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PROPULSION NOZZLE STUDIES

Volume I TRANSONIC FLOW IN AN ELLIPTIC NOZZLE

THERMAL SCIENCES AND PROPULSION CENTER
PURDUE UNIVERSITY
WEST LAFAYETTE, IND. 47907

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FOR THE COMMANDER

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→ nondimensional form. A general series for the radial component of velocity is then assumed along with a coordinate stretching in the axial direction. Corresponding series for the axial and tangential velocity components are then derived to satisfy two of the governing equations. The variable coefficients in the series are then determined such that the remaining governing equations and boundary conditions are satisfied. Constants arising in the solution are related to the boundary geometry. Only the first order solution is obtained herein; however the procedure can be used in a straightforward manner to obtain higher order solutions. Finally, it is shown that the two-dimensional axisymmetric solution is a special case of the three-dimensional solution.

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PREFACE

This final report was submitted by the Thermal Sciences and Propulsion Center of Purdue University, under Contract No. F33615-73-C-2010. The effort was sponsored by the Air Force Aero Propulsion Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command, Wright-Patterson AFB, Ohio under Project No. 3012, Task 301213, and Work Unit 30121306 with P. J. Hutchison/AFAPL/RJT as Project Engineer. Murray A. Epstein and H. Doyle Thompson of Purdue University were technically responsible for the work.

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SECTION I

INTRODUCTION

It can be said that a majority of modern high thrust propulsion nozzles obtain their thrust by means of a reactive force to the expelled fluid. Prediction and maximization of the performance of these devices require that the flow field be accurately analyzed in the three flow stages, namely the subsonic, the transonic, and the supersonic regimes.

In the present investigation, the transonic flow region of a propulsion nozzle is studied. In this region, high pressure fluid is received as a result of some combustion process. The second stage receives potential energy, associated with static fluid temperature, and relinquishes kinetic energy, associated with translational fluid velocity. Thus, in the transonic region, an increase in linear momentum is realized. The outflow of this stage is supersonic and, in fact, provides initial conditions for the third stage along which the flow is everywhere supersonic.

To this date there has been abundant literature published on the subject of transonic nozzle flow. One general method of solution, and the method of solution presented herein, is a series expansion technique. Generally, the expansion approach involves a parametric solution to a particular set of governing flow equations where the expansion parameter can be directly related to the flow geometry.

One of the pioneers of the expansion method of solution was Sauer. In his original paper (1) the governing equations are reduced to a convenient form by means of a perturbation. The flow velocity is then expanded in a power series as a function of the radial coordinate. The coefficients of the series are obtained by substituting the series into the perturbed governing equations. These coefficients turn out to be functions of the centerline velocity.

A different approach to the expansion technique was used by Hall.(2) In this report, the velocity components along the region of

(1) Sauer, R., "General Characteristics of the Flow Through Nozzles at Near Critical Speeds," NACA TM No. 1147 (1947).

(2) Hall, I.M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," Quart. Journ. Mech. and Applied Math., Vol. XV, Pt. 4 (1962) pp. 487-508.

the throat are calculated by means of an expansion in inverse powers of R , the radius of curvature of the nozzle profile at the throat. It can be shown⁽³⁾ that Sauer's solution is a special case of Hall's solution when terms of order R^{-2} are neglected. Two innovative aspects to Hall's work made it a unique contribution to the literature. One concept is to expand the wall contour in terms of a McLaurin series and the other is to employ a "coordinate stretch". Both of these ideas are used in the following investigation.

It is observed that the nozzle boundary is not a constant coordinate line in cylindrical coordinates and therefore must be evaluated as a power series. So, the wall boundary condition is never exactly satisfied in cylindrical coordinates. Also, the radial velocity component is proportional to the boundary slope, which can become quite large in the throat region of nozzles having small radii of curvature. Hence, Kliegel and Levine⁽⁴⁾ speculated that the accuracy of the solution could be improved by considering a solution in a coordinate system where the flow axis and wall are both coordinate lines. This suggests a toroidal coordinate system for nozzles having throats whose cross-section is circular. It happens that in toroidal coordinates, the solution now appears in powers of $1/(R + 1)$. In this system, the convergent properties of the expansion solution are in fact superior. In addition, in the limit of large radii of curvature, the flow field of Kliegel and Levine reduces to Hall's, as it must. Finally, it is noted⁽⁴⁾ that comparisons between theory and experiment are excellent.

Another expansion method which uses an inverse approach for solving the transonic flow problem, was proposed by Hopkins and Hill.⁽⁵⁾ In their development, a transformation to a velocity-potential, stream-function coordinate system was used in simplifying the governing equations. Four partial differential equations result from the basic flow characterization. The partial differential equations are solved by means of writing the unknowns in the form of a power series; the coefficients being functions of velocity potential only. The two authors make use of the solution to analyze annular type, unconventional nozzles.

⁽³⁾ May, R.J. Jr., Thompson, H.D., and Hoffman, J.D., "Comparison of Transonic Flow Solutions in C-D Nozzles," Air Force Aero Propulsion Laboratory Report No. AFAPL-TR-74-110, (October 1974).

⁽⁴⁾ Kliegel, J.R., and Levine, J.N., "Transonic Flow in Small Throat Radius of Curvature Nozzles," AIAA Journal, Vol. 7, No. 7 (July 1968) pp. 1375-1378.

⁽⁵⁾ Hopkins, D.F., and Hill, D.E., "Transonic Flow in Unconventional Nozzles," AIAA Journal, Vol. 6, No. 5 (November 1967), pp. 838-842.

Until now, almost all of the transonic flow analyses have dealt with either an axisymmetric or planar nozzle geometry. However, in present design, it is sometimes necessary to use non-circular propulsion nozzle to facilitate the nozzle, airframe integration. Ivanov and Ryl'ko⁽⁶⁾ consider a three-dimensional transonic flow problem for an elliptic geometry by writing the system of governing equations (gas dynamic equations) in integral form. The equations are solved numerically^(7,8) using a difference scheme of continuous calculations of the discontinuities of the solutions to the gas dynamic equations. To accomplish this, one fourth of the flow field (from symmetry considerations) was divided up into 3200 elemental cells. The initial distribution of flow variables in the nozzle was taken from the one-dimensional computations. Various ratios of semi-major to semi-minor axes were chosen and numerical results for the flow distribution are given.

This report concerns the analytical solution of the transonic flow in a converging-diverging nozzle having an elliptical cross-section and follows closely the M.S. Thesis of Epstein⁽⁹⁾. A power series expansion technique similar to that of Hall⁽²⁾ is employed. The procedure consists of:

1. Writing the governing equations and boundary conditions in a suitable non-dimensional form.
2. Writing the velocity components as a power series in terms of an expansion parameter ϵ .
3. Determining the first order governing equations and boundary conditions by substituting the power series form for the velocity components into the governing equations.
4. Assuming an appropriate (consistent) form for the solution to the first order governing equations.

⁽⁶⁾ Ivanov, M. Ya., and Ryl'ko, O.A., "Analysis of Transonic Flow in Elliptical Nozzles," *IZV. AKAO. NAUK SSSR, MEKH. ZHID. GAZA*, No. 3 (September 1972), pp. 161-163.

⁽⁷⁾ Godunov, S.K., "Difference Method of Numerical Solution of Different Problems of Fluid Mechanics." *Matem. Sb.*, Vol. 47(89), No. 3 (1959), pp. 271-306.

⁽⁸⁾ Godunov, S.K., Zabrodin, A.V., and Prokopov, G.P., "Difference Scheme for Two-Dimensional Unsteady Problems of Gas Dynamics and the Calculation of Flows with Detached Shock Waves," *Zh. Vychislit. Matem. i Matem. Fiz.*, Vol. 1, No. 6 (1961), pp. 1020-1050.

⁽⁹⁾ Epstein, M.A., "Transonic Flow in Nozzles Having an Elliptical Cross Section," M.S. Thesis, Purdue University, December 1976.

5. Using the boundary conditions to solve for the various constants that appear in the solution.

In the first appendix of the work done by May, Thompson, and Hoffman,⁽³⁾ a general transonic solution for an axisymmetric nozzle is presented. The relation between the work conducted herein and that of reference (3) is discussed in Section V.

SECTION II
GOVERNING EQUATIONS

The nozzle flow is considered to be irrotational, inviscid, and steady. Orthogonal cylindrical coordinates (x, r, θ) are used, and the origin of the coordinate system is taken at the center of the throat. x is measured in the downstream direction with r measured radially and θ measured tangentially.

The governing equations are the Euler equations,

$$-\nabla P = \rho \frac{D\vec{V}}{Dt} \quad (1)$$

and the continuity equation,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \quad (2)$$

where $\vec{V} = (\vec{i}_r v_r + \vec{i}_\theta v_\theta + \vec{i}_x v_x)$; v_x , v_r , and v_θ are the velocity components in the x , r , and θ directions respectively; D/Dt is the substantial derivative given by

$$\frac{D}{Dt} = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_x \frac{\partial}{\partial x} \quad (3)$$

for steady flow, and ρ is the density. Also, the "del" operator in cylindrical coordinates is written:

$$\nabla = \vec{i}_r \frac{\partial}{\partial r} + \vec{i}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{i}_x \frac{\partial}{\partial x} \quad (4)$$

where \vec{i}_r , \vec{i}_θ and \vec{i}_x are unit vectors in the coordinate direction.

Now the definition of the speed of sound, namely,

$$\left. \frac{dP}{d\rho} \right|_s = a^2 \quad (5)$$

can be written for an irrotational, steady, inviscid flow in the form,

$$\frac{DP}{Dt} = a^2 \frac{D\rho}{Dt} \quad (6)$$

Using equation (6) to eliminate density derivatives from equation (2) and expanding the result yields

$$a^2 \left(\frac{\partial v_r}{\partial r} + \frac{1}{r} v_r + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} \right) + v_r \frac{\partial P}{\partial r} + \frac{v_\theta}{r} \frac{\partial P}{\partial \theta} + v_x \frac{\partial P}{\partial x} = 0 \quad (7)$$

where a is the sound speed and P is the pressure.

The next step is to use the Euler equations to eliminate the pressure from equation (7). In cylindrical coordinates the Euler equations can be written

$$-\frac{\partial P}{\partial r} = \rho \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_x \frac{\partial v_r}{\partial x} \right) \quad (8)$$

$$-\frac{1}{r} \frac{\partial P}{\partial \theta} = \rho \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_x \frac{\partial v_\theta}{\partial x} \right) \quad (9)$$

$$-\frac{\partial P}{\partial x} = \rho \left(v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_x}{\partial \theta} + v_x \frac{\partial v_x}{\partial x} \right) \quad (10)$$

Equations (8) through (10) are substituted into equation (7) so that the pressure terms are eliminated:

$$\begin{aligned} & \frac{\partial v_r}{\partial r} (a^2 - v^2) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} (a^2 - v_\theta^2) + \frac{\partial v_x}{\partial x} (a^2 - v_x^2) - v_r v_\theta \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} \right) \\ & - v_r v_x \left(\frac{\partial v_r}{\partial x} + \frac{\partial v_x}{\partial r} \right) - v_\theta v_x \left(\frac{1}{r} \frac{\partial v_x}{\partial \theta} + \frac{\partial v_\theta}{\partial x} \right) + \frac{a^2}{r} v_r = 0 \end{aligned} \quad (11)$$

Now for a steady, irrotational, inviscid flow the Euler equations can be replaced by the irrotationality condition, namely

$$\nabla \times \vec{V} = 0 \quad (12)$$

Now, expanding the irrotationality condition in cylindrical coordinates yields

$$\frac{\partial v_x}{\partial \theta} = \frac{\partial v_\theta}{\partial x} r \quad (13)$$

$$\frac{\partial v_r}{\partial x} = \frac{\partial v_x}{\partial r} \quad (14)$$

$$\frac{\partial v_r}{\partial \theta} = r \frac{\partial v_\theta}{\partial r} + v_\theta = \frac{\partial (rv_\theta)}{\partial r} \quad (15)$$

Using equations (13) through (15) in equation (11) yields:

$$\begin{aligned} \frac{\partial v_r}{\partial r} (a^2 - v_r^2) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} (a^2 - v_\theta^2) + \frac{\partial v_x}{\partial x} (a^2 - v_x^2) - 2v_r v_\theta \frac{\partial v_\theta}{\partial r} - \frac{v_r v_\theta^2}{r} \\ - 2v_r v_x \frac{\partial v_x}{\partial r} - 2v_\theta v_x \frac{\partial v_\theta}{\partial x} + \frac{a^2 v_r}{r} = 0 \end{aligned} \quad (16)$$

The nozzle geometry near the throat is assumed to be elliptical in cross-section so that the boundary geometry is described by the equation

$$\frac{r^2 \cos^2 \theta}{\alpha^2(x)} + \frac{r^2 \sin^2 \theta}{\beta^2(x)} = 1 \quad (17)$$

where α and β are even functions of x such that the nozzle is symmetric about $x = 0$ in the throat region. Writing α and β in a power series for small values of x ,

$$\alpha(x) = A_0 + A_2 x^2 + O(x^4) \quad (18)$$

$$\beta(x) = B_0 + B_2 x^2 + O(x^4) \quad (19)$$

where $O(x^4)$ denotes terms of order x^4 and higher.

It is convenient to nondimensionalize the problem at this point. The length A_0 is used to nondimensionalize all distances and the critical sound speed, a^* , is used to nondimensionalize velocities. Thus,

$$\bar{r} \equiv \frac{r}{A_0} \text{ and } \bar{x} \equiv \frac{x}{A_0} \quad (20)$$

θ is already nondimensional since it is measured in radians. Similarly,

$$\bar{v}_r \equiv \frac{v_r}{a^*}, \quad \bar{v}_\theta \equiv \frac{v_\theta}{a^*}, \quad \text{and } \bar{v}_x \equiv \frac{v_x}{a^*} \quad (21)$$

For the adiabatic flow of an ideal gas Bernoulli's equation can be written

$$a^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} \left(v_r^2 + v_\theta^2 + v_x^2 \right) \quad (22)$$

Substituting equation (22) as well as the nondimensional coordinate and velocity components into equation (16) and simplifying yields:

$$\frac{\partial \bar{v}_r}{\partial \bar{r}} \left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2} (\bar{v}_r^2 + \bar{v}_\theta^2 + \bar{v}_x^2) - \bar{v}_r^2 \right] + \frac{1}{\bar{r}} \frac{\partial \bar{v}_\theta}{\partial \theta} \left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2} (\bar{v}_r^2 + \bar{v}_\theta^2 + \bar{v}_x^2) - \bar{v}_\theta^2 \right] + \frac{\partial \bar{v}_x}{\partial \bar{x}} \left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2} (\bar{v}_r^2 + \bar{v}_\theta^2 + \bar{v}_x^2) - \bar{v}_x^2 \right] - 2\bar{v}_r \bar{v}_\theta \frac{\partial \bar{v}_\theta}{\partial \bar{r}} \quad (23)$$

$$- \frac{1}{\bar{r}} \bar{v}_r \bar{v}_\theta^2 - 2\bar{v}_r \bar{v}_x \frac{\partial \bar{v}_x}{\partial \bar{r}} - 2\bar{v}_\theta \bar{v}_x \frac{\partial \bar{v}_\theta}{\partial \bar{x}} + \frac{1}{\bar{r}} \bar{v}_r \left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2} (\bar{v}_r^2 + \bar{v}_\theta^2 + \bar{v}_x^2) \right] = 0$$

Using the nondimensional expressions for the velocity components and coordinates in equations (13), (14), and (15) gives:

$$\frac{\partial \bar{v}_x}{\partial \theta} = \frac{\partial \bar{v}_\theta}{\partial \bar{x}} \bar{r} \quad (24)$$

$$\frac{\partial \bar{v}_r}{\partial \bar{x}} = \frac{\partial \bar{v}_x}{\partial \bar{r}} \quad (25)$$

$$\frac{\partial \bar{v}_r}{\partial \theta} = \bar{r} \frac{\partial \bar{v}_\theta}{\partial \bar{r}} + \bar{v}_\theta \quad (26)$$

Equations (23) through (26) are the nondimensional governing equations. The procedure now is to write the dimensionless velocity components in terms of an expansion parameter, ϵ , where ϵ is small compared to one. Then, since consideration is to be limited to a region close to the throat, \bar{v}_r and \bar{v}_θ are small and \bar{v}_x is approximately equal to unity. The assumed perturbation of the coordinate velocities are:

$$\bar{v}_x = 1 + \epsilon \bar{v}_{x1} + \epsilon^2 \bar{v}_{x2} + \epsilon^3 \bar{v}_{x3} + \dots \quad (27)$$

$$\bar{v}_r = \epsilon^{3/2} \bar{v}_{r1} + \epsilon^{5/2} \bar{v}_{r2} + \epsilon^{7/2} \bar{v}_{r3} + \dots \quad (28)$$

$$\bar{v}_\theta = \epsilon^{3/2} \bar{v}_{\theta1} + \epsilon^{5/2} \bar{v}_{\theta2} + \epsilon^{7/2} \bar{v}_{\theta3} + \dots \quad (29)$$

These particular powers of ϵ are used so that the boundary conditions can be satisfied and so that a reduction to the special case of axisymmetric flow is possible.

In order that the irrotationality equations (24), (25), and (26) be satisfied, the x-coordinate must be transformed to

$$\bar{x} = \bar{z} \epsilon^{1/2} \quad (30)$$

Substituting equations (27) through (30) into the governing equations (equations (23) through (26)) and retaining only the lowest order powers of epsilon yields

$$\frac{\partial v_{r1}}{\partial r} + \frac{1}{r} \frac{\partial v_{\theta 1}}{\partial \theta} - \frac{\partial v_{z1}}{\partial z} v_{z1} (\gamma+1) + \frac{1}{r} v_{r1} = 0 \quad (31)$$

$$\frac{\partial v_{z1}}{\partial \theta} = r \frac{\partial v_{\theta 1}}{\partial z} \quad (32)$$

$$\frac{\partial v_{r1}}{\partial z} = \frac{\partial v_{z1}}{\partial r} \quad (33)$$

$$r \frac{\partial v_{\theta 1}}{\partial r} + v_{\theta 1} = \frac{\partial v_{r1}}{\partial \theta} \quad (34)$$

For convenience, the bars over the velocities in the three coordinate directions and over r and z have been dropped. Equations (31) through (34) are the first order governing equations for the present problem.

SECTION III
BOUNDARY CONDITIONS

The first boundary condition arises from the fact that the velocity component normal to the nozzle boundary is zero. That is,

$$\vec{V} \cdot \vec{n} = 0 \quad (35)$$

where \vec{n} is a unit normal to the nozzle surface.

To write this condition in a suitable form for the present problem, it is necessary to recall the expression for the nozzle's geometry, equation (17). This can be nondimensionalized as follows. Let

$$\alpha(\bar{x})/A_0 \equiv \bar{\alpha}(\bar{x}) \equiv 1 + A_0 A_2 \bar{x}^2 \quad (36)$$

and

$$\beta(\bar{x})/A_0 \equiv \bar{\beta}(\bar{x}) \equiv \frac{B_0}{A_0} + A_0 B_2 \bar{x}^2 \quad (37)$$

Then,

$$\frac{r^2 \cos^2 \theta}{\bar{\alpha}^2(\bar{x})} + \frac{r^2 \sin^2 \theta}{\bar{\beta}^2(\bar{x})} = 1 \quad (38)$$

A_2 and B_2 can be related to the expansion parameter ϵ and will be discussed later. Again, for convenience, the bars over $\bar{\alpha}(\bar{x})$ and $\bar{\beta}(\bar{x})$ will be dropped.

For a wall contour defined by $r = h(x, \theta)$, a McLaurin expansion about $x = 0$, holding for all theta yields

$$r = h(x, \theta) = h(0, \theta) + m'x + \frac{1}{2!} m''x^2 + \frac{1}{3!} m'''x^3 + \dots \quad (39)$$

where

$$m' = \left. \frac{\partial h}{\partial x} \right|_{x=0}, \quad m'' = \left. \frac{\partial^2 h}{\partial x^2} \right|_{x=0}, \quad m''' = \left. \frac{\partial^3 h}{\partial x^3} \right|_{x=0}, \dots \quad (40)$$

From equation (38) and from the definition of the wall contour,

$$r = h(x, \theta) = \alpha\beta / \left(\beta^2 \cos^2 \theta + \alpha^2 \sin^2 \theta \right)^{1/2} \quad (41)$$

Then,

$$m' = \left. \frac{\partial h}{\partial x} \right|_{x=0} = 0 \quad (42)$$

and

$$m'' = \left. \frac{\partial^2 h}{\partial x^2} \right|_{x=0} = \frac{2A_0 \left(A_2 B_0^3 \cos^2 \theta + B_2 A_0^3 \sin^2 \theta \right)}{\left(B_0^2 \cos^2 \theta + A_0^2 \sin^2 \theta \right)^{3/2}} \quad (42)$$

The expansion of higher order derivatives, i.e., m''' , m^{iv} , etc. results in a series with terms of higher order in epsilon and are therefore neglected.

Now, returning to equation (38), observe that it can be written as

$$\beta^2 r^2 \cos^2 \theta + \alpha^2 r^2 \sin^2 \theta - \alpha^2 \beta^2 = 0 = f(r, \theta, x) \quad (44)$$

where f is a function of r , θ , and x and represents the nozzle surface. But,

$$\vec{\nabla} f = c \vec{n} \quad (45)$$

where c is a constant. In addition,

$$\vec{\nabla} f = \frac{\partial f}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{i}_\theta + \frac{\partial f}{\partial x} \vec{i}_x \quad (46)$$

Thus, equation (35) may be written as:

$$v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + v_x \frac{\partial f}{\partial x} = 0 \quad (47)$$

Next, the quantities v_r , v_θ , and v_x will be expanded about $r = h(0, \theta) \equiv h_0$:

$$v_r = v_r(x, h, \theta) = v_r(x, h_0, \theta) + (h-h_0) \left. \frac{\partial v_r}{\partial r} \right|_{x, h_0, \theta} + \frac{(h-h_0)^2}{2!} \left. \frac{\partial^2 v_r}{\partial r^2} \right|_{x, h_0, \theta} + \dots \quad (48)$$

$$v_\theta = v_\theta(x, h, \theta) = v_\theta(x, h_0, \theta) + (h-h_0) \left. \frac{\partial v_\theta}{\partial r} \right|_{x, h_0, \theta} + \frac{(h-h_0)^2}{2!} \left. \frac{\partial^2 v_\theta}{\partial r^2} \right|_{x, h_0, \theta} + \dots \quad (49)$$

$$v_x = v_x(x, h, \theta) = v_x(x, h_0, \theta) + (h-h_0) \left. \frac{\partial v_x}{\partial r} \right|_{x, h_0, \theta} + \frac{(h-h_0)^2}{2!} \left. \frac{\partial^2 v_x}{\partial r^2} \right|_{x, h_0, \theta} + \dots \quad (50)$$

Now, recall that the wall contour was defined by $r = h(x, \theta)$.
Then,

$$r - h(x, \theta) = 0 = f(r, \theta, x) \quad (51)$$

Differentiating this equality gives the following:

$$\frac{\partial f}{\partial r} = 1, \quad \frac{\partial f}{\partial \theta} = - \frac{\partial h}{\partial \theta}, \quad \text{and} \quad \frac{\partial f}{\partial x} = - \frac{\partial h}{\partial x} \quad (52)$$

This factor $1/r$ can be expanded in terms of the boundary by means of an expansion along the nozzle wall. Observe that

$$\frac{1}{r} = \frac{1}{h_0 + \delta} = \frac{1}{h_0(1 + \delta/h_0)} = \frac{1}{h_0} \left[1 - \frac{\delta}{h_0} + \frac{\delta^2}{h_0^2} - \dots \right] \quad (53)$$

Then, equation (47) becomes

$$v_r - v_\theta \frac{\partial h}{\partial \theta} \frac{1}{h_0} \left[1 - \frac{\delta}{h_0} + \frac{\delta^2}{h_0^2} - \dots \right] - v_x \frac{\partial h}{\partial x} = 0 \quad (54)$$

The expansions for v_r , v_θ , and v_x are substituted from equations (27), (28) and (29) and higher order terms are dropped. Thus,

$$\epsilon^{3/2} v_{r1}(z, h_0, \theta) - \epsilon^{3/2} v_{\theta 1}(z, h_0, \theta) \frac{\partial h}{\partial \theta} \frac{1}{h_0} - v_{x1}(z, h_0, \theta) \frac{\partial h}{\partial x} = 0 \quad (55)$$

But $\frac{\partial h}{\partial x}$ can be found directly from equation (39):

$$\frac{\partial h}{\partial x} = m' + m''x = m''\epsilon^{1/2}z \quad (56)$$

since m' was found to be zero. Therefore, equation (55) becomes

$$\begin{aligned} \epsilon^{3/2}v_{r1}(z, h_0, \theta) - \epsilon^{3/2}v_{\theta 1}(z, h_0, \theta) \frac{\partial h}{\partial \theta} \frac{1}{h_0} \\ - \epsilon^{1/2}m''v_{x1}(z, h_0, \theta) z = 0 \end{aligned} \quad (57)$$

where m'' is given by equation (43).

In order that the terms of equation (57) be of the same order in ϵ and reduce to the axisymmetric case the following relationship between A_2 , B_2 and ϵ is assumed.

$$A_2 = \epsilon/2; \quad B_2 = k\epsilon/2, \quad (58)$$

where k is a constant. The $\epsilon/2$ terms are essential for a reduction to the axisymmetric case but the selection of constant k is arbitrary. Moreover, it is observed that k has the effect of restricting the nozzle geometry by fixing the ratio between nozzle semi-major and semi-minor axes.

By the relationships of equations (58) m'' is of order ϵ and equation (57) reduces to

$$\epsilon^{3/2} \left[v_{r1}(z, h_0, \theta) - v_{\theta 1}(z, h_0, \theta) \frac{\partial h}{\partial \theta} \frac{1}{h_0} - \frac{m''z}{\epsilon} \right] = 0 \quad (59)$$

which is the first boundary condition.

A second boundary condition arises from the fact that the radial velocity at the nozzle centerline ($r = 0$) must be zero. That is,

$$v_{r1}(z, 0, \theta) = 0 \quad (60)$$

Finally, consideration of the tangential velocity component offers two additional boundary conditions. Since the two geometrical planes of symmetry are chosen as the planes of symmetry for the tangential velocity component, this velocity component vanishes at $\theta = 0, \pi/2, \pi,$ and $3\pi/2$ for all values of r and z . Due to elliptical symmetry only one-fourth of the flow region needs to be considered. So,

$$v_{\theta 1}(z, r, 0) = 0 \quad (61)$$

$$v_{\theta 1}(z, r, \pi/2) = 0 \quad (62)$$

Equations (59), (60), (61), and (62) are the four first order boundary conditions.

SECTION IV
PROBLEM SOLUTION

From Section II the first order governing equations are:

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{\partial v_z}{\partial z} v_z (\gamma+1) + \frac{v_r}{r} = 0 \quad (31)$$

$$\frac{\partial v_z}{\partial \theta} = r \frac{\partial v_\theta}{\partial z} \quad (32)$$

$$\frac{\partial v_r}{\partial z} = \frac{\partial v_z}{\partial r} \quad (33)$$

$$\frac{\partial}{\partial r} (r v_\theta) = \frac{\partial v_r}{\partial \theta} \quad (34)$$

and from Section III the first order boundary conditions are:

$$v_r(z, h_0, \theta) - \frac{V_\theta(z, h_0, \theta)}{h_0} \frac{\partial h_0}{\partial \theta} - \frac{m^2 z}{\epsilon} = 0 \quad (59)$$

$$v_r(z, 0, \theta) = 0 \quad (60)$$

$$v_\theta(z, r, 0) = 0 \quad (61)$$

$$v_\theta(z, r, \pi/2) = 0 \quad (62)$$

where the subscript one has been dropped from all velocities since consideration is limited only to first order. Note that the governing equations and boundary conditions for axisymmetric flow are a special case of the present formulation and are obtained by setting $v_\theta = \partial/\partial\theta = 0$. Further, it will be shown in Section V that the solution to the present three-dimensional transonic flow problem reduces to the axisymmetric solution.

To begin the solution, a series of the form

$$v_r = C_1 + E_1(r, \theta) + \frac{\partial E_2(r, \theta)}{\partial r} z \quad (63)$$

is assumed for the radial velocity component where C_1 is a constant and E_1 and E_2 are functions of r and θ . v_r is chosen in this fashion so that it can be represented as a linear yet completely general function of r , θ , and z , and at the same time reduce to the axisymmetric case when the θ dependence is suppressed.

Using this representation of v_r , it is seen that in order to satisfy equation (33) the axial component of velocity, v_z , must be of the form

$$v_z = E_2(r, \theta) + D_1(\theta) + D_2(\theta)z \quad (64)$$

where D_1 and D_2 are functions only of θ .

Similarly, in order that equation (32) be satisfied,

$$v_\theta = \frac{z}{r} \left(\frac{\partial E_2(r, \theta)}{\partial \theta} + \frac{\partial D_1(\theta)}{\partial \theta} \right) + \frac{1}{r} \frac{\partial D_2(\theta)}{\partial \theta} \frac{z^2}{2} + \frac{1}{r} E_3(r, \theta) \quad (65)$$

where E_3 is a function of r and θ .

Now, when equations (63), (64), and (65) are substituted into equation (31), one obtains a quadratic equation in z . Since that equation must hold for all z the coefficients of the various powers of z must be zero. The result is

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_1) + \frac{C_1}{r} + \frac{1}{r^2} \frac{\partial E_3}{\partial \theta} - D_2(\gamma+1)(E_2+D_1) = 0 \quad (66)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_2}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 E_2}{\partial \theta^2} + \frac{\partial^2 D_1}{\partial \theta^2} \right) - D_2^2 (\gamma+1) = 0 \quad (67)$$

$$\frac{1}{r^2} \frac{\partial^2 D_2}{\partial \theta^2} = 0 \quad (68)$$

Also, substituting equations (63) and (65) into equation (34) gives

$$\frac{\partial E_3}{\partial r} = \frac{\partial E_1}{\partial \theta} \quad (69)$$

Substituting the expressions for v_r , v_z , and v_θ into the four boundary conditions will yield additional relationships between the E's and D's. Applying boundary condition of equation (60)

$$C_1 + E_1 \Big|_{0,\theta} + \frac{\partial E_2}{\partial r} \Big|_{0,\theta} z = 0 \quad (70)$$

Since this expression must hold for all values of θ and z ,

$$C_1 = 0 \quad (71)$$

$$E_1 \Big|_{0,\theta} = 0 \quad (72)$$

$$\frac{\partial E_2}{\partial r} \Big|_{0,\theta} = 0 \quad (73)$$

Applying boundary condition of equation (61) and setting the coefficients of powers of z to zero gives

$$\left(\frac{\partial E_2}{\partial \theta} + \frac{\partial D_1}{\partial \theta} \right) \Big|_{\theta=0} = 0 \quad (74)$$

$$\frac{\partial D_2}{\partial \theta} \Big|_{\theta=0} = 0 \quad (75)$$

$$E_3 \Big|_{\theta=0} = 0 \quad (76)$$

Likewise for boundary equation (62),

$$\left(\frac{\partial E_2}{\partial \theta} + \frac{\partial D_1}{\partial \theta} \right) \Big|_{\theta=\pi/2} = 0 \quad (77)$$

$$\frac{\partial D_2}{\partial \theta} \Big|_{\theta=\pi/2} = 0 \quad (78)$$

$$E_3 \Big|_{\theta=\pi/2} = 0 \quad (79)$$

Finally, applying boundary condition of equation (59), and setting the coefficients of the powers of z to zero yields,

$$E_1 \Big|_{h_0, \theta} - \frac{1}{h_0^2} \frac{\partial h_0}{\partial \theta} E_3 \Big|_{h_0, \theta} = 0 \quad (80)$$

$$\frac{\partial E_2}{\partial r} \Big|_{h_0, \theta} - \frac{1}{h_0^2} \frac{\partial h_0}{\partial \theta} \left(\frac{\partial E_2}{\partial \theta} + \frac{\partial D_1}{\partial \theta} \right) \Big|_{h_0, \theta} - \frac{m''}{\epsilon} = 0 \quad (81)$$

$$\frac{1}{h_0^2} \frac{\partial h_0}{\partial \theta} \frac{\partial D_2}{\partial \theta} \Big|_{h_0, \theta} = 0 \quad (82)$$

The next step in deriving a solution is to choose a form for E_1 and E_2 . Again, the series technique is employed and general relationships depending on r and θ are assumed which will result in a reduction to the axisymmetric case when the theta dependence is suppressed. Then, let

$$\frac{\partial}{\partial r} E_2 = D_3(\theta)r + D_4(\theta) \quad (83)$$

where D_3 and D_4 are functions of θ only. But to satisfy equation (73),

$$D_4(\theta) = 0 \quad (84)$$

Thus after integration,

$$E_2 = D_3 \frac{r^2}{2} + D_5(\theta) \quad (85)$$

Equations (63), (64), and (65) indicate that D_5 can be set to zero since it can be combined with D_1 in equations (64) and (65) or is zero when differentiated with respect to r in equation (63). Then,

$$E_2 = D_3 \frac{r^2}{2} \quad (86)$$

Next, integration of equation (68) twice yields $D_2 = C_2\theta + C_3$ where C_2 and C_3 are constants. But, to satisfy equation (82) $C_2 = 0$ and

$$D_2 = C_3 \quad (87)$$

Substitution of the above expressions for B_2 and D_2 into equation (67) gives the following partial differential equation:

$$2D_3 + \frac{1}{2} \frac{\partial^2 D_3}{\partial \theta^2} = (\gamma+1)C_3^2 - \frac{1}{r^2} \frac{\partial^2 D_1}{\partial \theta^2} \quad (88)$$

Using the technique of series solution once more, the function E_1 is assumed to be of the form,

$$E_1 = D_6(\theta)r^3 + D_7(\theta)r^2 + D_8(\theta)r + D_9(\theta) \quad (89)$$

where D_6 , D_7 , D_8 , and D_9 are functions of θ only. Boundary equation (72) indicates that

$$D_9(\theta) = 0 \quad (90)$$

Substituting equations (88) and (89) into equation (66) gives

$$r^2 \left[4D_6 - \frac{C_3}{2}(\gamma+1)D_3 \right] + 3rD_7 + (2D_8 - C_3(\gamma+1)D_1) + \frac{1}{r^2} \frac{\partial E_3}{\partial \theta} = 0 \quad (91)$$

In addition, applying equation (89) to equation (69) and integrating,

$$E_3 = \frac{\partial D_6}{\partial \theta} \frac{r^4}{4} + \frac{\partial D_7}{\partial \theta} \frac{r^3}{3} + \frac{\partial D_8}{\partial \theta} \frac{r^2}{2} + D_{10}(\theta) \quad (92)$$

where D_{10} is a function of θ . Differentiating Equation (92) with respect to θ and substituting into equation (91) yields

$$r^3 \left(4D_6 - \frac{C_3}{2}(\gamma+1)D_3 + \frac{1}{4} \frac{\partial^2 D_6}{\partial \theta^2} \right) + r \left(3D_7 + \frac{1}{3} \frac{\partial^2 D_7}{\partial \theta^2} \right) + \left[2D_8 - C_3(\gamma+1)D_1 + \frac{1}{2} \frac{\partial^2 D_8}{\partial \theta^2} \right] + \frac{1}{r^2} \frac{\partial D_{10}}{\partial \theta} = 0 \quad (93)$$

which must hold for all values of r . Thus

$$\frac{\partial^2 D_6}{\partial \theta^2} + 16D_6 = 2C_3(\gamma+1)D_3 \quad (94)$$

$$\frac{\partial^2 D_7}{\partial \theta^2} + 9D_7 = 0 \quad (95)$$

$$\frac{\partial^2 D_8}{\partial \theta^2} + 4D_8 = 2C_3(\gamma+1)D_1 \quad (96)$$

$$\frac{\partial D_{10}}{\partial \theta} = 0 \quad (97)$$

Integration of equation (97) once gives

$$D_{10} = C_4 \quad (98)$$

where C_4 is a constant.

Inspection of the above group of equations shows that in order to solve differential equations (94) and (96), forms for D_3 and D_1 must be assumed. Since D_3 is a function only of θ , and, since the flow in the tangential direction is periodic, the function D_3 can be represented as a Fourier series (10)

$$D_3 = C_5 + \sum_{n=1}^{\infty} L_n \sin(\alpha_n \theta) + \sum_{n=1}^{\infty} M_n \cos(\alpha_n \theta) \quad (99)$$

where C_5 , L_n , M_n , and α_n are constants. Furthermore, in order that equation (88) be satisfied, D_1 must also have a Fourier series representation.

$$D_1 = C_6 + \sum_{n=1}^{\infty} N_n \sin(\beta_n \theta) + \sum_{n=1}^{\infty} R_n \cos(\beta_n \theta) \quad (100)$$

where C_6 , N_n , R_n , and β_n are constants.

(10) Spiegel, M.R., *Fourier Analysis*, Schaum's Outline Series, McGraw-Hill Book Co., 1974.

Since $\frac{\partial D_3}{\partial \theta}$ can be written in terms of $\frac{\partial E_2}{\partial \theta}$ by equation (86), boundary conditions (74) and (77) can be used in evaluating some of the constants in equation (99). That is,

$$\left. \frac{\partial D_3}{\partial \theta} \right|_{\theta=0} = \left(\sum_{n=1}^{\infty} \alpha_n L_n \cos(\alpha_n \theta) - \sum_{n=1}^{\infty} M_n \alpha_n \sin(\alpha_n \theta) \right) \Big|_{\theta=0} = 0 \quad (101)$$

Therefore,

$$L_n = 0 \quad (102)$$

Also,

$$\left. \frac{\partial D_3}{\partial \theta} \right|_{\theta=\pi/2} = \left(\sum_{n=1}^{\infty} \alpha_n L_n \cos(\alpha_n \theta) - \sum_{n=1}^{\infty} M_n \alpha_n \sin(\alpha_n \theta) \right) \Big|_{\theta=\pi/2} = 0 \quad (103)$$

Therefore,

$$\alpha_n = 2n \quad (104)$$

Similarly, boundary conditions (74) and (77) applied to equation (100) yields:

$$N_n = 0 \quad (105)$$

$$\beta_n = 2n \quad (106)$$

Thus, D_1 and D_3 are written as:

$$D_1 = \sum_{n=1}^{\infty} R_n \cos 2n\theta + C_6 \quad (107)$$

$$D_3 = \sum_{n=1}^{\infty} M_n \cos 2n\theta + C_5 \quad (108)$$

The above two expressions may now be substituted into equation (88) to obtain:

$$\sum_{n=1}^{\infty} M_n \cos 2n\theta \left(2 - \frac{(2n)^2}{2} \right) - \frac{1}{r^2} \sum_{n=1}^{\infty} 4n^2 R_n \cos 2n\theta + 2C_5 - (\gamma+1)C_3^2 = 0 \quad (109)$$

Since this expression must hold for all values of r and θ , the second two terms give the following information:

$$R_n = 0 \quad (110)$$

$$C_5 = \frac{(\gamma+1)}{2} C_3^2 \quad (111)$$

In addition, it is seen that in order to have the first term equal zero for all θ , $n = 1$. Then, equations (107) and (108) reduce to

$$D_1 = C_6 \quad (112)$$

and

$$D_3 = M_1 \cos 2\theta + \frac{(\gamma+1)}{2} C_3^2 \quad (113)$$

Expressions for both D_1 and D_3 are now known so that equations (94) and (96) can be solved. Equation (94) is written as

$$\frac{\partial^2 D_6}{\partial \theta^2} + 16D_6 = 2C_3(\gamma+1) \left[M_1 \cos 2\theta + \frac{(\gamma+1)}{2} C_3^2 \right] \quad (114)$$

Then, a particular solution⁽¹¹⁾ is given by

$$D_{6P} = \int_0^\theta \left[\frac{e^{(\theta-y)r_2} - e^{(\theta-y)r_1}}{(r_2 - r_1)} \right] g(y) dy \quad (115)$$

where $r_1 = 4i$, $r_2 = -4i$, and

$$g(y) = 2C_3(\gamma+1) \left[M_1 \cos 2y + \frac{\gamma+1}{2} C_3^2 \right] \quad (116)$$

Carrying out the indicated integration and adding the complementary solution to D_{6P} gives the complete form for D_6 :

⁽¹¹⁾ Bellman, R., and Cooke, K.L., Modern Elementary Differential Equations, Addison-Wesley Publishing Co., Inc., 1968.

$$D_6 = C_7 \sin 4\theta + C_8 \cos 4\theta - \frac{(\gamma+1)C_3 M_1}{6} (\cos 4\theta - \cos 2\theta) - \frac{(\gamma+1)^2 C_3^3}{16} (\cos 4\theta - 1) \quad (117)$$

where C_7 and C_8 are constants.

Looking to equation (95), its solution is simply

$$D_7 = C_9 \sin 3\theta + C_{10} \cos 3\theta \quad (118)$$

where C_9 and C_{10} are constants.

Similarly, the solution to equation (96) is seen to be

$$D_8 = C_{11} \sin 2\theta + C_{12} \cos 2\theta + C_{13} \quad (119)$$

where C_{11} , C_{12} , and C_{13} are constants. Upon substituting this expression for D_8 back into equation (96), C_{13} is found to be

$$C_{13} = \frac{C_3 C_6}{2} (\gamma+1) \quad (120)$$

Then,

$$D_8 = C_{11} \sin 2\theta + C_{12} \cos 2\theta + \frac{C_3 C_6}{2} (\gamma+1) \quad (121)$$

Expressions for D_6 , D_7 , D_8 , and D_{10} are now known in terms of θ and various constants. Some of these constants can be evaluated from boundary conditions (76) and (79). First, equations (98), (117), (118), and (121) are substituted into equation (92), the result being

$$E_3 = \frac{r^4}{4} \left[4C_7 \cos 4\theta - 4C_8 \sin 4\theta - \frac{2(\gamma+1)C_3 M_1}{12} (-4\sin 4\theta + 2\sin 2\theta) - \frac{(\gamma+1)^2 C_3^3}{16} (-4\sin 4\theta) \right] + \frac{r^3}{3} (3C_9 \cos 3\theta - 3C_{10} \sin 3\theta) + \frac{r^2}{2} (2C_{11} \cos 2\theta - 2C_{12} \sin 2\theta) + C_4 \quad (122)$$

When condition (76) is applied above,

$$C_7 = C_9 = C_{11} = C_4 = 0 \quad (123)$$

Likewise, applying condition (79), it is found that

$$C_{10} = 0 \quad (124)$$

Thus far the boundary equations (80) and (81) have not been satisfied. A summary of the results thus far follows:

$$v_r = E_1 + zrD_3 \quad (125)$$

$$v_z = C_6 + C_3z + D_3r^2/2 \quad (126)$$

$$v_\theta = E_3/r + (\partial D_3/\partial\theta)(zr/2) \quad (127)$$

where

$$E_1 = D_6r^3 + D_8r \quad (128)$$

$$E_3 = (\partial D_6/\partial\theta)(r^4/4) + (\partial D_8/\partial\theta)(r^2/2) \quad (129)$$

$$D_3 = M_1\cos 2\theta + C_3^2((\gamma+1)/2) \quad (130)$$

$$D_6 = C_8\cos 4\theta - \frac{\gamma+1}{6} C_3M_1(\cos 4\theta - \cos 2\theta) - \frac{(\gamma+1)^2 C_3^3}{16}(\cos 4\theta - 1) \quad (131)$$

and

$$D_8 = C_{12}\cos 2\theta + \frac{\gamma+1}{2} C_3C_6 \quad (132)$$

All that remains is to determine the constants C_3 , C_6 , C_8 , C_{12} and M_1 from the remaining boundary conditions. The first boundary condition to be used is equation (81). From Equation (41),

$$h_o = h(0,\theta) = B_o / \left(B_o^2 \cos^2\theta + A_o^2 \sin^2\theta \right)^{1/2} \quad (133)$$

and

$$\frac{\partial h_o}{\partial\theta} = \frac{-h_o^3 (A_o^2 - B_o^2) \sin 2\theta}{2B_o^2} \quad (134)$$

Also, from equations (43) and (58),

$$m'' = \frac{h_0 3A_0}{B_0^3} \epsilon \left[B_0^3 \cos^2 \theta + kA_0^3 \sin^2 \theta \right] \quad (135)$$

After equations (133), (134), (135), (112), and (86) are substituted into equation (81) and evaluated at h_0 , all resulting trigonometric functions can be written in terms of even powers of $\cos \theta$. Each respective power of $\cos \theta$ is factored, leaving

$$\cos^2 \theta \left[M_1 (A_0^2 + B_0^2) + \frac{\gamma+1}{2} C_3^2 (B_0^2 - A_0^2) - \left(B_0^2 A_0 - \frac{kA_0^4}{B_0} \right) \right] = 0, \quad (136)$$

and

$$C_3^2 \frac{(\gamma+1)}{2} A_0^2 - A_0^2 M_1 - \frac{kA_0^4}{B_0} = 0. \quad (137)$$

Since these two expressions must be true for all values of θ , equation (136) yields

$$M_1 = \frac{A_0}{2B_0} (B_0 - kA_0) \quad (138)$$

and equation (137) yields

$$C_3 = \left[\frac{A_0}{B_0(\gamma+1)} (B_0 + kA_0) \right]^{1/2} \quad (139)$$

Following the same procedure with equation (80) and after considerable algebraic manipulation one obtains

$$C_{12} = \frac{B_0^2}{A_0^2} \frac{[T_1 (T_6^2 - 4T_5^2) - 4T_2 T_5 T_6]}{4T_5} \quad (140)$$

$$C_6 = \frac{C_3}{2T_0(2T_5^2 + T_6^2)} (3T_1 T_6 - 4T_2 T_5 + 6T_5 T_6 C_{12}) \quad (141)$$

$$C_8 = T_1 + \frac{4T_2}{3} + \frac{T_0 C_6}{3C_3 T_5} \frac{A_0^2}{B_0^2} \quad (142)$$

where

$$T_0 \equiv \frac{\gamma+1}{2} C_3^2 \quad (143)$$

$$T_1 \equiv (T_0 M_1)/(3C_3) \quad (144)$$

$$T_2 \equiv (T_0^2)/(4C_3) \quad (145)$$

$$T_5 \equiv (A_0^2 + B_0^2)/2B_0^2 \quad (146)$$

$$T_6 \equiv (A_0^2 - B_0^2)/2B_0^2 \quad (147)$$

In summary the velocity components \bar{v}_{r1} , $\bar{v}_{\theta 1}$ and \bar{v}_{z1} at any point r , θ , and z can be easily calculated in terms of the geometric variables A_0 , B_0 , and k by calculating the ten constants M_1 , C_3 , T_0 , T_1 , T_2 , T_5 , T_6 , C_{12} , C_6 , and C_8 in that order from equations (138), (139), (143), (144), (145), (146), (147), (140), (141) and (142) respectively. The velocity components are:

$$\begin{aligned} \bar{v}_{r1} = & \cos 4\theta (C_8 - T_1 - T_2) \bar{r}^3 + \cos 2\theta (T_1 \bar{r}^3 + C_{12} \bar{r} + M_1 \bar{r} \bar{z}) + \\ & (T_2 \bar{r}^3 + T_0 C_6 \bar{r}/C_3 + T_0 \bar{z} \bar{r}) \end{aligned} \quad (148)$$

$$\bar{v}_{\theta 1} = -\sin 4\theta (C_8 - T_1 - T_2) \bar{r}^3 - \sin 2\theta (T_1 \bar{r}^3/2 + C_{12} \bar{r} + M_1 \bar{r} \bar{z}) \quad (149)$$

$$\bar{v}_{z1} = \cos 2\theta (M_1 \bar{r}^2/2) + T_0 \bar{r}^2/2 + C_3 \bar{z} + C_6 \quad (150)$$

where the original bar notation and subscript one have been revived to stress that the result is nondimensionalized and is only the first order solution.

SECTION V

REDUCTION TO THE AXISYMMETRIC CASE

In the first appendix of reference 3 the analogous solution for the flow field in a converging-diverging axisymmetric nozzle is derived. A necessary condition for the elliptic solution of this investigation is that it reduces to the axisymmetric solution when $A_0 = B_0$ and $k = 1$.

The first order solution of reference 3 for non-dimensional case (i.e. $A_0 = B_0 = 1$) is:

$$u_1 = y^2/2 - 1/4 + [2/(\gamma+1)]^{1/2} z \quad (151)$$

$$v_1 = (y/4)(y^2-1)[(\gamma+1)/2]^{1/2} + yz \quad (152)$$

For axisymmetric flow (i.e. $A_0 = B_0$ and $k = 1$) equations (138) through (147) reduce to:

$$M_1 = T_1 = T_6 = C_{12} = 0 \quad (153)$$

$$C_3 = [2A_0/(\gamma+1)]^{1/2} \quad (154)$$

$$T_0 = A_0; \quad T_2 = A_0^2/(4C_3); \quad T_5 = 1 \quad (155)$$

$$C_6 = -A_0/4; \quad C_8 = T_2 = A_0^2/(4C_3) \quad (156)$$

Thus for axisymmetric flow equations (148)-(150) become

$$\bar{v}_{r1} = (\bar{r}/4)(\bar{r}^2-1)[(\gamma+1)/2A_0]^{1/2} A_0^2 + A_0 \bar{r} \bar{z} \quad (157)$$

$$\bar{v}_{\theta 1} = 0 \quad (158)$$

$$\bar{v}_{z1} = A_0 \bar{r}^2/2 - A_0/4 + [2A_0/(\gamma+1)]^{1/2} \quad (159)$$

Equations (157)-(159) are identical to equations (151) and (152) (except for differences in notation) when A_0 is set to unity.

SECTION VI

SUMMARY AND RECOMMENDATIONS

1. The first order approximation to the transonic flow field for a converging-diverging propulsion nozzle having an elliptical cross section has been obtained.
2. The method presented is generalized such that it can be extended in a straight forward manner to obtain higher order approximations to the flow field.
3. The solution is periodic in the tangential velocity component, v_θ , as would be expected, and the solution reduces to the special case of axisymmetric flow when terms containing v_θ and $\partial/\partial\theta$ are suppressed.
4. The solution can be easily programmed. Parametric studies of the flow field as a function of geometric variables such as the ratio of major to minor axes of the elliptic cross-section could provide useful information for advanced propulsion nozzle design.

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