call n-fold dependence (Theorem 1.4.4).

Jointly SαS random variables $\xi_1,\ldots,\xi_n$ with spectral measure Γ are called *n-fold dependent* if

$$\Gamma\{x\in S: x_1\ldots x_n \neq 0\} > 0 .$$

This condition will often be satisfied and in fact fails to hold only when Γ is supported by a rather particular region of S having (n-1)-dimensional Lebesgue measure zero. It is clear from Theorem 1.2.1 that 2-fold dependence is equivalent to dependence, but that for $n \geq 3$, n-fold dependence is stronger than dependence (*i.e.*, non-independence). In Lemma 1.4.3 we prove an interesting characterization of n-fold dependence.

We shall begin with a result that gives a condition for the existence of moments in terms of the c.f., which is assumed only to be real-valued and not necessarily SαS. This theorem extends a special case of Theorem 2 of [Wolfe 1973] to a multivariate distribution and is proved by likewise extending the method of Wolfe's proof.

1.4.1 *THEOREM. Let* $\xi_1,\dots,\xi_n$ *be random variables with real-valued joint c.f.* ϕ *and suppose that* $0 < p_k < 2$ *for* $k=1,\dots,n$. *Then* $E(|\xi_1|^{p_1}\dots|\xi_n|^{p_n}) < \infty$ *if and only if*

$$\int_0^\varepsilon \cdots \int_0^\varepsilon \Big\{2^{n-1}\Big[1 - \sum_{k=1}^{n} \phi(\dots 0\dots, 2z_k, \dots 0 \dots)\Big] + 2^{n-2} \sum_{\substack{j<k \\ c_1\in\{0,1\}}} \phi(\dots 0\dots, 2z_j, \dots 0 \dots, (-1)^{c_1} 2z_k, \dots 0 \dots) \tag{*}$$

$$- \dots + (-1)^n \sum_{c_1,\dots,c_{n-1}\in\{0,1\}} \phi(2z_1, (-1)^{c_1} 2z_2, \dots, (-1)^{c_{n-1}} 2z_n)\Big\} \frac{dz_1 dz_2 \dots dz_n}{z_1^{1+p_1} z_2^{1+p_2} \dots z_n^{1+p_n}} < \infty$$

for some $\varepsilon > 0$.

Proof: We shall use the following elementary trigonometric identity:

$$2^{2n-1}\sin^2 z_1 \sin^2 z_2 \dots \sin^2 z_n$$

$$= 2^{n-1}\Big[1 - \sum_{k=1}^{n} \cos 2z_k\Big] + 2^{n-2} \sum_{\substack{j<k \\ c_1\in\{0,1\}}} \cos(2z_j + (-1)^{c_1} 2z_k)$$

$$- 2^{n-3} \sum_{\substack{i<j<k \\ c_1,c_2\in\{0,1\}}} \cos(2z_i + (-1)^{c_1} 2z_j + (-1)^{c_2} 2z_k)$$

$$+ \dots + (-1)^n \sum_{c_1,\dots,c_{n-1}\in\{0,1\}} \cos(2z_1 + (-1)^{c_1} 2z_2 + \dots + (-1)^{c_{n-1}} 2z_n) .$$

Thus if μ is the measure induced on R^n by $(\xi_1,\ldots,\xi_n)$, then the integral (*) can be written as

$$2^{2n-1}\int_0^{\varepsilon}\cdots\int_0^{\varepsilon}\int_{R^n}\sin^2 r_1z_1\sin^2 r_2z_2\ldots\sin^2 r_nz_n\,d\mu(r_1,\ldots,r_n)\,\frac{dz_1\,dz_2\ldots dz_n}{z_1^{1+p_1}z_2^{1+p_2}\ldots z_n^{1+p_n}}$$

$$= 2^{2n-1}\int_{R^n}|r_1|^{p_1}|r_2|^{p_2}\ldots|r_n|^{p_n}\int_0^{|r_1|\varepsilon}\cdots\int_0^{|r_n|\varepsilon}\sin^2 y_1\ldots\sin^2 y_n\,\frac{dy_1\ldots dy_n}{y_1^{1+p_1}\ldots y_n^{1+p_n}}\,d\mu(r_1,\ldots,r_n)\ .$$

Sufficiency. If the condition of the theorem holds, then

$$\int_{|r_1|\geq 1}\cdots\int_{|r_n|\geq 1}|r_1|^{p_1}\ldots|r_n|^{p_n}d\mu(r_1,\ldots,r_n)\int_0^{\varepsilon}\cdots\int_0^{\varepsilon}\sin^2 y_1\ldots\sin^2 y_n\,\frac{dy_1\ldots dy_n}{y_1^{1+p_1}\ldots y_n^{1+p_n}}$$

$$\leq \int_{|r_1|\geq 1}\cdots\int_{|r_n|\geq 1}|r_1|^{p_1}\ldots|r_n|^{p_n}\int_0^{|r_1|\varepsilon}\cdots\int_0^{|r_n|\varepsilon}\sin^2 y_1\ldots\sin^2 y_n\,\frac{dy_1\ldots dy_n}{y_1^{1+p_1}\ldots y_n^{1+p_n}}\,d\mu(r_1,\ldots,r_n)$$

$< \infty$, so that

$$\int_{R^n}|r_1|^{p_1}\ldots|r_n|^{p_n}d\mu(r_1,\ldots,r_n) < \infty\ .$$

Necessity. The integral (*) is less than or equal to

$$2^{2n-1}\int_{R^n}|r_1|^{p_1}\cdots|r_n|^{p_n}d\mu(r_1,\ldots,r_n)\int_0^\infty \sin^2 y_1 \frac{dy_1}{y_1^{1+p_1}}\cdots\int_0^\infty \sin^2 y_n \frac{dy_n}{y_n^{1+p_n}},$$

which is finite if $\int_{R^n} |r_1|^{p_1}\cdots|r_n|^{p_n}d\mu(r_1,\ldots,r_n) < \infty$. □

If the condition of the theorem holds and if we let ε increase to infinity, then the integral (*) converges to

$$2^{2n-1}E(|\xi_1|^{p_1}\cdots|\xi_n|^{p_n})\int_0^\infty \sin^2 y_1 \frac{dy_1}{y_1^{1+p_1}}\cdots\int_0^\infty \sin^2 y_n \frac{dy_n}{y_n^{1+p_n}},$$

and we therefore get an expression for $E(|\xi_1|^{p_1}\cdots|\xi_n|^{p_n})$ in terms of the c.f. ϕ.

For the analysis that follows we shall transform the rectangular coordinate system used in (*) to another coordinate system in R^n that is the familiar spherical coordinate system if $n = 3$. The details of this transformation are indicated in the following lemma.

1.4.2 LEMMA. *Condition (*) of Theorem 1.4.1 can be expressed as*

$$\int_0^{\pi/2}\cdots\int_0^{\pi/2}\int_0^{\varepsilon}\Big\{2^{n-1} - 2^{n-1}\sum_{k=1}^{n}\phi(\ldots 0\ldots,2rr_k(\theta),\ldots 0\ldots)$$

$$+ 2^{n-2}\sum_{\substack{j<k\\ c_1\in\{0,1\}}}\phi(\ldots 0\ldots,2rr_j(\theta),\ldots 0\ldots,(-1)^{c_1}2rr_k(\theta),\ldots 0\ldots)$$

$$+\cdots+(-1)^n\sum_{c_1,\ldots,c_{n-1}\in\{0,1\}}\phi(2rr_1(\theta),(-1)^{c_1}2rr_2(\theta),\ldots,(-1)^{c_{n-1}}2rr_n(\theta))\Big\}$$

$$\frac{dr}{r^{1+p_1+\cdots+p_n}}\;\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{\prod_{k=1}^{n-1}[(\sin\theta_k)^{1+\sum_{j=1}^k p_j}(\cos\theta_k)^{1+p_{k+1}}]} < \infty$$

for some $\varepsilon > 0$, *where* $\theta = (\theta_1, \ldots, \theta_{n-1})$ *and*

$$r_1(\theta) = \prod_{k=1}^{n-1} \sin \theta_k, \qquad r_2(\theta) = \cos \theta_1 \prod_{k=2}^{n-1} \sin \theta_k ,$$

$$r_3(\theta) = \cos \theta_2 \prod_{k=3}^{n-1} \sin \theta_k, \ldots, r_n(\theta) = \cos \theta_{n-1} .$$

Proof: Transform the region of integration of integral (*) as follows: $z_1 = rr_1(\theta)$, $z_2 = rr_2(\theta), \ldots, z_n = rr_n(\theta)$. The Jacobian of this transformation is $r^{n-1}\prod_{k=2}^{n-1}\sin^{k-1}\theta_k$, and the lemma follows by straightforward substitution. □

The next lemma characterizes n-fold dependence and has an interesting interpretation for bivariate SαS distributions.

1.4.3 *LEMMA. Let* Γ *be a finite symmetric measure on the Borel subsets of the unit sphere* S *in* R^n *and suppose that* $0 < \alpha < 2$. *Then* $\Gamma\{x \in S: x_1 \ldots x_n \neq 0\} > 0$ *if and only if*

$$2^{n-1} \sum_{k=1}^{n} \int_S |x_k r_k|^\alpha \Gamma(dx) - 2^{n-2} \sum_{\substack{j<k \\ c_1 \in \{0,1\}}} \int_S |x_j r_j + (-1)^{c_1} x_k r_k|^\alpha \Gamma(dx)$$

$$+ 2^{n-3} \sum_{\substack{i<j<k \\ c_1, c_2 \in \{0,1\}}} \int_S |x_i r_i + (-1)^{c_1} x_j r_j + (-1)^{c_2} x_k r_k|^\alpha \Gamma(dx)$$

(**)

$$+ \ldots + (-1)^{n-1} \sum_{c_1, \ldots, c_{n-1} \in \{0,1\}} \int_S |r_1 x_1 + (-1)^{c_1} r_2 x_2$$

$$+ \ldots + (-1)^{c_{n-1}} r_n x_n|^\alpha \Gamma(dx)$$

$$> 0$$

for all choices of $r_1, \ldots, r_n$ *such that* $r_1 \ldots r_n \neq 0$.

Proof: Similarly to what has been done before, define the σ-finite measure ρ on $(0,\infty)$ by $\rho(ds) = \frac{ds}{s^{1+\alpha}}$, define $\theta: (0,\infty) \times S \to R^n$ by

$\theta(s,x) = sx$, and let $G = (\rho\times\Gamma)\theta^{-1}$. Multiplying the left-hand side of inequality (**) by $\int_0^\infty (1-\cos s)\rho(ds)$, we obtain

$$\int_{R^n} \{2^{n-1}\sum_{k=1}^{n}[1-\cos r_k v_k] - 2^{n-2}\sum_{\substack{j<k \\ c_1\in\{0,1\}}}[1-\cos(r_j v_j+(-1)^{c_1} r_k v_k)]$$

$$+ 2^{n-3}\sum_{\substack{i<j<k \\ c_1,c_2\in\{0,1\}}}[1-\cos(r_i v_i+(-1)^{c_1} r_j v_j+(-1)^{c_2} r_k v_k)]$$

$$+\ldots+(-1)^{n-1}\sum_{c_1,\ldots,c_{n-1}\in\{0,1\}}[1-\cos(r_1 v_1+(-1)^{c_1} r_2 v_2+\ldots+(-1)^{c_{n-1}} r_n v_n)]\}G(dv)$$

$$= 2^{n-1}\int_{R^n}\{1 - \sum_{k=1}^{n}\cos r_k v_k + \sum_{j<k}\cos r_j v_j \cos r_k v_k$$

$$- \sum_{i<j<k}\cos r_i v_i \cos r_j v_j \cos r_k v_k$$

$$+\ldots+(-1)^n \prod_{k=1}^{n}\cos r_k v_k\}G(dv)$$

$$= 2^{n-1}\int_{R^n}(1-\cos r_1 v_1)(1-\cos r_2 v_2)\ldots(1-\cos r_n v_n)G(dv)$$

$$= 2^{n-1}\int_S\int_0^\infty (1-\cos r_1 s x_1)\ldots(1-\cos r_n s x_n)\rho(ds)\Gamma(dx)$$

$$= 2^{n-1}\int_{\{x\in S: x_1\ldots x_n\neq 0\}}\int_0^\infty (1-\cos r_1 s x_1)\ldots(1-\cos r_n s x_n)\rho(ds)\Gamma(dx) \quad ,$$

for all choices of $r_1,\ldots,r_n$ such that $r_1\ldots r_n \neq 0$.

It is then clear that (**) implies that $\Gamma\{x\in S\colon x_1\ldots x_n \neq 0\} > 0$. Conversely, for every $x\in S$ such that $x_1\ldots x_n \neq 0$ and every $r\in R^n$ such that

$r_1 \ldots r_n \neq 0$,

$$\int_0^\infty (1-\cos r_1 s x_1) \ldots (1-\cos r_n s x_n) \rho(ds) > 0 \ ,$$

so that $\Gamma\{x \in S: x_1 \ldots x_n \neq 0\} > 0$ clearly implies (**). □

It should be pointed out that the condition of Lemma 1.4.3 is not changed if we require inequality (**) to hold for only one choice of $r_1, \ldots, r_n$ such that $r_1 \ldots r_n \neq 0$, and it is also clear that the direction of inequality (**) is never reversed.

If we take $n = 2$ and $r_1 = r_2 = 1$, then Lemma 1.4.3 yields the following: $\Gamma\{x \in S: x_1 x_2 \neq 0\} = 0$ if and only if

$$\int_S |x_1+x_2|^\alpha \Gamma(dx) + \int_S |x_1-x_2|^\alpha \Gamma(dx)$$

$$= 2 \int_S |x_1|^\alpha \Gamma(dx) + 2 \int_S |x_2|^\alpha \Gamma(dx) \ .$$

If ξ_1 and ξ_2 are jointly SαS random variables with $1 < \alpha < 2$ and spectral measure Γ, this latter condition may be written in terms of a norm introduced in the next chapter:

$$||\xi_1+\xi_2||^\alpha + ||\xi_1-\xi_2||^\alpha = 2||\xi_1||^\alpha + 2||\xi_2||^\alpha \ ,$$

which is analogous to the parallelogram law for an inner product space. Indeed, if $\alpha = 2$ the equation is precisely the parallelogram law stated for two zero mean bivariate normal random variables, but if $1 < \alpha < 2$ we get that the "parallelogram law" holds for two bivariate SαS random variables if and only if they are independent (Theorem 1.2.1).

We now apply Theorem 1.4.1 to SαS random variables.

1.4.4 *THEOREM. Let* $\xi_1,\ldots,\xi_n$ *be n-fold dependent jointly* SαS *random variables with spectral measure* Γ *and* $0 < \alpha < 2$. *Then for positive numbers* $p_1,\ldots,p_n$ *we have* $E(|\xi_1|^{p_1}\ldots|\xi_n|^{p_n}) < \infty$ *if and only if* $p_1+\ldots+p_n < \alpha$.

Proof: Combining Theorem 1.4.1 and Lemma 1.4.2 and using the particular form of the c.f. ϕ, we have that $E(|\xi_1|^{p_1}\ldots|\xi_n|^{p_n} < \infty$ if and only if

$$\int_0^{\pi/2}\cdots\int_0^{\pi/2}\int_0^{\varepsilon}\{2^{n-1} - 2^{n-1}\sum_{k=1}^{n}\exp[-2^\alpha r^\alpha\int_S |r_k(\theta)x_k|^\alpha\Gamma(dx)]$$

$$+ 2^{n-2}\sum_{\substack{j<k\\ c_1\in\{0,1\}}}\exp[-2^\alpha r^\alpha\int_S |r_j(\theta)x_j+(-1)^{c_1}r_k(\theta)x_k|^\alpha\Gamma(dx)]+\ldots$$

$$+(-1)^n\sum_{c_1,\ldots,c_{n-1}\in\{0,1\}}\exp[-2^\alpha r^\alpha\int_S |r_1(\theta)x_1+(-1)^{c_1}r_2(\theta)x_2+\ldots$$

$$+(-1)^{c_{n-1}}r_n(\theta)x_n|^\alpha\Gamma(dx)]]\}$$

$$\frac{dr}{r^{1+p_1+\ldots+p_n}}\ \frac{d\theta_1\, d\theta_2\,\ldots\, d\theta_{n-1}}{\prod_{k=1}^{n-1}[(\sin\theta_k)^{1+\sum_{j=1}^{k}p_j}(\cos\theta_k)^{1+p_{k+1}}} < \infty$$

for some $\varepsilon > 0$. It is apparent from inspection of the development of this integral in Theorem 1.4.1 that the only region where convergence to a finite limit is in question is for points where r is small. At $r = 0$ the factor of the integrand in braces reduces to

$$2^{n-1} - 2^{n-1}\binom{n}{1} + 2^{n-2}\binom{n}{2}2 - 2^{n-3}\binom{n}{3}2^2+\ldots+(-1)^n 2^{n-1}$$

$$= 2^{n-1}[1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} +\ldots+(-1)^n]$$

$$= 2^{n-1}(1-1)^n = 0\ .$$

We now differentiate this factor in braces to assess the rate of convergence to zero as $r \to 0$. The resulting partial derivative with respect to r is

$$\alpha 2^{\alpha} r^{\alpha-1}[-2^{n-1}\sum_{k=1}^{n} \int_S |r_k(\theta)x_k|^{\alpha}\Gamma(dx)\phi(\ldots 0\ldots, 2rr_k(\theta), \ldots 0 \ldots)$$

$$+ 2^{n-2} \sum_{\substack{j<k \\ c_1\in\{0,1\}}} \{\int_S |r_j(\theta)x_j + (-1)^{c_1} r_k(\theta)x_k|^{\alpha}\Gamma(dx)$$

$$\times \phi(\ldots 0 \ldots, 2rr_j(\theta), \ldots 0 \ldots, (-1)^{c_1} 2rr_k(\theta), \ldots 0 \ldots)\}$$

$$\vdots$$

$$+(-1)^n \sum_{c_1,\ldots,c_{n-1}\in\{0,1\}} \{\int_S |r_1(\theta)x_1 + (-1)^{c_1} r_2(\theta)x_2 + \ldots$$

$$+(-1)^{c_{n-1}} r_n(\theta)x_n|^{\alpha}\Gamma(dx)\phi(2rr_1(\theta), (-1)^{c_1} 2rr_2(\theta), \ldots, (-1)^{c_{n-1}} 2rr_n(\theta))\}].$$

From the common factor $r^{\alpha-1}$ in this derivative it is evident that the factor of the integrand in braces is of order $O(r^{\alpha})$ as $r \to 0$. But it is possible that the convergence to zero with r is even faster if the other factor in the derivative converges to zero as $r \to 0$. However, at $r = 0$ this other factor in brackets is

$$-2^{n-1}\sum_{k=1}^{n} \int_S |r_k(\theta)x_k|^{\alpha}\Gamma(dx) + 2^{n-2} \sum_{\substack{j<k \\ c_1\in\{0,1\}}} \int_S |r_j(\theta)x_j + (-1)^{c_1} r_k(\theta)x_k|^{\alpha}\Gamma(dx)$$

$$-\ldots+(-1)^n \sum_{c_1,\ldots,c_{n-1}\in\{0,1\}} \int_S r_1(\theta)x_1 + (-1)^{c_1} r_2(\theta)x_2 + \ldots$$

$$+(-1)^{c_{n-1}} r_n(\theta)x_n|^{\alpha}\Gamma(dx) ,$$

which is nonzero for all $\theta = (\theta_1, \ldots, \theta_{n-1}) \in (0, \frac{\pi}{2})^{n-1}$ by Lemma 1.4.3 since the functional values $r_k(\theta)$ are nonzero for such θ (Lemma 1.4.2).

It is clear from Fubini's theorem that if the integral is finite then the factor of the integrand in braces is integrable over $(0,\varepsilon)$ with respect to the measure $dr/r^{1+p_1+\ldots+p_n}$. Since this factor of the integrand is of order $O(r^\alpha)$ and of no smaller order as $r \to 0$, if the integral is finite then $p_1+\ldots+p_n < \alpha$.

The sufficiency of the condition $p_1+\ldots+p_n < \alpha$ is clear, since

$$E(|\xi_1|^{p_1}\ldots|\xi_n|^{p_n})$$

$$\leq [E(|\xi_1|^{p_1+\ldots+p_n})]^{\frac{p_1}{p_1+\ldots+p_n}} \ldots [E(|\xi_n|^{p_1+\ldots+p_n})]^{\frac{p_n}{p_1+\ldots+p_n}}$$

by Hölder's inequality. □

The following corollary was conjectured by Holger Rootzén and is an immediate consequence of Theorem 1.4.4 for the case n = 2 and Theorem 1.2.1.

1.4.5 *COROLLARY. Let* ξ_1 *and* ξ_2 *be dependent jointly* SαS *random variables,* $0 < \alpha < 2$, *and let* $p_1 > 0$ *and* $p_2 > 0$ *be given. Then* $(|\xi_1|^{p_1}|\xi_2|^{p_2}) < \infty$ *if and only if* $p_1 + p_2 < \alpha$.

In Theorem 1.4.1 and the succeeding remark we have seen how to compute the absolute moments of monomials in $\xi_1,\ldots,\xi_n$ when their joint c.f. ϕ is real-valued. We shall conclude this section with an expression for $E[(\xi_1)^{p_1}(\xi_2)^{p_2}]$ in terms of ϕ when the corresponding absolute moment is finite. Such moments will be of interest in the next chapter for random variables ξ_1 and ξ_2 having finite absolute moments of order p, $1 < p < 2$, with $p_1 = 1$ and $p_2 = p-1$.

1.4.6 *THEOREM. Let* ξ_1 *and* ξ_2 *be random variables with real-valued joint c.f.* ϕ *such that* $E(|\xi_1|^{p_1}|\xi_2|^{p_2}) < \infty$, *where* $0 < p_1 < 2$ *and* $0 < p_2 < 2$, *and when* $0 < p < 2$ *let* $c(p) = 2\int_0^\infty \sin^3 y \frac{dy}{y^{1+p}}$. *Then*

$$E[(\xi_1)^{p_1}(\xi_2)^{p_2}]$$

$$= \frac{1}{32c(p_1)c(p_2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [9\phi(z_1,-z_2) - 9\phi(z_1,z_2)$$

$$+ \phi(3z_1,-3z_2) - \phi(3z_1,3z_2) - 3\phi(z_1,-3z_2)$$

$$+3\phi(z_1,3z_2) - 3\phi(3z_1,-z_2) + 3\phi(3z_1,z_2)]\frac{dz_1\, dz_2}{(z_1)^{1+p_1}(z_2)^{1+p_2}} .$$

<u>Proof</u>: As in Theorem 1.4.1 the proof is based on an elementary trigonometric identity:

$$32 \sin^3 z_1 \sin^3 z_2$$

$$= 9 \cos(z_1-z_2) - 9\cos(z_1+z_2) + \cos(3z_1-3z_2) - \cos(3z_1+3z_2)$$

$$- 3\cos(z_1-3z_2) + 3\cos(z_1+3z_2) - 3\cos(3z_1-z_2) + 3\cos(3z_1+z_2).$$

If μ is the measure induced on R^2 by (ξ_1,ξ_2), then

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} [9\phi(z_1,-z_2) - 9\phi(z_1,z_2) + \phi(3z_1,-3z_2) - \phi(3z_1,3z_2)$$

$$- 3\phi(z_1,-3z_2) + 3\phi(z_1,3z_2) - 3\phi(3z_1,-z_2) + 3\phi(3z_1,z_2)]\frac{dz_1\, dz_2}{(z_1)^{1+p_1}(z_2)^{1+p_2}}$$

$$= 32 \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{R^2} \sin^3 r_1 z_1 \sin^3 r_2 z_2 d\mu(r_1,r_2)\frac{dz_1\, dz_2}{(z_1)^{1+p_1}(z_2)^{1+p_2}}$$

$$= 32 \int_{R^2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \sin^3 r_1 z_1 \, \sin^3 r_2 z_2 \, \frac{dz_1 \, dz_2}{(z_1)^{1+p_1} (z_2)^{1+p_2}} \, d\mu(r_1, r_2)$$

$$= 32 \int_{R^2} (r_1)^{p_1} (r_2)^{p_2} \int_{-|r_1|\varepsilon}^{|r_1|\varepsilon} \int_{-|r_2|\varepsilon}^{|r_2|\varepsilon} \sin^3 y_1 \, \sin^3 y_2 \frac{dy_1 \, dy_2}{(y_1)^{1+p_1} (y_2)^{1+p_2}} \, d\mu(r_1, r_2).$$

Letting $\varepsilon \to \infty$, we get the stated expression for $E[(\xi_1)^{p_1} (\xi_2)^{p_2}]$. □

II. A FUNCTION SPACE APPROACH TO SαS PROCESSES

It is a customary method in the study of second order stochastic processes to establish an isomorphism between (a subspace of) the linear space of a process and a Hilbert space of functions and to translate problems formulated in terms of the process into problems in the more familiar function space. In this chapter we investigate an analogous approach to the study of p-th order processes by seeking an appropriate Banach space of functions that is isometric to (a subspace of) the linear space of a p-th order process (Section 2). When applied to a SαS process with independent increments (Section 3), our results yield the stochastic integral defined in [Schilder 1970], and when applied to an "absolutely continuous" process (Section 5), they yield a stochastic integral of more specific form which can be also regarded as a sample path integral. We conclude the chapter by exploring the possibility of using the statistical properties of the output process to identify a linear system with a known SαS input (Section 7).

1. The linear space of a process.

After defining the linear space of a p-th order process and the linear space of a SαS process, we show that the duals of both spaces have a kind of Riesz representation (Theorem 2.1.5) and that the two spaces coincide in the case of a SαS process (Proposition 2.1.2). These results are essential to our later development of stochastic integrals with respect to SαS processes.

Let $\xi_t = \{\xi_t, t\in T\}$ be a stochastic process with underlying probability space (Ω,F,P). If $\xi_t \in L_p(\Omega)$ for all $t\in T$ where $0 < p < \infty$, then we call ξ a p-th order process. Unless otherwise specified, throughout this chapter we make the restrictions $1 < p < \infty$ for p-th order processes and $1 < \alpha < 2$ for SαS processes.

Let $\ell(\xi)$ be the space of all finite linear combinations of elements of $\{\xi_t, t\in T\}$. If ξ is a p-th order process, then define a norm on $\ell(\xi)$ by

$$||\zeta|| = (E|\zeta|^p)^{1/p}$$

for all $\zeta\in\ell(\xi)$. If ξ is a SαS process, then for every $\zeta\in\ell(\xi)$ it follows from result 1.1.2 that there exists some $b \geq 0$ such that $E(e^{ir\zeta}) = e^{-b_\zeta |r|^\alpha}$ for all $r\in R$. Then

$$||\zeta|| = b_\zeta^{1/\alpha}$$

defines a norm on $\ell(\xi)$ ([Schilder 1970, Corollary 2.1]). It is a consequence of the continuity theorem for characteristic functions that convergence with respect to this norm is equivalent to convergence in probability. For a p-th order or a SαS process ξ we let $L(\xi)$ denote the completion of $\ell(\xi)$ with respect to its norm. If ξ is SαS, we shall show that $L(\xi)$ is a SαS family by using characterization 1.1.3 of the SαS c.f. due to Kuelbs.

2.1.1 *PROPOSITION. If ξ is a* SαS *process, then $L(\xi)$ is a family of jointly* SαS *random variables.*

Proof: For each $j = 1,\dots,k$ let $\{\xi_n^{(j)}\}$ be a Cauchy sequence in $\ell(\xi)$.

Then $\{\xi_n^{(j)}\}$ is Cauchy in probability and hence converges in probability to some real-valued random variable $\xi^{(j)}$. It follows that the sequence of random vectors $\{(\xi_n^{(1)},\ldots,\xi_n^{(k)})\}$ converges in probability to $(\xi^{(1)},\ldots,\xi^{(k)})$. For each n let Λ_n be a homogeneous negative-definite function of order α on R^k such that $\exp\{-\Lambda_n^\alpha(y)\}$ is the joint c.f. of $(\xi_n^{(1)},\ldots,\xi_n^{(k)})$. By the continuity theorem for c.f.'s,

$$\Lambda_n^\alpha(y) \to \Lambda^\alpha(y)$$

for all $y \in R^k$, where $\exp\{-\Lambda^\alpha(y)\}$ is the joint c.f. of $(\xi^{(1)},\ldots,\xi^{(k)})$. It is easy to verify that Λ is a homogeneous negative-definite function of order α which is continuous on R^k, and the result follows from 1.1.3.

□

Clearly we may regard a SαS process ξ as a p-th order process where $1 \le p < \alpha$, and in fact, an application of Theorem 2 of [Wolfe 1973] shows that the two norms on $L(\xi)$ are equivalent.

2.1.2 *PROPOSITION. If ζ is a* SαS *random variable with c.f.* $\phi(r) = e^{-||\zeta||^\alpha |r|^\alpha}$, *then*

$$(E|\zeta|^p)^{1/p} = c(p,\alpha)||\zeta|| \ ,$$

where $1 \le p < \alpha$ *and*

$$c(p,\alpha) = \left[\frac{2^{p-1}\int_0^\infty s^{-\frac{p}{\alpha}-1}(1-e^{-s})ds}{\alpha\int_0^\infty v^{-p-1}\sin^2 v\, dv}\right]^{1/p} .$$

This proposition shows that for a SαS process ξ, $L(\xi)$ is the completion of $\ell(\xi)$ with respect to either norm.

<u>Proof</u>: By Theorem 2 of [Wolfe 1973],

$$(*) \qquad E|\zeta|^p = \frac{\int_0^\infty r^{-p-1}[1-\phi(2r)]dr}{2\int_0^\infty v^{-p-1}\sin^2 v\, dv} .$$

Now

$$\int_0^\infty r^{-p-1}[1-\phi(2r)]dr = \int_0^\infty r^{-p-1}(1 - e^{-||\zeta||^\alpha |2r|^\alpha})dr$$

$$= \frac{2^p}{\alpha} ||\zeta||^p \int_0^\infty s^{-\frac{p}{\alpha}-1}(1-e^{-s})ds .$$

Substituting back into (*), we get the stated result. □

If M is a Banach space of p-th order random variables, then for each $\zeta \in M$ we define a continuous linear functional A_ζ on M by

$$A_\zeta(\eta) = E[\eta(\zeta)^{p-1}]$$

for all $\eta \in M$.

If M is a Banach space of SαS random variables, then for each $\zeta \in M$ we define a functional $A_\zeta : M \to R$ by

$$A_\zeta(\eta) = C_{\eta,\zeta}$$

for all $\eta \in M$. The linearity of A_ζ in this case follows from Corollary 1.3.4 and the linearity of conditional expectations. For the continuity of A_ζ note that

$$|A_\zeta(\eta)| = |\int_S x_1(x_2)^{\alpha-1}\Gamma_{\eta,\zeta}(dx)|$$

$$\leq \left[\int_S |x_1|^\alpha \Gamma_{\eta,\zeta}(dx)\right]^{1/\alpha} \left[\int_S |x_2|^\alpha \Gamma_{\eta,\zeta}(dx)\right]^{\frac{\alpha-1}{\alpha}} = ||\eta||\ ||\zeta||^{\alpha-1}$$

by Hölder's inequality. Thus A_ζ is a continuous linear functional on $L(\xi)$ with $||A_\zeta||_{L(\xi)*} = ||\zeta||^{\alpha-1}$.

We now show that the continuous linear functionals thus defined on $L(\xi)$ represent the dual space $L(\xi)*$. Although no reference for this result is known to us, its proof is analogous to the argument used for the Riesz representation for continuous linear functionals on a Hilbert space.

2.1.4 *LEMMA.* [Singer 1970, Corollary 3.5 and Theorem 1.11]. *If* M *is a closed linear subspace of* $L(\xi)$ *and* $\eta_1 \epsilon L(\xi) - M$, *then there exists a unique* $\eta_2 \epsilon M$ *such that* $||\eta_1 - \eta_2|| = \inf_{\eta \epsilon M} ||\eta_1 - \eta||$. *Moreover,* $A_{\eta_1 - \eta_2}(\eta) = 0$ *for every* $\eta \epsilon M$.

2.1.5 *THEOREM. Let* ξ *be either an* α*-th order process or a* SαS *process. If* A *is a continuous linear functional on* $L(\xi)$, *then there exists a unique* $\zeta \epsilon L(\xi)$ *such that* $A = A_\zeta$ *(and hence* $||A||_{L(\xi)*} = ||\zeta||^{\alpha-1}$).

Proof: Consider $M = \{\eta \epsilon L(\xi) : A(\eta) = 0\}$, a subspace of $L(\xi)$. If $M = L(\xi)$, take $\zeta = 0$. Otherwise, choose $\eta_1 \epsilon L(\xi) - M$, let η_2 be the best approximation to η_1 in M, define $\eta_3 = \dfrac{\eta_1 - \eta_2}{||\eta_1 - \eta_2||}$, and take $\zeta = [A(\eta_3)]^{\frac{1}{\alpha-1}} \eta_3$. For every $\eta \epsilon L(\xi)$ write

$$\eta = \eta - \frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} \zeta + \frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} \zeta .$$

Note that $\eta - \frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} \zeta \epsilon M$. Thus

$$A_\zeta(\eta) = A_\zeta\left(\frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} \zeta\right)$$

by Lemma 2.1.4. Therefore

$$A_\zeta(\eta) = \frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} A_\zeta(\zeta) = \frac{A(\eta)}{|A(\eta_3)|^{\frac{\alpha}{\alpha-1}}} ||\zeta||^\alpha = A(\eta) .$$

The uniqueness of ζ follows from Hölder's inequality, since $A_\zeta = A_{\zeta_0}$ implies that

$$A_{\zeta_0}\left(\frac{\zeta}{||\zeta||}\right) = A_\zeta\left(\frac{\zeta}{||\zeta||}\right) = ||\zeta||^{\alpha-1} = ||\zeta_0||^{\alpha-1} . \qquad \square$$

We now obtain some auxiliary results which will be useful in the development of the stochastic integral.

2.1.6 *PROPOSITION. Let* $\{\zeta_n\}_{n=1}^\infty$ *be a sequence in* $L(\xi)$ *such that* $\{A_\eta(\zeta_n)\}_{n=1}^\infty$ *converges for every* $\eta \epsilon L(\xi)$. *Then* $\{\zeta_n\}$ *converges weakly to some* $\zeta \epsilon L(\xi)$.

Proof: For every n let $Q_n \epsilon L(\xi)^{**}$ be defined by

$$Q_n(\xi^*) = \xi^*(\zeta_n)$$

for all $\xi^* \epsilon L(\xi)^*$. By [Rudin 1973, Theorem 2.8], Theorem 2.1.5, and our hypothesis, we can define another element Q of $L(\xi)^{**}$ by

$$Q(\xi^*) = \lim_{n\to\infty} Q_n(\xi^*) ,$$

for all $\xi^* \epsilon L(\xi)^*$. Since $L(\xi)$ is reflexive, it follows that $\{\zeta_n\}$ converges weakly to ζ, where ζ corresponds to Q under the isomorphism between $L(\xi)$ and $L(\xi)^{**}$. □

In the development which follows we apply a result found in [Cudia 1964] to show that the map $\zeta \mapsto A_\zeta$ from $L(\xi)$ onto $L(\xi)^*$ is continuous with respect to the norm topologies (Proposition 2.1.7 and Theorem 2.1.10). This result is used to obtain sufficient conditions for a SαS process to have weak right limits (Proposition 2.1.11).

2.1.7 *PROPOSITION. For every* $f \epsilon L_p(Y,T,\nu) = L_p(\nu)$, $1 < p < \infty$, *let*

$$A_f(g) = \int_Y g(f)^{p-1} d\nu$$

for all $g \epsilon L_p(\nu)$ *define a continuous linear functional on* $L_p(\nu)$. *Then the map* $L_p(\nu) \to L_p(\nu)^*$ *defined by* $f \mapsto A_f$ *is continuous with respect to the norm topologies of* $L_p(\nu)$ *and* $L_p(\nu)^*$.

Proof: We can easily check that the function

$$\frac{|(x)^{p-1} - (y)^{p-1}|}{|x-y|^{p-1}}$$

is bounded on $R^2 - \{x=y\}$ by $c = 2^{2-p}$; so

$$|(x)^{p-1} - (y)^{p-1}| \leq c|x-y|^{p-1}$$

for all $(x,y) \epsilon R^2$. The result then follows from the following inequality:

$$||A_f - A_g||_{L_p^*(\nu)} = \left(\int_Y |(f)^{p-1} - (g)^{p-1}|^{\frac{p}{p-1}} d\nu\right)^{\frac{p-1}{p}}$$

$$\leq \left(\int_Y c^{\frac{p}{p-1}} |f-g|^p d\nu\right)^{\frac{p-1}{p}} = c||f-g||_{L_p(\nu)}^{p-1} .$$ □

Of course this result applies to the map $\zeta \mapsto A_\zeta$ from $L_p(\Omega,F,P)$ to $L_p(\Omega,F,P)^*$ where

$$A_\zeta(\eta) = E[\eta(\zeta)^{p-1}] ,$$

but we shall also need the norm continuity of the map $\zeta \mapsto A_\zeta$ when $\zeta \epsilon L(\xi)$, ξ is a SαS process, and

$$A_\zeta(\eta) = \int_S x_1(x_2)^{\alpha-1}\Gamma_{\eta,\zeta}(dx)$$

for all $\eta \epsilon L(\xi)$. In [Cudia 1964] the continuity of such maps is related to properties of the norm, and by applying the development there together with Proposition 2.1.2 we shall obtain the continuity of $\zeta \mapsto A_\zeta$ for ζ belonging to a SαS family such as $L(\xi)$.

Let X be a Banach space, let $U = \{x\epsilon X: ||x|| \leq 1\}$, let $C = \{x\epsilon X: ||x|| = 1\}$, and let $C^* = \{f\epsilon X^*: ||f|| = 1\}$. For any $x\epsilon C$ let E_x be the set of elements $f\epsilon C^*$ such that $\{y: f(y) = r\}$ is a hyperplane of support of U at x for some $r > 0$. Then the set E_x is called the *spherical image* of x. We say that the norm on X is *Fréchet differentiable* if for all $x\epsilon C$ the limit

$$\lim_{r\to 0} \frac{||x+ry|| - ||x||}{r}$$

exists and the convergence is uniform as y varies in C.

2.1.8 *LEMMA*. [Cudia 1964, Corollary 4.12]. *The norm on* X *is Fréchet differentiable if and only if the spherical image map* E *defined on* C *is single-valued and continuous from the norm topology on* C *into the norm topology on* C*.

Notice that the set-valued function E in the lemma is taken to be a mapping into C^* when the image is a singleton set. We shall apply the result when $X = M$, a space of jointly SαS random variables, and begin by examining the spherical image map in this setting.

2.1.9 *PROPOSITION. If* M *is a linear space of* α*-th order random variables or jointly* SαS *random variables, then* $E_\zeta = A_\zeta$ *for all* $\zeta \in M$, $||\zeta|| = 1$.

Proof: Let $\hat{U} = \{\eta \in M: ||\eta|| \leq 1\}$ and recall that $||A_\zeta||_{M^*} = ||\zeta||^{\alpha-1} = 1$. Given any $\eta \in U$,

$$A_\zeta(\eta) \leq |A_\zeta(\eta)| \leq ||\zeta||^{\alpha-1}||\eta|| = ||\eta|| \leq 1 ,$$

with equality implying that $\zeta = \eta$ by Hölder's inequality. Therefore, $A_\zeta(\eta) < 1$ for $\eta \in U$, $\eta \neq \zeta$, and hence

$$\{\eta \in M: A_\zeta(\eta) = 1\}$$

is a hyperplane of support of U at ζ. □

2.1.10 *THEOREM. If* M *is a linear space of jointly* SαS *random variables, then the map* $\zeta \mapsto A_\zeta$ *from* M *into* M* *is continuous with respect to the norm topologies.*

Proof: If we regard M as a subspace of $L_p(\Omega,F,P)$ for some p such that $1 < p < \alpha$, then Propositions 2.1.7 and 2.1.9 together with Lemma 2.1.8 imply that the $L_p(\Omega)$ norm on M is Fréchet differentiable. It is therefore clear from Proposition 2.1.2 that the norm $||\cdot||$ on M satisfying $||\eta|| = -\log[E(e^{i\eta})]$ is Fréchet differentiable, and thus the map $\zeta \mapsto A_\zeta$ from C into C^* is norm continuous, again by Lemma 2.1.8 and Proposition 2.1.9.

We complete the proof by showing that the map $\zeta \mapsto A_\zeta$ from M into M^* is also norm continuous. Indeed, if $\zeta_n \to \zeta \neq 0$ in M, then

$$||A_\zeta - A_{\zeta_n}||_{M^*}$$

$$= \left|\left| ||\zeta||^{\alpha-1}\left(A_{\frac{\zeta}{||\zeta||}} - A_{\frac{\zeta_n}{||\zeta_n||}}\right) + (||\zeta||^{\alpha-1} - ||\zeta_n||^{\alpha-1})A_{\frac{\zeta_n}{||\zeta_n||}}\right|\right|_{M^*}$$

$$\leq ||\zeta||^{\alpha-1}\left|\left|A_{\frac{\zeta}{||\zeta||}} - A_{\frac{\zeta_n}{||\zeta_n||}}\right|\right|_{M^*} + \left|||\zeta||^{\alpha-1} - ||\zeta_n||^{\alpha-1}\right|\left|\left|A_{\frac{\zeta_n}{||\zeta_n||}}\right|\right|_{M^*}$$

$\to 0$ as $n \to \infty$; and if $\zeta_n \to 0$ in M, then

$$||A_{\zeta_n}||_{M^*} = ||\zeta_n||^{\alpha-1} \to 0 \text{ as } n \to \infty. \qquad \square$$

Under the conditions of Theorem 2.1.10 it can in fact be shown that the map $\zeta \mapsto A_\zeta$ is uniformly continuous, but we shall have no use for this stronger result.

We conclude this section by obtaining sufficient conditions for a SαS process to have weak right limits (assumption (a1) of Section 2). This proposition will be used in Section 3 to define $\int f d\xi$ when ξ is SαS with independent increments.

2.1.11 *PROPOSITION. Suppose that* $\xi = \{\xi_t,\ a \leq t \leq b\}$ *is a* SαS *process with* $1 < \alpha < 2$. *Let* $\xi_{t'} = 0$ *for some* t' *in order that we may direct our attention toward the elements, rather than the increments, of the process. Then the following two conditions imply that for each* $\zeta \in L(\xi)$, *the right limit* $F_\zeta(t+0)$ *exists at every* $t \in [a,b)$, *where* $F_\zeta(t) = A_\zeta(\xi_t)$.

($\alpha 1$) For every $s \in [a,b)$ *there exists an* $\varepsilon_s > 0$ *and an* $M_s < \infty$ *such that* $||\xi_t|| \leq M_s$ *whenever* $s < t \leq s + \varepsilon_s$.

($\alpha 2$) For every $\zeta \in \ell(\xi)$, F_ζ *is of bounded variation on* T.

Notice that condition ($\alpha 1$) is necessary for the existence of the right limits $F_\zeta(t+0)$ and that conditions ($\alpha 1$) and ($\alpha 2$) together are apparently weaker than assuming that ξ is of weak bounded variation.

Proof: Fix $\zeta \in L(\xi)$ and let $\{\zeta_n\} \subset \ell(\xi)$ be such that $\zeta_n \to \zeta$. By ($\alpha 2$), $A_{\zeta_n}(\xi_t)$ is a function of bounded variation on [a,b] for each n; so $D_n(t) = \lim_{\varepsilon_m \downarrow 0} A_{\zeta_n}(\xi_{t+\varepsilon_m})$ exists for all $t \in [a,b)$. For fixed $s \in [a,b)$, apply ($\alpha 1$) to get $\varepsilon_s > 0$ and $M_s < \infty$ such that $||\xi_t|| \leq M_s$ whenever $s < t \leq s + \varepsilon_s$. We shall use Theorem 2.1.10 to show that $\{A_{\zeta_n}\}$ is uniformly convergent on $\{\xi_t\colon s < t \leq s + \varepsilon_s\}$. Indeed, let $\varepsilon > 0$ be given and choose N such that $n \geq N$ implies $||A_{\zeta_n} - A_\zeta||_{L(\xi)^*} < \frac{\varepsilon}{M_s}$. Then for every $t \in (s, s+\varepsilon_s]$ and $n \geq N$,

$$|A_{\zeta_n}(\xi_t) - A_\zeta(\xi_t)| \leq ||A_{\zeta_n} - A_\zeta||_{L(\xi)^*}||\xi_t|| < \varepsilon,$$

and hence the desired uniform convergence. Now by a standard result for a uniformly convergent sequence of functions,

$$\lim_{\varepsilon_m \downarrow 0} A_\zeta(\xi_{s+\varepsilon_m}) = \lim_{\varepsilon_m \downarrow 0} \lim_{n\to\infty} A_{\zeta_n}(\xi_{s+\varepsilon_m}) = \lim_{n\to\infty} D_n(s)$$

exists, and thus the right limit $F_\zeta(s+0)$ exists. □

2. The integral $\int_T f(t)d\xi_t$.

We begin this section by defining the stochastic integral $\int_T f(t)d\xi_t$ for an appropriate class of (deterministic) "functions" f and obtain, under certain (smoothness) conditions, an integral representation for the elements of $L(\xi)$. Our approach is motivated by [Huang 1975] where the case $\alpha = 2$ is treated.

Let $\xi = \{\xi_t, t\in T\}$ be either an α-th order or a SαS process with $T = [a,b]$. For each $\zeta\in L(\xi)$ define the real-valued function F_ζ on T by $F_\zeta(t) = A_\zeta(\xi_t)$ for all $t\in T$. We shall begin with two assumptions, (a1) and (a2), on ξ .

(a1) For every $\zeta\in L(\xi)$, assume that the right limit $F_\zeta(t+0)$ exists for all $t\in[a,b)$.

In other words, $\lim_{\varepsilon_n\downarrow 0} A_\zeta(\xi_{t+\varepsilon_n})$ exists for every $\zeta\in L(\xi)$, so that the sequence $\{\xi_{t+\varepsilon_n}\}$ converges weakly in $L(\xi)$ by Proposition 2.1.6 and we denote its limit by ξ_{t+0}. Hence $F_\zeta(t+0) = A_\zeta(\xi_{t+0})$. Let ξ_{b+0} be equal to ξ_b and let

$$I = \left\{\zeta\in L(\xi):\ \zeta = \sum_{k=1}^{n} a_k(\xi_{t_k+0}-\xi_{t_{k-1}+0}) \text{ where } n\geq 1,\ a_k\in R, \text{ and } a=t_0<t_1<\ldots<t_n=b\right\} .$$

(a2) For every $\zeta\in I$, assume that $F_\zeta(t)$ is of bounded variation on T.

Let S be the linear space of all step functions on T of the form $f(t) = \sum_{k=1}^n f_k\chi_{(t_{k-1},t_k]}(t)$. For each such f define $\int f d\xi$ to be $\sum_{k=1}^n f_k(\xi_{t_k+0}-\xi_{t_{k-1}+0})$. Define a norm on S in terms of a Lebesgue-Stieltjes integral by

$$||f||_S^\alpha = \int_T f(t)dF_{\int f d\xi}(t) = \sum_{k=1}^n f_k \int_T \chi_{(t_{k-1},t_k]}(t)dF_{\int fd\xi}(t)$$

$$= \sum_{k=1}^n f_k A_{\int fd\xi}(\xi_{t_k+0} - \xi_{t_{k-1}+0}) = A_{\int fd\xi}(\int fd\xi) = ||\int fd\xi||^\alpha .$$

Let Λ_α be the completion of S with respect to this norm.

Every element f of Λ_α can be represented as $f = \{f_n\}$, a Cauchy sequence in S. It follows that $\{\int f_n d\xi\}$ is a Cauchy sequence in $L(\xi)$, and we will denote its limit by $\int fd\xi$. Then the map $\Lambda_\alpha \to L(\xi)$ defined by $f \mapsto \int fd\xi$ is an isometry from Λ_α onto a closed subspace of $L(\xi)$. If we assume that $\xi_{t'} = 0$ for some $t' \in T$ and that the process is weakly continuous from the right, then this isometry will be onto $L(\xi)$.

Suppose now that the process ξ is of *weak bounded variation* (see [Shachtman 1970]); *i.e.*, we assume that $F_\zeta(t)$ is of bounded variation on T for all $\zeta \in L(\xi)$. This condition is clearly stronger than (a1) and (a2), since functions of bounded variation have left and right limits at all points. Let S' be the space of all bounded measurable functions $f: T \to R$ for which there exists some $\eta \in L(\xi)$ such that the Lebesgue-Stieltjes integral $\int_T f(t)dF_\zeta(t)$ equals $A_\zeta(\eta)$ for every $\zeta \in L(\xi)$. Since the dual separates points on $L(\xi)$, we see by Theorem 2.1.5 that f uniquely determines η and we denote the latter by $\int fd\xi$. It is immediate that S' is a linear space containing S and that the definition of $\int fd\xi$ on S' extends our previous definition of $\int fd\xi$ on S. As before, define a norm on S' by

$$||f||_{S'}^\alpha = \int_T f(t)dF_{\int fd\xi}(t) = A_{\int fd\xi}(\int fd\xi) = ||\int fd\xi||^\alpha ,$$

and let Λ'_α be the completion of S' with respect to this norm. Then

$\Lambda_\alpha \subset \Lambda'_\alpha$ and as done above we can define an isometry from Λ'_α onto a closed subspace of $L(\xi)$. Our next result shows that S is *strictly* contained in S'.

2.2.1 *PROPOSITION. Let* $\xi = \{\xi_t, a \le t \le b\}$ *be an* α*-th order or a* SαS *process of weak bounded variation. Then all continuous functions on* $[a,b]$ *belong to* S'.

Proof: If f is a function on T and π is a partition of $[a,b]$ defined by $a = t_0 < t_1 < \ldots < t_m = b$, let $f_\pi(t) = \sum_{k=1}^m f(t'_k)\chi_{(t_{k-1},t_k]}(t)$, where t'_k is an arbitrary point in $[t_{k-1},t_k]$, and $\Delta(\pi) = \max_{1 \le k \le m}(t_k - t_{k-1})$. If f is a continuous function and $\{\pi_n\}$ is a sequence of partitions of $[a,b]$ with $\Delta(\pi_n) \to 0$, then $f_{\pi_n} \in S'$ for every n and

$$\int_T f(t)dF_\zeta(t) = \lim_{n\to\infty} \int_T f_{\pi_n}(t)dF_\zeta(t) = \lim_{n\to\infty} A_\zeta(\int f_{\pi_n} d\xi)$$

for all $\zeta \in L(\xi)$. Letting η be the weak limit of $\{\int f_{\pi_n} d\xi\}$ in $L(\xi)$ (Proposition 2.1.6), we see that $\int_T f(t)dF_\zeta(t) = A_\zeta(\eta)$ for all $\zeta \in L(\xi)$ and hence $f \in S'$. □

Although we have failed to resolve whether it is possible that $\Lambda_\alpha \neq \Lambda'_\alpha$ when ξ is of weak bounded variation, we now show that $\Lambda_\alpha = \Lambda'_\alpha$ when ξ is of strong bounded variation, which is defined in the usual way:

For every $t \in [a,b]$ define

$$V_\xi(t) = \sup_{a=s_0<\ldots<s_m=t} \sum_{k=1}^m ||\xi_{s_k} - \xi_{s_{k-1}}|| ,$$

where the supremum is taken over all finite partitions of $[a,t]$. If

$V_\xi(b) < \infty$, then the stochastic process ξ is said to be of *strong bounded variation* (see [Brézis 1973, p. 141]). It is easily seen that strong bounded variation implies weak bounded variation.

2.2.2 *PROPOSITION. If ξ is of strong bounded variation, then $\Lambda_\alpha = \Lambda'_\alpha$.*

Proof: For every $\zeta \in L(\xi)$ define the total variation function $|F_\zeta|$ by

$$|F_\zeta|(t) = \sup_{a=s_0<\ldots<s_m=t} \sum_{k=1}^{m} |F_\zeta(s_k) - F_\zeta(s_{k-1})| ,$$

and note that for $t_1 < t_2$,

$$|F_\zeta|(t_2) - |F_\zeta|(t_1) \le ||\zeta||^{\alpha-1}[V_\xi(t_2) - V_\xi(t_1)] .$$

Given any $f \in S'$ let $\{f_n\}_{n=1}^{\infty}$ be a sequence of step functions converging to f in $L_1(T, \mathcal{B}_T, dV_\xi)$. Then

$$\begin{aligned} ||f-f_n||_{S'}^{\alpha} &= \int_T [f(t) - f_n(t)]dF_{\int(f-f_n)d\xi}(t) \\ &\le \int_T |f(t) - f_n(t)|d|F_{\int(f-f_n)d\xi}|(t) \\ &\le ||\int(f-f_n)d\xi||^{\alpha-1}\int_T |f(t)-f_n(t)|dV_\xi(t) \\ &= ||f-f_n||_{S'}^{\alpha-1}\int_T |f(t)-f_n(t)|dV_\xi(t) . \end{aligned}$$

Therefore

$$||f-f_n||_{S'} \le \int_T |f(t) - f_n(t)|dV_\xi(t)$$

and the right-hand side converges to zero as $n \to \infty$. It follows that $S \supset S'$, so that $\Lambda_\alpha = \Lambda'_\alpha$. □

If additional conditions are placed on the process, it can be shown that other classes of functions belong to the function space Λ_α. For instance, if the process ξ is weakly continuous, *i.e.*, if $F_\zeta(t)$ is a continuous function on [a,b] for all $\zeta \in L(\xi)$, then S' contains all bounded functions that are continuous when restricted to a subset containing all but countably many points of [a,b].

3. The integral $\int_T f(t)d\xi_t$ when ξ is SαS with independent increments.

In this section we show that when ξ is a SαS process with independent increments, the function space Λ_α is an L_α space ([Schilder 1970, Theorem 3.1]) and, if M is the subspace of $L(\xi)$ isometric to Λ_α, we relate the dual elements of M to dual elements of the L_α space (Proposition 2.3.1). As a corollary, we identify a separating family of continuous linear functionals on M (Corollary 2.3.2).

Suppose that $\xi = \{\xi_t, a \le t \le b\}$ is a SαS process, $1 < \alpha < 2$, with independent increments and such that $\xi_a = 0$. By Lemma 3.2 of [Schilder 1970],

$$||\xi_{t_2} - \xi_{t_1}||^\alpha = ||\xi_{t_2}||^\alpha - ||\xi_{t_1}||^\alpha$$

if $a \le t_1 \le t_2 \le b$ and therefore $||\xi_t||^\alpha$ is an increasing function on $[a,b]$. Notice that condition (α1) of Proposition 2.1.11 is satisfied since $||\xi_t|| \le ||\xi_b||$ for all $t \in [a,b]$.

To check that condition (α2) holds, consider $\zeta \in \ell(\xi)$ and $a = t_0 < t_1 < \ldots < t_n = b$. We can clearly choose $a = s_0 < s_1 < \ldots < s_m = b$ such that $\{t_k\}_{k=0}^n \subset \{s_j\}_{j=0}^m$ and $\zeta = \sum_{j=1}^m a_j(\xi_{s_j} - \xi_{s_{j-1}})$. Write X_j for $\xi_{s_j} - \xi_{s_{j-1}}$, $1 \le j \le m$, and let $M_\zeta = ||\xi_b||^\alpha \max_{1 \le j \le m} |a_j|^{\alpha-1}$. Applying Corollary 1.2.3, we get that

$$A_\zeta(X_{j_0}) = \int_{S_2} x_1(x_2)^{\alpha-1} \Gamma_{X_{j_0}, \sum_{j=1}^m a_j X_j}(dx)$$

$$= \int_{S_m} x_{j_0} \left(\sum_{j=1}^m a_j x_j\right)^{\alpha-1} \Gamma_{X_1,\ldots,X_m}(dx)$$

$$= (a_{j_0})^{\alpha-1} ||X_{j_0}||^\alpha,$$

where $1 \le j_0 \le m$. Thus

$$\sum_{k=1}^{n} |A_\zeta(\xi_{t_k} - \xi_{t_{k-1}})| \le \sum_{j=1}^{m} |A_\zeta(\xi_{s_j} - \xi_{s_{j-1}})|$$

$$= \sum_{j=1}^{m} |A_\zeta(X_j)| = \sum_{j=1}^{m} |a_j|^{\alpha-1} ||X_j||^\alpha \le M_\zeta \quad ,$$

and the bounded variation of $F_\zeta(t)$ follows.

Because these conditions are satisfied, we know that the SαS process $\{\xi_{t+0}, a \le t \le b\}$ is well-defined by Proposition 2.1.11, and we see, after a moment's reflection, that it has independent increments. It is clear then that conditions (a1) and (a2) are satisfied; so the integral $\int f d\xi$ is defined as in the previous section.

Let f and g be two step functions in S:

$$f(t) = \sum_{j=1}^{n} f_j \chi_{(t_{j-1}, t_j]}(t) \quad \text{and} \quad g(t) = \sum_{k=1}^{n} g_k \chi_{(t_{k-1}, t_k]}(t)$$

where f and g are defined over the same intervals with no loss of generality. Let G(t) be the increasing function $||\xi_{t+0}||^\alpha$ on T. Writing X_j for $\xi_{t_j+0} - \xi_{t_{j-1}+0}$, $1 \le j \le n$, and recalling the notation of the previous section, we find

$$\int f(t) dF_{\int g d\xi}(t) = \sum_{j=1}^{n} f_j A_{\int g d\xi}(X_j) = \sum_{j=1}^{n} f_j \int x_1 (x_2)^{\alpha-1} \Gamma_{X_j, \Sigma_{k=1}^n g_k X_k}(dx)$$

$$= \sum_{j=1}^{n} f_j \int x_j \Big(\sum_{k=1}^{n} g_k x_k\Big)^{\alpha-1} \Gamma_{X_1, \ldots, X_n}(dx)$$

$$= \sum_{j=1}^{n} f_j (g_j)^{\alpha-1} ||X_j||^\alpha$$

$$= \int \sum_{j=1}^{n} f_j (g_j)^{\alpha-1} \chi_{(t_{j-1}, t_j]}(t) dG(t)$$

$$= \int f(g)^{\alpha-1} dG(t) \quad .$$

In particular, $||f||_{\Lambda_\alpha}^{\alpha} = \int f(t)dF_{\int fd\xi}(t) = \int |f(t)|^{\alpha} dG(t)$. We conclude that $\Lambda_\alpha = L_\alpha(dG)$, since the step functions are dense in both spaces.

In summary, if ξ is a SαS process with independent increments and $1 < \alpha < 2$, then we have shown that the integral $\int fd\xi$ is defined and yields an isometry between a closed subspace of $L(\xi)$ and an L_α space. We obtain therefore the result found in [Schilder 1970, Theorem 3.1] without any continuity assumptions on the process. Schilder also obtains an analogous result for the case $0 < \alpha \le 1$.

Our definition of $\int fd\xi$ in this section easily extends to the case where the stochastic process ξ is indexed by an infinite interval T. If we let S be the set of all step functions that are zero outside a compact subinterval, then we can define the norm on S in the same way as before, and the completion of S with respect to this norm will be $L_\alpha(T, \mathcal{B}_T, dG)$.

It will be useful to have an expression for certain continuous linear functionals evaluated at points of the form $\int fd\xi$.

2.3.1 *PROPOSITION. Let* $\eta = \int fd\xi$ *and* $\zeta = \int gd\xi$, *where* f *and* g *belong to* $L_\alpha(dG)$. *Then*

$$A_\zeta(\eta) = \int_T f(g)^{\alpha-1} dG .$$

<u>Proof</u>: Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences of step functions in $L_\alpha(dG)$ such that $f_n \to f$ and $g_n \to g$. Then $\int f_n d\xi \to \eta$ and $\int g_n d\xi \to \zeta$, and therefore

$$\begin{aligned} A_\zeta(\eta) &= \lim_{m\to\infty} A_\zeta(\int f_m d\xi) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} A_{\int g_n d\xi}(\int f_m d\xi) \end{aligned}$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \int_T f_m(g_n)^{\alpha-1} dG$$

$$= \lim_{m\to\infty} \int_T f_m(g)^{\alpha-1} dG$$

$$= \int_T f(g)^{\alpha-1} dG ,$$

by Theorem 2.1.10. □

Let $\xi = \{\xi_t, t\epsilon T\}$ be as above with $T = [a,\infty)$ and $\xi_{a+0} = 0$. Then if M is the closed subspace of $L(\xi)$ which is isometric to $L_\alpha(dG)$, we get from Proposition 2.3.1 that the set of continuous linear functionals $\{A_{\xi_{t+0}} : t\epsilon T\}$ separates points on M.

2.3.2 *COROLLARY. If* $\zeta\epsilon M$ *and* $A_{\xi_{t+0}}(\zeta) = 0$ *for all* $t\epsilon T$, *then* $\zeta = 0$ *in* M.

Proof: Let $f\epsilon L_\alpha(dG)$ be such that $\zeta = \int_T f(t)d\xi_t$. Then

$$A_{\xi_{t+0}}(\zeta) = 0 \qquad \text{for all } t\epsilon T$$

$$\Longrightarrow \int_{(a,t]} f(s)dG(s) = 0 \text{ for all } t\epsilon T$$

$$\Longrightarrow f = 0 \text{ a.e. } [dG]$$

$$\Longrightarrow \zeta = 0 \text{ in } M.$$ □

If ξ is weakly continuous from the right, then $L_\alpha(T,\mathcal{B}_T,dG)$ is isometric to $L(\xi)$ and Corollary 2.3.2 implies that the set of continuous linear functionals $\{A_{\xi_t} : t\epsilon T\}$ separates points on $L(\xi)$.

As a further application of the developments in this section, we reexamine a regression problem introduced in Chapter I.

2.3.3 *PROPOSITION. Let* $\{\eta, \xi_t, a<t\le b\}$ *be a family of SαS random variables,* $1 < \alpha < 2$, *such that the process* $\{\xi_t, a<t\le b\}$ *has independent increments and* $\xi_{a+0} = 0$. *For every Borel subset* B *of* (a,b] *define*

$$X(B) = \int \chi_B(t)d\xi_t ,$$

$$\mu_{\eta X}(B) = C_{\eta X(B)} ,$$

$$\mu_{XX}(B) = C_{X(B)X(B)} .$$

Then $\mu_{\eta X}$ *is a f.s.m. which is absolutely continuous with respect to the measure* μ_{XX}. *Moreover, the Radon-Nikodym derivative* $d\mu_{\eta X}/d\mu_{XX}$ *belongs to* $L_\alpha(dF)$, *where* $F(t) = ||\xi_t||^\alpha$, *and*

$$E(\eta|\xi_t, a<t\le b) = \int \frac{d\mu_{\eta X}}{d\mu_{XX}}(t)d\xi_t \quad \text{a.s.}$$

<u>Proof:</u> To see that $\mu_{\eta X}$ is countably additive, let $B = \bigcup_{i=1}^{\infty} B_i$, where the B_i's are disjoint measurable subsets of (a,b]. Then using Theorem 2.1.10 and the techniques of this section, we have that

$$\mu_{\eta X}(B) = C_{\eta X(B)} = \lim_{n\to\infty} C_{\eta \sum_{i=1}^{n} X(B_i)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} C_{\eta X(B_i)} = \sum_{i=1}^{\infty} \mu_{\eta X}(B_i) .$$

It is clear that $d\mu_{XX} = dF$ and $\mu_{\eta X} << \mu_{XX}$. Let T_∞ be a countable subset of (a,b] such that

$$E(\eta|\xi_t, a<t\le b) = E(\eta|\xi_t, t\in T_\infty) .$$

Without loss of generality, we may assume that the points in T_∞ are dense in (a,b], order them, and let T_n be the set containing the first n points.

We know from the discussion in Chapter I and Proposition 2.1.2 that

$$E(\eta|\xi_t, t\epsilon T_n) \to E(\eta|\xi_t, t\epsilon T_\infty)$$

in $L(\xi)$. If $T_n = \{t_1,\ldots,t_n\}$ where $t_1<t_2<\ldots<t_n$, then $\sigma(\xi_t, t\epsilon T_n) = \sigma(\xi_{t_1}, \xi_{t_2}-\xi_{t_1}, \ldots, \xi_{t_n}-\xi_{t_{n-1}})$ and (letting $t_0 = a$)

$$E(\eta|\xi_t, t\epsilon T_n) = \sum_{k=1}^{n} \frac{\mu_{\eta X}((t_{k-1},t_k])}{\mu_{XX}((t_{k-1},t_k])} (\xi_{t_k}-\xi_{t_{k-1}})$$

$$= \int \sum_{k=1}^{n} \frac{\mu_{\eta X}((t_{k-1},t_k])}{\mu_{XX}((t_{k-1},t_k])} \chi_{(t_{k-1},t_k]}(t)d\xi_t$$

by Corollary 1.3.6. Therefore by the isometry between $L_\alpha(dF)$ and the subspace M of $L(\xi)$, the latter "integrands" form a Cauchy sequence in $L_\alpha(dF)$ which converges to $d\mu_{\eta X}/d\mu_{XX}$ in $L_\alpha(dF)$ and a.e. [dF] ([Hewitt and Stromberg 1965]). Hence

$$E(\eta|\xi_t, a<t\le b) = \int \frac{d\mu_{\eta X}}{d\mu_{XX}}(t)d\xi_t \quad \text{a.s.}$$ □

4. Spectral representation of a SαS process.

In this section we state an interesting result due to Kuelbs (2.4.1) and derive Schilder's characterization of independence (Theorem 2.4.2) from the characterization in Theorem 1.2.1. A simple result on estimation of SαS random variables is also included (Proposition 2.4.3).

In [Schilder 1970] the integral $\int f d\zeta$, where ζ is a SαS process with independent increments, was used to obtain a "spectral representation" for a finite set of jointly SαS variables. Kuelbs has extended this result to processes indexed by an infinite set.

Two stochastic processes $\{\xi_t, t\in T\}$ and $\{\eta_t, t\in T\}$ having the same index set are called *indistinguishable* if their finite dimensional distributions are the same.

2.4.1 [Kuelbs 1973] *Let* $\{\xi_t, t\in T\}$ *be a* SαS *process that is continuous in probability with* T *an interval and* $1 < \alpha < 2$. *Then there exist a* SαS *process* $\{\zeta_\lambda, -1/2 \le \lambda \le 1/2\}$ *with independent increments and a family of functions* $\{f_t, t\in T\}$ *in* $L_\alpha([-\frac{1}{2}, \frac{1}{2}], dF)$, *where* $F(\lambda) = ||\zeta_\lambda||^\alpha$, *such that* $\{\int f_t(\lambda) d\zeta_\lambda, t\in T\}$ *and* $\{\xi_t, t\in T\}$ *are indistinguishable.*

Let us consider two jointly SαS random variables ξ_1 and ξ_2, $1 < \alpha < 2$. By 2.4.1 these variables have the same joint distribution as two random variables $\int f_1(\lambda) d\zeta(\lambda)$ and $\int f_2(\lambda) d\zeta(\lambda)$, where $\{\zeta_\lambda, -1/2 \le \lambda \le 1/2\}$ is a SαS process with independent increments, $F(\lambda) = ||\zeta_\lambda||^\alpha$, and $f_i \in L_\alpha([-1/2, 1/2], dF)$ for i = 1,2. The following condition for independence of ξ_1 and ξ_2 is expressed in terms of f_1 and f_2.

2.4.2 *THEOREM.* [Schilder 1970] *The random variables* $\int f_1(\lambda)d\zeta_\lambda$ *and* $\int f_2(\lambda)d\zeta_\lambda$ *are independent if and only if* $f_1 f_2 = 0$ a.e. [dF].

<u>Proof</u>: The random vector $(\int f_1(\lambda)d\zeta_\lambda, \int f_2(\lambda)d\zeta_\lambda)$ has c.f.

$$E\, e^{i[r_1\int f_1(\lambda)d\zeta_\lambda + r_2\int f_2(\lambda)d\zeta_\lambda]} = \exp\{-\int |r_1 f_1(\lambda)+r_2 f_2(\lambda)|^\alpha dF(\lambda)\} .$$

Let $f(\lambda) = [f_1^2(\lambda) + f_2^2(\lambda)]^{1/2}$, and define

$$g_1(\lambda) = \begin{cases} \dfrac{f_1(\lambda)}{f(\lambda)} & \text{if } f(\lambda) > 0, \\ 1 & \text{if } f(\lambda) = 0, \end{cases} \qquad g_2(\lambda) = \begin{cases} \dfrac{f_2(\lambda)}{f(\lambda)} & \text{if } f(\lambda) > 0 , \\ 0 & \text{if } f(\lambda) = 0 . \end{cases}$$

Let $S = \{(x_1,x_2): x_1^2 + x_2^2 = 1\}$, define $T: [-1/2,1/2] \to S$ by $T(\lambda) = (g_1(\lambda),g_2(\lambda))$, and define a finite measure ν on $[-1/2,1/2]$ by $\nu(d\lambda) = f^\alpha(\lambda)dF(\lambda)$. Then

$$\int |r_1 f_1(\lambda)+r_2 f_2(\lambda)|^\alpha dF(\lambda) = \int |\langle r,T(\lambda)\rangle|^\alpha \nu(d\lambda)$$

$$= \int_S |\langle r,x\rangle|^\alpha \nu T^{-1}(dx) .$$

By Theorem 1.2.1 we have independence if and only if

$$0 = \nu T^{-1}\{x\in S:\ x_1x_2 \neq 0\} = \nu\{\lambda\in[-1/2,1/2]:\ g_1(\lambda)g_2(\lambda)\neq 0\}$$

$$= \nu\{\lambda\in[-1/2,1/2]:\ f_1(\lambda)f_2(\lambda)\neq 0\}$$

$$= \int_{\{\lambda\in[-1/2,1/2]: f_1(\lambda)f_2(\lambda)\neq 0\}} f^\alpha(\lambda)dF(\lambda) .$$

Since $f(\lambda) > 0$ whenever $f_1(\lambda)f_2(\lambda) \neq 0$, the result follows. □

Notice in this theorem that the index set $[-\frac{1}{2},\frac{1}{2}]$ plays no essential role and could be any interval.

In Theorem 5.2 of [Schilder 1970] a necessary and sufficient condition is given for the best estimate of a SαS random variable by a linear combination from a finite set of SαS random variables. In the following result we solve an estimation problem in an infinite-dimensional space of random variables and provide an explicit expression for the estimator.

2.4.3 *PROPOSITION. Let* $\{\xi_t, t\in T\}$ *be a* SαS *process with independent increments,* $1 < \alpha < 2$, $\xi_{t+0} = \xi_t$ *for all* $t\in T$, *and* $F(t) = ||\xi_t||^\alpha$. *Let* I *be a subinterval of* T *and denote by* L(I) *the linear space of the increments of the process formed from all subintervals of* I. *If* $\eta = \int_T f(t)d\xi_t$, *where* $f\in L_\alpha(dF)$, *then the best approximation to* η *in* L(I) *is given by*

$$\hat{\eta} = \int_I f(t)d\xi_t .$$

<u>Proof</u>: If $I' = T-I$, then for any fixed $\zeta\in L(I)$ we can write $\zeta = \int_T g(t)d\xi_t$ where $g\in L_\alpha(dF)$ and $g(t) = 0$ if $t\in I'$. Observe that

$$\eta - \hat{\eta} = \int_{I'} f(t)d\xi_t ,$$

and therefore $\eta - \hat{\eta}$ and ζ are independent by Theorem 2.4.2. Thus $A_{\eta-\hat{\eta}}(\zeta) = \int_S x_1(x_2)^{\alpha-1}\Gamma_{\zeta,\eta-\hat{\eta}}(dx) = 0$, so that $A_{\eta-\hat{\eta}}$ annihilates L(I). Using a standard argument,

$$\begin{aligned} ||\eta-\hat{\eta}||^\alpha &= A_{\eta-\hat{\eta}}(\eta-\hat{\eta}) = A_{\eta-\hat{\eta}}(\eta) \\ &= A_{\eta-\hat{\eta}}(\eta-\zeta) \le ||\eta-\hat{\eta}||^{\alpha-1}||\eta-\zeta||, \end{aligned}$$

whence

$$||\eta-\hat{\eta}|| \le ||\eta-\zeta||$$

for all $\zeta\in L(I)$. □

5. The integral $\int_T f(t)\xi_t \nu(dt)$.

The integral $\int_T f(t)d\xi_t$ discussed in Section 2 requires two assumptions on the process ξ and consequently is somewhat restricted in its applicability. Even when these assumptions hold (such as when ξ is SαS with independent increments), there is no direct relation between the sample paths of ξ and the realizations of $\int_T f(t)d\xi_t$. In contrast, the integral $\int_T f(t)\xi_t \nu(dt)$ which is defined and discussed in this section requires less stringent assumptions on ξ and for a large class of functions f can be interpreted as a sample path integral (Theorem 2.5.5).

Throughout this section we shall consider ξ to be a general p-th order process, $p > 1$, and q to be such that $1/p + 1/q = 1$. A stochastic process $\{\xi_t, t \in T\}$ on a probability space (Ω, F, P) is called *measurable* if $(t,\omega) \mapsto \xi(t,\omega)$ is a product measurable map from $T \times \Omega$ into R. We shall investigate conditions for the existence of measurable p-th order processes in Chapter III. The following argument is taken from [Cambanis and Masry 1971] where it is applied to a second order process.

2.5.1 *PROPOSITION. Let* $\xi = \{\xi_t, t \in T\}$ *be a measurable, p-th order process with index set* T *an arbitrary interval of the real line, and let* $\alpha > p$ *be given. Then there exists a finite measure* ν *on* $(T, \mathcal{B}_T)$ *such that* ν *is equivalent to Lebesgue measure on* T *and*

$$\int_T ||\xi_t||^\alpha \nu(dt) < \infty .$$

<u>Proof</u>: Choose $g_1 \in L_1(T, \mathcal{B}_T, \text{Leb})$ such that $g_1 > 0$ a.e. [Leb] on T, and define g_2 on T by

$$g_2(t) = \begin{cases} 1 & \text{if } 0 \le ||\xi_t|| \le 1 , \\ \dfrac{1}{||\xi_t||^\alpha} & \text{if } 1 < ||\xi_t|| . \end{cases}$$

Then the measure ν on $(T,\mathcal{B}_T)$ defined by $\nu(dt) = g_1(t)g_2(t)dt$ is clearly equivalent to Lebesgue measure on T and

$$\int_T ||\xi_t||^\alpha \nu(dt) = \int_T ||\xi_t||^\alpha g_1(t)g_2(t)dt \le \int_T g_1(t)dt < \infty . \qquad \square$$

Under the conditions of Proposition 2.5.1,

$$E\int_T |\xi_t(\omega)|^p \nu(dt) = \int_T ||\xi_t||^p \nu(dt) < \infty ,$$

since $p < \alpha$ and ν is a finite measure; so the sample paths $\xi_t(\omega)$ belong to $L_p(T,\mathcal{B}_T,\nu)$ with probability one, by the measurability of ξ and Fubini's theorem. We can therefore define a stochastic process $\eta = \{\eta_t, t\epsilon T\}$ by

$$\eta_t^{(u)} = \int_{(-\infty,t)\cap T} \xi_s(\omega)\nu(ds) \quad \text{a.s.}$$

for each $t\epsilon T$ and observe that $\eta_t \epsilon L_p(\Omega)$, since

$$E|\eta_t|^p = E|\int_{(-\infty,t)\cap T} \xi_s(\omega)\nu(dt)|^p \le \{\nu[(-\infty,t)\cap T]\}^{p/q} E\int_T |\xi_s(\omega)|^p \nu(dt) < \infty .$$

2.5.2 *LEMMA. If* $t_1 < t_2$, *then*

$$||\int_{t_1}^{t_2} \xi_s(\omega)\nu(ds)|| \le \int_{t_1}^{t_2} ||\xi_s||\nu(ds) .$$

Proof: Let $\zeta(\omega) = \int_{t_1}^{t_2} \xi_s(\omega)\nu(ds)$ and observe that

$$||\zeta||^p = A_\zeta\left(\int_{t_1}^{t_2} \xi_s(\omega)\nu(ds)\right) = \int_{t_1}^{t_2} A_\zeta(\xi_s)\nu(ds)$$
$$\le ||\zeta||^{p-1}\int_{t_1}^{t_2} ||\xi_s||\nu(ds) . \qquad \square$$

2.5.3 *PROPOSITION. The stochastic process* η *is of strong bounded variation.*

Proof: For every partition $t_0 < t_1 < \ldots < t_n$ of T and every $\zeta \in L(\xi)$,

$$\sum_{k=1}^{n} ||\eta_{t_k} - \eta_{t_{k-1}}|| \le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} ||\xi_s||\nu(ds)$$

$$\le \int_T ||\xi_s||\nu(ds) < \infty .$$ □

It follows that the stochastic integral $\int f(t)d\eta_t$ is defined for $f \in \Lambda'_p(\eta)$ as in Section 2 and that $\Lambda'_p(\eta) = \Lambda_p(\eta)$ (Proposition 2.2.2). We now define $\lambda_p(\xi) \triangleq \Lambda_p(\eta)$ and for every $f \in \lambda_p(\xi)$

$$\int_T f(t)\xi_t\nu(dt) \triangleq \int_T f(t)d\eta_t .$$

We shall see that the "stochastic integral" $\int_T f(t)\xi_t\nu(dt)$ can be expressed as a sample path integral for a large class of functions in $\lambda_p(\xi)$ (Lemma 2.5.4) and that sample path integrals of the form $\int_T f(t)\xi_t(\omega)\nu(dt)$ belong to $L(\eta)$ for all $f \in L_q(T, \mathcal{B}_T, \nu)$ (Theorem 2.5.5). In addition, these sample path integrals are dense in $L(\xi)$ when ξ is a weakly continuous process (Theorem 2.5.6).

We begin our investigation of $\lambda_p(\xi)$ by recalling the space of functions S' which generates $\Lambda'_p(\eta)$: S' is the space of all bounded measurable functions $f: T \to R$ for which there exists some $\eta_0 \in L(\eta)$ such that the Lebesgue-Stieltjes integral $\int_T f(t)dF_\zeta(t)$ equals $A_\zeta(\eta_0)$ for all $\zeta \in L(\eta)$. Since the stochastic process η is suppressed in our notation $F_\zeta(t)$ here, we emphasize that $F_\zeta(t) = A_\zeta(\eta_t)$.

2.5.4 *LEMMA. For every* $f \in S'$ *the sample path integral* $\int_T f(t)\xi_t(\omega)\nu(dt)$ *equals* $\int_T f(t)\xi_t\nu(dt)$ *with probability one.*

Proof: It is clear that the sample path integral exists since f is a bounded function and $\xi_t(\omega)\epsilon L_p(\nu)$ a.s. For any given $\zeta\epsilon L_p(\Omega)$ there exists some $\eta_1 \epsilon L(\eta)$ such that $A_\zeta = A_{\eta_1}$ on $L(\eta)$ by Theorem 2.1.5. Note that for $t_1 \le t_2$,

$$\int_{t_1}^{t_2} dF_{\eta_1}(t) = A_{\eta_1}(\eta_{t_2} - \eta_{t_1}) = A_\zeta(\eta_{t_2} - \eta_{t_1})$$

$$= A_\zeta\left[\int_{t_1}^{t_2} \xi_t(\omega)\nu(dt)\right] = \int_{t_1}^{t_2} A_\zeta(\xi_t)\nu(dt) .$$

Thus

$$A_\zeta\left(\int_T f(t)\xi_t\nu(dt) - \int_T f(t)\xi_t(\omega)\nu(dt)\right)$$

$$= A_\zeta\left(\int_T f(t)\xi_t\nu(dt)\right) - E\left[(\zeta)^{p-1}\int_T f(t)\xi_t(\omega)\nu(dt)\right]$$

$$= A_{\eta_1}\left(\int_T f(t)\xi_t\nu(dt)\right) - \int_T f(t)E[(\zeta)^{p-1}\xi_t]\nu(dt)$$

$$= \int_T f(t)dF_{\eta_1}(t) - \int_T f(t)A_\zeta(\xi_t)\nu(dt)$$

$$= \int_T f(t)dF_{\eta_1}(t) - \int_T f(t)dF_{\eta_1}(t) = 0 .$$ □

We have seen that $\xi_t(\omega)\epsilon L_p(T,\mathcal{B}_T,\nu)$ with probability one, so that for all $f\epsilon L_q(T,\mathcal{B}_T,\nu)$ the sample path integral $\int_T f(t)\xi_t(\omega)\nu(dt)$ is defined a.s. and is easily seen to belong to $L_p(\Omega)$. In fact, it belongs to $L(\eta)$, and the function space $\lambda_p(\xi)$ contains $L_q(\nu)$ in a sense which we now make precise.

2.5.5 *THEOREM. Every function* $f\epsilon L_q(\nu)$ *determines uniquely an element* $\tilde{f}\epsilon\lambda_p(\xi)$ *such that*

$$\int_T f(t)\xi_t(\omega)\nu(dt) = \int_T \tilde{f}(t)\xi_t\nu(dt) \text{ a.s. },$$

where the left-hand side is a sample path integral and the right-hand side a stochastic integral.

Proof: Given any $g \epsilon L_q(\nu)$ let $\zeta = \int_T g(t)\xi_t(\omega)\nu(dt)$, and observe that

$$||\zeta||^p = A_\zeta\left(\int_T g(t)\xi_t(\omega)\nu(dt)\right) = \int_T g(t)A_\zeta(\xi_t)\nu(dt)$$

$$\leq ||g||_{L_q(\nu)}\left(\int_T |A_\zeta(\xi_t)|^p\nu(dt)\right)^{1/p}$$

$$\leq ||g||_{L_q(\nu)}||\zeta||^{p-1}\left(\int_T ||\xi_t||^p\nu(dt)\right)^{1/p} .$$

Thus we have the relationship

$$||\int_T g(t)\xi_t(\omega)\nu(dt)|| \leq ||g||_{L_q(\nu)}\left(\int_T ||\xi_t||^p\nu(dt)\right)^{1/p}$$

for all $g \epsilon L_q(\nu)$.

Given any $f \epsilon L_q(\nu)$ let $\{f_n\}_{n=1}^\infty$ be a sequence of step functions converging to f in $L_q(\nu)$ and recall that $f_n \epsilon S'$ for each n. Thus the sample path integral $\int_T f_n(t)\xi_t(\omega)\nu(dt)$ belongs to $L(\eta)$ by Lemma 2.5.4, and

$$||\int_T [f(t)-f_n(t)]\xi_t(\omega)\nu(dt)|| \leq ||f-f_n||_{L_q(\nu)}\left(\int_T ||\xi_t||^p\nu(dt)\right)^{1/p},$$

which converges to zero as $n \to \infty$, so that the sample path integral $\int_T f(t)\xi_t(\omega)\nu(dt)$ belongs to $L(\eta)$.

Note that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in S', since

$$||f_m-f_n||_{S'} = ||\int_T [f_m(t)-f_n(t)]\xi_t(\omega)\nu(dt)||$$

$$\leq ||f_m-f_n||_{L_q(\nu)}\left(\int_T ||\xi_t||^p\nu(dt)\right)^{1/p} \to 0$$

as $m,n \to \infty$, and denote its limit in $\lambda_p(\xi)$ by $\tilde{f}$. Then

$$||\int_T f(t)\xi_t(\omega)\nu(dt) - \int_T \tilde{f}(t)\xi_t\nu(dt)||$$

$$= \lim_{m,n\to\infty} ||\int_T [f_m(t)-f_n(t)]\xi_t(\omega)\nu(dt)|| = 0 ,$$

so that

$$\int_T f(t)\xi_t(\omega)\nu(dt) = \int_T \tilde{f}(t)\xi_t\nu(dt) \quad \text{a.s.}$$

It is immediate that f determines $\tilde{f}$ uniquely, since if g_1, g_2 in $\lambda_p(\xi)$ satisfy

$$\int_T g_i(t)\xi_t\nu(dt) = \int_T f(t)\xi_t(\omega)\nu(dt) \quad \text{a.s.}$$

for $i = 1,2$, then

$$||g_1-g_2||_{\lambda_p(\xi)} = ||\int_T [g_1(t)-g_2(t)]\xi_t\nu(dt)|| = 0 ,$$

whence $g_1 = g_2$ in $\lambda_p(\xi)$. □

Identifying f with $\tilde{f}$ we can then consider $L_q(\nu)$ as a subset of $\lambda_p(\xi)$. It is straightforward to check that the process η is continuous in p-th mean and consequently that $\lambda_p(\xi)$ is isometric to all of $L(\eta)$. Since $S' \subset L_q(\nu)$, it is clear that $L_q(\nu)$ is always dense in $\lambda_p(\xi)$ and hence that $L_q(\nu)$ is isometric to a dense subset of $L(\eta)$. The final result in this section shows that $L_q(\nu)$ is isometric to a dense subset of $L(\xi)$ when ξ is weakly continuous.

2.5.6 *THEOREM. Suppose that* ξ *is weakly continuous from the right and that* $T = [a,\infty)$. *Then the closure of* $\{\int_T f(t)\xi_t(\omega)\nu(dt)$: f *is a step function} in* $L_p(\Omega)$ *is* $L(\xi)$.

Proof: Fix $t \in T$, and for every integer $n \geq 1$ define

$$g_n(s) = \nu^{-1}\{(t,t+n^{-1})\}\chi_{(t,t+n^{-1})}(s) ,$$

so that

$$\int_T g_n(s)\nu(ds) = 1 \quad .$$

Given any $\varepsilon > 0$ and any $\zeta \in L_p(\Omega)$, use weak right continuity to choose an N such that $n \geq N$ implies that $|A_\zeta(\xi_t - \xi_s)| < \varepsilon$ for $t < s < t + 1/n$. Then for $n \geq N$,

$$|A_\zeta[\xi_t - \int_T g_n(s)\xi_s(\omega)\nu(ds)]| = |\int_T g_n(s)A_\zeta(\xi_t - \xi_s)\nu(ds)|$$

$$\leq \int_T g_n(s)|A_\zeta(\xi_t - \xi_s)|\nu(ds) \leq \varepsilon\int_T g_n(s)\nu(ds) = \varepsilon \ .$$

Hence $\int_T g_n(s)\xi_s(\omega)\nu(ds)$ converges weakly to ξ_t by Proposition 2.1.6 and therefore ξ_t belongs to the closure of $\{\int_T f(t)\xi_t(\omega)\nu(dt)$: f is a step function$\}$ by [Rudin 1973, Theorem 3.1.2]. □

Thus when ξ is weakly continuous, every element of $L(\xi)$ can be expressed as a limit in L_p (and hence also a.s.) of sample path integrals. Specifically, if $\zeta \in L(\xi)$, then there exists a sequence $\{f_n\}_{n=1}^\infty \subset L_q(\nu)$ such that

$$\zeta(\omega) = \lim_{n\to\infty} \int_T f_n(t)\xi_t(\omega)\nu(dt) \quad \text{a.s.}$$

6. The integral $\int_T f(t)\xi_t\nu(dt)$ when ξ is SαS with independent increments.

When ξ is SαS with independent increments and $f \in L_q$ we find the relationship between the two integrals of Sections 2 and 5 (Theorem 2.6.1), and we characterize the independence of $\int f_1(t)\xi_t\nu(dt)$ and $\int f_2(t)\xi_t\nu(dt)$ (Theorem 2.6.3).

Let $\xi = \{\xi_t, t\in T\}$ be a SαS process with independent increments, index set $T = [a,\infty)$, $\xi_a = 0$, weak continuity from the right, and $F(t) = ||\xi_t||^\alpha$. The process ξ is therefore a p-th order process for any p such that $1 < p < \alpha$, and the integral $\int_T f(t)\xi_t\nu(dt)$ is defined as in the previous section. Under the conditions we have placed on ξ, we know in fact by Theorem 2.5.5 that integrals of the form $\int_T f(t)\xi_t\nu(dt)$, where $f\in L_q(T,\mathcal{B}_T,\nu)$, $1/p + 1/q = 1$, are dense in $L(\xi)$. The following Fubini-type result relates this integral to the one in Section 3.

2.6.1 *THEOREM. If* $f\in L_q(T,\mathcal{B}_T,\nu)$, *then*

$$\int_T f(s)\xi_s\nu(ds) = \int_T \int_u^\infty f(s)\nu(ds)d\xi_u .$$

<u>Proof</u>: We begin by showing that the right-hand integral exists, *i.e.*, that $\int_u^\infty f(s)\nu(ds)\in L_\alpha(dF)$. Since ν is a finite measure (Proposition 2.5.1), we can choose M so large that $\nu[M,\infty) < 1$. Thus

$$\int \left|\int_u^\infty f(s)\nu(ds)\right|^\alpha dF(u)$$

$$\le \int_T \left(\int_T |f(s)|^q\nu(ds)\right)^{\alpha/q}\left(\int_u^\infty \nu(ds)\right)^{\alpha/p}dF(u)$$

$$= ||f||^\alpha_{L_q(\nu)}\int_T[\nu[u,\infty)]^{\alpha/p}dF(u)$$

$$\le ||f||^\alpha_{L_q(\nu)}\left[\int_a^M [\nu[u,\infty)]^{\alpha/p}dF(u) + \int_M^\infty \nu[u,\infty)dF(u)\right]$$

which is finite since

$$\int_a^M [\nu[u,\infty)]^{\alpha/p} dF(u) \le \nu(R)^{\alpha/p} F(M) < \infty$$

and

$$\int_M^\infty \nu[u,\infty) dF(u) \le \int_T \int_T \chi_{[u,\infty)}(s)\nu(ds)dF(u)$$

$$= \int_T \int_T \chi_{(a,s]}(u)dF(u)\nu(ds) \le \int_T ||\xi_s||^\alpha \nu(ds) < \infty,$$

by Proposition 2.5.1.

We complete the proof by showing that

$$A_{\xi_t}\left(\int_T f(s)\xi_s \nu(ds)\right) = A_{\xi_t}\left(\int_T \int_u^\infty f(s)\nu(ds)d\xi_u\right)$$

for all $t \in T$ and applying the remark following Corollary 2.3.2. Indeed,

$$A_{\xi_t}\left(\int_T \int_u^\infty f(s)\nu(ds)d\xi_u\right) = \int_T \chi_{(a,t]}(u) \int_u^\infty f(s)\nu(ds)dF(u)$$

by Proposition 2.3.1. Also,

$$A_{\xi_t}\left(\int_T f(s)\xi_s \nu(ds)\right)$$

$$= \int_T f(s) A_{\xi_t}(\xi_s)\nu(ds)$$

$$= \int_T f(s) \int_T \chi_{(a,t]}(u)\, \chi_{(a,s]}(u) dF(u)\nu(ds)$$

$$= \int_T f(s) \int_T \chi_{(a,t]}(u)\, \chi_{[u,\infty)}(s) dF(u)\nu(ds)$$

$$= \int_T \chi_{(a,t]}(u) \int_u^\infty f(s)\nu(ds)dF(u). \qquad \square$$

The following corollary is immediate in view of Proposition 2.3.1.

2.6.2 *COROLLARY. If* f *and* g *belong to* $L_q(T,\mathcal{B}_T,\nu)$, *then*

$$ {}^A\!\int_T f(s)\xi_s \nu(ds)\left(\int_T g(t)\xi_t\nu(dt)\right) $$

$$ = \int_T\left(\int_u^\infty f(s)\nu(ds)\right)^{\alpha-1}\left(\int_u^\infty g(t)\nu(dt)\right)dF(u) \ . $$

Also,

$$ \left|\left|\int_T f(s)\xi_s\nu(ds)\right|\right|^\alpha = \int_T\left|\int_u^\infty f(s)\nu(ds)\right|^\alpha dF(u) \ . $$

As another application of Theorem 2.6.1 we obtain necessary and sufficient conditions for independence of such integrals.

2.6.3 *THEOREM. Let* F(t) *be strictly increasing, and for* i = 1,2 *consider* $f_i \in L_q(T,\mathcal{B}_T,\nu)$ *with*

$$ B_i = \{u\in T: \int_u^\infty f_i(t)\nu(dt) = 0\} $$

and $B_i' = T - B_i$. *Then* $\int_T f_1(t)\xi_t\nu(dt)$ *and* $\int_T f_2(t)\xi_t\nu(dt)$ *are independent if and only if one of the following two (equivalent) conditions hold:*

(i) $f_1 = 0$ *a.e.* $[\nu]$ *on* B_2' *and for each* $u\in B_2'$

$$ \int_u^\infty \chi_{B_2}(t) f_1(t)\nu(dt) = 0; $$

(ii) $f_2 = 0$ *a.e.* $[\nu]$ *on* B_1' *and for each* $u\in B_1'$

$$ \int_u^\infty \chi_{B_1}(t) f_2(t)\nu(dt) = 0 \ . $$

Proof: If $\int_T f_1(t)\xi_t\nu(dt)$ and $\int_T f_2(t)\xi_t\nu(dt)$ are independent, then

$$ (*) \qquad \int_u^\infty f_1(t)\nu(dt)\int_u^\infty f_2(t)\nu(dt) = 0 $$

a.e. [dF] by Theorem 2.4.2 (with a slight modification of the index set of the process). Thus (*) holds for all $u \in T$, since the integrals are continuous functions of u and F is strictly increasing. Also by continuity of the integrals, B_2 is a closed set and hence $(a,\infty) - B_2$ is an open set on which $\int_u^\infty f_1(t)\nu(dt)$ is zero. Therefore $f_1 = 0$ a.e. [ν] on B_2' and consequently

$$\int_u^\infty \chi_{B_2}(t)f_1(t)\nu(dt) = 0$$

for all $u \in B_2'$.

The necessity of (ii) follows in a similar manner, and the converse can be seen by reversing the argument. □

It is possible to obtain similar conditions for independence for slightly more general F, allowing say F to be constant on a closed subset of (a,∞) but strictly increasing elsewhere.

7. System identification.

In earlier sections it has been convenient to use A_ζ to denote a continuous linear functional defined by $A_\zeta(\eta) = E[\eta(\zeta)^{p-1}]$ on a p-th order family and by $A_\zeta(\eta) = C_{\eta\zeta}$ on a SαS family. We shall now refer to $A_\zeta(\eta)$ in both cases as the covariation of η with ζ and likewise extend the use of the symbol $C_{\eta\zeta}$ to the p-th order case. The covariation function $C_{\xi\xi}$ of a stochastic process $\xi = \{\xi_t, t\in T\}$ is defined by

$$C_{\xi\xi}(s,t) = C_{\xi_s\xi_t},$$

and the cross covariation function $C_{X\xi}$ of a stochastic process $X = \{X_v, v\in V\}$ with ξ is defined by

$$C_{X\xi}(v,t) = C_{X_v\xi_t}.$$

Let the p-th order process ξ be the input to a linear system and let its output X be given by

(1) $$X_v = \int_T f_v(t)d\xi_t$$

where $f = \{f_v(\cdot), v\in V\} \subset S'$, or by

(2) $$X_v(\omega) = \int_T f_v(t)\xi_t(\omega)\nu(dt)$$

where $f = \{f_v(\cdot), v\in V\} \subset L_q(\nu)$. In both cases f is called the (time varying) inpulse response of the system. Our purpose is to investigate what can be determined about the system from knowledge of the statistical relationship between ξ and X, specifically from knowledge of the covariation function of ξ and either the covariation function of X or the cross covariation function of X with ξ.

When ξ is SαS with independent increments we find that the impulse response of the system, in both cases considered above, can be identified from the cross covariation of the output with the input process (Propositions 2.7.3 and 2.7.5). With additional restrictions it is also possible to determine the impulse response of system (2) from the covariation function of the output process (Proposition 2.7.6) and from the cross covariation of the output with a known stationary sub-Gaussian input process (Example 2.7.1).

For a p-th order process ξ of weak bounded variation, the cross covariation function for system (1) is given by

$$C_{X\xi}(v,t) = A_{\xi_t}\left(\int_T f_v(s) d\xi_s \right) = \int_T f_v(s) d_s C_{\xi\xi}(s,t) ,$$

and it is clear that the function $C_{X\xi}(v,t)$, $t \in T$, determines f_v if and only if the signed measures determined by $C_{\xi\xi}(\cdot,t)$, $t \in T$, separate points on S'. Similarly, for a measurable p-th order process ξ, the cross covariation function for system (2) is given by

$$C_{X\xi}(v,t) = A_{\xi_t}\left(\int_T f_v(s) \xi_s \nu(ds) \right) = \int_T f_v(s) C_{\xi\xi}(s,t) \nu(ds) ,$$

and $C_{X\xi}(v,t)$, $t \in T$, determines f_v if and only if the functions $C_{\xi\xi}(\cdot,t)$, $t \in T$, separate points on $L_q(\nu)$.

One situation in which system identification is possible is when the covariation function $C_{\xi\xi}(s,t)$ of the input process is a stationary covariance function, an interesting circumstance which arises for certain stationary sub-Gaussian processes. To see the form of $C_{\xi\xi}(s,t)$ for a general α-sub-Gaussian process, let

$$\Lambda^{\alpha}_{s,t}(r_1,r_2) = 2^{-\frac{\alpha}{2}}[r_1^2 R(s,s) + 2r_1 r_2 R(s,t) + r_2^2 R(t,t)]^{\frac{\alpha}{2}} ,$$

for every $s,t\in T$, where $R(s,t)$ is a covariance function and $1 < \alpha < 2$. Then observe that

$$C_{\xi\xi}(s,t) = \frac{1}{\alpha}\frac{\partial\Lambda_{s,t}(r_1,r_2)}{\partial r_1}\Bigg|_{\substack{r_1=0\\ r_2=1}}$$

$$= \frac{2^{-\frac{\alpha}{2}}[r_1R(s,s)+r_2R(s,t)]}{[r_1^2R(s,s)+2r_1r_2R(s,t)+r_2^2R(t,t)]^{\frac{2-\alpha}{2}}}\Bigg|_{\substack{r_1=0\\ r_2=1}}$$

$$= \frac{2^{-\frac{\alpha}{2}}}{R(t,t)^{\frac{2-\alpha}{2}}}R(s,t) .$$

2.7.1 *EXAMPLE*. Suppose that ξ is a p-th order process of weak bounded variation having covariation function

$$C_{\xi\xi}(s,t) = cR(s,t) ,$$

where $c > 0$ and R is a covariance function of the form

$$R(s,t) = \int_{-\infty}^{\infty} e^{ir(t-s)}g(r)dr$$

with $g(r) > 0$, $g\in L_1$ (Leb). Assuming that $\int_{-\infty}^{\infty}|r|g(r)dr < \infty$, we obtain for system (1) the cross covariation

$$C_{\chi\xi}(v,t) = \int_{-\infty}^{\infty} f_v(s)d_sC_{\xi\xi}(s,t)$$

$$= \int_{-\infty}^{\infty} f_v(s)(c\int_{-\infty}^{\infty}(-ir)e^{ir(t-s)}g(r)dr)ds$$

$$= -ic\int_{-\infty}^{\infty} re^{itr}g(r)(\int_{-\infty}^{\infty} e^{-irs}f_v(s)ds)dr .$$

Thus knowledge of $C_{\chi\xi}(v,t)$ for all t and of g determines $\int_{-\infty}^{\infty}e^{-irs}f_v(s)ds$

for all r and hence $f_v(s)$ a.e. [Leb]. Assuming moreover that $C_{X\xi}(v,t)\epsilon L_1$ (or L_2) as a function of t and that $\frac{1}{rg(r)}\int_{-\infty}^{\infty} e^{-irt}C_{X\xi}(v,t)\epsilon L_1$ (or L_2) as a function of r, we can express the impulse response as

$$f_v(s) = \frac{i}{c(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{irs}}{rg(r)}\Big(\int_{-\infty}^{\infty} e^{-irt}C_{X\xi}(v,t)dt\Big)dr \quad \text{a.e. [Leb].}$$

If we suppose instead that ξ is a measurable p-th order process with the same covariation function as above, then we obtain for system (2) the cross covariation

$$\begin{aligned} C_{X\xi}(v,t) &= \int_{-\infty}^{\infty} f_v(s)C_{\xi\xi}(s,t)\nu(ds) \\ &= c\int_{-\infty}^{\infty} f_v(s)\int_{-\infty}^{\infty} e^{ir(t-s)}g(r)dr\;\nu(ds) \\ &= c\int_{-\infty}^{\infty} g(r)e^{irt}\int_{-\infty}^{\infty} f_v(s)e^{-irs}\nu(ds)dr\ . \end{aligned}$$

Thus knowledge of $C_{X\xi}(v,t)$ for all t and of g determines $\int_{-\infty}^{\infty}e^{-irs}f_v(s)\nu(ds)$ for all r and hence $f_v(s)$ a.e. [Leb]. Assuming as in the previous case that $C_{X\xi}(v,t)\epsilon L_1$ (or L_2) as a function of t and that $\frac{1}{g(r)}\int_{-\infty}^{\infty} e^{-irt}C_{X\xi}(v,t)dt$ ϵL_1 (or L_2) as a function of r, we can express the impulse response as

$$f_v(s) = \frac{1}{(2\pi)^2 c[\frac{d\nu}{d\,\text{Leb}}](s)}\int_{-\infty}^{\infty}\frac{e^{irs}}{g(r)}\Big(\int_{-\infty}^{\infty} e^{-irt}C_{X\xi}(v,t)dt\Big)dr$$

a.e. [Leb].

Throughout the remainder of this section we assume that ξ is a SαS process and use the special properties of SαS processes to obtain more concrete results. We begin with a simple proposition relating covariations in the output space with the finite-dimensional distributions of X.

2.7.2 *PROPOSITION. Knowing* $A_Y(X_v)$ *for all* $Y\epsilon L(X)$ *and all* $v\epsilon V$ *is equivalent to knowing the finite-dimensional distributions of* X.

Proof: Suppose that the finite-dimensional distributions of X are known and fix $Y \in L(X)$. Then $Y = \lim_{n\to\infty} Y_n$ where each Y_n is a finite linear combination of $\{X_v, v \in V\}$. For every $v \in V$ we know the joint distribution of X_v and Y_n, and therefore we know

$$A_{Y_n}(X_v) = \int_S x_1 (x_2)^{\alpha-1} \Gamma_{X_v, Y_n}(dx)$$

and thus also

$$\lim_{n\to\infty} A_{Y_n}(X_v) = A_Y(X_v) ,$$

by Theorem 2.1.10.

Conversely, if $A_Y(X_v)$ is known for all $Y \in L(X)$, then for any integer $n \geq 1$ and any $v_1, \ldots, v_n \in V$ we know

$$A_{r_1 X_{v_1} + \ldots + r_n X_{v_n}}(r_1 X_{v_1} + \ldots + r_n X_{v_n})$$

$$= \int_S |r_1 x_1 + \ldots + r_n x_n|^{\alpha} \Gamma_{X_{v_1}, \ldots, X_{v_n}}(dx)$$

for all $r_1, \ldots, r_n$ and hence the joint c.f. of $X_1, \ldots, X_n$. □

Now the dual of $L(X)$ separates points on $L(X)$; so each X_v is determined in $L(X)$ by knowledge of $A_Y(X_v)$ for all $Y \in L(X)$ (Theorem 2.1.5), hence by the finite-dimensional distributions of X (Proposition 2.7.2). However, $L(X)$ in general may be strictly contained in M, the subspace of $L(\xi)$ isometric to Λ_α, and the set of linear functionals $\{A_Y: Y \in L(X)\}$ may not separate points on M. In such cases the elements of the output process need not be determined in M by the finite-dimensional distributions of X. (If $\{A_Y: Y \in L(X)\}$ does not separate points on M, then there exists $\zeta \in M$, $\zeta \neq 0$, such that $A_Y(\zeta) = 0$ for all $Y \in L(X)$. Thus for any X_v,

$$A_Y(X_v+\zeta) = A_Y(X_v)$$

for all $Y\epsilon L(X)$, and therefore knowledge of the finite-dimensional distributions of X does not distinguish between X_v and $X_v+\zeta$ in M.)

Since M and Λ_α are isometric, knowing the output variable X_v in M is equivalent to knowing the impulse response function f_v in Λ_α. Hence *the finite-dimensional distributions of the output process* X *determine* $f = \{f_v, v\epsilon V\}$ *if and only if the set* $\{A_Y, Y\epsilon L(X)\}$ *separates points on* M.

An interesting example for which the system is determined only to within an equivalence class is treated in [Kanter 1973]. In that paper Kanter considers a SαS process $\{\xi_t, t\epsilon R\}$ with independent increments and $||\xi_t||^\alpha = t$ and a time-invariant system of type (1) with impulse response $f\epsilon L_\alpha(R)$ and output

$$X_v = \int_R f(v+t)d\xi_t \ .$$

Then the finite-dimensional distributions of X determine f up to translation and multiplication by ± 1.

To see how system (1) can be determined from covariation functions in the SαS case, we assume that ξ has independent increments, is right continuous, $T = [a,\infty)$, $\xi_a = 0$, and $||\xi_t||^\alpha = F(t)$. Then

$$C_{\xi\xi}(s,t) = \int \chi_{(a,\min(s,t)]}(r)dF(r) = F(\min(s,t)) \ .$$

Therefore by Proposition 2.3.1,

$$\begin{aligned} C_{X\xi}(v,t) &= A_{\xi_t}\Big(\int_T f_v(s)d\xi_s\Big) \\ &= \int_{(a,t]} f_v(s)dF(s) = \int_T f_v(s)d_sC_{\xi\xi}(s,t) \end{aligned}$$

for every $f_v\epsilon L_\alpha(dF)$, with $C_{X\xi}(v,t)$, $t\epsilon T$, determining f_v if and only if the signed measures determined by $C_{\xi\xi}(\cdot,t)$, $t\epsilon T$, separate points on $L_\alpha(dF)$. In addition,

$$C_{XX}(u,v) = A_{X_v}\left(\int_T f_u(s)d\xi_s\right) = \int_T f_u(s)(f_v(s))^{\alpha-1}dF(s),$$

and consequently f_u is determined by the covariation function $C_{XX}(u,v)$, $v\epsilon V$, if and only if the functions $f_v^{\alpha-1}$, $v\epsilon V$, separate points on $L_\alpha(dF)$.

2.7.3 *PROPOSITION. Suppose that* ξ *has independent increments,* $T = [a,\infty)$, *and* $\xi_{a+0} = 0$. *Then each impulse response function* f_v *is determined by the cross covariation function* $C_{X\xi}(v, t+0)$, $t\epsilon T$. *Moreover, assuming for simplicity of expression that* ξ *is weakly continuous from the right,*

$$f_v(t) = \lim_{n\to\infty} \frac{C_{X\xi}(v, t^{(n)}_{k^n(t)+1}) - C_{X\xi}(v, t^{(n)}_{k^n(t)})}{F(t^{(n)}_{k^{(n)}(t)+1}) - F(t^{(n)}_{k^n(t)})} \quad \text{a.e. } [dF],$$

where $F(t) = ||\xi_t||^\alpha$, $\{I_k^{(n)}\}_{k=1}^\infty$ *is a partition of* (a,∞) *into semiclosed intervals* $I_k^{(n)} = (t_k^{(n)}, t_{k+1}^{(n)}]$ *such that the partitions become finer as* n *increases and*

$$\delta^{(n)} = \sup_k \text{Leb}(I_k^{(n)}) \to 0$$

as $n \to \infty$, *and* $k^{(n)}(t)$ *is the unique* k *such that* $t\epsilon I_k^{(n)}$.

Proof: The first part of the proposition is immediate from Corollary 2.3.2, and the second part follows by an exercise similar to (20.61)(b) of [Hewitt and Stromberg 1965].

For a discussion of the estimation of $C_{X\xi}(v,t)$, see [Kanter and Steiger 1974]. □

2.7.4 *EXAMPLE.* Suppose that ξ satisfies the conditions in Proposition 2.7.3 with dF a finite measure such that dF ~ Leb and

$$\int_a^\infty t^2 dF(t) = c < \infty .$$

Let the impulse response of system (1) be given by

$$f_v(t) = (\cos vbt)^{\frac{1}{\alpha-1}}$$

for all $v \geq 0$, where the "frequency" $b > 0$ is unknown. Then for each $u \geq 0$, the covariation function

$$C_{XX}(u,v) = \int_a^\infty f_u(t)\cos(vbt)dF(t), \quad \forall v \geq 0 ,$$

determines f_u a.e. [Leb] and hence b. To obtain an explicit expression for b, observe that

$$\frac{d^2C_{XX}(0,v)}{dv^2} = -b^2\int_a^\infty t^2\cos(vbt)dF(t) ,$$

and therefore

$$b^2 = -\frac{1}{c}\left.\frac{d^2C_{XX}(0,v)}{dv^2}\right|_{v=0} .$$

We next consider how system (2) can be determined from covariation functions and therefore assume that ξ is a measurable SαS process. Then

$$C_{X\xi}(v,t) = \int_T f_v(s)C_{\xi\xi}(s,t)\nu(ds) ,$$

and as before $C_{X\xi}(v,t)$, $t\epsilon T$, determines f_v if and only if the functions $C_{\xi\xi}(\cdot,t)$, $t\epsilon T$, separate points on $L_q(\nu)$. Assuming that ξ has independent increments, is weakly continuous from the right, $T = [a,\infty)$, $\xi_a = 0$, and $F(t) = ||\xi_t||^\alpha$, then

$$C_{XX}(u,v) = \int_T \left(\int_r^\infty f_u(s)\nu(ds)\right)\left(\int_r^\infty f_v(t)\nu(dt)\right)^{\alpha-1}dF(r)$$

by Corollary 2.6.2, and we shall see in Example 2.7.7 that the covariation function $C_{XX}(u,v)$, $v\epsilon V$, sometimes determines f_u.

2.7.5 *PROPOSITION. Let* ξ *be* SαS *with independent increments, weakly continuous from the right,* $T = [a,\infty)$, $\xi_a = 0$, *and* $F(t) = ||\xi_t||^\alpha$ *strictly increasing. Then for each fixed* $v \in V$ *the cross covariation function* $C_{X\xi}(v,t)$, $t \in T$, *determines* f_v *in* $L_q(T,\mathcal{B}_T,\nu)$ *by the relationship*

$$f_v(t) = -\left[\frac{d\,\mathrm{Leb}}{d\nu}\right](t) \times \frac{d}{dt}\left[\frac{dC_{X\xi}(v,\cdot)}{dF}\right](t) \quad \text{a.e. [Leb]} .$$

<u>Proof</u>: Observe that

$$\begin{aligned} C_{X\xi}(v,t) &= A_{\xi_t}\left(\int_T f_v(s)\xi_s \nu(ds)\right) \\ &= A_{\int_a^t d\xi_r}\left(\int_T \int_r^\infty f_v(s)\nu(ds)d\xi_r\right) \\ &= \int_a^t \int_r^\infty f_v(s)\nu(ds)dF(r) \end{aligned}$$

by Theorem 2.6.1 and Proposition 2.3.1. Thus the covariations $C_{X\xi}(v,t)$ for all $t \in T$ determine $\int_r^\infty f_v(s)\nu(ds)$ for all r since F is strictly increasing, and likewise f_v is determined a.e. [Leb] since ν is equivalent to Lebesgue measure. □

Applying Proposition 2.7.2 to system (2) we see that knowledge of the finite-dimensional distributions of the output process X is equivalent to knowing $A_Y(X_v)$ for all $Y \in L(X)$ and all $v \in V$. In this case,

$$\begin{aligned} A_Y(X_v) &= A_Y\left(\int_T f_v(t)\xi_t \nu(dt)\right) \\ &= \int_T f_v(t) A_Y(\xi_t)\nu(dt) \ , \end{aligned}$$

where for each $Y \in L(X)$, $A_Y(\xi_t) \in L_p(T,\mathcal{B}_T,\nu)$ as we saw in the proof of Theorem 2.5.5. Thus the finite-dimensional distributions of X determine $f_v \in L_q(T,\mathcal{B}_T,\nu)$ if and only if the family $\{A_Y(\xi_t) : Y \in L(X)\}$, a subset of $L_p(\nu)$, separates points on $L_q(\nu)$.

As an example we show that for certain systems the finite-dimensional distributions of X are more than is necessary, and in fact each impulse response function f_u is determined in $L_q(\nu)$ by the covariation function $C_{XX}(u,v)$, $v \in V$.

2.7.6 *PROPOSITION. Let* ξ *be a* SαS *process with independent increments, weakly continuous from the right,* $T = [0,\infty)$, $F(t) = ||\xi_t||^\alpha = t$ *for all* $t \in T$, *and* $\nu(dt) = h(t)dt$ *where* h *is chosen as in Proposition 2.5.1. Let* $\{G_v, v \in V\}$ *be a set of twice differentiable functions in* $L_p(\nu)$ *that separate points on* $L_q(\nu)$ *and satisfy* $G_v(0) = 0$. *Let* $\{g_v, v \in V\}$ *be a set of differentiable functions related to the functions* G_v *by*

$$g_v(r) = \left(\frac{dG_v(r)}{dr}\right)^{\frac{1}{\alpha-1}}$$

and such that $\lim_{s\to\infty} g_v(s) = 0$ *and*

$$\frac{1}{h(s)} \frac{dg_v(s)}{ds} \in L_q(\nu)$$

for all $v \in V$. *If the system is defined by*

$$f_v(s) = -\frac{1}{h(s)} \frac{dg_v(s)}{ds}$$

for all $v \in V$, *then each* f_u *is determined by the covariation function* $C_{XX}(u,v)$, $v \in V$.

<u>Proof</u>: Given $u \in V$, then

$$C_{XX}(u,v) = A_{X_v}\left(\int_T f_u(t)\xi_t \nu(dt)\right)$$

$$= \int_T f_u(t) A_{X_v}(\xi_t)\nu(dt)$$

$$= \int_T f_u(t) A_{\int_T \int_r^\infty f_v(s)\nu(ds)d\xi_r}(\xi_t)\nu(dt)$$

$$= \int_T f_u(t) \int_0^t \left(\int_r^\infty f_v(s)\nu(ds)\right)^{\alpha-1} dr\ \nu(dt)$$

$$= -\int_T f_u(t) \int_0^t \left(\int_r^\infty \left(\frac{dg_v(s)}{ds}\right) ds\right)^{\alpha-1} dr\ \nu(dt)$$

$$= \int_T f_u(t) \int_0^t (g_v(r))^{\alpha-1} dr\ \nu(dt)$$

$$= \int_T f_u(t) \int_0^t \frac{dG_v(r)}{dr}\ dr\ \nu(dt)$$

$$= \int_T f_u(t) G_v(t)\nu(dt) ,$$

for each $v \in V$. Since $\{G_v, v \in V\}$ separates points on $L_q(\nu)$, it follows that f_u is determined by $C_{XX}(u,v)$, $v \in V$. □

2.7.7 *EXAMPLE*. Let $G_v(t) = e^{-t}\sin vt$ for $v \in [0,\infty)$ and choose $h(t) = \min(1,t^{-3})$ (Proposition 2.5.1). Then

$$g_v(r) = \left(\frac{dG_v(r)}{dr}\right)^{\frac{1}{\alpha-1}} = e^{-\frac{r}{\alpha-1}} (v \cos vr - \sin vr)^{\frac{1}{\alpha-1}}$$

and

$$f_v(s) = \frac{1}{h(s)} \frac{dg_v(s)}{ds}$$

$$= \frac{1}{\min(1,s^{-3})} \frac{e^{-\frac{s}{\alpha-1}}}{\alpha-1} \Big[(\sin vs - v \cos vs)^{\frac{1}{\alpha-1}}$$

$$+ (v^2 \sin vs + v \cos vs)(\sin vs - v \cos vs)^{\frac{2-\alpha}{\alpha-1}} \Big] ,$$

which clearly belongs to $L_q(\nu)$ and converges to zero as $s \to \infty$.

An alternative approach to the system identification problem, when the statistics of the output are known, is to investigate which systems will produce the same output (in distribution) to the same input. For each system f, write

$$X_v^{(f)} = \int_T f_v(t)d\xi_t,\ v\epsilon V ,$$

where ξ is a SαS process, and let L(f) be the space of finite linear combinations of functions in $\{f_v,\ v\epsilon V\}$. Then two output processes $X^{(f)}$ and $X^{(g)}$ have the same finite dimensional distributions if and only if the map from L(f) onto L(g) defined by

$$\sum_{k=1}^{n} r_k f_{v_k} \mapsto \sum_{k=1}^{n} r_k g_{v_k}$$

for all integers $n \geq 1$, $r_1,\ldots,r_n \epsilon R$, and $v_1,\ldots,v_n \epsilon V$, is an isometry. We shall content ourselves here with discussing the simplified problem when V is a singleton:

$$(1) \qquad X_f = \int_T f(t)d\xi_t \quad ,$$

$$(2) \qquad X_f = \int_T f(t)\xi_t \nu(dt) \quad .$$

In system (1) with ξ a SαS process, we have $X_f \overset{d}{=} X_g$ if and only if $||f||_{S'} = ||g||_{S'}$. Also, if f and g are bounded functions vanishing outside a bounded set, then $X_f \overset{d}{=} X_g$ for all SαS ξ with independent increments if and only if $|f(t)| = |g(t)|$ for all t. For system (2) with ξ as in Theorem 2.6.1, we have that $X_f \overset{d}{=} X_g$ if and only if

$$||\int_r^\infty f\ d\nu||_{L_\alpha(dF)} = ||\int_r^\infty g\ d\nu||_{L_\alpha(dF)} \quad .$$

Moreover, for bounded functions f and g, $X_f \overset{d}{=} X_g$ for all such inputs ξ if and only if

$$\left| \int_r^\infty f d\nu \right| = \left| \int_r^\infty g d\nu \right|$$

for all r.

2.7.8 *PROPOSITION. Consider system (2) with*

$$\xi_t = \int \cos(t\lambda) d\zeta_\lambda \ , \quad t \geq 0 \ ,$$

where $\zeta = \{\zeta_\lambda, \lambda \geq 0\}$ *is a* SαS *process with independent increments, weakly continuous from the right, such that* $F(\lambda) = ||\zeta_\lambda||^\alpha$ *is a bounded function and* $F(0) = 0$. *Then*

$$X_f = \int \phi_f(\lambda) d\zeta_\lambda \ ,$$

where

$$\phi_f(\lambda) = \int f(t)\cos(t\lambda)\nu(dt) \ .$$

Consequently, $X_f \overset{d}{=} X_g$ *for a given* ζ *if and only if*

$$||\phi_f||_{L_\alpha(dF)} = ||\phi_g||_{L_\alpha(dF)} \ ,$$

and $X_f \overset{d}{=} X_g$ *for all such* ζ *if and only if*

$$|\phi_f(\lambda)| = |\phi_g(\lambda)|$$

for all $\lambda \geq 0$.

The proof follows by a familiar line of argument.

III. PATH PROPERTIES

Some known path properties of Gaussian processes can be easily extended to SαS processes. For instance, the zero-one laws for Gaussian processes in [Cambanis and Rajput 1973] hold as well for SαS processes with no change in proof in view of the zero-one law for stable measures in [Dudley and Kanter 1974] (see also [Fernique 1973]). This chapter extends to p-th order or SαS processes certain results known for 2nd order or Gaussian processes. Specifically we give necessary and sufficient conditions for the measurability of a p-th order or a SαS process (Section 1), necessary and sufficient conditions for the integrability of almost all paths of a SαS process (Section 2), and sufficient and necessary and sufficient conditions for almost sure path absolute continuity for p-th order and for SαS processes, respectively (Section 3). These results are obtained by appropriate modification of the proofs of similar results for second order or Gaussian processes.

1. Measurability of p-th order processes.

Let $\xi = \{\xi_t : t \in T\}$ and $\eta = \{\eta_t : t \in T\}$ be stochastic processes on the probability space (Ω, F, P), where T is a Borel subset of a complete separable metric space and $B(T)$ denotes the Borel subsets of T. The process η is a *modification* of ξ if $P\{\xi_t = \eta_t\} = 1$ for all $t \in T$; η is called *measurable* if $(t,\omega) \mapsto \eta_t(\omega)$ is a product measurable map from $T \times \Omega$ into R. The existence of a measurable modification is frequently of interest

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in the study of path properties.

The following condition for the existence of a measurable modification is contained in [Cohn 1972]. We shall use ξ to denote the map $t \mapsto \xi_t$ and shall specify the range space when required for clarity. Let M be the space of all real-valued random variables on (Ω, F, P), and define a metric ρ on M by

$$\rho(\zeta_1, \zeta_2) = \left(\frac{|\zeta_1 - \zeta_2|}{1 + |\zeta_1 - \zeta_2|}\right)$$

for all $\zeta_1, \zeta_2 \in M$. Then ρ metrizes the topology of convergence in probability. *The process ξ has a measurable modification if and only if the map ξ from* T *to* M *is Borel measurable.*

For p-th order processes we now obtain further equivalent conditions for the existence of a measurable modification. Our line of proof follows that in [Cambanis 1975a] where the case p = 2 is treated.

3.1.1 *THEOREM. Let* $\{\xi_t, t \in T\}$ *be a p-th order process with* $p > 1$ *or a* SαS *process with* $1 < \alpha < 2$, *and let* $L(\xi)$ *be the linear space of the process. Then the following are equivalent:*

(i) The process ξ has a measurable modification.

(ii) The map $\xi: T \to L(\xi)$ *has separable range and is such that, for every* $t_0 \in T$, $||\xi_t - \xi_{t_0}||$ *is* $B(T)$*-measurable.*

(iii) The map $\xi: T \to L(\xi)$ *is Borel measurable.*

(iv) $L(\xi)$ *is separable and for every* $\zeta \in L(\xi)$ *the function* $F_\zeta(t)$ *is Borel measurable.*

It suffices by Proposition 2.1.2 to prove the result for p-th order processes.

Proof: (i) *implies* (ii). Consider $\{\zeta_t, t\in T\}$, a measurable modification of ξ. Then for every $t_0 \in T$, $||\xi_t - \xi_{t_0}|| = (E|\zeta_t - \zeta_{t_0}|^p)^{1/p}$ is $\mathcal{B}(T)$-measurable by Fubini's theorem. We next show that $\xi(T)$ is separable with respect to the norm topology, *i.e.*, in $L_p(\Omega,F,P)$.

Assume for the moment that $||\xi_t||$ is uniformly bounded on T. Now by (i) it follows that $\xi(T)$ is a separable subset of M with respect to the toplogy of convergence in probability. Thus there is a countable subset N of $\xi(T)$ such that, for each $t\in T$, there exists a sequence $\{\xi_{t_n}\}_{n=1}^{\infty} \subset N$ converging in probability to ξ_t. By [Hewitt and Stromberg 1965, p. 207] the sequence $\{\xi_{t_n}\}_{n=1}^{\infty}$ converges weakly to ξ_t in $L_p(\Omega,F,P)$, and therefore the linear manifold generated by N is dense in $L(\xi)$ with respect to the norm topology ([Rudin 1973, p. 65]). It follows that both $L(\xi)$ and $\xi(T)$ are separable.

Relaxing the above assumption, we let $T_N = \{t\in T: ||\xi_t|| \le N\}$. Then $T_N \in \mathcal{B}(T)$ and $\xi(T) = \bigcup_{N=1}^{\infty} \xi(T_N)$ is clearly separable with respect to the norm topology.

(ii) *implies* (iii). [Hoffmann-Jørgensen 1973, p. 206]. If V is an open subset of $\xi(T)$ in the norm topology, choose $\{t_N\}_{N=1}^{\infty} \subset T$ and $a_N > 0$ such that

$$V = \bigcup_{N=1}^{\infty} \{\xi_t: ||\xi_t - \xi_{t_N}|| < a_N\} .$$

Then

$$\xi^{-1}(V) = \bigcup_{N=1}^{\infty} \{t\in T: ||\xi_t - \xi_{t_N}|| < a_N\} \in \mathcal{B}(T) .$$

(iii) *implies* (iv) is clear.

(iv) *implies* (iii) follows from Theorem 2.1.1 and a theorem due to B.J. Pettis ([Hille 1973, Theorem 7.5.10]).

(iii) *implies* (i) is clear since convergence in $L_p(\Omega,F,P)$ implies convergence in probability. □

It should be noted that this result does not reflect the fact that the existence of a measurable modification is a property of the two-dimensional distributions of the process, as shown in [Hoffman-Jørgensen 1973].

3.1.2 *COROLLARY. A stochastic process* ξ *as in Theorem 3.1.1 has a measurable modification under each of the following three conditions:*

(i) ξ *is a weakly continuous process.*

(ii) T *is an arbitrary interval and the strong left (right) limit of* ξ *exists at all but countably many* $t \in T$.

(iii) T *is an arbitrary interval and* ξ *is an* SαS *process with independent increments.*

Proof: (i) If ξ is a weakly continuous process, then $F_\zeta(t)$ is a continuous (hence measurable) function of t for every $\zeta \in L(\xi)$. To see the separability of $L(\xi)$, let T* be a countable dense subset of T and let N be the space of all rational linear combinations of elements in $\{\xi_t, t \in T^*\}$. Then N is a countable dense subset of $L(\xi)$ by [Rudin 1973, Theorem 3.12], and the existence of a measurable modification follows from (iv) of Theorem 3.1.1.

(ii) Parts (i) and (ii)(a) of the proof in [Bulatović and Ašić 1976] for second order processes hold with no alteration for the process ξ and show that the set T_1 of all points of discontinuity of ξ is countable. Let $T_2 \subset T-T_1$ be a countable dense subset of T. Then the space of all rational linear combinations of elements in $\{\xi_t, t\in T_1\cup T_2\}$ is a countable dense subset of $L(\xi)$. It is clear that $F_\zeta(t)$ is Borel measurable for every $\zeta\in L(\xi)$ since it is a continuous function on $T-T_1$. Thus ξ has a measurable modification again by (iv) of Theorem 3.1.1.

(iii) If ξ is SαS with independent increments, then $F(t) = ||\xi_t||^\alpha$ is an increasing function which therefore has at most countably many points of discontinuity. Let T_1 be the set of all points of discontinuity of F, and let $T_2 \subset T-T_1$ be a countable dense subset of T. From the relationship $||\xi_s-\xi_t||^\alpha = |F(s)-F(t)|$ for all $s,t\in T$, it is easy to see that the space of all rational linear combinations of elements in $\{\xi_t, t\in T_1\cup T_2\}$ is a countable dense subset of $L(\xi)$. For the measurability of $F_\zeta(t)$, $\zeta\in L(\xi)$, recall from Section 3 of Chapter II that the right limit $F_\zeta(t+0)$ exists for all t. □

The next corollary provides a result for sub-Gaussian processes analogous to the corresponding result for Gaussian processes ([Cambanis 1975a]).

3.1.3 *COROLLARY. If ξ is a sub-Gaussian process, then it has a measurable modification if and only if $L(\xi)$ is separable and $C_{\xi\xi}(s,t)$ is product measurable.*

<u>Proof</u>: The two-dimensional c.f.'s of ξ are given by

$$\phi_{s,t}(r_1,r_2) = \exp\{-||r_1\xi_s+r_2\xi_t||^\alpha\}$$
$$= \exp\{-2^{-\frac{\alpha}{2}}[R(s,s)r_1^2 + 2R(s,t)r_1r_2+R(t,t)r_2^2]^{\frac{\alpha}{2}}\} ,$$

where R(s,t) is a covariance function, and clearly

$$||\xi_t - \xi_{t_0}|| = 2^{-\frac{1}{2}}[R(t,t) - 2R(t,t_0) + R(t_0,t_0)]^{\frac{1}{2}} .$$

Using the relationship preceding Example 2.7.1, it is straightforward to show that

$$R(s,t) = 2\, C_{\xi\xi}(t,t)^{\frac{2-\alpha}{\alpha}} C_{\xi\xi}(s,t) \quad .$$

Thus if $C_{\xi\xi}(s,t)$ is product measurable, then so is $R(s,t)$, and hence $||\xi_t - \xi_{t_0}||$ is measurable in t for each fixed $t_0 \epsilon T$. Assuming $L(\xi)$ separable, the existence of a measurable modification therefore follows from (ii) of Theorem 3.1.1.

Conversely, if ξ has a measurable modification η, then we can see from Proposition 2.1.2 and Fubini's theorem that $||\xi_s + \xi_t|| = ||\eta_s + \eta_t||$ is a product measurable function on $T \times T$ and that $||\xi_s|| = ||\eta_s||$ is a measurable function on T. From

$$2||r_1 \xi_s + r_2 \xi_t||^2 = r_1^2 R(s,s) + 2r_1 r_2 R(s,t) + r_2^2 R(t,t)$$

for all r_1, r_2, one has

$$2||\xi_s + \xi_t||^2 = R(s,s) + R(t,t) + 2R(s,t)$$

$$2||\xi_s||^2 = R(s,s) \ ,$$

and thus

$$R(s,t) = ||\xi_s + \xi_t||^2 - ||\xi_s||^2 - ||\xi_t||^2 \ .$$

Consequently, $R(s,t)$ is product measurable and the product measurability of $C_{\xi\xi}(s,t)$ now follows from

$$C_{\xi\xi}(s,t) = \frac{2^{-\frac{\alpha}{2}} R(s,t)}{R(t,t)^{\frac{2-\alpha}{2}}} \quad . \qquad \square$$

For a general SαS process ξ it appears that $C_{\xi\xi}(s,t)$ product measurable and $L(\xi)$ separable are not sufficient for the existence of a measurable modification, though we do not have a counterexample. Since $||\xi_t - \xi_{t_0}||$ cannot be expressed in terms of $C_{\xi\xi}(t,t_0)$ for $1 < \alpha < 2$, we cannot express the condition for a measurable modification in terms of $C_{\xi\xi}(t,t_0)$ except in the sub-Gaussian case (Gaussian when $\alpha = 2$), where

$$||\xi_t - \xi_{t_0}||^2 = C_{\xi\xi}(t,t)^{\frac{2}{\alpha}} - 2C_{\xi\xi}(t_0,t_0)^{\frac{2-\alpha}{\alpha}} C_{\xi\xi}(t,t_0) + C_{\xi\xi}(t_0,t_0)^{\frac{2}{\alpha}} .$$

If we set $\sigma(s,t) = ||\xi_s - \xi_t||$, then of course product measurability of $\sigma(s,t)$ implies measurability of $\sigma(t,t_0)$ in t, for each fixed t_0. Conversely, if $L(\xi)$ is separable and $\sigma(\cdot,t_0)$ is measurable for each $t_0 \in T$, then (ii) of Theorem 3.1.1 implies the existence of a measurable modification η and we can apply Fubini's theorem to show that $\sigma(s,t) = ||\eta_s - \eta_t||$ is product measurable. Hence condition (ii) in Theorem 3.1.1 may be written in the more symmetric form:

(ii)' *The map* $\xi: T \to L(\xi)$ *has separable range and the function* $\sigma(s,t) = ||\xi_s - \xi_t||$ *is* $B(T)\times B(T)$*-measurable.*

Finally, we note that if ξ is SαS and $L(\xi)$ is separable, then ξ has an integral representation of the type

$$\xi_t = \int f_t(u) d\zeta_u ,$$

where $\{\zeta_u, -\frac{1}{2} \le u \le \frac{1}{2}\}$ is a SαS process with independent increments, $||\zeta_u||^\alpha = F(u)$, and $f_t(\cdot) \in L_\alpha(dF)$ for all t ([Kuelbs 1973, Theorem 4.2]). And conversely, if ξ has such a spectral representation, then $L(\xi)$ is separable since $L(\zeta)$ is separable (Corollary 3.1.2 (iii)). In particular, every measurable SαS process has such a spectral representation.

2. Integrability of sample paths of SαS processes.

In this section we apply a result due to DeAcosta to obtain a necessary and sufficient condition for almost all sample paths of an SαS process to belong to $L_p(T,A,\nu)$, $1 < p < \alpha$ (Theorem 3.2.4). We also show that a sample path integral of an SαS process is an SαS random variable (Lemma 3.2.3).

3.2.1 [DeAcosta 1975] *If* μ *is an* SαS *Borel probability measure on* $L_p(T,A,\nu)$ *with measurable seminorm* w *, then for every* $r < \alpha$,

$$\int w^r d\mu < \infty \quad .$$

We begin by proving a lemma involving a p-th order process.

3.2.2 *LEMMA. Let* (T,A,ν) *be a finite measure space, and let* $\xi = \{\xi_t, t\epsilon T\}$ *be a measurable p-th order process with* $1 < p < \infty$ *and with* $||\xi_t|| \le M < \infty$ *for all* $t\epsilon T$. *For any element* $f\epsilon L_q(T,A,\nu)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *the sample path integral*

$$\int_T f(t)\xi_t(\omega)\nu(dt)$$

belongs to $L(\xi)$.

Proof: From the measurability of ξ and Fubini's theorem, we get that $\xi_t(\omega)\epsilon L_p(T,A,\nu)$ a.s., since

$$E\int_T |\xi_t(\omega)|^p \nu(dt) = \int_T ||\xi_t||^p \nu(dt) \le M^p \nu(T) < \infty \ .$$

Therefore $\int f(t)\xi_t(\omega)\nu(dt)\epsilon L_p(\Omega)$, since

$$E\left|\int_T f(t)\xi_t(\omega)\nu(dt)\right|^p \le \left[\int_T |f(t)|^q \nu(dt)\right]^{\frac{p}{q}} E\int|\xi_t(\omega)|^p\nu(dt) .$$

In particular, the random variable $\int f(t)\xi_t(\omega)\nu(dt)$ defines a continuous linear functional on $L(\xi)^*$, and it follows from Theorem 2.1.5 and the reflexivity of $L(\xi)$ that there exists a unique $\eta \in L(\xi)$ satisfying

$$A_\zeta(\eta) = A_\zeta\left(\int_T f(t)\xi_t \nu(dt)\right)$$

for all $\zeta \in L(\xi)$.

Given any $\zeta' \in L_p(\Omega)$, we apply Theorem 2.1.5 to obtain $\zeta \in L(\xi)$ such that $A_\zeta = A_{\zeta'}$ on $L(\xi)$. Then

$$\begin{aligned} &A_{\zeta'}\left(\eta - \int_T f(t)\xi_t(\omega)\nu(dt)\right) \\ &= A_{\zeta'}(\eta) - E\left[(\zeta')^{p-1}\int_T f(t)\xi_t\nu(dt)\right] \\ &= A_{\zeta'}(\eta) - \int_T f(t)E[(\zeta')^{p-1}\xi_t]\nu(dt) \\ &= A_\zeta(\eta) - \int_T f(t)A_\zeta(\xi_t)\nu(dt) \\ &= A_\zeta(\eta) - A_\zeta\left(\int_T f(t)\xi_t\nu(dt)\right) = 0 . \end{aligned}$$

Thus $\eta = \int_T f(t)\xi_t(\omega)\nu(dt)$ a.s., whence $\int_T f(t)\xi_t(\omega)\nu(dt) \in L(\xi)$. □

We next prove a stronger version of Lemma 3.2.2 for an SαS process by using a truncation suggested by L.A. Shepp and used in [Rajput 1972, Proposition 3.2], where a similar result is proved for Gaussian processes.

3.2.3 *LEMMA. Let* (T,A,ν) *be a* σ*-finite measure space, and let* $\xi = \{\xi_t, t\in T\}$ *be a measurable* SαS *process with* $\xi(\cdot,\omega)$ $L_p(T,A,\nu)$ *a.s.,* *where* $1 < p < \alpha$. *Then for every element* $f \in L_q(T,A,\nu)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *the sample path integral*

$$\int_T f(t)\xi_t(\omega)\nu(dt)$$

belongs to $L(\xi)$ *and is thus a* SαS *random variable.*

Proof: For each integer $n \geq 1$ define a truncated process by

$$\xi^{(n)}(t,\omega) = \xi(t,\omega)\chi_{\{s:\|\xi_s\|\leq n\}}(t) \ ,$$

and assume for the moment that $\nu(T) < \infty$. Then the sample path integral $\int_T f(t)\xi_t^{(n)}(\omega)\nu(dt)$ belongs to $L(\xi)$ by Lemma 3.2.2, and, by the dominated convergence theorem,

$$\int_T f(t)\xi_t^{(n)}(\omega)\nu(dt) \to \int_T f(t)\xi_t(\omega)\nu(dt) \quad \text{a.s.}$$

as $n \to \infty$. Because almost sure convergence implies convergence in $L(\xi)$ for an SαS process ξ, we therefore have that $\int_T f(t)\xi_t(\omega)\nu(dt)$ belongs to $L(\xi)$.

If ν is a σ-finite measure, let $\{T_m\}_{m=1}^{\infty}$ be a monotone increasing sequence of elements of A such that $\nu(T_m) < \infty$ for every m and $\bigcup_{m=1}^{\infty} T_m = T$. Then

$$\{\int_T f(t)\xi_t(\omega)\chi_{T_m}(t)\nu(dt)\}_{m=1}^{\infty}$$

in a sequence in $L(\xi)$ that converges to $\int_T f(t)\xi_t(\omega)\nu(dt)$ by the dominated convergence theorem. □

3.2.4 THEOREM. *Let* (T,A,ν) *be a* σ*-finite measure space, let* $\{\xi_t, t\in T\}$ *be a measurable* SαS *process, and suppose that* $L_p(T,A,\nu)$ *is separable,* *where* $1 < p < \alpha$. *Then* $\int_T |\xi_t|^p \nu(dt) < \infty$ *a.s. if and only if* $\int_T E(|\xi_t|^p)\nu(dt) < \infty$.

This result for $\alpha = 2$ is contained in [Rajput 1972, Proposition 3.4]. Note that Proposition 2.1.2 provides an expression for $E(|\xi_t|^p)$ in terms of the covariation of ξ on the diagonal.

Proof: Sufficiency is clear. For the necessity, define the map $\Phi\colon \Omega \to L_p(T,A,\nu)$ by

$$\Phi(\omega) = \begin{cases} \xi(\cdot,\omega) & \text{if } \xi(\cdot,\omega)\in L_p(T,A,\nu) , \\ 0 & \text{otherwise} . \end{cases}$$

Then Φ is a measurable map from (Ω,F) to $(L_p, B(L_p))$, since the process ξ is measurable and $L_p(T,A,\nu)$ is separable. Thus ξ induces a measure μ on $L_p(T,A,\nu)$ by

$$\mu(B) = P\Phi^{-1}(B)$$

for all $B\in B(L_p)$. If $f\in L_q(T,A,\nu)$, $\frac{1}{p} + \frac{1}{q} = 1$, and if f^* denotes the element of $L_p^*(T,A,\nu)$ corresponding to $f\in L_q(T,A,\nu)$, then we have that $f^*(\xi(\cdot,\omega)) = \int_T f(t)\xi_t(\omega)\nu(dt)$ defines an element of $L(\xi)$ by Lemma 3.2.3. Therefore $\mu(f^*)^{-1}$ is an SαS distribution on R, since for every Borel subset B of R,

$$\mu(f^*)^{-1}(B) = P\Phi^{-1}\{x\in L_p(T,A,\nu)\colon f^*(x)\in B\}$$

$$= P\{\omega\in\Omega\colon f^*(\xi(\cdot,\omega))\in B\} .$$

Thus μ is an SαS measure on $L_p(T,A,\nu)$, so that by result 3.2.1 of DeAcosta

$$\int_T E(|\xi_t|^p)\nu(dt) = E\int_T |\xi_t|^p \nu(dt)$$

$$= \int_\Omega ||\xi(\cdot,\omega)||^p_{L_p(T)} P(d\omega)$$

$$= \int_{L_p(T)} ||x||^p_{L_p(T)} \mu(dx) < \infty . \qquad \square$$

If $(T,A,\nu) = ([a,b],B, \text{Leb})$, then Theorem 3.2.4 holds for $p = 1$. The only alteration required to the proof is to take $q = \infty$ and use a result in [Doob 1953, p. 64] instead of Lemma 3.2.3.

3. Absolute continuity of sample paths.

In this section we obtain sufficient conditions for the sample paths of a p-th order process to be absolutely continuous (Theorem 3.3.1) and show these conditions to be also necessary when the process is SαS (Theorem 3.3.3).

If $\mathbb{B}$ is a Banach space with norm $||\cdot||$ and $T = [a,b]$ is an interval of R, then we write $L_1[T,\mathbb{B}]$ for the space of Borel measurable functions $f: T \to \mathbb{B}$ such that $||f(t)|| \in L_1(T,\text{Leb})$. We call f absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every disjoint family $\{(s_k,t_k)\}_{k=1}^n$ of subintervals of T, $\sum_{k=1}^n (t_k - s_k) \le \delta$ implies that $\sum_{k=1}^n ||f(t_k) - f(s_k)|| \le \varepsilon$. The following characterization of absolute continuity is given in [Brézis 1973, Appendice]: the function $f: T \to \mathbb{B}$ is absolutely continuous if and only if for every $t \in T$ it can be expressed in terms of a Bochner integral

$$f(t) = f(a) + \int_a^t \hat{f}(s)ds$$

where $\hat{f} \in L_1[T,\mathbb{B}]$.

We now present sufficient conditions for a p-th order process to have absolutely continuous paths. The argument used is from [Cambanis 1975b] where second order processes are considered.

3.3.1 *THEOREM. Let* $\xi = \{\xi_t, t \in T\}$ *be a separable p-th order process on* (Ω,F,P), *where* $p > 1$ *and* $T = [a,b]$. *Then each of the following two equivalent conditions is sufficient for the sample paths of* ξ *to be absolutely continuous with probability one (and have a measurable p-th order process as derivative):*

(i) The map $T \to L(\xi)$ *defined by* $t \mapsto \xi_t$ *is absolutely continuous.*

(ii) The function $C_{\zeta\xi}(t) = A_\zeta(\xi_t)$ *is absolutely continuous for all* $\zeta \in L(\xi)$, *for all* $t \in T - T_0$ *with* $Leb(T_0) = 0$ *the derivative* $C'_{\zeta\xi}(t)$ *exists for all* $\zeta \in L(\xi)$, *and*

$$\int_T ||\hat{\xi}_t|| dt < \infty \quad ,$$

where for each $t \in T - T_0$, $\hat{\xi}_t$ *is the unique element of* $L(\xi)$ *satisfying* $A_\zeta(\hat{\xi}_t) = C'_{\zeta\xi}(t)$ *for all* $\zeta \in L(\xi)$.

Proof: The equivalence of (i) and (ii) is contained in [Brézis 1973, p. 145]. If $t \mapsto \xi_t$ is absolutely continuous, we have

$$\xi_t = \xi_a + \int_a^t \hat{\xi}_s ds$$

for all $t \in T$ where $\hat{\xi} \in L_1[T, L(\xi)]$. By Theorem 3.1.1, $\hat{\xi}$ has a measurable modification, say η. Observe that

$$E \int_a^b |\eta(t,\omega)| dt \le \int_a^b ||\eta(t,\omega)||_{L_p(\Omega)} dt = \int_a^b ||\hat{\xi}_t||_{L_p(\Omega)} dt < \infty \; ;$$

so $\eta(\cdot,\omega) \in L_1(T, Leb)$ a.s., *i.e.*, for every $\omega \in \Omega - \Omega_0$ with $P(\Omega_0) = 0$. Define X by

$$X(t,\omega) = \begin{cases} \xi(a,\omega) + \int_a^t \eta(s,\omega) ds, & t \in T, \; \omega \in \Omega - \Omega_0 \; , \\ 0 & , t \in T, \; \omega \in \Omega_0 \; , \end{cases}$$

and note that the sample paths of X are absolutely continuous.

Let $\hat{\xi}_n$ be a sequence of simple functions $T \to L(\xi)$ such that

$$\int_a^t ||\hat{\xi}_n(s) - \hat{\xi}(s)||_{L_p(\Omega)} ds \to 0$$

for all $t \in T$. Then

$$E|\xi_t - X_t| = E|\int_a^t \hat{\xi}_s ds - \int_a^t \eta_s ds|$$

$$= \lim_{n\to\infty} E|\int_a^t \hat{\xi}_n(s)ds - \int_a^t \eta(s)ds|$$

$$\leq \lim_{n\to\infty} \int_a^t E|\hat{\xi}_n(s) - \eta(s)|ds$$

$$\leq \lim_{n\to\infty} \int_a^t ||\hat{\xi}_n(s) - \eta(s)||_{L_p(\Omega)} ds = 0 .$$

Thus $P\{\xi_t = X_t\} = 1$ for all $t\epsilon T$. Let S be a separating set for ξ with $N\epsilon F$, $P(N) = 0$, such that if $\omega\epsilon\Omega-N$ and $t\epsilon T-S$, then $\xi(t,\omega) = \lim_{\substack{s\to t\\ s\epsilon S}} \xi(s,\omega)$. Now $P\{\xi_t = X_t, \forall t\epsilon S\} = 1$. Thus there exists $\Omega_1\epsilon F$ with $P(\Omega_1) = 0$ such that $\xi(t,\omega) = X(t,\omega)$ for all $t\epsilon S$ and $\omega\epsilon\Omega-\Omega_1$. Given any $t\epsilon T-S$ and any $\omega\epsilon\Omega-(N\cup\Omega_1)$,

$$\xi(t,\omega) = \lim_{\substack{s\to t\\ s\epsilon S}} \xi(s,\omega) = \lim_{\substack{s\to t\\ s\epsilon S}} X(s,\omega) = X(t,\omega) ,$$

since the paths of X are continuous. Thus, if $\omega\epsilon\Omega-(N\cup\Omega_1)$, then $\xi(t,\omega) = X(t,\omega)$ for all $t\epsilon T$, and hence the sample paths of ξ are absolutely continuous with probability one. □

In [Cambanis 1973] it is shown for a separable Gaussian process ξ that at every fixed $t\epsilon T$ the paths of ξ are continuous, or differentiable, with probability zero or one. Also, if ξ is measurable, then with probability one its paths have essentially the same points of differentiability and continuity. These same results follow for SαS processes (with no change in argument) by applying a zero-one law for stable measures from [Dudley and Kanter 1974]. We now state in detail one of these results which will be used later.

3.3.2 *THEOREM.* [Cambanis] *Let* $\xi = \{\xi_t : t \in T\}$ *be a separable* SαS *process, where* T *is any interval.*

(i) At every fixed point $t \in T$ *the paths of* ξ *are differentiable with probability zero or one.*

(ii) Let T_d *be the set of points* t *in* T *where the paths of* ξ *are differentiable with probability one, and* $T_d(\omega)$ *be the set of points* t *in* T *where the path* $\xi(\cdot,\omega)$ *is differentiable. If* ξ *is measurable, then with probability one*

$$\mathrm{Leb}\{T_d(\omega) \,\Delta\, T_d\} = 0 \ .$$

The next result shows that the conditions in Theorem 3.3.1 are necessary for a SαS process ξ. Once again, we use the proof of a similar result in [Cambanis 1975b] for Gaussian processes, certain passages being lifted in their entirety.

3.3.3 *THEOREM. Let* $\xi = \{\xi_t, t \in T\}$ *be a separable* SαS *process with* $T = [a,b]$ *and* $\alpha > 1$. *Then the following are equivalent:*

(i) The paths of ξ *are absolutely continuous with probability one.*

(ii) The map $T \to L(\xi)$ *defined by* $t \mapsto \xi_t$ *is absolutely continuous.*

(iii) The function $C_{\zeta\xi}(t) = A_\zeta(\xi_t)$ *is absolutely continuous for all* $\zeta \in L(\xi)$, *for all* $t \in T - T_0$ *with* $\mathrm{Leb}(T_0) = 0$ *the derivative* $C'_{\zeta\xi}(t)$ *exists for all* $\zeta \in L(\xi)$, *and*

$$\int_T ||\hat{\xi}_t|| dt < \infty \ ,$$

where for each $t \in T - T_0$, $\hat{\xi}_t$ *is the unique element of* $L(\xi)$ *satisfying* $A_\zeta(\hat{\xi}_t) = C'_{\zeta\xi}(t)$ *for all* $\zeta \in L(\xi)$.

Proof: (i) $\Rightarrow$ (ii). Absolute continuity of a path implies differentiability a.e. [Leb], and therefore $\text{Leb}[T-T_d(\omega)] = 0$ a.s. Since ξ has continuous paths with probability one, it is measurable. Thus by Theorem 3.3.2, $\text{Leb}[T_d(\omega) \Delta T_d] = 0$ a.s., and hence $\text{Leb}(T-T_d) = 0$. Take $T_0 = T-T_d$, and let $\Omega_0 \in F$ with $P(\Omega_0) = 0$ be such that $\xi(\cdot,\omega)$ is absolutely continuous for all $\omega\in\Omega-\Omega_0$. Define ζ by

$$\zeta(t,\omega) = \begin{cases} \limsup\limits_{n\to\infty} n[\xi(t+\frac{1}{n},\omega) - \xi(t,\omega)], & t\in[a,b),\ \omega\in\Omega-\Omega_0\ , \\ 0\ , & t\in[a,b),\ \omega\in\Omega_0;\ t=b,\ \omega\in\Omega\ . \end{cases}$$

Then ζ is measurable and for all $\omega\in\Omega-\Omega_0$ we have $\zeta(t,\omega) = \xi'(t,\omega)$ for $t\in T_d(\omega) - \{b\}$, where $\xi'(\cdot,\omega)$ denotes the path derivative of $\xi(\cdot,\omega)$. Also, for all $t\in T_d - \{b\}$, $\zeta(t,\omega) = \xi'(t,\omega)$ a.s. Now define η by

$$\eta(t,\omega) = \begin{cases} \zeta(t,\omega),\ t\in T_d,\ \omega\in\Omega\ , \\ 0\ ,\ t\in T_0,\ \omega\in\Omega\ , \end{cases}$$

and note that η is measurable and SαS. Also, for all $\omega\in\Omega-\Omega_0$,

$$\eta(\cdot,\omega) = \zeta(\cdot,\omega) = \xi'(\cdot,\omega) \quad \text{a.e. [Leb]}$$

on T, and $\xi'(\cdot,\omega)\in L_1(T, \text{Leb})$ since $\xi(\cdot,\omega)$ is absolutely continuous. It follows that $\eta(\cdot,\omega)\in L_1(T,\text{Leb})$, and hence $\int_T ||\eta_s||_{L_1(\Omega)} ds < \infty$ by the remark following Theorem 3.2.4. In addition, for $\omega\in\Omega-\Omega_0$, we have

$$\xi(t,\omega) = \xi(a,\omega) + \int_a^t \eta(s,\omega)ds$$

for all $t\in T$.

Let $\{(s_k,t_k)\}_{k=1}^n$ be a family of disjoint subintervals of T. Then

$$\sum_{k=1}^n ||\xi_{t_k} - \xi_{s_k}||_{L_1(\Omega)} = \sum_{k=1}^n E\left|\int_{s_k}^{t_k} \eta(s,\omega)ds\right|$$

$$\leq \sum_{k=1}^{n} \int_{s_k}^{t_k} ||\eta_s||_{L_1(\Omega)}(ds) = \int_{\bigcup_{k=1}^{n}(s_k,t_k)} ||\eta_s||_{L_1(\Omega)} ds .$$

Therefore the map $t \mapsto \xi_t$ is absolutely continuous by the absolute continuity of the indefinite integral since $\int_T ||\eta_s||_{L_1(\Omega)} ds < \infty$.

(ii) => (i) and (ii) <=> (iii) follow from Theorem 3.3.1. □

Theorems 3.3.1 and 3.3.3 with appropriate modifications give conditions for paths to be absolutely continuous with derivatives in $L_p(T,\text{Leb})$. They can also be extended to give conditions for paths to have (n-1) continuous derivatives with the (n-1)-th derivative absolutely continuous with derivative in $L_p(T,\text{Leb})$.

The following corollary generalizes a well known result for stationary Gaussian processes to the (nonstationary) SαS case.

3.3.4 *COROLLARY. Let* $\{\zeta_\lambda, -\infty < \lambda < \infty\}$ *be a* SαS *process with independent increments and* $F(\lambda) = ||\zeta_\lambda||^\alpha$ *a bounded function. Then a separable stochastic process* $\{\xi_t, a \leq t \leq b\}$ *defined by*

$$\xi_t = \int_{-\infty}^{\infty} \cos(t\lambda) d\zeta_\lambda$$

has absolutely continuous sample paths with probability one if and only if

$$\int_{-\infty}^{\infty} |\lambda|^\alpha dF(\lambda) < \infty .$$

Proof: If ξ has absolutely continuous sample paths, then for every $g \in L_\alpha(dF)$ and $\zeta = \int_{-\infty}^{\infty} g(\lambda) d\zeta_\lambda$,

$$C_{\zeta\xi}(t) = \int_{-\infty}^{\infty} \cos(t\lambda)(g(\lambda))^{\alpha-1} dF(\lambda)$$

is absolutely continuous on [a,b] by Theorem 3.3.3. Thus for any

$g_0 \epsilon L_{\alpha/\alpha-1}(dF)$, $(g_0)^{1/\alpha-1} \epsilon L_\alpha(dF)$ and

$$\int_{-\infty}^{\infty} \cos(t\lambda) g_0(\lambda) dF(\lambda)$$

is absolutely continuous on $[a,b]$. It is known that a function $f(t) = \int_{-\infty}^{\infty} \cos(t\lambda) d\mu(\lambda)$, μ finite, is absolutely continuous if and only if $\int_{-\infty}^{\infty} |\lambda| d\mu(\lambda) < \infty$. Therefore $\int_{-\infty}^{\infty} |\lambda| g_0(\lambda) dF(\lambda) < \infty$ for every positive function $g_0 \epsilon L_{\alpha/\alpha-1}(dF)$, and it follows that $\int_{-\infty}^{\infty} |\lambda|^{\alpha} dF(\lambda) < \infty$.

For the converse, note that every $\zeta \epsilon L(\xi)$ can be expressed as $\zeta = \int_{-\infty}^{\infty} g(\lambda) d\zeta_\lambda$, where $g \epsilon L_\alpha(dF)$. If $\lambda \epsilon L_\alpha(dF)$, then

$$\int_{-\infty}^{\infty} |\lambda| |g(\lambda)|^{\alpha-1} dF(\lambda) < \infty$$

and therefore

$$C_{\zeta\xi}(t) = \int_{-\infty}^{\infty} \cos(t\lambda) (g(\lambda))^{\alpha-1} dF(\lambda)$$

is absolutely continuous. It is easily seen that $C'_{\zeta\xi}(t)$ exists at every $t \epsilon (a,b)$ and that

$$\hat{\xi}_t = - \int_{-\infty}^{\infty} \lambda \sin(t\lambda) d\zeta_\lambda$$

satisfies $C'_{\zeta\xi}(t) = A_\zeta(\hat{\xi}_t)$ for all $\zeta \epsilon L(\xi)$. Thus

$$\int_a^b ||\hat{\xi}_t|| dt = \int_a^b ||\lambda \sin(t\lambda)||_{L_\alpha(dF)} dt < \infty ,$$

and therefore the paths of ξ are absolutely continuous with probability one by Theorem 3.3.3. □

It should be remarked that a SαS process $\xi = \{\xi_t, a \le t \le b\}$ with independent increments cannot have absolutely continuous sample paths (except in the trivial case where $F(t) = ||\xi_t||^\alpha$ is a constant function).

For, the claim is obvious if F is not absolutely continuous with respect to Lebesgue measure. In the case of absolutely continuous F, given any $\varepsilon > 0$ and $\delta > 0$ it is possible to choose a finite family of disjoint subintervals $\{(s_k,t_k)\}_{k=1}^n$ such that $\sum_{k=1}^n (t_k - s_k) \le \delta$, but

$$\sum_{k=1}^n ||\xi_{t_k} - \xi_{s_k}|| = \sum_{k=1}^n |F(t_k) - F(s_k)|^{1/\alpha} > \varepsilon .$$

To see this case, let (a_1,b_1) be a subinterval of $[a,b]$ such that $b_1 - a_1 < \delta$, $F(b_1) - F(a_1) > 0$, and define

$$h = \left(\frac{F(b_1) - F(a_1)}{\alpha\varepsilon}\right)^{\alpha/\alpha-1} .$$

By the uniform continuity of F we can choose n so large that $|t-s| < \frac{\delta}{n}$ implies $|F(t)-F(s)| < h$, for all $s,t \in [a,b]$. Let $\{(s_k,t_k)\}_{k=1}^n$ be a partition of (a_1,b_1) into n subintervals of equal length. Then $\sum_{k=1}^n (t_k - s_k) = b_1 - a_1 < \delta$, but

$$\sum_{k=1}^n |F(t_k) - F(s_k)|^{1/\alpha} > \sum_{k=1}^n \frac{F(t_k) - F(s_k)}{\alpha h^{\alpha-1/\alpha}} = \varepsilon .$$

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