AD-A042 346 INSTITUTE FOR DEFENSE ANALYSES ARLINGTON VA SCIENCE A--ETC F/G 20/5
FEEDBACK AND ACOUSTO OPTIC ISOLATION EFFECTS ON LASER STABILITY--ETC(U)
MAR 77 V J CORCORAN DAHC15-73-C-0200
UNCLASSIFIED P-1240 IDA/HQ-76-18850 NL

1 OF 1
AD A042346



END
DATE FILMED
8-77
DDC

Copy 181 of 225 copies

AD A042346

AD No. ____
DDC FILE COPY

12

IDA PAPER P-1240

FEEDBACK AND ACOUSTO OPTIC ISOLATION EFFECTS ON LASER STABILITY

Vincent J. Corcoran

March 1977

DDC
RECEIVED
AUG 2 1977
C

Prepared for

Office of the Director of Defense Research & Engineering

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

INSTITUTE FOR DEFENSE ANALYSES
SCIENCE AND TECHNOLOGY DIVISION

IDA Log No. HQ 76-18850

The work reported in this document was conducted under contract DAHC15 73 C 0200 for the Department of Defense. The publication of this IDA Paper does not indicate endorsement by the Department of Defense, nor should the contents be construed as reflecting the official position of that agency.

Approved for public release; distribution unlimited.

(18) IDA/HQ

(19) 76-18850

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER Paper P-1240	2 GOVT ACCESSION NO.	3 RECIPIENT'S CATALOG NUMBER
4 TITLE (and Subtitle) Feedback and Acousto Optic Isolation Effects on Laser Stability.		5 TYPE OF REPORT & PERIOD COVERED FINAL June 1975-June 1976
		6 PERFORMING ORG. REPORT NUMBER P-1240
7 AUTHOR(s) Vincent J. Corcoran		8 CONTRACT OR GRANT NUMBER(s) DAHC15-73-C-0200
9 PERFORMING ORGANIZATION NAME AND ADDRESS INSTITUTE FOR DEFENSE ANALYSES 400 Army-Navy Drive Arlington, Virginia 22202		10 PROGRAM ELEMENT PROJECT TASK AREA & WORK UNIT NUMBERS Task T-136
11 CONTROLLING OFFICE NAME AND ADDRESS Defense Advanced Research Projects Agency 1400 Wilson Boulevard Arlington, Virginia 22209		12 REPORT DATE March 1977 13 NUMBER OF PAGES 12
14 MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Director of Defense Research and Engineering (DDR&E), Electronics & Physical Sciences The Pentagon, Washington, D.C. 20301		15 SECURITY CLASS (of this report) UNCLASSIFIED 15a DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16 DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DDC
AUG 2 1977
C

17 DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

N/A

18 SUPPLEMENTARY NOTES

N/A

19 KEY WORDS (Continue on reverse side if necessary and identify by block number)

Acousto-optics, Lasers, Frequency Stabilization, Laser Instability, Isolators

20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper analyzes the effect of optical feedback on laser frequency stability and the acousto optical isolator concept, which was demonstrated previously. The frequency pulling effect associated with feedback of a signal at a frequency other than the oscillator frequency has been shown to require a nonlinearity such as saturation in the laser medium. The analysis mathematically corroborates the initial acousto optic isolator concept and the limited experimental data avail-

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

403108

20.

able.

In the study of the acousto optic isolator, it was determined that an acceptable analytic expression for the imaginary part of the susceptibility of an unsaturated inhomogeneously broadened line, and the expression for the susceptibility of a saturated inhomogeneously broadened line, were not available. The expressions which were desired for the analysis in this paper were obtained for what is believed to be the first time.

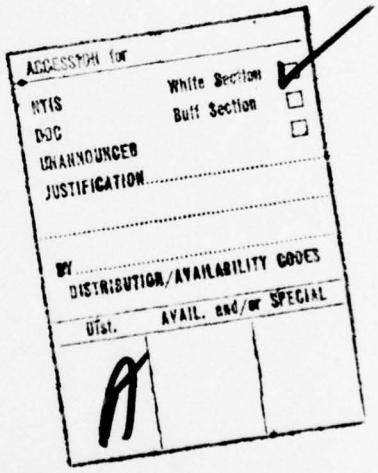
The results of this paper provide a foundation for the acousto optic isolator. Applicability of these findings should increase as coherent laser radars, which require isolation to insure accuracy of the Doppler information in the return signal, are developed. Additionally, the results reported herein should prove useful in view of the increasing need for stable amplitude optically pumped submillimeter sources.

IDA PAPER P-1240

FEEDBACK AND ACOUSTO OPTIC ISOLATION EFFECTS ON LASER STABILITY

Vincent J. Corcoran

March 1977



INSTITUTE FOR DEFENSE ANALYSES
SCIENCE AND TECHNOLOGY DIVISION
400 Army-Navy Drive, Arlington, Virginia 22202

Contract DAHC15 73 C 0200
Task T-136

ACKNOWLEDGEMENT

The author wishes to acknowledge the useful discussions with W. Wasylkiwskyj, pertaining to the solution of the integral which is designated "I" in the text, and L. Seekamp, for computations that resulted in some of the figures in the text.

PRECEDING PAGE BLANK-NOT FILMED

ABSTRACT

This paper analyzes the effect of optical feedback on laser frequency stability and the acousto optic isolator concept, which was demonstrated previously. The frequency pulling effect associated with feedback of a signal at a frequency other than the oscillator frequency has been shown to require a nonlinearity such as saturation in the laser medium. The analysis mathematically corroborates the initial acousto optic isolator concept and the limited experimental data available.

In the study of the acousto optic isolator, it was determined that an acceptable analytic expression for the imaginary part of the susceptibility of an unsaturated inhomogeneously broadened line, and the expression for the susceptibility of a saturated inhomogeneously broadened line, were not available. The expressions which were desired for the analysis in this paper were obtained for what is believed to be the first time.

The results of this paper provide a foundation for the acousto optic isolator. Applicability of these findings should increase as coherent laser radars, which require isolation to insure accuracy of the Doppler information in the return signal, are developed. Additionally, the results reported herein should prove useful in view of the increasing need for stable amplitude optically pumped submillimeter sources.

PRECEDING PAGE BLANK-NOT FILMED

CONTENTS

	Acknowledgement	iii
	Abstract	v
	Summary	1
I.	Introduction	3
II.	The Acousto Optic Isolator Concept	7
III.	Theory	11
	A. Unperturbed Oscillator Frequency	12
	B. Saturation Effects	17
	C. Effect of Feedback	24
	References	29
APPENDIX A -	Derivation of Equations	A-1
APPENDIX B -	Unsuccessful Attempts to Find I	B-1
APPENDIX C -	Table of Dawson's Integral	C-1
APPENDIX D -	Table of Error Function for Complex Arguments	D-1

FIGURES

1.	Illustration of 180° Rotational Symmetry of an Acousto Optic Material	8
2.	Illustration of Double Frequency Shift of Optical Beam Due to Double Pass Through an Acousto Optic Device	8
3.	Illustration of Shift of Feedback Signal of the Gain Curve of the Oscillator	9

PRECEDING PAGE BLANK-NOT FILMED

4. (No title) 12

5. Real and Imaginary Parts of the Complex Lorentzian Lineshape Factor 15

6. Frequency Pulling Effects for Cases Where Molecular and Cavity Bandwidths Differ Substantially 16

7. Comparison of Gaussian and Lorentzian Lines of the Same Linewidth 16

8a. Real and Imaginary Parts of the Complex Susceptibility of an Inhomogeneously Broadened Line 18

8b. Comparison of Real Susceptibility of an Inhomogeneously Broadened Line 18

9. Absorption Constant Under Saturating Conditions for a Homogeneously Broadened Line 20

10a. Real Susceptibility 22

10b. Imaginary Susceptibility 23

11. Comparison of Gain Line Profile and Laser Isolation Achieved versus Feedback Offset Frequency 26

SUMMARY

This paper analyzes the effect of optical feedback on laser oscillator frequency stability and provides a foundation for the acousto optic isolator. This should find increased applicability as coherent laser radars are developed which need isolation to insure accuracy of the Doppler information in the return signal, and also as requirements develop for stable amplitude submillimeter lasers that are optically pumped.

Expressions are obtained from the frequency of oscillation for a laser without external feedback, and with feedback of a signal at a frequency other than the oscillator frequency. These expressions are directly applicable to the analysis of the acousto optic concept. Both homogeneously broadened lines and inhomogeneously broadened lines are considered.

The frequency of oscillation for the unsaturated condition and the saturated condition is determined. Saturation is found to change the oscillator frequency for both the homogeneously and inhomogeneously broadened lines.

The effect of feedback on the oscillator frequency is determined for the unsaturated condition and found to not influence the oscillator frequency because of the lack of coupling between the feedback signal and the oscillator frequency.

For the homogeneously broadened, saturated condition, however, the frequency of oscillation is seen to be pulled by the feedback signal, and the effect is proportional to the intensity of the feedback and the lineshape of the transition at the feedback frequency. This condition is considered to be a good approximation to the medium in a CO_2 laser.

For the inhomogeneously broadened saturated condition, the expression for the pulling effect can be obtained, but no simple relation between the frequency of oscillation and the feedback characteristics is obvious.

In the process of the analysis in this paper, some expressions were derived that were needed to determine the operating frequency. These include the complex susceptibility for an unsaturated inhomogeneously broadened line and the complex susceptibility for a saturated inhomogeneously broadened line.

I. INTRODUCTION

After the usual sources of laser frequency instability, such as power supply instabilities, mechanical vibrations, acoustical effects, temperature variations, and discharge instabilities have been reduced to a satisfactory level, there sometimes remains another source of frequency instability--feedback of laser radiation from components external to the laser cavity. This feedback can degrade the performance of systems that require stable frequency laser sources.

In some laser radar applications, a low power stable laser oscillator is used to drive a high power laser amplifier or to injection lock a high power-pulsed laser source. The low power laser provides a frequency stable signal which is especially desirable in systems using heterodyne detection to extract Doppler information, or simply for increased sensitivity in detection. In such systems, isolation is needed between the stable oscillator and the high power device to prevent unwanted feedback from the high power source to the stable oscillator. The signal feedback has a tendency to pull the frequency of the stable oscillator, or in extreme cases, to damage the oscillator.

Optically pumped submillimeter lasers use gases that have very narrow absorption lines at the typical operating pressure. Consequently, the pumps, which are often CO_2 lasers, must be frequency stable; otherwise, absorption by the lasing gas varies and the submillimeter signal fluctuates in intensity. Therefore, close coupling is required to insure a stable submillimeter signal.

The standard technique used to achieve isolation in the microwave or optical region is to use a nonreciprocal device. The design of most isolators in these regions is based on the principle of Faraday rotation (Refs. 1 and 2). In the IR region, it is usually necessary to cool a material to liquid nitrogen or liquid helium temperatures in order to observe the Faraday rotation effect (Ref. 3). As a result, the Faraday rotator is not used extensively in the IR region. The currently used alternatives, however, are quite unsatisfactory (Ref. 4), or are of limited applicability (Refs. 5 and 6). These include using a polarizer alone, which provides isolation from stray radiation of the opposite polarization only, or a quarter wave plate, which generates circularly polarized light after one trip through the plate and linearly polarized light of the opposite polarization to the input polarization after the return trip.

The acousto optic isolator (AOI) is a concept developed to provide nonreciprocal isolation between stages in a master oscillator/power amplifier (MOPA) chain. The concept is based on the fact that optical or IR radiation passing through an acousto optic device is partially deflected, and the frequency is shifted by the interaction with the acoustical wave. If the shifted wave is returned back through the acousto optic device, the wave is again partially deflected and the frequency shifted so that the wave has a total shift of twice the acoustical frequency. This effect, which has been demonstrated, can be used to prevent frequency pulling of a signal by spurious radiation which may be fed back in a MOPA chain.

AOIs may become important in future laser radar systems as the output power increases in sophisticated systems which must accurately measure the frequency content of the return signal, and in submillimeter systems requiring stable amplitude transmitters which are optically pumped.

Previous work in this area has qualitatively predicted the AOI effect and experiments have been performed to verify the qualitative predictions. The objectives of this paper are to quantitatively assess the effect of optical feedback on laser frequency stability and to provide a quantitative analysis of the AOI. The quantitative analysis of the AOI can then be used to predict the performance of a device as a function of input parameters.

In Section II the acousto optic concept is explained in more detail. In Section III-A, the general equation for frequency of an oscillator is determined, followed by a determination of the actual frequency equations for homogeneously and inhomogeneously broadened lines. The effects of saturation are examined in Section III-B.

Finally, the effects of the amplitude and the frequency of the feedback signal on the oscillator frequency are determined in Section III-C, and these results are related to the performance of an AOI.

Because of the extensive number of equations involved, the body of this paper essentially summarizes the results which are derived in Appendix A.

II. THE ACOUSTO OPTIC ISOLATOR CONCEPT

The AOI concept is based on the fact that an acousto optic modulator shifts the frequency of the radiation that is deflected from the input optical beam by the acousto optic interaction. Because an acousto optic modulator has 180° rotational symmetry about the axis corresponding to the acoustic beam direction, as illustrated in Fig. 1, an optical wave incident from the output side of the modulator will be frequency shifted in the same direction as an optical wave incident on the input side of the modulator. Therefore, radiation reflected back through the modulator will be shifted by twice the modulation frequency of the device, as indicated in Fig. 2.

The double frequency shift can be easily observed by homodyne detection of the radiation from a gas laser that has been passed twice through an acousto optic modulator. The output of the homodyne detection, when the local oscillator radiation and the signal to be detected impinge on the detection, is twice the frequency of the acoustical input, thus verifying the double frequency shift (Ref. 7).

Isolation can be achieved by means of a modulator, the acoustic frequency of which is such that the feedback signal is shifted outside of the gain curve of the oscillator medium as shown in Fig. 3. In the case of a CO_2 oscillator, which has a fluorescent linewidth Δf of approximately 50 MHz, the reflected signal can be shifted outside of the gain curve of the medium with an acoustical signal whose frequency is approximately 30 MHz. In this frequency range, acoustical signals are only slightly attenuated in the acousto optic medium and the coupling

PRECEDING PAGE BLANK-NOT FILMED

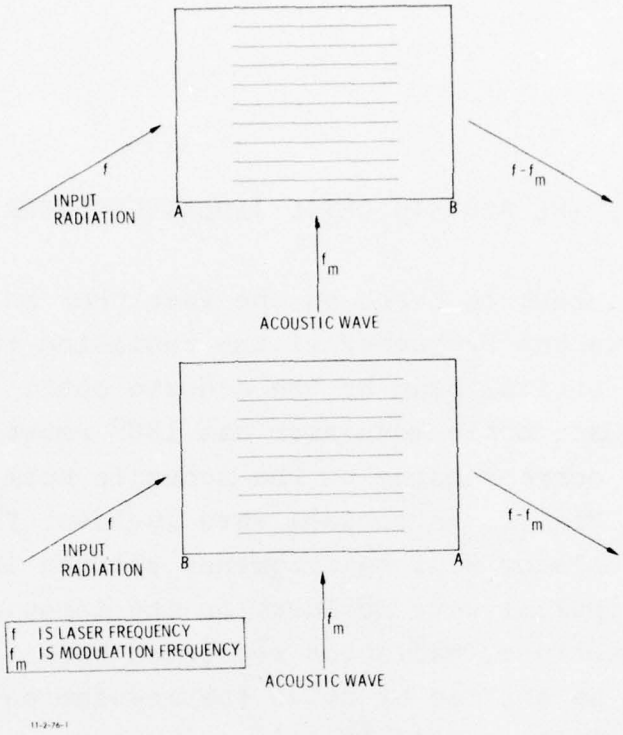


FIGURE 1. Illustration of 180^{o} Rotational Symmetry of an Acousto Optic Material

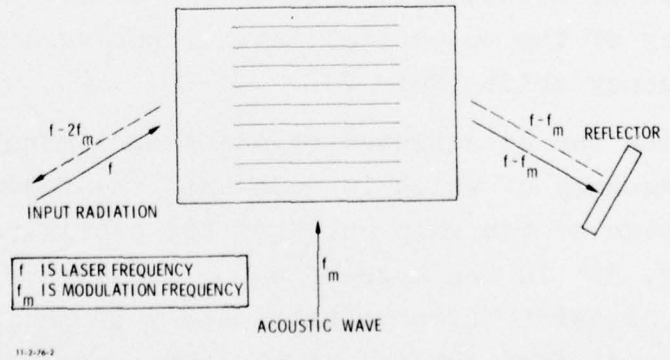


FIGURE 2. Illustration of Double Frequency Shift of Optical Beam Due to Double Pass Through an Acousto Optic Device

from electrical to acoustical energy by the transducer can be achieved easily.

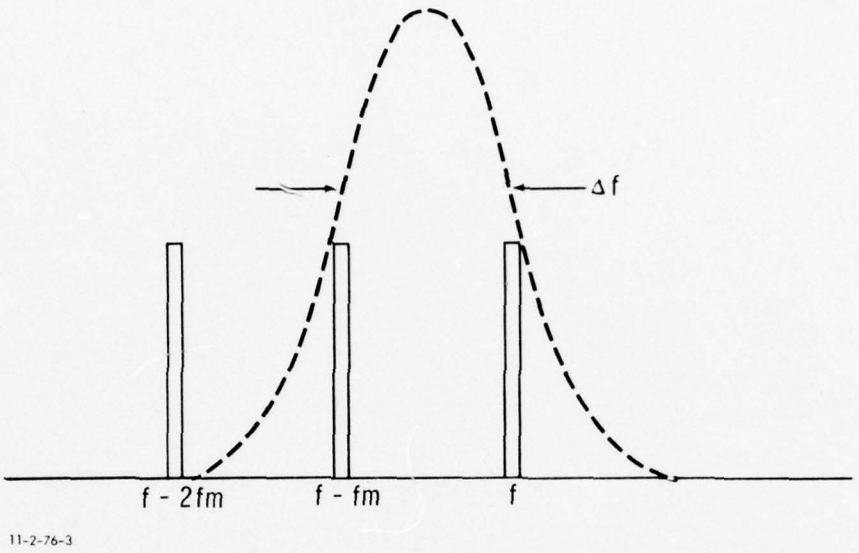


FIGURE 3. Illustration of Shift of Feedback Signal of the Gain Curve of the Oscillator. For a typical CO_2 oscillation Δf = 50 MHz, fm = 30 MHz.

An extraneous signal that enters the laser medium acts as a loss as far as the oscillator signal is concerned because the extraneous signal depopulates the gain medium. This loss results in pulling of the oscillator frequency. When the feedback frequency is outside of the gain transition, the loss due to depopulation is reduced, thereby decreasing the frequency pulling, or increasing the stability of the laser oscillator.

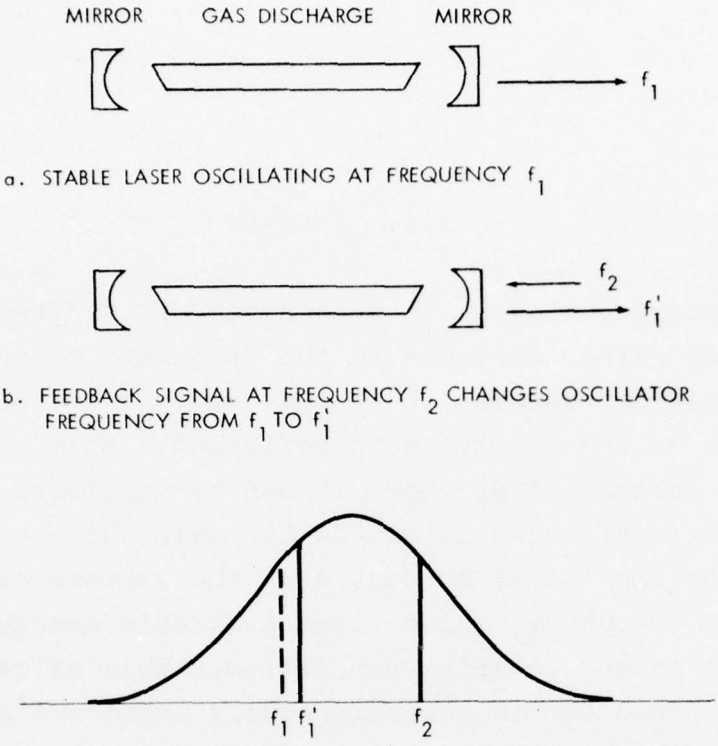
III. THEORY

The detailed problem to be analyzed is diagramed in Fig. 4. When a signal other than one at the frequency of oscillation hits the front mirror (assumed to be the only port of entry for feedback), some of the radiation is reflected. Also, part of it enters the laser cavity, where it can be amplified if the frequency of the radiation is within the gain curve of the laser medium. The amplification occurs at the expense of the laser oscillating frequency, which loses available energy for amplification stored in the inverted population. This effect should not be as pronounced for an inhomogeneously broadened line, since there are inverted states that are not affected by the laser oscillation frequency in the inhomogeneously broadened case.

The problem is one in which a cavity exists that determines an oscillation frequency. A driving function represented by the feedback signal is input to the cavity, and the effect of the feedback signal whose frequency is f_2, on the oscillator frequency, f_1, is to be determined.

To solve the problem, one must first derive the general equation for the oscillator frequency without feedback, and then determine the frequency for the type of transition involved. The oscillator frequency equation with feedback is derived and compared to the equation without feedback to determine the effect of the feedback. This equation is then related to the effect that the magnitude and frequency of the input AOI signal has on the optical feedback signal.

PRECEDING PAGE BLANK-NOT FILMED



a. STABLE LASER OSCILLATING AT FREQUENCY f_1

b. FEEDBACK SIGNAL AT FREQUENCY f_2 CHANGES OSCILLATOR FREQUENCY FROM f_1 TO f_1'

c. ILLUSTRATION OF FREQUENCY PULLING OF OSCILLATOR FROM f_1 TO f_1' BY FEEDBACK SIGNAL f_2

11-2-76-4

FIGURE 4.

A. UNPERTURBED OSCILLATOR FREQUENCY

The frequency of oscillation of a laser has been approximately determined by considering the frequency pulling, caused by the phase shift per transit produced by the refractive laser medium on the radiation (Ref. 8), or by considering the frequency pulling due to dielectric losses which are equated to conductivity cavity losses for a homogeneously broadened line (Ref. 9).

The oscillating frequency can be determined more accurately by solving the eigenvalue equation. It is derived from the wave

equation that describes the electromagnetic field in the laser cavity, by substituting the material equation that relates the polarization to the electric field into the wave equation. When this procedure is followed, as shown in Appendix A, the angular frequency is given by

$$(\omega_c^2 - \omega^2)\, \chi_{IM} = \frac{\omega}{\tau_c}\, \chi_{RE} \quad , \qquad (2)^{\dagger}$$

where

ω_c is the cavity angular frequency
ω is the angular frequency of oscillation
τ_c is the cavity lifetime
χ_{RE} is the real part of the linear susceptibility

and

χ_{IM} is the imaginary part of the linear susceptibility.

This equation provides the general solution for the angular frequency, ω, of the oscillator whose cavity satisfies Eq. 1B in Appendix A. The angular frequency depends upon both the real and imaginary parts of the susceptibility. (The real part of the susceptibility of a material causes a phase shift of an electromagnetic wave as it propagates through the material. The fact that the phase shift is frequency dependent results in the dispersion of light as it passes through the material. The imaginary part of the susceptibility results in absorption or amplification of waves.) Therefore, in order to evaluate this equation it is necessary to determine the equation for the susceptibility.

Two generic types of transitions occur that give rise to different equations for susceptibility. They are homogeneously broadened and inhomogeneously broadened transitions.

---

†Appendix A contains the entire sequence of mathematical computations for this paper, from which specific equations have been repeated in this Section for the convenience of the reader. Therefore, number identification of equations in this Section corresponds to the placement of these equations in Appendix A.

1. Homogeneous Broadening

When the macroscopic polarization of a material is due to the sum of the contributions from individual atoms which have the same center transition frequency and linewidth, the absorption or spontaneous emission line is said to be homogeneously broadened. From another viewpoint, the effect of a group of the atoms interacting with radiation is the same as for one atom interacting with the radiation except for a multiplicative constant. Homogeneous broadening results whenever the relaxation processes are the same for all atoms. These relaxation processes include coupling to lattice vibrations, collisions, and spin-spin coupling.

The values of χ_{RE} and χ_{IM} are found by solving the differential equation for the polarization in terms of the electric field. When this is done for a homogeneously broadened line (Appendix A), the results are as shown in Fig. 5. When the values for χ_{RE} and χ_{IM} are substituted into Eq. 2, the general equation for the oscillator frequency, then the equation for the oscillating frequency is

$$\omega^2 - \omega_c^2 = \frac{1/\tau_c}{2/T_2} (\Omega^2 - \omega^2) \qquad (6)$$

$$= T_2/\tau_c \; \omega(\Omega^2 - \omega^2) \text{ for } \omega \approx \Omega \quad ,$$

which shows that the frequency of oscillation assumes a value between the center of the transition and the cavity frequency. This fact is illustrated in Fig. 6, which indicates the position of the oscillation frequency compared to the spontaneous emission linewidth and the cavity linewidth.

2. Inhomogeneous Broadening

When the lineshape of a collection of atoms or molecules is different for each particle because of slightly different transition frequencies and/or different relaxation times, then the

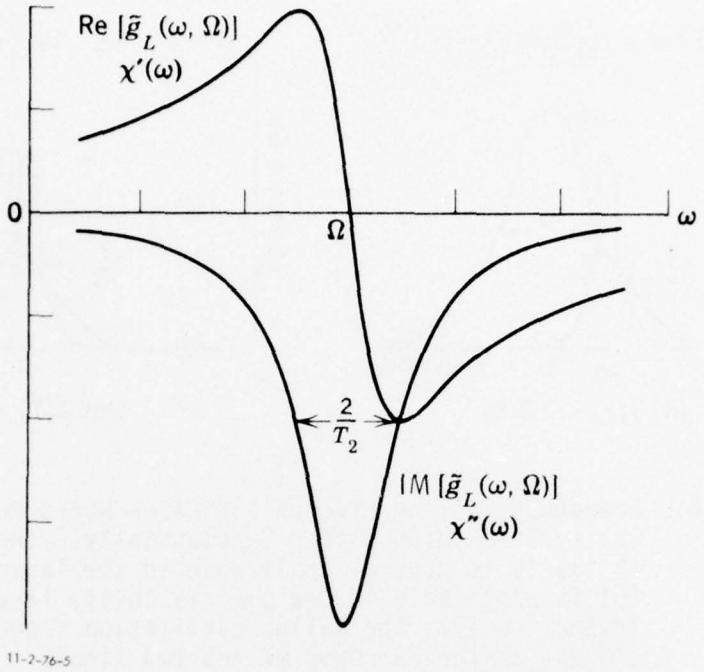


FIGURE 5. Real and Imaginary Parts of the Complex Lorentzian Lineshape Factor

line is said to be inhomogeneously broadened. Crystalline inhomogeneities that produce different local fields at different lattice points in a crystal, nonuniform magnetic fields enveloping a spin system, or the Doppler effect due to motion of atoms or molecules in a gas, all give rise to inhomogeneous broadening. The resulting lineshape for an inhomogeneously broadened line is determined by summing the homogeneous lineshapes of the individual particles involved in the process. The result of summing a large number of Lorentzian lines, as would be expected by the central limit theorem, is a Gaussian lineshape (Ref. 10) (cf. Fig. 7).

The same procedure is followed for the inhomogeneously broadened case as was used for the homogeneously broadened case (Appendix A) to obtain the susceptibility.

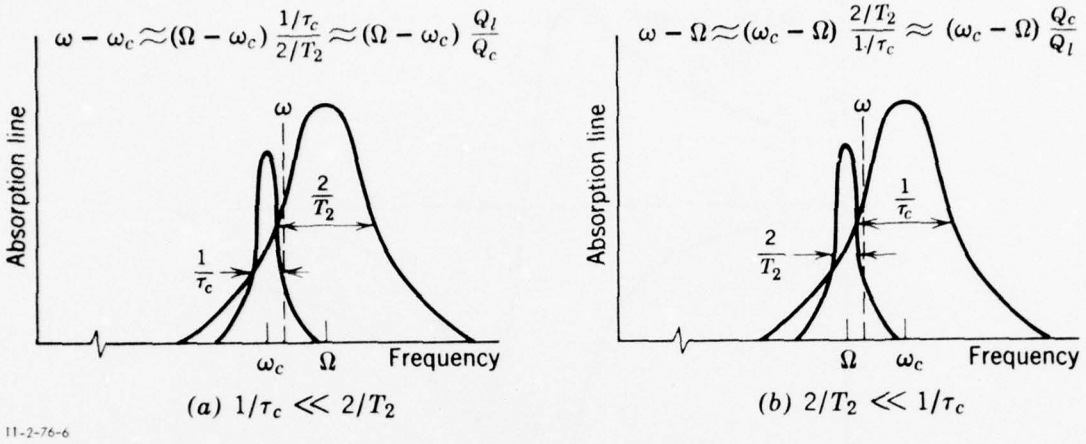


FIGURE 6. Frequency Pulling Effects for Cases Where Molecular and Cavity Bandwidths Differ Substantially. The case shown in (a) is in general applicable to the laser, whereas (b) is applicable to the ammonia cavity laser discussed in Section IV. The pulled oscillation frequency, ω, lies closest to the narrower of the two lines. Q_1 and Q_c are, respectively, the molecular line and cavity Q is defined by $Q_1 \equiv \Omega T_2/2$, $Q_c \equiv \omega_c T_c$.

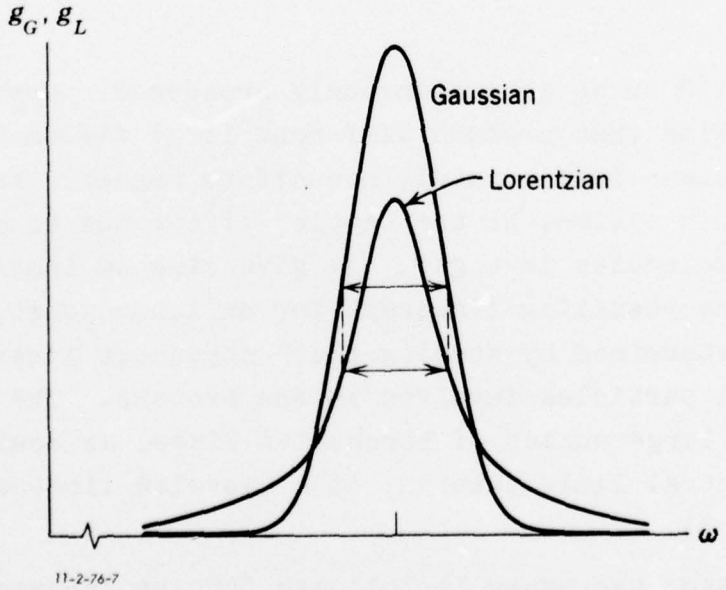


FIGURE 7. Comparison of Gaussian and Lorentzian Lines of the Same Linewidth

A plot of the real and imaginary parts of the susceptibility for an inhomogeneously broadened line is given in Fig. 8a and a comparison of the real susceptibility for a homogeneously broadened line and an inhomogeneously broadened line is made in Fig. 8b.

The equation for the frequency of oscillation for the frequency of an inhomogeneously broadened line is obtained when these values for the susceptibility are substituted into Eq. 2. The equation is then

$$(\omega_c^2-\omega^2)\pi = \frac{\omega}{\tau_c}\, 2\sqrt{\pi} \int_0^{\frac{\omega-\omega_o}{\Delta\omega_G}[4\ell n2]^{1/2}} e^{X^2}\, dX \quad . \qquad (9)$$

Although this equation is considerably more complicated than Eq. 6, it nevertheless shows that the frequency of oscillation assumes a value between the center of the transition and the cavity frequency.

B. SATURATION EFFECTS

If there were no radiation in the laser cavity, then the population inversion would be determined by the pumping level and the relaxation rates for transitions from the upper to the lower laser level. In the presence of electromagnetic radiation, stimulated transitions occur so that the population inversion in equilibrium is reduced from the value obtained without radiation in the cavity. As the level of radiation increases, the value of the population inversion decreases. For a noninverted population, energy is absorbed and then is dissipated in relaxation processes; however, for a large energy absorption the system may not relax to equilibrium as fast as it can for small signals, so the system saturates. These ideas can be made quantitative by considering the differential equation for the population difference and including the effect of the electric field on this population difference.

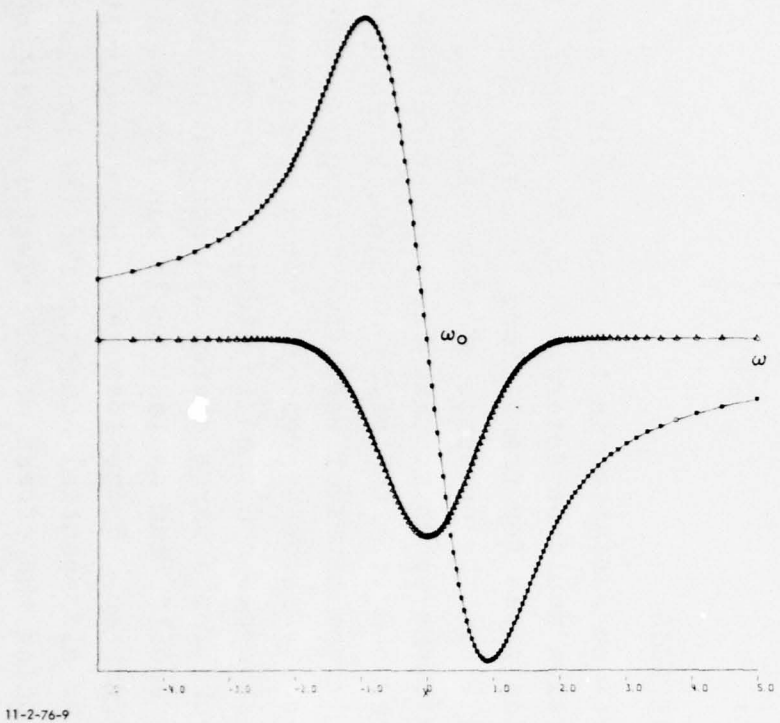


FIGURE 8a. Real and Imaginary Parts of the Complex Susceptibility of an Inhomogeneously Broadened Line. (See Fig. 5 for homogeneously broadened line.)

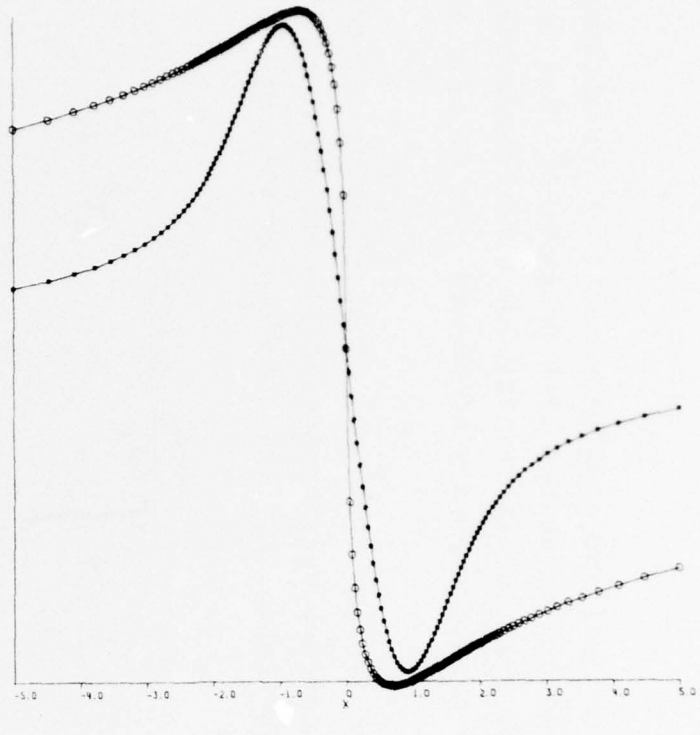


FIGURE 8b. Comparison of Real Susceptibility of an Inhomogeneously Broadened Line. (See Fig. 7 for comparison of imaginary parts.)

If the effects of the electromagnetic field are included, then the steady state solution for the population difference is obtained by setting the time dependent parts equal to zero. The result (Appendix A) is:

$$\frac{(N_1-N_2)-(N_1-N_2)^e}{T_1} = \frac{\omega\ \varepsilon_o}{\hbar\Omega}\ \chi_{IM}\ |E|^2 \quad . \tag{10}$$

Equation 10 is the general equation for the steady state population difference. It shows that the population difference is a function of the imaginary part of the susceptibility of the medium and the field intensity. The value of the susceptibility, of course, is itself a function of the population difference.

This general equation can be used to determine the effects of saturation on the frequency of oscillation by substituting the specific expressions for the imaginary part of the susceptibility, solving for the population difference, and substituting the population difference into Eq. 2 to determine the oscillation frequency. The values of the susceptibility for homogeneously and inhomogeneously broadened lines can now be substituted.

1. Homogeneous Broadening

The final form for the saturation effect depends upon the type of transition involved, since the form of the susceptibility depends upon the character of the transition.

The effect of saturation is determined by substituting the imaginary part of the susceptibility for the specific transition into Eq. 10 and substituting Eq. 10 into the equation for the complex susceptibility of the transition. The saturation effect obtained by this procedure for a homogeneously broadened line (Appendix A) is illustrated in Fig. 9.

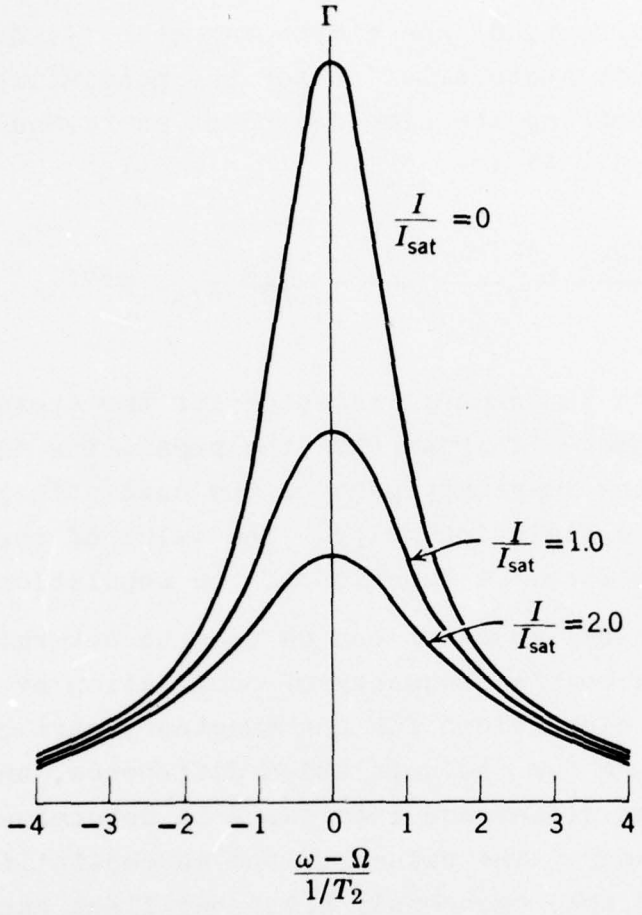


FIGURE 9. Absorption Constant Under Saturating Conditions for a Homogeneously Broadened Line

The complex susceptibility is affected only in the denominator by the change in $|E|^2$ or ω, and the substitution into Eq. 2 gives

$$(\omega_c^2 - \omega^2)\, \chi_{IM} = \frac{\omega}{\tau_c}\, \chi_{RE} \qquad (2)$$

$$(\omega_c^2 - \omega^2)\, g_{LIM} = \frac{\omega}{\tau_c}\, g_{LRE} \quad . \qquad (13)$$

Consequently, the equation for the operating frequency is unchanged explicitly by saturation effects on the susceptibility.

A change, however, does occur in the cavity frequency ω_c, since the cavity frequency is related to the index of refraction, which depends on the susceptibility. When this factor is considered, Eq. 2, for the case where saturation of a homogeneously broadened line occurs, becomes

$$\left\{\left[\omega_{co}\left(1 - \frac{\Delta\chi}{\eta_o}\right)\right]^2 - \omega^2\right\} \chi_{IM} = \frac{\omega}{\tau_c} \chi_{RE} \cdot \qquad (15)$$

Saturation, therefore, does change the oscillator frequency from the frequency obtained in the unsaturated condition for a homogeneously broadened line.

2. Inhomogeneous Broadening

For an inhomogeneously broadened line, the equation for the population difference that is used for a homogeneously broadened line is applicable to those atoms or molecules whose transitions are centered at a specified frequency. The total population difference is the integral of the specific population difference over all possible frequencies. Ordinarily, when the integration is performed the result for the imaginary part of the susceptibility is the same as for the homogeneously broadened line, except that the saturation effect depends on the square root of intensity rather than the first power. In order to calculate the frequency of oscillation, however, the real term is needed as well.

If the same procedure as for the homogeneously broadened line is followed, the complex susceptibility is obtained (Appendix A), and the real and imaginary parts of the susceptibility are plotted, as in Figs. 10a and 10b.

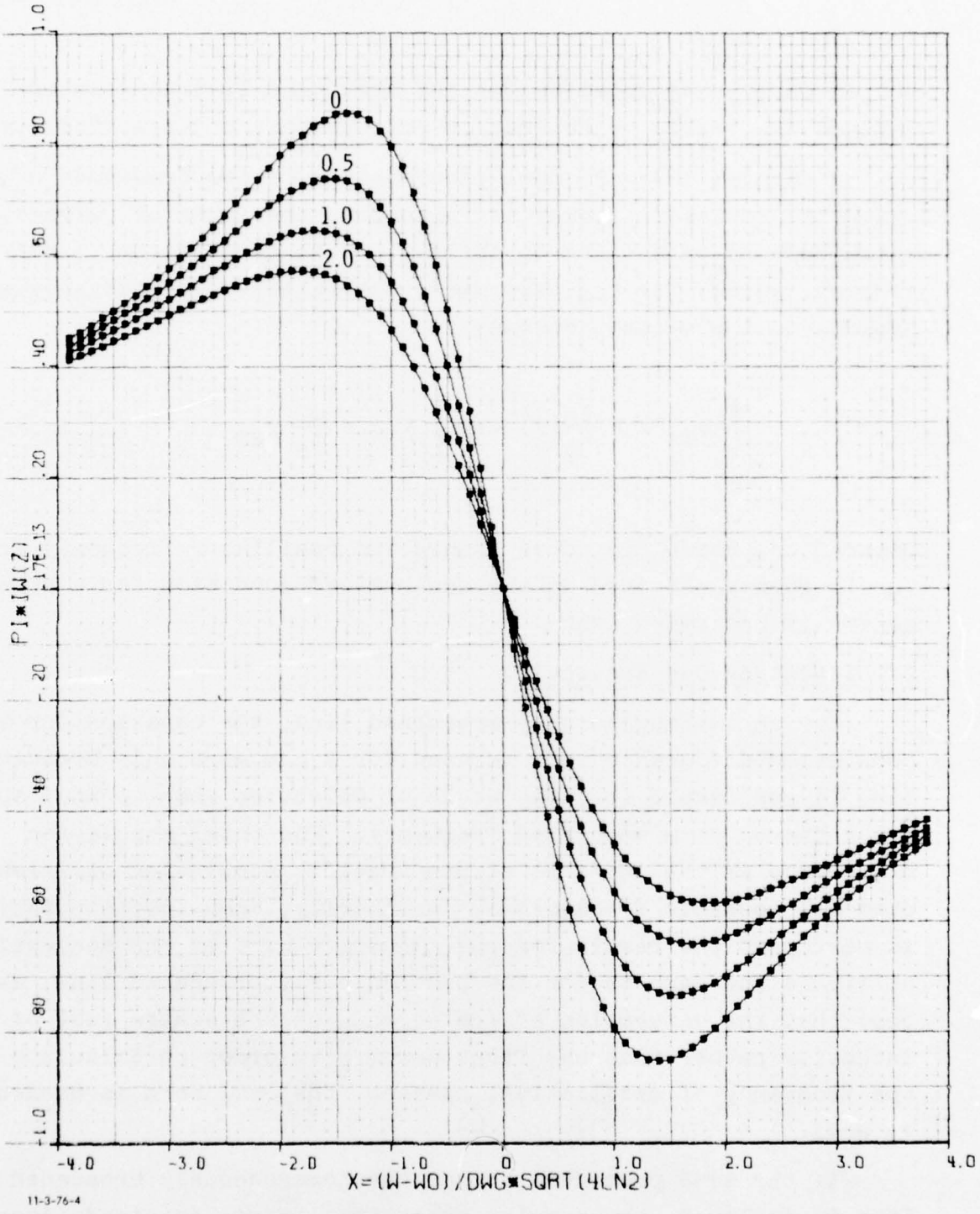


FIGURE 10a. Real Susceptibility. Illustration of saturation effect for three values of I/I_{SAT} on the susceptibility of an inhomogeneously broadened line.

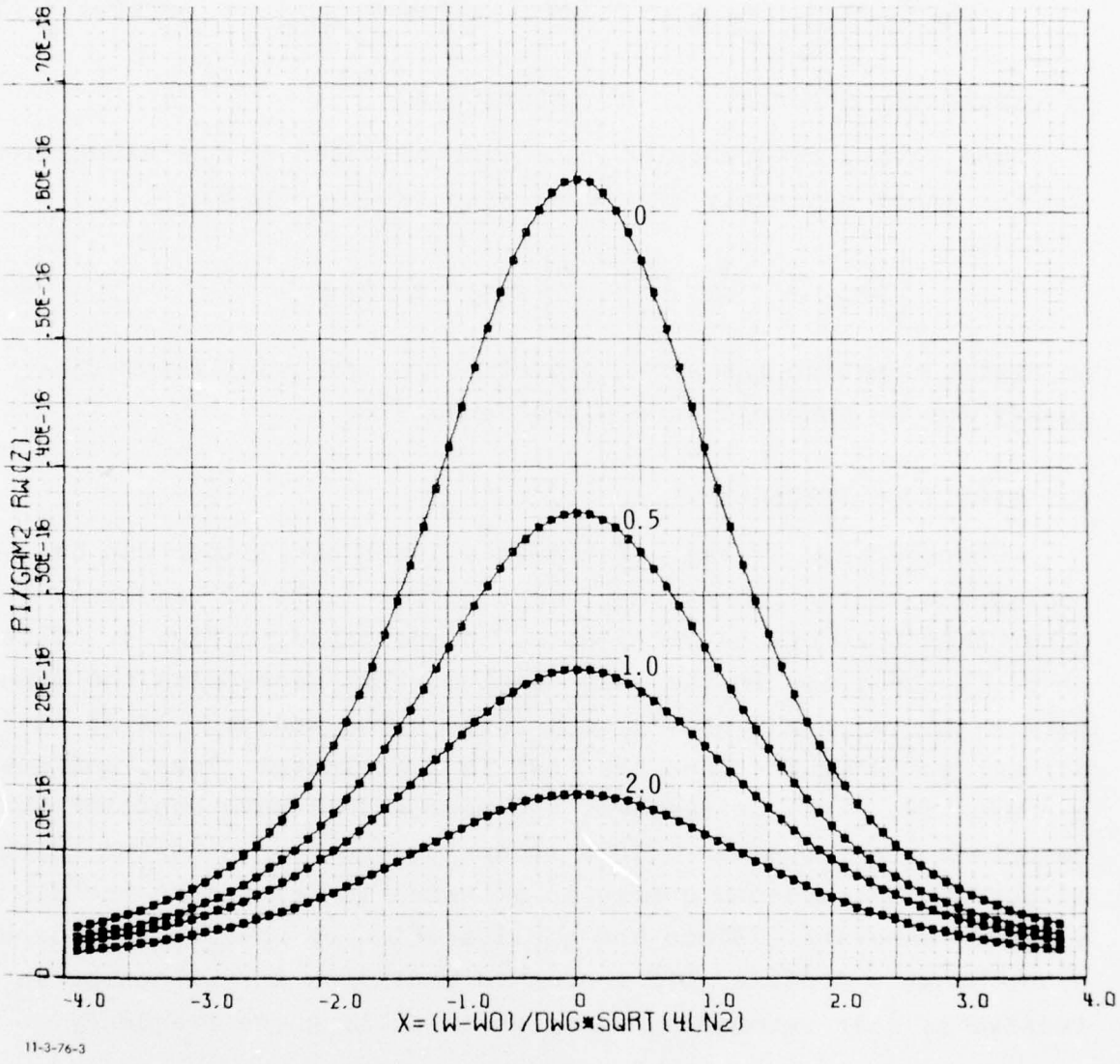


FIGURE 10b. Imaginary Susceptibility

This equation for the susceptibility is considerably more complicated than the susceptibility for the homogeneously broadened case. In this case, the effect of the angular frequency, ω, and the intensity effect given by $|E|^2$, are different for the real and imaginary parts of the susceptibility. Consequently the frequency equation

$$(\omega_c^2-\omega^2)\left[\frac{1}{\gamma^2}\mathcal{R}_e\, e^{-z^2}\,\mathrm{erfc}(-z)\right] = \frac{\omega}{\tau_c}\left[\mathcal{I}_m\, e^{-z^2}\,\mathrm{erfc}(-z)\right] \quad (19)$$

is explicitly affected by the saturation.

The cavity frequency, ω_c, is also affected by the saturation for the inhomogeneously broadened line as given by Eq. 15:

$$\left\{[\omega_{co}(1-\frac{\Delta\chi}{n_o})]^2 - \omega^2\right\}\chi_{IM} = \frac{\omega}{\tau_c}\chi_{RE} \quad , \quad (15)$$

so that saturation has a twofold effect on the oscillator frequency for an inhomogeneously broadened line.

C. EFFECT OF FEEDBACK

The feedback of radiation into a laser oscillator can be considered as the insertion of an electric field of frequency other than that of the oscillator, as indicated in Fig. 4. This field is amplified in the laser medium, thus decreasing the population inversion and acting as a loss in the medium. It is important to recognize that the loss is a dielectric loss, and not a conductive loss, in terms of the macroscopic wave equation that describes the processes. This is due to the fact that the change in population causes a change in χ, which is related to the dielectric constant. Since the amplification of the feedback depends upon the gain profile, off resonance radiation should not be as effective as near resonance radiation in pulling the frequency.

The effect of feedback on the oscillating frequency can be analyzed by the method that was used to determine the oscillating frequency of a laser without external feedback, except that now the feedback signal must be included. That is, the differential equation for the polarization is used to determine the susceptibility, and this result is substituted into the cavity equation to obtain the canonical equations that specify the frequency of oscillation.

1. Unsaturated Transitions

For the unsaturated case, the analysis shows the oscillator frequency equation can be written as

$$(\omega_c^2-\omega_n^2)\ \chi_{IM}\ (\omega_n) = \frac{\omega_n}{\tau_c}\ \chi_{RE}\ (\omega_n) \quad ; \qquad (20)$$

but χ_{IM} and χ_{RE} were seen to be functions of only one frequency, so ω_1 is independent of ω_2. That is, when the medium is linear no coupling exists between the two frequencies; therefore, they act independently. Mathematically, this is to be expected since a linear differential equation with a sum of linear inputs has a solution which is the sum of the solutions due to the individual inputs; however, physically it is somewhat surprising since the inverted population is changed by the feedback signal. The dilemma is resolved by recognizing that the change in population of the laser medium by the feedback signal requires saturation. Otherwise, the population supplying energy to the oscillator frequency is not changed by the feedback signal.

2. Saturated Transitions

In order for the feedback wave to interact with the oscillator frequency, in general a nonlinearity is required in the system. This nonlinearity is provided by the saturation effect. The effect of the feedback signal on the laser oscillation frequency can be determined by considering the equation for the population difference, which, it will be recalled, is applicable to a homogeneous line or to a set of particles in an inhomogeneous line with identical energy levels.

Again, the analysis shows (Appendix A) that the change in the population does not affect the frequency of oscillation through the explicit susceptibility. The cavity frequency, however, does change so that

$$\omega_1 = \omega_{10} - \frac{\pi L |\mu_{12}|^2}{3\epsilon_o \hbar} \frac{(N_1 - N_2)^e}{\eta_o} \tilde{g}_L(\omega, \Omega) \left[\frac{1}{\delta} \frac{1}{\omega_{co}} \frac{\chi_{RE}}{\tau_c \chi_{IM}} - \omega_{co} \right]$$

$$\cdot \frac{g_L(\omega_2, \Omega)}{I_{SAT} T_2 / \pi} I_2 \ , \qquad (25)$$

where ω_{10} is the frequency of oscillation without feedback. From Eq. 25 the oscillator frequency can be seen to be linearly related to the value of the transition curve at the feedback frequency and the intensity of the feedback. Both of these results have been experimentally verified as indicated by Fig. 11.

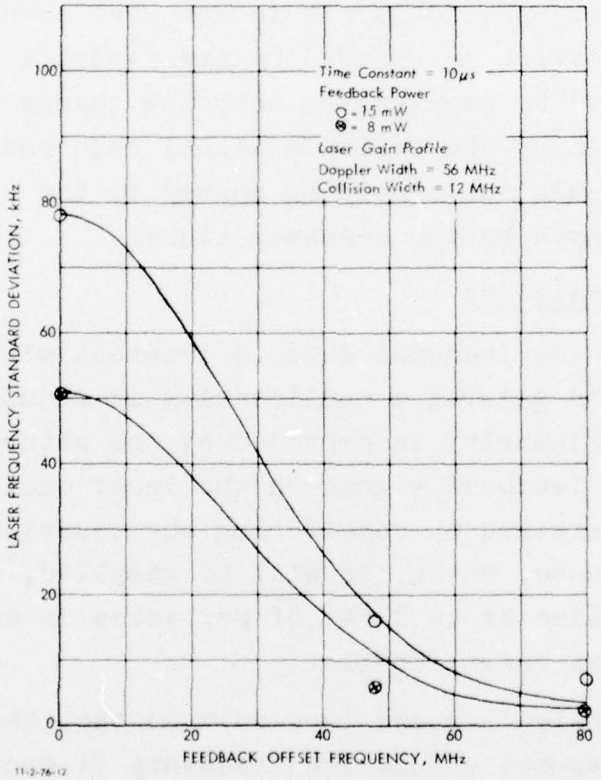


Figure 11. Comparison of Gain Line Profile and Laser Isolation Achieved versus Feedback Offset Frequency

For the inhomogeneously broadened line, the complex susceptibility is considerably more complicated, as can be seen by the analysis in Appendix A.

Fortunately, for the case of a CO_2 laser, the medium is not inhomogeneously broadened. This is due to the fact that the Doppler line of the CO_2 laser appears to be homogeneously broadened even at low pressures because of the short cross relaxation time of the molecules across the velocity profile (Ref. 11). The analysis for the homogeneously broadened case is, therefore, more likely to be applicable.

REFERENCES

1. M. Skolnik, *Radar Handbook*, McGraw-Hill, New York (1970).

2. I. J. Aplet and J. W. Carson, "A Faraday Effect Isolator," *Appl. Opt.*, 3,544 (1964).

3. J. O. Dummock, C. E. Hurwitz, and T. B. Reed, "Infrared Transmission, Magnetic Birefringence, and Faraday Rotation in FeO," *Appl. Phys. Letters*, 14, 49 (1969).

4. C. Freed, Lincoln Laboratory, Massachusetts Institute of Technology, private communications.

5. N. Skribanowitz, et. al., "Possibility of a Unidirectional Laser Amplifier Produced by Monochromatic Optical Pumping of a Coupled Doppler-Broadened Transition," *Appl. Phys. Letters*, 19, 161 (1971).

6. N. Skribanowitz, et. al., "Anisotropic Ultrahigh Gain Emission Observed in Rotational Transitions in Optically Pumped HF Gas," *Appl. Phys. Letters*, 20, 428 (1972).

7. V. J. Corcoran, J. M. Martin, and W. T. Smith, "The Application of the Effect to Feedback Isolation," paper presented at Electro-optic System Design Conference, New York (September 1973).

8. W. R. Bennett, Jr., "Hole Burning Effects of a He Ne Optical Maser," *Phys. Rev.*, 126,580 (1962).

9. T. G. Polanyi and I. Tobias, "The Frequency Stabilization of Gas Lasers," *Lasers*, 2,373 (1968).

10. A. Yariv, *Quantum Electronics*, John Wiley & Sons, Inc., New York, 457 (1967).

11. H. Franek, et. al., "Experiment on Cross Relaxation in CO_2," *IEEE Journal of Quantum Electronics*, QE-8, 404 (1972).

PRECEDING PAGE BLANK-NOT FILMED

APPENDIX A

DERIVATION OF EQUATIONS

PRECEDING PAGE BLANK-NOT FILMED

APPENDIX A

DERIVATION OF EQUATIONS

This appendix contains the mathematical derivations of the equations that are contained in the body of this paper. The sections in this appendix correspond to the sections of the main text. For the convenience of the reader, some of the material from the main text is repeated below.

III. THEORY

The approach to solve the problem diagramed in Fig. 4 (p. 12) requires deriving the general equation for the oscillator frequency without feedback, and then determining the frequency for the type of transition involved. The oscillator frequency equation with feedback is derived and compared to the equation without feedback to determine the effect of the feedback.

A. UNPERTURBED OSCILLATOR FREQUENCY

The frequency of oscillation of a laser has been approximately determined by considering the frequency pulling, caused by the phase shift per transit produced by the refractive laser medium on the radiation, or by considering the frequency pulling due to dielectric losses which are equated to conductivity cavity losses for a homogeneously broadened line. (See Refs. 8 and 9.)

The laser oscillating frequency can be determined more accurately than in Refs. 8 and 9 by solving the eigenvalue

PRECEDING PAGE BLANK-NOT FILMED

equation which is derived from the wave equation that describes the electromagnetic field in the laser cavity. The equations of interest are the differential equations for the polarization, the electric field, and the population inversion. When these equations are derived, as in Ref. A-1, the results are

$$\ddot{P} + \frac{2}{T_2}\dot{P} + \Omega^2 P = \frac{2\Omega}{\hbar}\frac{|\mu_{12}|}{3}(N_1 - N_2)\,E^{LOC} \qquad (1A)$$

$$\ddot{E} + \frac{1}{\tau_c}\dot{E} + \omega_c^2 E = -\frac{1}{\varepsilon}\ddot{P}^s \qquad (1B)$$

$$\frac{\partial}{\partial t}(N_1 - N_2) + \frac{(N_1 - N_2) - (N_1 - N_2)^e}{T_1} = -\frac{2}{\hbar\Omega}\dot{P}\cdot E^{LOC}, \qquad (1C)$$

where

- P is the polarization
- T_2 is the transverse relaxation time
- Ω is the angular transition frequency of the medium
- $\hbar$ = $h/2\pi$, where h is Planck's constant
- $|\mu_{12}|$ is the magnitude of the induced dipole moment
- N_1 is the number of molecules/volume in the lower energy state
- N_2 is the number of molecules/volume in the upper energy state

E^{LOC} is the local electric field

E is the macroscopic electric field

τ_c is the cavity lifetime

ω_c is the cavity angular frequency

ε is the permittivity of the medium

P^s is the polarization for the transition of interest

N_1^e is the number of molecules/VOL in the lower energy state in equilibrium

N_2^e is the number of molecules/VOL in the upper energy state of equilibrium.

If the polarization is expressed in terms of a linear susceptibility by the equation

$$\tilde{P} = \varepsilon_o \chi(\omega) \tilde{E}$$

$$= (\chi_{RE} + i \chi_{IM})\ \varepsilon_o \tilde{E} \quad ,$$

where

$\sim$ denotes a complex quantity

ε_o is the permittivity of free space

χ_{RE} is the real part of the linear susceptibility χ

χ_{IM} is the imaginary part of the susceptibility,

and the field and polarization are expressed by

$$E = \tilde{E}\, e^{i(\omega t - kz)}\,^{\dagger}$$

$$P = \tilde{P}\, e^{i(\omega t - kz)} \quad ,$$

---

†Because of the linearity of the equations involved, the complex form for the electric field and polarization can be used. Later in the paper this will not be possibile.

then substituting into the cavity wave Eq. 1B for the electric field gives

$$(i\omega)^2 + \frac{i\omega}{\tau_c} + \omega_c^2 = -\frac{\varepsilon_o}{\varepsilon}\chi(\omega)(i\omega)^2$$

$$\left[\omega_c^2 - \omega^2 + i\frac{\omega}{\tau_c}\right]\frac{1}{\chi(\omega)} = \frac{\omega^2\,\varepsilon_o}{\varepsilon}$$

$$\left[\omega_c^2 - \omega^2 + i\frac{\omega}{\tau_c}\right]\frac{(\chi_{RE} - i\chi_{IM})}{|\chi(\omega)|^2} = \frac{\omega^2\,\varepsilon_o}{\varepsilon} \quad .$$

Since the right side of this equation is real, the imaginary part of the left side must be zero, i.e.,

$$(\omega_c^2 - \omega^2)\,\chi_{IM} = \frac{\omega}{\tau_c}\,\chi_{RE} \quad . \tag{2}$$

This equation provides the general solution for the angular frequency, ω, of the oscillator whose cavity satisfies Eq. 1B. The angular frequency depends upon both the real and imaginary parts of the susceptibility. (The real part of the susceptibility of a material causes a phase shift of an electromagnetic wave as it propagates through the material. The fact that the phase shift is frequency dependent results in the dispersion of light as it passes through the material. The imaginary part of the susceptibility results in absorption or amplification of waves.) In order to evaluate this equation, therefore, it is necessary to determine the equation for the susceptibility.

Two generic types of transitions occur that give rise to different equations for susceptibility. They are homogeneously broadened and inhomogeneously broadened transitions.

1. Homogeneous Broadening

When the macroscopic polarization of a material is due to the sum of the contributions from individual atoms which have the same center transition frequency and linewidth, the absorption or spontaneous emission line is said to be homogeneously broadened. From another viewpoint the effect of a group of the atoms interacting with radiation is the same as for one atom interacting with the radiation except for a multiplicative constant. Homogeneous broadening results whenever the relaxation processes are the same for all atoms. These relaxation processes include coupling to lattice vibrations, collisions, and spin-spin coupling.

The polarization for the homogeneously broadened line is given by

$$\ddot{P} + \frac{2}{T_2}\dot{P} + \Omega^2 P = \frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} (N_1 - N_2) E \quad ,$$

where $L = [\eta^2 + 2/3]^2$ is the Lorentz correction factor for an isotroptic medium. When a solution of the form

$$E = \tilde{E}\, e^{i(\omega t - kz)}$$

$$P = \tilde{P}\, e^{i(\omega t - kz)}$$

is assumed, the solution to the differential equation is

$$\tilde{P} = \frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} (N_1 - N_2) \frac{1}{(\Omega^2 - \omega^2) + i\,(2\omega/T_2)} \tilde{E}$$

$$= \varepsilon_o\, \chi\,(\omega)\, \tilde{E} \quad ,$$

where

$$\chi(\omega) = \frac{\pi}{\hbar \varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1 - N_2)\, \tilde{g}_L\,(\omega, \Omega) \qquad (3)$$

and $\tilde{g}_L(\omega, \Omega)$ is the complex frequency dependent line shape. When $\tilde{g}_L(\omega, \Omega)$ is written in terms of real and imaginary parts,

$$\tilde{g}_L(\omega, \Omega) = \frac{2\Omega}{\pi} \frac{(\Omega^2-\omega^2)}{(\Omega^2-\omega^2)^2+(2\omega/T_2)^2} - i\frac{2\Omega}{\pi} \frac{2\omega/T_2}{(\Omega^2-\omega^2)^2+(2\omega/T_2)^2} \quad (4)$$

so that the linear susceptibility is

$$\chi = \chi_{RE} + i\,\chi_{IM} \quad ,$$

where

$$\chi_{RE} = \frac{1}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1 - N_2) \frac{2\Omega(\Omega^2-\omega^2)}{(\Omega^2-\omega^2)^2+(2\omega/T_2)^2}$$

$$\chi_{IM} = \frac{-1}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1-N_2) \frac{2\Omega 2\omega/T_2}{(\Omega^2-\omega^2)^2+(2\omega/T_2)^2} \quad .$$

Ordinarily the assumption of near reasonance, $\omega \approx \Omega$, is made, in which case,

$$\tilde{P} = \frac{1}{\hbar} L \frac{|\mu_{12}|^2}{3} (N_1 - N_2) \frac{1}{(\Omega-\omega) + i\,1/T_2} \tilde{E}$$

so

$$\tilde{g}_L(\omega, \Omega) = \frac{1}{\pi} \frac{\Omega - \omega}{(\Omega-\omega)^2 + (1/T_2)^2} - i\,\frac{1}{\pi} \frac{1/T_2}{(\Omega-\omega)^2 + (1/T_2)^2} \quad . \quad (5)$$

These susceptibility terms corresponding to $\tilde{g}_L(\omega, \Omega)$ are plotted in Fig. 5 (p. 15). When the values for χ_{RE} and χ_{IM} are substituted into Eq. 2, the general equation for the oscillator frequency, then the equation for the oscillating frequency is

$$\omega^2 - \omega_c^2 = \frac{1/\tau_c}{2/T_2} (\Omega^2 - \omega^2) \quad (6)$$

$$= T_2/\tau_c\, \omega(\Omega^2 - \omega^2) \text{ for } \omega \approx \Omega \quad ,$$

which shows that the frequency of oscillation assumes a value between the center of the transition and the cavity frequency. This fact is illustrated in Fig. 6 (p. 16), which indicates the position of the oscillation frequency compared to the spontaneous emission linewidth and the cavity linewidth.

2. *Inhomogeneous Broadening*

When the lineshape of a collection of atoms or molecules is different for each particle because of slightly different transition frequencies and/or different relaxation times, then the line is said to be inhomogeneously broadened. Crystalline inhomogeneities that produce different local fields at different lattice points in a crystal, nonuniform magnetic fields enveloping a spin system, or the Doppler effect due to motion of atoms or molecules in a gas all give rise to inhomogeneous broadening. The resulting lineshape for an inhomogeneously broadened line is determined by summing the homogeneous lineshapes of the individual particles involved in the process. The result of summing a large number of Lorentzian lines, as would be expected by the central limit theorem, is a Gaussian lineshape (Ref. 10).

For the Doppler broadened case where the number of molecules per unit volume dM in a transition frequency range of $d\Omega_1$ is determined by the Maxwellian thermal equilibrium velocity distribution, dM is

$$dM = N_V \, g_G \, (\Omega_i, \, \omega_o) d\Omega_i \ ,$$

where N_V is the total number of molecules per unit volume and g_G is the Gaussian lineshape factor. The equation for g_G is

$$g_G \, (\Omega_i, \, \omega_o) = \frac{[(4/\pi) \ell n 2]^{1/2}}{\Delta\omega_G} \, e^{-4\ell n 2 \frac{(\Omega_i - \omega_o)^2}{(\Delta\omega_G)^2}} \ ,$$

where

ω_o is the transition frequency of the stationary molecule and $\Delta\omega_G$ is the 1/2 width of the Gaussian line.

The Gaussian line is compared to the Lorentzian line in Fig. 7 (p. 16). The Lorentzian line is seen to have broader skirts than the Gaussian line, but a lower peak value, so that the energy is more concentrated for the Gaussian line.

The susceptibility of the Doppler broadened line is

$$d\tilde{P} = \varepsilon_o \tilde{E}\, d\chi\ (\omega,\ \Omega_i) \quad ,$$

where now different molecules have different properties according to the Maxwellian distribution. The linear susceptibility is, therefore,

$$d\chi(\omega,\ \Omega_i) = \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (\overline{\rho_{11}-\rho_{22}})\ \tilde{g}_L(\omega,\ \Omega_i)\cdot N_V\ g_G(\Omega_i,\omega_o)d\Omega_i ,$$

where ρ_{ii} is the probability of occupation of the ith state.

Now $N_V\ (\overline{\rho_{11}-\rho_{22}}) = N_1-N_2$, so that

$$d\chi\ (\omega,\ \Omega_i) = \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1-N_2)\ \tilde{g}_L(\omega,\Omega_i)g_G(\Omega_i,\ \omega_o)d\Omega_i \ .$$

The total susceptibility is

$$\chi(\omega) = \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1-N_2)\int_{\infty}^{\infty} \tilde{g}_L(\omega,\Omega_i)g_G(\Omega_i,\omega_o)d\Omega_i \ .$$

For the case where the Lorentzian lines that make up the Gaussian line are much narrower than the overall width of the Gaussian lines, $\tilde{g}_L(\omega,\Omega_i)$ can be approximated by

$$\tilde{g}_L(\omega,\Omega_i) = \frac{1}{\pi}\frac{1}{\Omega_1-\omega} - i\,\delta(\Omega_i - \omega)\,,$$

so that

$$\chi_{IM}(\omega) = -\frac{\pi}{\hbar\epsilon_o} L \frac{|\mu_{12}|^2}{3}(N_1 - N_2)\, g_G(\omega, \omega_o)\,.$$

From the Kramer-Kronig relations

$$\chi_{RE}(\omega) = -\frac{1}{\pi} PP \int_{-\infty}^{\infty} \frac{\chi_{IM}}{\Omega_i-\omega}\, d\Omega_i\,,$$

where PP is the principal part of the contour integral. The absorption is seen to have the same value as for the homogeneously broadened case, except for the lineshape difference.

The real part of the susceptibility is

$$\chi_{RE}(\omega) = -\frac{1}{\pi} PP \int_{-\infty}^{\infty} \frac{\chi_{IM}}{\Omega_i-\omega}\, d\Omega_i$$

$$= -\frac{1}{\pi} PP \int_{-\infty}^{\infty} \frac{\frac{-\pi}{\hbar\epsilon_o} L\frac{|\mu_{12}|^2}{3}(N_1 - N_2)\, g_G(\Omega_i,\omega_o)}{\Omega_i-\omega} d\Omega_i$$

when substitution for $\chi_{IM}(\Omega_i)$ is made. Substituting for $g_G(\Omega_i, \omega_o)$ yields

$$\chi_{RE}(\omega) = \frac{L}{\hbar\epsilon_o}\frac{|\mu_{12}|^2}{3}(N_1-N_2)\, PP\int_{-\infty}^{\infty} \frac{[(4/\pi)\ell n2]}{\Delta\omega_G}^{1/2} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{(\Delta\omega_G)^2}}}{\Omega_i-\omega}\, d\Omega_i$$

$$= \frac{L}{\hbar\epsilon_o}\frac{|\mu_{12}|^2}{3}(N_1-N_2)\frac{[(4/\pi)\ell n2]}{\Delta\omega_G}^{1/2} PP\int_{-\infty}^{\infty} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{(\Delta\omega_G)^2}}}{\Omega_i-\omega}\, d\Omega_i.$$

The solution for the integral (Ref. A-2)

$$I = PP \int_{-\infty}^{\infty} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G^2}}}{\Omega_i-\omega} d\Omega_i \quad .$$

can be found by the substitutions

$$\frac{4\ell n2}{\Delta\omega_G^2} = C, \qquad \Omega_i - \omega_o = S, \qquad \omega - \omega_o = b \quad ,$$

which simplify the integral to

$$I = PP \int_{-\infty}^{\infty} \frac{e^{-CS^2}}{S-b} dS \quad .$$

Now the denominator is

$$\frac{1}{S-b} = \frac{S+b}{S^2-b^2} \quad ,$$

so the integral can be written as

$$I = PP \int_{-\infty}^{\infty} \frac{e^{-CS^2}}{S-b} dS = PP \int_{-\infty}^{\infty} (S+b) \frac{e^{-CS^2}}{S^2-b^2} dS$$

$$= PP \int_{-\infty}^{\infty} b \frac{e^{-CS^2}}{S^2-b^2} dS + PP \int_{-\infty}^{\infty} S \frac{e^{-CS^2}}{S^2-b^2} dS \quad .$$

Since the integral of an odd function over limits that are symmetrical about the origin is zero, the second term vanishes so that

$$I(C,b) = \int_{-\infty}^{\infty} b \frac{e^{-CS^2}}{S^2-b^2} dS \quad ,$$

which can be written as

$$= be^{-b^2C} \int_{-\infty}^{\infty} \frac{e^{-C(S^2-b^2)}}{S^2-b^2} dS \quad .$$

If $I(C,b)$ is multiplied by e^{b^2C}, and the product is differentiated with respect to C,

$$\frac{d}{dC}\{I(C,b)e^{b^2C}\} = \frac{d}{dC} b \int_{-\infty}^{\infty} \frac{e^{-C(S^2-b^2)}}{S^2-b^2} dS$$

$$= -b \int_{-\infty}^{\infty} e^{-C(S^2-b^2)} dS$$

$$= -be^{Cb^2} \int_{-\infty}^{\infty} e^{-CS^2} dS \quad .$$

The integral can be written as the error function whose value can be found in a table of definite integrals. The integral is equal to

$$\int_{-\infty}^{\infty} e^{-CX^2}\, dX = \frac{\sqrt{\pi}}{2\sqrt{C}} \quad ,$$

so that

$$\frac{d\,\{I(C,b)e^{b^2C}\}}{dC} = -be^{Cb^2}\sqrt{\frac{\pi}{C}} \quad .$$

When the integration is performed

$$I(C,b)\, e^{b^2C} - I(0,b) = -b\sqrt{\pi}\int_0^C \frac{e^{Cb^2}}{\sqrt{C}}\, dC \quad .$$

But $I(0,b) = 0$, so

$$I(C,b) = -b\sqrt{\pi}\, e^{-b^2C}\int_0^C \frac{e^{Cb^2}}{\sqrt{C}}\, dC \quad .$$

Let

$$Cb^2 = X^2$$

$$b^2 dC = 2X dX \quad ,$$

so that

$$I(C,b) = -2\sqrt{\pi}\, e^{-b^2C}\int_0^{b\sqrt{C}} e^{X^2}\, dX \quad .$$

This equation can be identified as Dawson's Integral, which is tabulated. (See Appendix C and Ref. A-3.) Notice that the

integrand is an even function, so the integral is an odd function, i.e., $I(-b) = -I(b)$.

The real part of the susceptibility is, therefore,

$$\chi_{RE}(\omega) = \frac{L}{\hbar\varepsilon_o} \frac{|\mu_{12}|^2}{3} (N_1-N_2) \frac{[(4/\pi)\ell n2]^{1/2}}{\Delta\omega_G} I(c,b)$$

$$= \frac{L}{\hbar\varepsilon_o} \frac{|\mu_{12}|^2}{3} (N_1-N_2) (-2)\sqrt{\pi} \int_0^{\frac{\omega-\omega_o}{\Delta\omega_G}[4\ell n2]^{1/2}} e^{X^2} dX \; g_G(\omega,\omega_o), \tag{7A}$$

while the imaginary part has been given by

$$\chi_{IM}(\omega) = -\pi \frac{L}{\hbar\varepsilon_o} \frac{|\mu_{12}|^2}{3} (N_1-N_2) g_G(\omega,\omega_o) , \tag{7B}$$

where

$$g_G(\omega_1\omega_o) = \frac{[(4/\pi)\ell n2]^{1/2}}{\Delta\omega_G} e^{-(\omega-\omega_o)^2 \frac{4\ell n2}{\Delta\omega_G^2}} . \tag{8}$$

A plot of the real and imaginary parts of the susceptibility for an inhomogeneously broadened line is given in Fig. 8a (p. 18), and a comparison of the real susceptibility for a homogeneously broadened line and an inhomogeneously broadened line is given in Fig. 8b (p. 18).

The equation for the frequency of oscillation for the frequency of an inhomogeneously broadened line is obtained when these values for the susceptibility are substituted into Eq. 2. The equation is then

$$(\omega_c^2-\omega^2)\pi = \frac{\omega}{\tau_c} 2\sqrt{\pi} \int_0^{\frac{\omega-\omega_o}{\Delta\omega_G}[4\ell n2]^{1/2}} e^{X^2} dX . \tag{9}$$

Although this equation is considerably more complicated than Eq. 6, it nevertheless shows that the frequency of oscillation assumes a value between the center of the transition and the cavity frequency.

B. SATURATION EFFECTS

If there were no radiation in the laser cavity, then the population inversion would be determined by the pumping level and the relaxation rates for transitions from the upper to the lower laser level. In the presence of electromagnetic radiation, stimulated transitions occur so that the population inversion in equilibrium is reduced from the value obtained without radiation in the cavity. As the level of radiation increases, the value of the population inversion decreases. For a noninverted population, energy is absorbed and then is dissipated in relaxation processes; however, for a large energy absorption the system may not relax to equilibrium as fast as it can for small signals, so the system saturates. These ideas can be made quantitative by considering the differential equation for the population difference

$$\frac{\partial}{\partial t}(N_1-N_2) + \frac{(N_1-N_2) - (N_1-N_2)^e}{T_1} = -\frac{2}{\hbar\Omega}\,\dot{P}\cdot E \quad .$$

The first term is the power delivered to the dipole medium, the second is the dissipation of the power to the surroundings, and the right side of the equation represents power delivered by the electromagnetic field.

When no field is present, the steady state solution is obtained by setting the first term equal to zero, so that

$$(N_1-N_2) = (N_1-N_2)^e \quad ,$$

i.e., the population difference is the equilibrium value, as is expected. In thermal equilibrium this is the Boltzmann distribution.

If the effects of the electromagnetic field are included, then the steady state solution for the population difference is obtained by setting the time dependent parts equal to zero. For

$$P = 1/2\ \tilde{P}\ e^{i(\omega t - kz)} + cc^{+}$$

$$E = 1/2\ \tilde{E}\ e^{i(\omega t - kz)} + cc\ ,$$

the equation becomes

$$\frac{(N_1-N_2) - (N_1-N_2)^e}{T_1} = -\frac{i\omega}{2\hbar\Omega}\ (\tilde{P}\cdot\tilde{E}^* - \tilde{P}^*\cdot\tilde{E})\ .$$

Since

$$\tilde{P} = \varepsilon_o\ \chi(\omega)\tilde{E} = \varepsilon_o\ \left[\chi_{RE}(\omega) + i\ \chi_{IM}(\omega)\right]\tilde{E}\ ,$$

$$\frac{(N_1-N_2)-(N_1-N_2)^e}{T_1} = \frac{-i\omega}{2\hbar\Omega}\left\{\varepsilon_o\left[\chi_{RE}+i\ \chi_{IM}\right]\tilde{E}\cdot\tilde{E}^* - \varepsilon_o\left[\chi_{RE}-i\ \chi_{IM}\right]\tilde{E}^*\cdot\tilde{E}\right\}$$

$$= \frac{\omega\ \varepsilon_o}{\hbar\Omega}\ \chi_{IM}\ |E|^2\ . \qquad (10)$$

---

†At this point it is necessary to use the complex conjugate terms because of the terms involving products of the polarization and the electric field. Without the complex conjugates, no steady state term would be obtained.

Equation 10 is the general equation for the steady state population difference. It can be used to determine the effects of saturation on the frequency of oscillation by substituting the specific expressions for the imaginary part of the susceptibility, solving for the population difference, and substituting the population difference into Eq. 2 to determine the oscillation frequency. The values of the susceptibility for homogeneously and inhomogeneously broadened lines can now be substituted.

1. Homogeneous Broadening

The final form for the saturation effect depends upon the type of transition involved, since the form of the susceptibility depends upon the character of the transition. The imaginary part of the susceptibility for a homogeneously broadened line from Eq. 3 is found to be

$$\chi_{IM} = - \frac{\pi L |\mu_{12}|^2 (N_1 - N_2)}{3\varepsilon_o \hbar} g_L \ ,$$

where

$$g_L = \frac{2\Omega}{\pi} \frac{2\omega/T_2}{(\Omega^2-\omega^2)^2 + \left(\frac{2\omega}{T_2}\right)^2}$$

$$\approx \frac{1}{\pi} \frac{1/T_2}{(\Omega-\omega)^2 + \left(1/T_2\right)^2} \qquad \text{for } \omega \approx \Omega \ .$$

When this equation is substituted into Eq. 10, the result is

$$\frac{(N_1-N_2)-(N_1-N_2)^e}{T_1} = -\frac{\pi L|\mu_{12}|^2(N_1-N_2)}{3\hbar^2}\,|E|^2\,\frac{\omega g_L}{\Omega}$$

$$\frac{N_1-N_2}{T_1} + \frac{\pi L|\mu_{12}|^2}{3\hbar^2}\frac{\omega}{\Omega}g_L(\omega)(N_1-N_2)\,|E|^2 = \frac{(N_1-N_2)^e}{T_1}$$

$$N_1-N_2 = \frac{(N_1-N_2)^e}{1 + \frac{\pi L|\mu_{12}|^2}{3\hbar^2}\frac{\omega}{\Omega}T_1\,g_L(\omega)|E|^2} \quad . \tag{11}$$

This equation shows that the change in population, due to a change in g_L, occurs in the denominator of the equation. Substitution of the population difference from Eq. 11 into the complex susceptibility, which is given in Eq. 3

$$\tilde{\chi}(\omega) = \frac{\pi L|\mu_{12}|^2(N_1-N_2)}{3\varepsilon_o\hbar}\,\tilde{g}_L(\omega,\Omega) \tag{3}$$

yields

$$\tilde{\chi}(\omega) = \frac{\pi L|\mu_{12}|^2}{3\varepsilon_o\hbar}\,\tilde{g}_L\,\frac{(N_1-N_2)^e}{1 + \frac{\pi L|\mu_{12}|^2}{3\hbar^2}\,\frac{\omega}{\Omega}\,T_1\,g_L|E|^2} \quad , \tag{12}$$

which shows that both terms of the susceptibility, the imaginary part, which causes the absorption, or gain, and the real part, which produces a phase shift, are dependent on the frequency of operation ω and the strength of the field E. The saturation effect for a homogeneously broadened line is illustrated in Fig. 9 (p. 20).

The complex susceptibility is affected only in the denominator by the change in $|E|^2$ or ω, and the substitution into Eq. 2 gives

$$(\omega_c^2 - \omega^2)\, \chi_{IM} = \frac{\omega}{\tau_c}\, \chi_{RE} \qquad (2)$$

$$(\omega_c^2 - \omega^2)\, g_{LIM} = \frac{\omega}{\tau_c}\, g_{LRE} \quad . \qquad (13)$$

Consequently, the equation for the operating frequency is unchanged explicitly by saturation effects on the susceptibility.

A change, however, does occur in the cavity frequency ω_c, which is given by

$$\omega_c = \frac{\pi n c}{\eta L} \quad , \qquad (14)$$

where

n is the longitudinal mode number
c is the velocity of light
η is the index of refraction of the medium
L is the resonator length.

Since

$$\eta = 1 + \chi(\omega) \quad ,$$

the cavity frequency is

$$\omega_c = \frac{\pi n c}{L(1+\chi)} = \frac{\pi n c}{L(1+\chi_o+\Delta\chi)}$$

$$\approx \frac{\pi n c}{L}\,(1 - \chi + \chi^2 + \ldots)$$

$$\approx \omega_{co}\left(1 - \frac{\Delta\chi}{\eta_o}\right)$$

for small values of $\Delta\chi$. This change in ω_c, due to the field dependent change in χ, results in a change in the operating frequency. Equation 2, for the case where saturation of a homogeneously broadened line occurs, becomes

$$(\omega_c^2 - \omega^2)\,\chi_{IM} = \frac{\omega}{\tau_c}\,\chi_{RE}$$

$$\left\{\left[\omega_{co}\left(1 - \frac{\Delta\chi}{\eta_o}\right)\right]^2 - \omega^2\right\}\,\chi_{IM} = \frac{\omega}{\tau_c}\,\chi_{RE} \quad . \tag{15}$$

Saturation, therefore, does change the oscillator frequency from the frequency obtained in the unsaturated condition for a homogeneously broadened line.

2. Inhomogeneous Broadening

For an inhomogeneously broadened line, the equation for the population difference that is used for a homogeneously broadened line is applicable to those atoms or molecules whose transitions are centered at a specified frequency. The total population difference is the integral of the specific population difference over all possible frequencies. Ordinarily, when the integration is performed the result for the imaginary part of the susceptibility is the same as for the homogeneously broadened line except that the saturation effect depends on the square root of intensity rather than the first power. In order to calculate the frequency of oscillation, however, the real term is needed as well.

The polarization equation for the set of particles whose transitions are all centered at the same frequency is

$$\tilde{P}^k = \varepsilon_o \chi^k(\omega) \tilde{E}^k,$$

where

$$\chi^k(\omega) = \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1-N_2)^k \tilde{g}_L^k$$

for a homogeneously broadened line. The total effect due to all particles is the summation over each set of particles with different center transition frequencies. For a continuous distribution of transition frequencies, the result is

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi^k(\omega)\, p(k)\, dk$$

$$= \int_{-\infty}^{\infty} \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} (N_1-N_2)^k \tilde{g}_L^k\, p(k)\, dk \qquad (16)$$

where p(k) is the probability that a particle belongs to the set whose center frequency corresponds to k. When the change in population due to the saturation effect for a homogenous line

$$(N_1-N_2)^k = \frac{(N_1-N_2)^{ek}}{1 + \frac{\pi L|\mu_{12}|^2}{3\hbar^2} \frac{\omega}{\Omega_k} T_1 g_L^k |E|^2}$$

is substituted, then the susceptibility is

$$\chi(\omega) = \frac{\pi}{\hbar\varepsilon_o} L \frac{|\mu_{12}|^2}{3} \int_{-\infty}^{\infty} \frac{(N_1-N_2)^{ek} \tilde{g}_L^k}{1 + \frac{\pi L|\mu_{12}|^2}{3\hbar^2} \frac{\omega}{\Omega_k} T_1 g_L^k |E|^2} p(k)\, dk ,$$

where

$$\tilde{g}_L^k(\omega,\Omega_k) = \frac{2\Omega_k}{\pi}\frac{(\Omega_k^2-\omega^2)}{(\Omega_k^2-\omega^2)^2+(2\omega/T_2)^2} - \frac{i2\Omega}{\pi}\frac{2\omega/T_2}{(\Omega_k^2-\omega^2)^2+(2\omega/T_2)^2}$$

and

$$g_L^k(\omega,\Omega_k) = \frac{2\Omega}{\pi}\frac{2\omega/T_2}{(\Omega_k^2-\omega^2)^2+(2\omega/T_2)^2} \quad .$$

When $\omega \approx \Omega_k$, then these expressions become

$$\tilde{g}_L^k(\omega,\Omega_k) = \frac{1}{\pi}\frac{\Omega_k-\omega}{(\Omega_k-\omega)^2+(1/T_2)^2} - \frac{i}{\pi}\frac{1/T_2}{(\Omega_k-\omega)^2+(1/T_2)^2}$$

$$g_L^k(\omega,\Omega_k) = \frac{1}{\pi}\frac{1/T_2}{(\Omega_k-\omega)^2+(1/T_2)^2} \quad .$$

These are the expressions that are usually used, and when they are substituted into the expression for the susceptibility Eq. 16, it becomes

$$\chi(\omega) = \frac{\pi}{\hbar\epsilon_o}\frac{L|\mu_{12}|^2}{3}(N_1-N_2)^{ek}\int_{-\infty}^{\infty}\frac{\frac{1}{\pi}\frac{(\Omega_k-\omega)}{(\Omega_k-\omega)^2+(1/T_2)^2} - \frac{i}{\pi}\frac{1/T_2}{(\Omega_k-\omega)^2+(1/T_2)^2}}{1+\frac{\pi L|\mu_{12}|^2}{3}T_1\frac{1}{\pi}\frac{1/T_2}{(\Omega_k-\omega)^2+(1/T_2)^2}|E|^2}\,p(\Omega_k)\,d\Omega_k$$

$$= \frac{\pi}{\hbar\epsilon_o}\frac{L|\mu_{12}|^2}{3}(N_1-N_2)^{ek}\left\{\int_{-\infty}^{\infty}\frac{\frac{1}{\pi}\frac{\Omega_k-\omega}{(\Omega_k-\omega)^2+(1/T_2)^2+\frac{L|\mu_{12}|^2}{3}\frac{T_1}{T_2}|E|^2}}{}\,p(\Omega_k)\,d\Omega_k\right. \quad (17A)$$

$$\left. + \int_{-\infty}^{\infty}\frac{-\frac{i}{\pi}\frac{1/T_2}{(\Omega_k-\omega)^2+(1/T_2)^2+\frac{L|\mu_{12}|^2}{3}\frac{T_1}{T_2}|E|^2}}{}\,p(\Omega_k)\,d\Omega_k\right\} .$$

Ordinarily it is assumed that $p(\Omega_k)$ is essentially constant over the region of integration and $\omega \approx \Omega_k$. For this assumption these equations can be seen to be in the form

$$\chi(\omega) = \frac{\pi}{\hbar\varepsilon_o} \frac{L|\mu_{12}|^2}{3} (N_1 - N_2)^{ek} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X dX}{X^2+\gamma^2} - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{dX}{X^2+\gamma^2} \right\} ,$$

where

$$\gamma^2 = \left(\frac{1}{T_2}\right)^2 + \frac{L|\mu_{12}|^2}{3} \frac{T_1}{T_2} |E|^2 \quad .$$

The evaluation of the imaginary part of the susceptibility in this case presents no problem, so little difficulty is ordinarily encountered since the imaginary part of the susceptibility, corresponding to the absorption or amplification, is usually all that is desired. However, the real part of the susceptibility is zero under these assumptions since the integrand is odd. The approach of assuming $p(\Omega_k)$ is a constant, therefore, cannot be used for this problem; a more rigorous approach is necessary.

The transitions that occur at any frequency can be considered to be the sum of the transitions that can occur at that frequency due to all of the homogeneously broadened lines that make up the inhomogeneously broadened curve, weighted by the shape of the inhomogeneously broadened curve. The weighting function for this case is

$$g_G(\Omega_k, \omega_o) = \frac{[(4/\pi)\ell n 2]^{1/2}}{\Delta\omega_G} e^{\frac{-4\ell n 2}{(\Delta\omega_G)^2} (\Omega_k - \omega_o)^2} \quad .$$

The equation for the susceptibility is, then,

$$\chi(\omega) = \frac{\pi}{\hbar\varepsilon_o}\frac{L|\mu_{12}|^2}{3}(N_1-N_2)ek\left\{\frac{1}{\pi}\frac{[(4/\pi)\ell n2]^{1/2}}{\Delta\omega_G}\int_{-\infty}^{\infty}\frac{Xe^{\frac{-4\ell n2}{(\Delta\omega_G)^2}(X+\omega-\omega_o)^2}}{X^2+\gamma^2}\,dX\right.$$

$$\left. - \frac{i}{\pi}\frac{[(4/\pi)\ell n2]^{1/2}}{\Delta\omega_G}\int_{-\infty}^{\infty}\frac{e^{\frac{-4\ell n2}{(\Delta\omega_G)^2}(X+\omega-\omega_o)^2}}{X^2+\gamma^2}\,dX\right\}. \qquad (17B)$$

It would appear that both integrals could be evaluated by a contour integration where the contour includes the real axis and the upper half plane. The exponential term, however, diverges for large imaginary values so that a reasonable contour is difficult to find.

Various attempts to alter the equation to fit the tabulated integrals failed. A complicated solution using a power series expansion was found. The complexity encouraged search for another solution, which was finally successful.

The solution is to make a substitution which causes the integrals to take a form given in Ref. 11.

The first integral

$$\int_{-\infty}^{\infty}\frac{e^{-a(X+\omega-\omega_o)^2}}{X^2+\gamma^2}\,dX$$

is solved by substituting

$$t = \sqrt{a}\ (X+\omega-\omega_o) \qquad dX = dt/\sqrt{a}$$

so that the first integral becomes

$$\int_{-\infty}^{\infty} \frac{e^{-t^2}}{\left[\frac{t}{\sqrt{a}} - (\omega-\omega_o)\right]^2 + \gamma^2} \frac{dt}{\sqrt{a}}$$

$$= \int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{a}}}{\frac{1}{a}\{[t - \sqrt{a}(\omega-\omega_o)]^2 + a\,\gamma^2\}} e^{-t^2}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{a}\, e^{-t^2}}{\left[\sqrt{a}\,(\omega-\omega_o) - t\right]^2 + a\,\gamma^2}\, dt \quad .$$

When the values

$$x = \sqrt{a}\,(\omega-\omega_o) \qquad y = \sqrt{a}\,\gamma$$

are substituted, the integral is

$$\frac{1}{\gamma^2} \int_{-\infty}^{\infty} \frac{y\ e^{-t^2}}{(x-t)^2 + y^2}\, dt = \frac{1}{\gamma^2}\, \pi\, \Re\, w(x+iy) \qquad \begin{matrix} x \text{ real} \\ y > o \end{matrix}$$

$$= \frac{\pi}{\gamma^2}\, \Re\, e^{-(x+iy)^2}\ \operatorname{erfc}\,[-i(x+iy)] \quad .$$

The second integral is found in a similar way:

$$\int_{-\infty}^{\infty} \frac{X\, e^{-a(X+\omega-\omega_o)^2}}{X^2 + \gamma^2}\, dX \quad .$$

Let

$$t = \sqrt{a}\ (X+\omega-\omega_o) \qquad\qquad dX = dt/\sqrt{a} \quad .$$

Then

$$\int_{-\infty}^{\infty} \frac{\left[\frac{t}{\sqrt{a}} - (\omega-\omega_o)\right] e^{-t^2}}{\left[\frac{t}{\sqrt{a}} - (\omega-\omega_o)\right]^2 + \gamma^2} \quad \frac{dt}{\sqrt{a}}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \frac{[t - \sqrt{a}\ (\omega-\omega_o)]\ e^{-t^2}}{\frac{1}{a}\left\{[t - \sqrt{a}\ (\omega-\omega_o)]^2 + a\gamma^2\right\}} \quad \frac{dt}{\sqrt{a}}$$

$$= \int_{-\infty}^{\infty} \frac{[t - \sqrt{a}\ (\omega-\omega_o)]\ e^{-t^2}}{[t - \sqrt{a}\ (\omega-\omega_o]^2 + a\gamma^2} \quad dt \qquad ;$$

and if the values

$$x = \sqrt{a}\ (\omega-\omega_o) \qquad\qquad y^2 = a\gamma^2$$

are substituted,

$$\int_{-\infty}^{\infty} \frac{(t-x)\ e^{-t^2}}{(t-x)^2 + y^2}\ dt$$

$$= \int_{-\infty}^{\infty} \frac{(x-t)\ e^{-t^2}}{(x-t)^2 + y^2}\ dt = \pi i\omega(x+iy) \qquad\qquad \begin{array}{l} x \text{ real} \\ y > o \end{array}$$

$$= \pi i e^{-(x+iy)^2}\ \text{erfc}\ [-i(x+iy)] \ ,$$

where

$$\text{erfc}\ z = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\ dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{x+iy}^{\infty} e^{-t^2}\ dt \ .$$

The equation for the susceptibility is, therefore,

$$\chi(\omega) = \frac{\pi}{\hbar\varepsilon_o}\frac{L|\mu_{12}|^2}{3}(N_1-N_2)ek\left\{-\frac{1}{\pi}\frac{[(4/\pi)\ln 2]^{1/2}}{\Delta\omega_G}\pi e^{-z^2}\,\mathrm{erfc}(-z)\right.$$

$$\left. -\frac{i}{\pi}\frac{[(4/\pi)\ln 2]^{1/2}}{\Delta\omega_G}\frac{\pi}{\gamma^2}\,\mathrm{Re}\,e^{-z^2}\,\mathrm{erfc}(-z)\right\}, \tag{18}$$

where

$$z = x+iy$$

$$x = \sqrt{a}\,(\omega-\omega_o)$$

$$y = \sqrt{a}\,\gamma$$

$$a = \frac{4\ln 2}{(\Delta\omega_G)^2}$$

$$\gamma^2 = \left(\frac{1}{T_2}\right)^2 + \frac{L|\mu_{12}|^2}{3}\frac{T_1}{T_2}|E|^2 .$$

The real and imaginary parts of Eq. 18 are plotted in Figs. 10a and 10b (pps. 22 and 23).

This equation for the susceptibility is considerably more complicated than the susceptibility for the homogeneously broadened case. In this case the effect of the angular frequency, ω, and the intensity effect given by $|E|^2$, are different for the real and imaginary parts of the susceptibility. Consequently the frequency equation

$$(\omega_c^2-\omega^2)\,\chi_{IM} = \frac{\omega}{\tau_c}\chi_{RE} \tag{2}$$

$$(\omega_c^2-\omega)\left[\frac{1}{\gamma^2}\,\mathrm{Re}\,e^{-z^2}\,\mathrm{erfc}(-z)\right] = \frac{\omega}{\tau_c}\left[\mathrm{Im}\,e^{-z^2}\mathrm{erfc}(-z)\right] \tag{19}$$

is explicitly affected by the saturation, since

$$z = \sqrt{a}\,(\omega-\omega_o) + i\sqrt{a}\,\gamma$$

$$\gamma^2 = \left(\frac{1}{T_2}\right)^2 + \frac{L|\mu_{12}|^2}{3}\frac{T_1}{T_2}|E|^2 .$$

The cavity frequency, ω_c, is also affected by the saturation for the inhomogeneously broadened line as given by Eq. 15

$$\left\{ [\omega_{co}(1 - \frac{\Delta\chi}{\eta_o})]^2 - \omega^2 \right\} \chi_{IM} = \frac{\omega}{\tau_c} \chi_{RE} , \tag{15}$$

so that saturation has a twofold effect on the oscillator frequency for an inhomogeneously broadened line.

C. EFFECT OF FEEDBACK

The feedback of radiation into a laser oscillator can be considered as the insertion of an electric field of frequency other than that of the oscillator, as indicated in Fig. 4 (p. 12). The effect of feedback on the oscillating frequency can be analyzed by the method that was used to determine the oscillating frequency of a laser without external feedback, except that now the feedback signal must be included. That is, the differential equation for the polarization is used to determine the susceptibility, and this result is substituted into the cavity equation to obtain the canonical equations that specify the frequency of oscillation.

1. Unsaturated Transitions

For the unsaturated case, the equation for the polarization is

$$\ddot{P} + \frac{2}{T_2} \dot{P} + \Omega^2 P = - \frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} NE ,$$

where $N = N_2 - N_1$.

If
$$P = \tilde{P}_1 e^{i(\omega_1 t - k_1 z)} + \tilde{P}_2 e^{i(\omega_2 t - k_2 z)}$$
$$E = \tilde{E}_1 e^{i(\omega_1 t - k_1 z)} + \tilde{E}_2 e^{i(\omega_2 t - k_2 z)} ,$$

then substitution into the polarization equation yields

$$\left[(\Omega^2 - \omega_1^2) + \frac{2}{T_2} i\omega_1\right] \tilde{P}_1 e^{i(\omega_1 t - k_1 z)} + \left[\Omega^2 - \omega_2^2 + \frac{2}{T_2} i\omega_2\right] \tilde{P}_2 e^{i(\omega_2 t - k_2 z)}$$
$$= -\frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} N \left[\tilde{E}_1 e^{i(\omega_1 t - k_1 z)} + \tilde{E}_2 e^{i(\omega_2 t - k_2 z)}\right] .$$

Coefficients of the same frequency can be equated because of the orthogonality of the two input frequencies.

The results are

$$\left[(\Omega^2 - \omega_1^2) + \frac{i2\omega_1}{T_2}\right] \tilde{P}_1 = -\frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} N \tilde{E}_1$$

$$\left[(\Omega^2 - \omega_2^2) + \frac{i2\omega_2}{T_2}\right] \tilde{P}_2 = -\frac{2\Omega}{\hbar} L \frac{|\mu_{12}|^2}{3} N E_2 ,$$

so that

$$\tilde{P}_n = \varepsilon_o \chi(\omega_n) \tilde{E}_n ,$$

where

$$\chi(\omega_n) = -\frac{2\Omega}{\varepsilon_o \hbar} L \frac{|\mu_{12}|^2}{3} N \frac{1}{(\Omega^2 - \omega_n^2) + \frac{i2\omega_n}{T_2}} .$$

When the polarization and electric field are substituted into the differential equation for the field in the cavity

$$\ddot{E} + \frac{1}{\tau_c}\dot{E} + \omega_c^2 E = -\frac{1}{\varepsilon}\ddot{P} ,$$

the result is

$$\left[(\omega_c^2-\omega_1^2) + \frac{i\omega_1}{\tau_c}\right]\tilde{E}_1 e^{i(\omega_1 t-k_1 z)} + \left[(\omega_c^2-\omega_2^2)+\frac{i\omega_2}{\tau_c}\right]\tilde{E}_2 e^{i(\omega_2 t-k_2 z)}$$

$$= \frac{\omega_1^2}{\varepsilon}\tilde{P}_1 e^{i(\omega_1 t-k_1 z)} + \frac{\omega_2^2}{\varepsilon}\tilde{P}_2 e^{i(\omega_2 t-k_2 z)} .$$

Again, the terms of the same frequency can be equated and the expression for the susceptibility can be substituted so that the canonical equations become

$$\left[(\omega_c^2-\omega_1^2) + \frac{i\omega_1}{\tau_c}\right]\tilde{E}_1 = \frac{\omega_1^2}{\varepsilon}\varepsilon_o\,\chi(\omega_1)\,\tilde{E}_1$$

$$\left[(\omega_c^2-\omega_2^2) + \frac{i\omega_2}{\tau_c}\right]\tilde{E}_2 = \frac{\omega_2^2}{\varepsilon}\varepsilon_o\,\chi(\omega_2)\,\tilde{E}_2 .$$

Each of these equations is identical in form to the oscillator frequency equation so that the frequency equation can be written as

$$(\omega_c^2-\omega_n^2)\,\chi_{IM}(\omega_n) = \frac{\omega_n}{\tau_c}\,\chi_{RE}(\omega_n) \quad ; \qquad (20)$$

but χ_{IM} and χ_{RE} were seen to be functions of only one frequency, so ω_1 is independent of ω_2. That is, when the medium is linear no coupling exists between the two frequencies; therefore they act independently. Mathematically, this is to be expected since a linear differential equation with a sum of linear inputs has a solution which is the sum of the solutions due to the individual inputs; however, physically it is somewhat surprising since the inverted population is changed by the feedback signal. The dilemma is resolved by recognizing that the change in population of the laser medium by the feedback signal requires saturation. Otherwise, the population supplying energy to the oscillator frequency is not changed by the feedback signal.

2. Saturated Transitions

In order for the feedback wave to interact with the oscillator frequency, in general a nonlinearity is required in the system. This nonlinearity is provided by the saturation effect. The effect of the feedback signal on the laser oscillation frequency can be determined by considering the equation for the population difference

$$\frac{\partial (N_1-N_2)}{\partial t} + \frac{(N_1-N_2) - (N_1-N_2)^e}{T_1} = -\frac{2}{\hbar\Omega} \dot{P} \cdot E \quad ,$$

where P and E each consist of two frequencies. This equation, it will be recalled, is applicable to a homogeneous line or to a set of particles in an inhomogeneous line with identical energy levels.

The polarization and electric field are

$$P = 1/2\ \tilde{P}_1 e^{i(\omega_1 t - k_1 z)} + 1/2\ \tilde{P}_1^* e^{-i(\omega_1 t - k_1 z)}$$

$$+ 1/2\ \tilde{P}_2 e^{i(\omega_2 t - k_2 z)} + 1/2\ \tilde{P}_2^* e^{-i(\omega_2 t - k_2 z)} ,$$

and

$$E = 1/2\ \tilde{E}_1 e^{i(\omega_1 t - k_1 z)} + 1/2\ \tilde{E}_1^* e^{-i(\omega_1 t - k_1 z)}$$

$$+ 1/2\ \tilde{E}_2 e^{i(\omega_2 t - k_2 z)} + 1/2\ \tilde{E}_1^* e^{-1(\omega_2 t - k_2 z)} ,$$

so that

$$\dot{P} \cdot E = \left\{ \frac{i\omega_1}{2} \tilde{P}_1 e^{i(\omega_1 t - k_1 z)} - \frac{i\omega_1}{2} \tilde{P}_1^* e^{-i(\omega_1 t - k_1 z)} \right.$$

$$\left. + \frac{i\omega_2}{2} \tilde{P}_2 e^{i(\omega_2 t - k_2 z)} - \frac{i\omega_2}{2} \tilde{P}_2^* e^{-i(\omega_2 t - k_2 z)} \right\}$$

$$\cdot \left\{ 1/2\ \tilde{E}_1 e^{i(\omega_1 t - k_1 z)} + 1/2\ \tilde{E}_1^* e^{-i(\omega_1 t - k_1 z)} \right.$$

$$\left. + 1/2\ \tilde{E}_2 e^{i(\omega_2 t - k_2 z)} + 1/2\ \tilde{E}_2^* e^{-i(\omega_2 t - k_2 z)} \right\} .$$

The time independent terms are

$$\frac{i\omega_1}{4} \left(\tilde{P}_1 \cdot \tilde{E}_1^* - \tilde{P}_1^* \cdot \tilde{E}_1 \right) + \frac{i\omega_2}{4} \left(\tilde{P}_2 \cdot \tilde{E}_2^* - \tilde{P}_2^* \cdot \tilde{E}_2 \right) .$$

The steady state or time independent part of the equation for the population difference is, therefore,

$$\frac{(N_1 - N_2) - (N_1 - N_2)^e}{T_1} = - \frac{i}{2\ \Omega} \left[\omega_1 \left(\tilde{P}_1 \cdot \tilde{E}_1^* - \tilde{P}_1^* \cdot \tilde{E}_1 \right) + \omega_2 \left(\tilde{P}_2 \cdot \tilde{E}_2^* - \tilde{P}_2^* \cdot \tilde{E}_2 \right) \right] .$$

Now

$$\tilde{P} = \tilde{P}_1 + \tilde{P}_2 = \chi_1 \tilde{E}_1 + \chi_2 \tilde{E}_2$$

so

$$\frac{(N_1-N_2)-(N_1-N_2)^e}{T_1} = -\frac{i}{2\hbar\Omega}\left[\omega_1\left(\chi_1 \tilde{E}_1 \tilde{E}_1^* - \chi_1^* \tilde{E}_1^* \tilde{E}_1\right) + \omega_2\left(\chi_2 \tilde{E}_2 \tilde{E}_2^* - \chi_2^* \tilde{E}_2^* \tilde{E}_2\right)\right]$$

$$= \frac{1}{\hbar\Omega}\left[\omega_1 \chi_{IM}(\omega_1)|E_1|^2 + \omega_2 \chi_{IM}(\omega_2)|E_2|^2\right] \quad ,$$

where all time dependent terms, including the low frequency components generated by the mixing of two optical frequencies, have been ignored. Substitution of the value of $\chi_{IM}(\omega_n)$ from Eqs. 3 and 5,

$$\chi_{IM}(\omega_n) = -\frac{\pi}{\varepsilon_o \hbar} L \frac{|\mu_{12}|^2}{3} (N_1-N_2)\, g_{Ln} \quad ,$$

where

$$g_{Ln} = \frac{2\Omega}{\pi} \frac{2\omega_n/T_2}{(\Omega^2-\omega_n^2)^2 + \left(\frac{2\omega_n}{T_2}\right)^2}$$

$$\approx \frac{1}{\pi} \frac{1/T_2}{(\Omega-\omega_n)^2 + (1/T_2)^2} \qquad \text{for } \omega \approx \Omega$$

into the above equation yields

$$\frac{N_1-N_2}{T_1} + \frac{\pi L}{\varepsilon_o \hbar^2} \frac{|\mu_{12}|^2}{3\Omega}\left[\omega_1 g_{L1}|E_1|^2 + \omega_2 g_{L2}|E_2|^2\right](N_1-N_2) = \frac{(N_1-N_2)^e}{T_1} \quad .$$

The new population difference is

$$N_1-N_2 = \frac{(N_1-N_2)^e}{1 + \frac{\pi L T_1}{\varepsilon_o \hbar^2}\frac{|\mu_{12}|^2}{3\Omega}\left[\omega_1 g_{L1}|E_1|^2 + \omega_2 g_{L2}|E_2|^2\right]}$$

$$= \frac{(N_1-N_2)^e}{1 + K\,(\omega_1 g_{L1} I_1 + \omega_2 g_{L2} I_2)} \quad , \tag{21}$$

where $I_n = C_n|E_n|^2$ is the intensity of the field at frequency ω_n. This equation can be seen to be a generalization of Eq. 11 for the case where two signals are influencing the oscillator. If, therefore, saturation has occurred without feedback, as it does in gas lasers, then the population is determined by $g_{L1}(g_{L2} = 0)$, so the frequency is fixed. Now feedback by another signal changes the population due to g_{L2}. If the saturation term is small, then

$$N_1-N_2 = (N_1-N_2)^e \left[1 - K\,(\omega_1 g_{L1} I_1 + \omega_2 g_{L2} I_2)\right]$$

$$\approx (N_1-N_2)^e \left[1 - \frac{I_1 g_L(\omega_1,\Omega) + I_2 g_L(\omega_2,\Omega)}{I_{SAT}\, T_2/\pi}\right] \quad \text{for } \omega_1 \approx \omega_2 \,, \tag{22}$$

where $K\omega = I_{SAT}\, T_2/\pi$, which shows that the value of N_1-N_2 is linearly related to I_2, the intensity of the feedback signal, and $g_L(\omega_2,\Omega)$, the lineshape of the transition. When the new population difference is substituted into Eq. 3, the expression for the complex susceptibility becomes

$$\chi(\omega) = \frac{\pi L|\mu_{12}|^2}{3\varepsilon_o \hbar} (N_1 - N_2)^e \left[1 - \frac{I_1 g_L(\omega_1,\Omega) + I_2 g_L(\omega_2,\Omega)}{I_{SAT}\ T_2/\pi}\right] \tilde{\varepsilon}_L(\omega,\Omega)$$

$$= \chi_o(\omega) + \Delta\chi(\omega) \ . \qquad (23)$$

The exact expression for the frequency pulling or pushing of the oscillator due to the feedback is obtained by substituting the change in susceptibility due to the population change into the general equation

$$(\omega_c^2 - \omega_1^2)\ \chi_{IM}(\omega_1) = \frac{\omega_1}{\tau_c}\ \chi_{RE}(\omega_1) \ , \qquad (20A)$$

where

$$\omega_c = \frac{nc}{\eta 2L} \approx \frac{nc}{2L}\ (1 - \chi_{TOTAL}) \ .$$

Again, the change in the population does not affect the frequency of oscillation through the explicit susceptibility expression since, from the general frequency equation,

$$\omega_1^2 + \frac{\chi_{RE}}{\tau_c \chi_{IM}}\ \omega_1 - \omega_c^2 = 0 \ ,$$

and the ratio χ_{RE}/χ_{IM} cancels any change in the real and imaginary parts of the susceptibility due to population changes. The cavity frequency, however, does change so that

$$\omega_1 = -\frac{\frac{\chi_{RE}}{\tau_c \chi_{IM}} \pm \sqrt{\left(\frac{\chi_{RE}}{\tau_c \chi_{IM}}\right)^2 + 4\ \omega_c^2}}{2} \ , \qquad (24)$$

which for small changes in the frequency from the cavity frequency is given by

$$\omega_1 \approx \omega_c + \frac{1}{\delta}\frac{1}{\omega_c}\frac{\chi_{RE}}{\tau_c\chi_{IM}} - \frac{\chi_{RE}}{2\tau_c\chi_{IM}}$$

$$\approx \frac{nc}{\eta 2L} + \frac{1}{\delta}\frac{1}{\frac{nc}{\eta 2L}}\frac{\chi_{RE}}{\tau_c\chi_{IM}} - \frac{\chi_{RE}}{2\tau_c\chi_{IM}}$$

$$\approx \frac{nc}{2L(1+\chi)} + \frac{1}{\delta}\frac{1+\chi}{\frac{nc}{2L}}\frac{\chi_{RE}}{\tau_c\chi_{IM}} - \frac{\chi_{RE}}{2\tau_c\chi_{IM}}$$

$$\approx \omega_{10} + \frac{\Delta\chi}{\eta_o}\left[\frac{1}{\delta}\frac{1}{\omega_{co}}\frac{\chi_{RE}}{\tau_c\chi_{IM}} - \omega_{co}\right] \quad ,$$

which, from Eq. 22, becomes

$$\omega_1 = \omega_{10} - \frac{\pi L|\mu_{12}|^2}{3\varepsilon_o\hbar}\frac{(N_1-N_2)^e}{\eta_o}\tilde{g}_L(\omega,\Omega)\left[\frac{1}{\delta}\frac{1}{\omega_{co}}\frac{\chi_{RE}}{\tau_c\chi_{IM}} - \omega_{co}\right]$$

$$\cdot \frac{g_L(\omega_2,\Omega)}{I_{SAT}T_2/\pi} I_2 \quad , \qquad (25)$$

where ω_{10} is the frequency of oscillation without feedback. From Eq. 25 the oscillator frequency can be seen to be linearly related to the value of the transition curve at the feedback frequency and the intensity of the feedback. Both of these results have been experimentally verified as indicated by Fig. 11 (p. 26).

For the inhomogeneously broadened line, the complex susceptibility is considerably more complicated. In this case the complex susceptibility is found by substituting Eq. 21 into Eq. 16, so that

$$\chi(\omega_1) = \int_{-\infty}^{\infty} \frac{\pi}{\hbar\varepsilon_o} \frac{L|\mu_{12}|^2}{3} \frac{(N_1-N_2)^{ek}\ \tilde{g}_L^k\ p(k)dk}{1 + \frac{\pi L T_1}{\varepsilon_o \hbar^2} \frac{|\mu_{12}|^2}{3\Omega} [\omega_1 g_L^k |E_1|^2 + \omega_2\ g_{L2}^k |E_2|^2]}$$

$$= \frac{\pi}{\hbar\varepsilon_o} \frac{L|\mu_{12}|^2}{3} (N_1-N_2)^{ek}$$

$$\int_{-\infty}^{\infty} \frac{\frac{1}{\pi}\frac{(\Omega_k-\omega_1)}{(\Omega_k-\omega_1)^2 + (1/T_2)^2} - \frac{i}{\pi}\frac{1/T_2}{(\Omega_k-\omega_1)^2 + (1/T_2)^2}\ p(k)dk}{1 + \frac{\pi L|\mu_{12}|^2}{\pi\ 3\ \Omega} T_1 \frac{1/T_2}{(\Omega_k-\omega_1)^2 + (1/T_2)^2} \frac{\omega_1}{} |E_1|^2 + \frac{\pi L|\mu_{12}|^2}{\pi\ 3\ \Omega} T_1 \frac{\omega_2\ 1/T_2 |E_2|^2}{(\Omega_k-\omega_2)^2+(1/T_2)^2}},$$

if $\omega_1 \sim \omega_2$ then the values of the denominators are such that

$$(\Omega_i - \omega_1)^2 + (1/T_2)^2 \approx (\Omega_i - \omega_2)^2 + (1/T_2)^2$$

for $\omega_1, \omega_2 \sim \Omega_i$.

The equation for the complex susceptibility can then be simplified to

$$\chi(\omega_1) = \frac{\pi}{\hbar\varepsilon_o} \frac{L|\mu_{12}|^2}{3} (N_1-N_2)^{ek}$$

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{(\Omega_k-\omega_1)\ p(\Omega_k)\ dk}{(\Omega_k-\omega_1)^2 + (1/T_2)^2 + \frac{L|\mu_{12}|^2}{3} \frac{T_1}{T_2} \frac{1}{\Omega_k} (\omega_1|E_1|^2 + \omega_2|E_2|^2)}$$

$$+ \int_{-\infty}^{\infty} -\frac{i}{\pi} \frac{1/T_2\ p(\Omega_k)dk}{(\Omega_k-\omega_1)^2 + (1/T_2)^2 + \frac{L|\mu_{12}|^2}{3} \frac{T_1}{T_2} \frac{1}{\Omega_k} (\omega_1|E_1|^2 + \omega_2|E_2|^2)}.$$

If Ω_k in the right side of the denominator can be assumed to be constant over the range of integration, then $\chi(\omega_1)$ is the same form as Eq. 17B, where

$$\gamma = \left(\frac{1}{T_2}\right)^2 + \frac{L|\mu_{12}|^2}{3} \frac{T_1}{T_2} \frac{1}{\Omega_k} (\omega_1|E_1|^2 + \omega_2|E_2|^2) \quad ,$$

so that the final solution is Eq. 19, with γ as given above. This equation is difficult to evaluate compared to the equation for the homogeneously broadened case.

Fortunately, for the case of a CO_2 laser, the medium is not inhomogeneously broadened, so the analysis for the homogeneously broadened case is more likely to be applicable.

REFERENCES

A-1. R. H. Pantell and H. E. Puthoff, *Fundamentals of Quantum Electronics*, John Wiley & Sons, Inc., New York, p. 40 (1969).

A-2. L. B. Felsen and N. Marcovitz, *Radiation and Scattering of Waves*, Prentice Hall, Inc., Englewood Cliffs, N.J., p. 399 (1973).

A-3. R. V. Churchill, *Introduction to Complex Variables*, McGraw-Hill Book Co. Inc., New York, p. 126 (1948).

APPENDIX B

UNSUCCESSFUL ATTEMPTS TO FIND I

PRECEDING PAGE BLANK-NOT FILMED

APPENDIX B

UNSUCCESSFUL ATTEMPTS TO FIND I

The first approach toward solving this equation is to search for a definite integral in a table. After no satisfactory tabular form is found, a straightforward contour integral is attempted.

The integral

$$I = PP\int_{-\infty}^{\infty} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{(\Delta\omega_G)^2}}}{\Omega_i-\omega}\, d\Omega_i$$

is equal to

$$I = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G{}^2}}}{\Omega_i-\omega}\, d\Omega_i \quad .$$

This integral has a pole on the positive real axis because ω, the angular frequency, is positive and real. Since an optical frequency is involved, it is also apparent that $\omega>>0$. The function can be visualized, as indicated in Fig. B-1, as the product of a Gaussian Line centered at a fixed value ω_o, and the term $1/(\Omega_i-\omega)$ where ω is variable but close to ω_o. Because the integral has a real pole, the theory of residues cannot be used

PRECEDING PAGE BLANK-NOT FILMED

since the denominator of an integral to be evaluated by the theory of residues cannot have a real zero of any order (Ref. B-1).

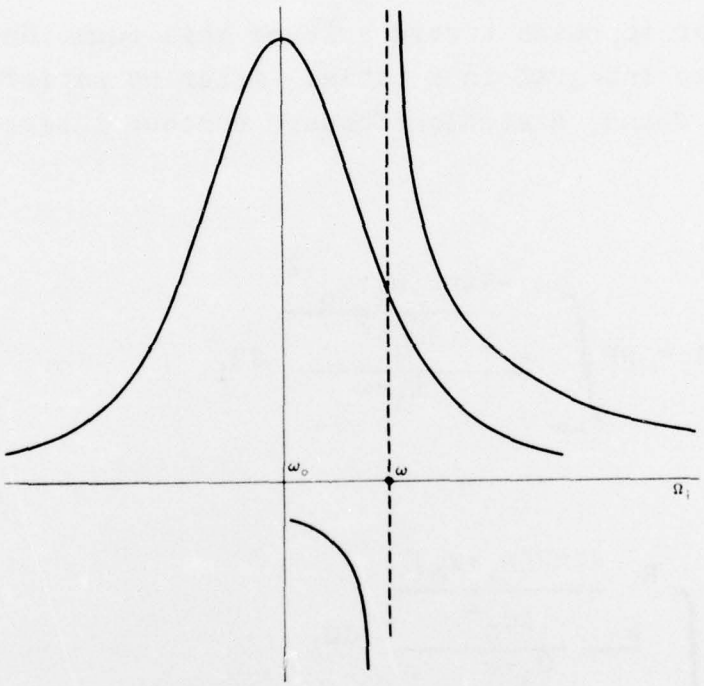


FIGURE B-1. Illustration of Two Functions of Which Integrand is Composed. Because Gaussian function is symmetrically about ω_0, and $(1/\Omega_i-\omega)$ is antisymmetrical about ω_0, the integral vanishes when $\omega = \omega_0$.

Rather than a contour integral which includes the pole, a contour integral excluding the pole can be attempted. If a contour is chosen that excludes the pole, then the integral is

$$I = \int_{-\infty}^{\omega-r} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G^2}}}{\Omega_i-\omega}\, d\Omega_i$$

$$+ \int_{-\pi}^{0} \frac{e^{\frac{-4\ell 2(re^{i\theta}-\omega_o)^2}{\Delta\omega_G^2}}}{re^{i\theta}-\omega}\, rd\theta + \int_{\omega+r}^{\infty} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G^2}}}{\Omega_i-\omega}\, d\Omega_i$$

$$+ \int_{0}^{\pi} \frac{e^{\frac{-4\ell n2(Re^{i\theta}-\omega_o)^2}{\Delta\omega_G^2}}}{Re^{i\theta}-\omega}\, Rd\theta \ ,$$

which becomes

$$I = \int_{-\infty}^{\omega} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G^2}}}{\Omega_i-\omega}\, d\Omega_i + \int_{\omega}^{\infty} \frac{e^{\frac{-4\ell n2(\Omega_i-\omega_o)^2}{\Delta\omega_G^2}}}{\Omega_i-\omega}\, d\Omega_i$$

$$= 0 \ .$$

By inspection of Fig. B-1, it is clear that the integral is zero for $\omega = \omega_o$, but must have a value for $\omega \neq \omega_o$. Also, the value of the integral is antisymmetrical for values of ω about ω_o.

Another approach is to change the form of the integral so that it takes the form of a tabulated integral. If the substitution $X = \Omega_i - \omega$ is made

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-4\ell n2(\Omega_i-\omega_o)^2/\Delta\omega_G{}^2}}{\Omega_i-\omega}\, d\Omega_i = \lim_{R\to\infty} \int_{-R-\omega}^{R-\omega} \frac{e^{-4\ell n2(X+\omega-\omega_o)^2/\Delta\omega_G{}^2}}{X}\, dX$$

$$= \lim_{R\to\infty} \int_{-R-\omega}^{R-\omega} \frac{e^{-4\ell n2[X^2+2X(\omega-\omega_o)\ +\ (\omega-\omega_o)^2]/\Delta\omega_G{}^2}}{X}\, dX \ ,$$

which is not available in tables either.

REFERENCE

B-1. M. Abromowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series, 55, U.S. Government Printing Office, Washington, D.C., p. 319 (1970).

APPENDIX C

TABLE OF DAWSON'S INTEGRAL

This Table is reproduced from Ref. 11 for the convenience of the reader.

DAWSON'S INTEGRAL

Table 7.5

x	$e^{-x^2}\int_0^x e^{t^2}dt$	x	$e^{-x^2}\int_0^x e^{t^2}dt$	x^{-2}	$xe^{-x^2}\int_0^x e^{t^2}dt$	$\langle x\rangle$
0.00	0.00000 00000	1.00	0.53807 95069	0.250	0.60268 0777	2
0.02	0.01999 46675	1.02	0.53637 44359	0.245	0.60046 6027	2
0.04	0.03995 73606	1.04	0.53431 71471	0.240	0.59819 8606	2
0.06	0.05985 62071	1.06	0.53192 50787	0.235	0.59588 1008	2
0.08	0.07965 95389	1.08	0.52921 57454	0.230	0.59351 6018	2
0.10	0.09933 59924	1.10	0.52620 66800	0.225	0.59110 6724	2
0.12	0.11885 46083	1.12	0.52291 53777	0.220	0.58865 6517	2
0.14	0.13818 49287	1.14	0.51935 92435	0.215	0.58616 9107	2
0.16	0.15729 70920	1.16	0.51555 55409	0.210	0.58364 8516	2
0.18	0.17616 19254	1.18	0.51152 13448	0.205	0.58109 9080	2
0.20	0.19475 10334	1.20	0.50727 34964	0.200	0.57852 5444	2
0.22	0.21303 68833	1.22	0.50282 85611	0.195	0.57593 2550	2
0.24	0.23099 28865	1.24	0.49820 27897	0.190	0.57332 5618	2
0.26	0.24859 34747	1.26	0.49341 20827	0.185	0.57071 0126	2
0.28	0.26581 41727	1.28	0.48847 19572	0.180	0.56809 1778	2
0.30	0.28263 16650	1.30	0.48339 75174	0.175	0.56547 6462	2
0.32	0.29902 38575	1.32	0.47820 34278	0.170	0.56287 0205	2
0.34	0.31496 99336	1.34	0.47290 38898	0.165	0.56027 9114	2
0.36	0.33045 04051	1.36	0.46751 26208	0.160	0.55770 9305	3
0.38	0.34544 71562	1.38	0.46204 28368	0.155	0.55516 6829	3
0.40	0.35994 34819	1.40	0.45650 72375	0.150	0.55265 7582	3
0.42	0.37392 41210	1.42	0.45091 79943	0.145	0.55018 7208	3
0.44	0.38737 52812	1.44	0.44528 67410	0.140	0.54776 0994	3
0.46	0.40028 46599	1.46	0.43962 45670	0.135	0.54538 3766	3
0.48	0.41264 14572	1.48	0.43394 20135	0.130	0.54305 9774	3
0.50	0.42443 63835	1.50	0.42824 90711	0.125	0.54079 2591	3
0.52	0.43566 16609	1.52	0.42255 51804	0.120	0.53858 5013	3
0.54	0.44631 10184	1.54	0.41686 92347	0.115	0.53643 8983	3
0.56	0.45637 96813	1.56	0.41119 95842	0.110	0.53435 5529	3
0.58	0.46586 43551	1.58	0.40555 40424	0.105	0.53233 4747	3
0.60	0.47476 32037	1.60	0.39993 98943	0.100	0.53037 5810	3
0.62	0.48307 58219	1.62	0.39436 39058	0.095	0.52847 7031	3
0.64	0.49080 32040	1.64	0.38883 23346	0.090	0.52663 5967	3
0.66	0.49794 77064	1.66	0.38335 09429	0.085	0.52484 9575	3
0.68	0.50451 30066	1.68	0.37792 50103	0.080	0.52311 4393	4
0.70	0.51050 40576	1.70	0.37255 93490	0.075	0.52142 6749	4
0.72	0.51592 70382	1.72	0.36725 83182	0.070	0.51978 2972	4
0.74	0.52078 93010	1.74	0.36202 58410	0.065	0.51817 9571	4
0.76	0.52509 93152	1.76	0.35686 54206	0.060	0.51661 3369	4
0.78	0.52886 66089	1.78	0.35178 01580	0.055	0.51508 1573	4
0.80	0.53210 17071	1.80	0.34677 27691	0.050	0.51358 1788	4
0.82	0.53481 60684	1.82	0.34184 56029	0.045	0.51211 1971	5
0.84	0.53702 20202	1.84	0.33700 06597	0.040	0.51067 0372	5
0.86	0.53873 26921	1.86	0.33223 96091	0.035	0.50925 5466	5
0.88	0.53996 19480	1.88	0.32756 38080	0.030	0.50786 5903	6
0.90	0.54072 43187	1.90	0.32297 43193	0.025	0.50650 0473	6
0.92	0.54103 49328	1.92	0.31847 19293	0.020	0.50515 8078	7
0.94	0.54090 94485	1.94	0.31405 71655	0.015	0.50383 7717	8
0.96	0.54036 39857	1.96	0.30973 03141	0.010	0.50253 8471	10
0.98	0.53941 50580	1.98	0.30549 14372	0.005	0.50125 9494	14
1.00	0.53807 95069	2.00	0.30134 03889	0.000	0.50000 0000	∞
	$\left[\begin{smallmatrix}(-5)7\\4\end{smallmatrix}\right]$		$\left[\begin{smallmatrix}(-5)4\\4\end{smallmatrix}\right]$		$\left[\begin{smallmatrix}(-6)8\\6\end{smallmatrix}\right]$	

See **Example 3.**

$\langle x\rangle$ = nearest integer to x.

Compiled from J. B. Rosser, Theory and application of $\int_0^z e^{-x^2}dx$ and $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$. Mapleton House, Brooklyn, N.Y., 1948; and B. Lohmander and S. Rittsten, Table of the function $y=e^{-x^2}\int_0^x e^{t^2}dt$, Kungl. Fysiogr. Sallsk. i Lund Forh. **28**, 45–52, 1958 (with permission).

PRECEDING PAGE BLANK-NOT FILMED

APPENDIX D

TABLE OF ERROR FUNCTION FOR COMPLEX ARGUMENTS

This Table is reproduced from Ref. 11 for the convenience of the reader.

PRECEDING PAGE BLANK-NOT FILMED

$w(z) = e^{-z^2}\,\mathrm{erfc}(-iz) \qquad z = x+iy$

	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$
y	$x = 0$		$x = 0.1$		$x = 0.2$		$x = 0.3$		$x = 0.4$	
0.0	1.000000	0.000000	0.990050	0.112089	0.960789	0.219753	0.913931	0.318916	0.852144	0.406153
0.1	0.896457	0.000000	0.888479	0.094332	0.864983	0.185252	0.827246	0.269600	0.777267	0.344688
0.2	0.809020	0.000000	0.802567	0.080029	0.783538	0.157403	0.752895	0.229653	0.712146	0.294653
0.3	0.734599	0.000000	0.729337	0.068410	0.713801	0.134739	0.688720	0.197037	0.655244	0.253613
0.4	0.670788	0.000000	0.666463	0.058897	0.653680	0.116147	0.632996	0.170203	0.605295	0.219706
0.5	0.615690	0.000000	0.612109	0.051048	0.601513	0.100782	0.584333	0.147965	0.561252	0.191500
0.6	0.567805	0.000000	0.564818	0.044524	0.555974	0.087993	0.541605	0.129408	0.522246	0.167880
0.7	0.525930	0.000000	0.523423	0.039064	0.515991	0.077275	0.503896	0.113821	0.487556	0.147975
0.8	0.489101	0.000000	0.486982	0.034465	0.480697	0.068235	0.470452	0.100647	0.456579	0.131101
0.9	0.456532	0.000000	0.454731	0.030566	0.449383	0.060563	0.440655	0.089444	0.428808	0.116714
1.0	0.427584	0.000000	0.426044	0.027242	0.421468	0.054014	0.413989	0.079864	0.403818	0.104380
1.1	0.401730	0.000000	0.400406	0.024392	0.396470	0.048393	0.390028	0.071628	0.381250	0.093752
1.2	0.378537	0.000000	0.377393	0.021934	0.373989	0.043542	0.368412	0.064510	0.360799	0.084547
1.3	0.357643	0.000000	0.356649	0.019805	0.353691	0.039336	0.348839	0.058329	0.342206	0.076538
1.4	0.338744	0.000000	0.337876	0.017951	0.335294	0.035671	0.331054	0.052936	0.325248	0.069538
1.5	0.321585	0.000000	0.320825	0.016329	0.318561	0.032463	0.314839	0.048210	0.309736	0.063393
1.6	0.305953	0.000000	0.305284	0.014905	0.303290	0.029643	0.300009	0.044051	0.295506	0.057978
1.7	0.291663	0.000000	0.291072	0.013648	0.289309	0.027154	0.286406	0.040377	0.282417	0.053186
1.8	0.278560	0.000000	0.278035	0.012536	0.276470	0.024948	0.273892	0.037118	0.270346	0.048931
1.9	0.266509	0.000000	0.266042	0.011547	0.264648	0.022987	0.262350	0.034217	0.259186	0.045139
2.0	0.255396	0.000000	0.254978	0.010664	0.253732	0.021236	0.251677	0.031626	0.248844	0.041748
2.1	0.245119	0.000000	0.244745	0.009874	0.243628	0.019669	0.241783	0.029304	0.239239	0.038706
2.2	0.235593	0.000000	0.235256	0.009165	0.234251	0.018260	0.232592	0.027217	0.230300	0.035968
2.3	0.226742	0.000000	0.226438	0.008526	0.225531	0.016991	0.224033	0.025335	0.221963	0.033498
2.4	0.218499	0.000000	0.218224	0.007949	0.217404	0.015845	0.216047	0.023633	0.214172	0.031263
2.5	0.210806	0.000000	0.210557	0.007427	0.209813	0.014806	0.208582	0.022090	0.206879	0.029234
2.6	0.203613	0.000000	0.203387	0.006952	0.202710	0.013862	0.201589	0.020687	0.200039	0.027389
2.7	0.196874	0.000000	0.196668	0.006520	0.196050	0.013002	0.195028	0.019409	0.193613	0.025706
2.8	0.190549	0.000000	0.190360	0.006125	0.189796	0.012216	0.188861	0.018241	0.187566	0.024168
2.9	0.184602	0.000000	0.184429	0.005764	0.183912	0.011498	0.183056	0.017172	0.181868	0.022759
3.0	0.179001	0.000000	0.178842	0.005433	0.178368	0.010839	0.177581	0.016192	0.176491	0.021466
y	$x = 0.5$		$x = 0.6$		$x = 0.7$		$x = 0.8$		$x = 0.9$	
0.0	0.778801	0.478925	0.697676	0.535713	0.612626	0.576042	0.527292	0.600412	0.444858	0.610142
0.1	0.717588	0.408474	0.651076	0.459665	0.580698	0.497744	0.509299	0.522932	0.439421	0.536087
0.2	0.663273	0.350751	0.608322	0.396852	0.549739	0.432442	0.489710	0.457569	0.430271	0.472773
0.3	0.614[illegible]	0.303124	0.569238	0.344645	0.520192	0.377688	0.469480	0.402194	0.418736	0.418491
0.4	0.571717	0.263563	0.533581	0.300989	0.492289	0.331535	0.449244	0.355082	0.405763	0.371813
0.5	0.533157	0.230488	0.501079	0.264268	0.466127	0.292432	0.429418	0.314828	0.392021	0.331544
0.6	0.498591	0.202666	0.471453	0.233206	0.441712	0.259136	0.410264	0.280290	0.377977	0.296692
0.7	0.467521	0.179123	0.444434	0.206787	0.418998	0.230646	0.391936	0.250532	0.363957	0.266427
0.8	0.439512	0.159087	0.419766	0.184200	0.397906	0.206155	0.374518	0.224789	0.350182	0.240057
0.9	0.414191	0.141945	0.397216	0.164793	0.378341	0.185005	0.358043	0.202429	0.336799	0.217004
1.0	0.391234	0.127202	0.376571	0.148036	0.360200	0.166660	0.342511	0.182932	0.323899	0.196783
1.1	0.370363	0.114460	0.357637	0.133501	0.343375	0.150681	0.327900	0.165868	0.311537	0.178990
1.2	0.351335	0.103395	0.340241	0.120838	0.327766	0.136706	0.314176	0.150877	0.299741	0.163281
1.3	0.333942	0.093744	0.324229	0.109759	0.313273	0.124435	0.301294	0.137661	0.288519	0.149370
1.4	0.318001	0.085288	0.309463	0.100026	0.299804	0.113620	0.289208	0.125971	0.277865	0.137012
1.5	0.303355	0.077851	0.295820	0.091443	0.287274	0.104054	0.277869	0.115594	0.267766	0.126002
1.6	0.289866	0.071283	0.283192	0.083845	0.275602	0.095563	0.267228	0.106355	0.258203	0.116164
1.7	0.277412	0.065461	0.271479	0.077096	0.264718	0.088001	0.257237	0.098103	0.249151	0.107348
1.8	0.265890	0.060283	0.260598	0.071081	0.254554	0.081245	0.247851	0.090710	0.240586	0.099427
1.9	0.255205	0.055661	0.250469	0.065701	0.245050	0.075190	0.239027	0.084068	0.232482	0.092291
2.0	0.245276	0.051521	0.241025	0.060876	0.236152	0.069748	0.230724	0.078085	0.224813	0.085845
2.1	0.236031	0.047804	0.232204	0.056534	0.227810	0.064842	0.222905	0.072680	0.217552	0.080009
2.2	0.227407	0.044454	0.223952	0.052617	0.219978	0.060409	0.215535	0.067785	0.210676	0.074712
2.3	0.219347	0.041428	0.216219	0.049073	0.212616	0.056391	0.208581	0.063342	0.204160	0.069894
2.4	0.211800	0.038686	0.208961	0.045859	0.205686	0.052741	0.202013	0.059298	0.197982	0.065500
2.5	0.204723	0.036196	0.202139	0.042936	0.199155	0.049417	0.195804	0.055610	0.192120	0.061486
2.6	0.198074	0.033929	0.195717	0.040271	0.192992	0.046384	0.189928	0.052238	0.186554	0.057811
2.7	0.191818	0.031859	0.189664	0.037836	0.187170	0.043608	0.184362	0.049150	0.181265	0.054439
2.8	0.185924	0.029966	0.183950	0.035607	0.181662	0.041064	0.179084	0.046315	0.176237	0.051339
2.9	0.180361	0.028231	0.178549	0.033561	0.176447	0.038728	0.174074	0.043708	0.171452	0.048485
3.0	0.175105	0.026636	0.173437	0.031680	0.171502	0.036577	0.169315	0.041306	0.166895	0.045851

See Examples 12–19.

$$w(x) = e^{x^2} + \frac{2i}{\sqrt{\pi}} e^{-x^2} \int_0^x e^{t^2}\,dt$$

$$w(-x+iy) = \overline{w(x+iy)} \qquad w(x-iy) = 2e^{y^2-x^2}(\cos 2xy + i \sin 2xy) - \overline{w(x+iy)}$$

$$w(iy) = e^{y^2}\,\mathrm{erfc}\, y \qquad w[(1+i)u] = e^{-2iu^2}\left\{1 + (i-1)\left[C\left(\frac{2u}{\sqrt{\pi}}\right) + iS\left(\frac{2u}{\sqrt{\pi}}\right)\right]\right\}$$

PRECEDING PAGE BLANK-NOT FILMED

Table 7.9 ERROR FUNCTION FOR COMPLEX ARGUMENTS

$w(z) = e^{-z^2}\operatorname{erfc}(-iz) \qquad z = x+iy$

y	$\mathcal{R}w(z)$	$\mathcal{I}w(z)$	$\mathcal{R}w(z)$	$\mathcal{I}w(z)$	$\mathcal{R}w(z)$	$\mathcal{I}w(z)$	$\mathcal{R}w(z)$	$\mathcal{I}w(z)$	$\mathcal{R}w(z)$	$\mathcal{I}w(z)$
	$x = 1.0$		$x = 1.1$		$x = 1.2$		$x = 1.3$		$x = 1.4$	
0.0	0.367879	0.607158	0.298197	0.593761	0.236928	0.572397	0.184520	0.545456	0.140858	0.515113
0.1	0.373170	0.538555	0.312136	0.532009	0.257374	0.518283	0.209431	0.499216	0.168407	0.476535
0.2	0.373153	0.478991	0.319717	0.477439	0.270928	0.469488	0.227362	0.456555	0.189247	0.440005
0.3	0.369386	0.427225	0.322586	0.429275	0.279199	0.425667	0.239793	0.417491	0.204662	0.405823
0.4	0.363020	0.382166	0.321993	0.386777	0.283443	0.386412	0.247908	0.381908	0.215711	0.374110
0.5	0.354900	0.342872	0.318884	0.349266	0.284638	0.351299	0.252654	0.349611	0.223262	0.344868
0.6	0.345649	0.308530	0.313978	0.316128	0.283540	0.319910	0.254784	0.320368	0.228026	0.318022
0.7	0.335721	0.278445	0.307816	0.286815	0.280740	0.291851	0.254895	0.293927	0.230578	0.293453
0.8	0.325446	0.252024	0.300807	0.260847	0.276693	0.266757	0.253461	0.270040	0.231385	0.271015
0.9	0.315064	0.228759	0.293259	0.237800	0.271752	0.244295	0.250858	0.248462	0.230826	0.250549
1.0	0.304744	0.208219	0.285402	0.217306	0.266189	0.224168	0.247381	0.228967	0.229205	0.231897
1.1	0.294606	0.190036	0.277407	0.199046	0.260213	0.206108	0.243266	0.211343	0.226767	0.214902
1.2	0.284731	0.173896	0.269401	0.182742	0.253985	0.189878	0.238695	0.195398	0.223710	0.199416
1.3	0.275174	0.159531	0.261476	0.168151	0.247628	0.175271	0.233813	0.180957	0.220192	0.185299
1.4	0.265967	0.146712	0.253697	0.155066	0.241233	0.162100	0.228733	0.167863	0.216340	0.172423
1.5	0.257128	0.135242	0.246112	0.143305	0.234870	0.150205	0.223542	0.155975	0.212253	0.160668
1.6	0.248665	0.124954	0.238752	0.132711	0.228592	0.139441	0.218309	0.145167	0.208014	0.149927
1.7	0.240578	0.115702	0.231635	0.123147	0.222436	0.129684	0.213086	0.135326	0.203684	0.140103
1.8	0.232861	0.107361	0.224775	0.114495	0.216428	0.120822	0.207912	0.126353	0.199315	0.131106
1.9	0.225503	0.099824	0.218176	0.106650	0.210587	0.112760	0.202818	0.118158	0.194947	0.122858
2.0	0.218493	0.092998	0.211839	0.099523	0.204926	0.105411	0.197827	0.110662	0.190608	0.115286
2.1	0.211816	0.086801	0.205760	0.093035	0.199452	0.098700	0.192953	0.103795	0.186324	0.108125
2.2	0.205457	0.081162	0.199935	0.087116	0.194166	0.092562	0.188208	0.097495	0.182112	0.101919
2.3	0.199402	0.076021	0.194356	0.081706	0.189072	0.086936	0.183599	0.091706	0.177985	0.096015
2.4	0.193634	0.071324	0.189014	0.076753	0.184165	0.081773	0.179131	0.086378	0.173954	0.090567
2.5	0.188139	0.067024	0.183901	0.072208	0.179444	0.077024	0.174805	0.081467	0.170024	0.085532
2.6	0.182903	0.063080	0.179009	0.068031	0.174903	0.072651	0.170623	0.076933	0.166201	0.080873
2.7	0.177910	0.059456	0.174324	0.064186	0.170538	0.068617	0.166582	0.072742	0.162487	0.076557
2.8	0.173147	0.056118	0.169840	0.060639	0.166342	0.064890	0.162681	0.068863	0.158883	0.072553
2.9	0.168602	0.053041	0.165546	0.057363	0.162310	0.061440	0.158916	0.065266	0.155389	0.068834
3.0	0.164261	0.050197	0.161434	0.054331	0.158435	0.058243	0.155285	0.061926	0.152005	0.065375
y	$x = 1.5$		$x = 1.6$		$x = 1.7$		$x = 1.8$		$x = 1.9$	
0.0	0.105399	0.483227	0.077305	0.451284	0.055576	0.420388	0.039164	0.391291	0.027052	0.364437
0.1	0.134049	0.451763	0.105843	0.426168	0.083112	0.400743	0.065099	0.376214	0.051038	0.353066
0.2	0.156521	0.421076	0.128895	0.400837	0.105929	0.380161	0.087090	0.359721	0.071811	0.340004
0.3	0.173865	0.391665	0.147272	0.375911	0.124612	0.359313	0.105522	0.342479	0.089592	0.325873
0.4	0.186984	0.363828	0.161702	0.351803	0.139717	0.338676	0.120793	0.324985	0.104641	0.311161
0.5	0.196636	0.337720	0.172820	0.328777	0.151751	0.318584	0.133288	0.307609	0.117233	0.296240
0.6	0.203461	0.313397	0.181177	0.306990	0.161171	0.299261	0.143369	0.290613	0.127644	0.281392
0.7	0.207990	0.290847	0.187245	0.286517	0.168379	0.280846	0.151366	0.274180	0.136134	0.266823
0.8	0.210664	0.270016	0.191423	0.267378	0.173725	0.263418	0.157578	0.258431	0.142949	0.252681
0.9	0.211846	0.250823	0.194049	0.249556	0.177513	0.247012	0.162268	0.243439	0.148310	0.239067
1.0	0.211837	0.233171	0.195407	0.233009	0.180002	0.231630	0.165667	0.229244	0.152418	0.226046
1.1	0.210881	0.216954	0.195734	0.217678	0.181414	0.217253	0.167977	0.215857	0.155452	0.213656
1.2	0.209182	0.202067	0.195228	0.203494	0.181938	0.203847	0.169373	0.203272	0.157569	0.201914
1.3	0.206902	0.188403	0.194053	0.190384	0.181733	0.191366	0.170003	0.191471	0.158906	0.190821
1.4	0.204177	0.175862	0.192347	0.178275	0.180933	0.179762	0.169997	0.180425	0.159585	0.180367
1.5	0.201115	0.164349	0.190222	0.167092	0.179651	0.168980	0.169465	0.170099	0.159709	0.170534
1.6	0.197806	0.153773	0.187772	0.156765	0.177983	0.158969	0.168500	0.160457	0.159369	0.161300
1.7	0.194320	0.144054	0.185073	0.147226	0.176008	0.149674	0.167183	0.151458	0.158641	0.152637
1.8	0.190717	0.135113	0.182189	0.138412	0.173792	0.141045	0.165579	0.143063	0.157593	0.144516
1.9	0.187043	0.126883	0.179172	0.130262	0.171390	0.133033	0.163746	0.135234	0.156282	0.136908
2.0	0.183335	0.119298	0.176064	0.122723	0.168849	0.125590	0.161733	0.127931	0.154757	0.129781
2.1	0.179623	0.112302	0.172901	0.115744	0.166206	0.118674	0.159580	0.121118	0.153059	0.123108
2.2	0.175930	0.105842	0.169710	0.109277	0.163493	0.112243	0.157320	0.114761	0.151224	0.116858
2.3	0.172276	0.099870	0.166513	0.103280	0.160737	0.106260	0.154982	0.108827	0.149281	0.111003
2.4	0.168674	0.094343	0.163330	0.097713	0.157958	0.100689	0.152591	0.103285	0.147256	0.105519
2.5	0.165136	0.089222	0.160175	0.092541	0.155175	0.095499	0.150165	0.098107	0.145172	0.100378
2.6	0.161669	0.084472	0.157060	0.087732	0.152402	0.090660	0.147732	0.093265	0.143045	0.095558
2.7	0.158281	0.080061	0.153993	0.083254	0.149649	0.086143	0.145274	0.088735	0.140892	0.091037
2.8	0.154975	0.075960	0.150981	0.079082	0.146927	0.081925	0.142834	0.084493	0.138725	0.086794
2.9	0.151753	0.072142	0.148030	0.075191	0.144243	0.077982	0.140411	0.080519	0.136555	0.082809
3.0	0.148618	0.068585	0.145144	0.071558	0.141602	0.074293	0.138012	0.076794	0.134391	0.079065

See Examples 12–19.

$w(x) = e^{-x^2} + \frac{2i}{\sqrt{\pi}} e^{-x^2} \int_0^x e^{t^2}\,dt$

$w(-x+iy) = \overline{w(x+iy)} \qquad w(x-iy) = 2e^{y^2-x^2}(\cos 2xy - i\sin 2xy) - \overline{w(x+iy)}$

$w(iy) = e^{y^2}\operatorname{erfc} y \qquad w[(1+i)u] = e^{-2iu^2}\left\{1 + (i-1)\left[C\left(\frac{2u}{\sqrt{\pi}}\right) - iS\left(\frac{2u}{\sqrt{\pi}}\right)\right]\right\}$

ERROR FUNCTION FOR COMPLEX ARGUMENTS Table 7.9

$w(z) = e^{-z^2}\operatorname{erfc}(-iz) \qquad z = x+iy$

	$\mathscr{R}w(z)$	$\mathscr{I}w(z)$	$\mathscr{R}w(z)$	$\mathscr{I}w(z)$	$\mathscr{R}w(z)$	$\mathscr{I}w(z)$	$\mathscr{R}w(z)$	$\mathscr{I}w(z)$	$\mathscr{R}w(z)$	$\mathscr{I}w(z)$
y	x = 2.0		x = 2.1		x = 2.2		x = 2.3		x = 2.4	
0.0	0.018316	0.340026	0.012155	0.318073	0.007907	0.298468	0.005042	0.281026	0.003151	0.265522
0.1	0.040201	0.331583	0.031936	0.311886	0.025678	0.293982	0.020958	0.277795	0.017397	0.263201
0.2	0.059531	0.321332	0.049726	0.303894	0.041927	0.287771	0.035728	0.272968	0.030792	0.259435
0.3	0.076396	0.309831	0.065521	0.294574	0.056586	0.280232	0.049248	0.266865	0.043211	0.254478
0.4	0.090944	0.297529	0.079385	0.284327	0.069655	0.271710	0.061473	0.259775	0.054585	0.248566
0.5	0.103359	0.284786	0.091422	0.273482	0.081182	0.262499	0.072408	0.251953	0.064890	0.241914
0.6	0.113836	0.271881	0.101765	0.262308	0.091245	0.252844	0.082092	0.243617	0.074132	0.234714
0.7	0.122574	0.259031	0.110558	0.251016	0.099943	0.242947	0.090585	0.234952	0.082345	0.227129
0.8	0.129768	0.246396	0.117948	0.239772	0.107383	0.232968	0.097963	0.226111	0.089576	0.219302
0.9	0.135600	0.234096	0.124081	0.228703	0.113679	0.223037	0.104309	0.217219	0.095884	0.211349
1.0	0.140240	0.222213	0.129097	0.217904	0.118941	0.213253	0.109709	0.208376	0.101336	0.203368
1.1	0.143840	0.210805	0.133125	0.207442	0.123277	0.203692	0.114251	0.199660	0.105999	0.195438
1.2	0.146541	0.199904	0.136286	0.197366	0.126788	0.194410	0.118019	0.191133	0.109942	0.187620
1.3	0.148466	0.189529	0.138689	0.187705	0.129570	0.185446	0.121092	0.182840	0.113232	0.179965
1.4	0.149725	0.179687	0.140432	0.178478	0.131709	0.176827	0.123548	0.174814	0.115935	0.172510
1.5	0.150415	0.170371	0.141604	0.169691	0.133284	0.168569	0.125454	0.167078	0.118109	0.165281
1.6	0.150622	0.161572	0.142283	0.161343	0.134367	0.160680	0.126877	0.159646	0.119812	0.158299
1.7	0.150418	0.153274	0.142540	0.153429	0.135021	0.153161	0.127873	0.152526	0.121096	0.151576
1.8	0.149870	0.145457	0.142434	0.145938	0.135305	0.146009	0.128495	0.145721	0.122010	0.145120
1.9	0.149032	0.138100	0.142021	0.138855	0.135269	0.139217	0.128792	0.139229	0.122597	0.138933
2.0	0.147953	0.131180	0.141347	0.132164	0.134959	0.132773	0.128805	0.133045	0.122897	0.133015
2.1	0.146675	0.124674	0.140453	0.125849	0.134414	0.126667	0.128574	0.127161	0.122945	0.127363
2.2	0.145234	0.118558	0.139375	0.119891	0.133669	0.120885	0.128130	0.121569	0.122773	0.121972
2.3	0.143660	0.112810	0.138145	0.114272	0.132755	0.115413	0.127506	0.116258	0.122411	0.116834
2.4	0.141982	0.107408	0.136789	0.108973	0.131699	0.110236	0.126726	0.111218	0.121884	0.111942
2.5	0.140220	0.102329	0.135331	0.103977	0.130524	0.105339	0.125814	0.106436	0.121215	0.107286
2.6	0.138395	0.097554	0.133791	0.099265	0.129252	0.100709	0.124792	0.101901	0.120424	0.102858
2.7	0.136523	0.093062	0.132187	0.094822	0.127900	0.096330	0.123676	0.097601	0.119530	0.098648
2.8	0.134619	0.088837	0.130533	0.090631	0.126483	0.092189	0.122484	0.093523	0.118548	0.094646
2.9	0.132693	0.084859	0.128842	0.086677	0.125016	0.088273	0.121229	0.089658	0.117492	0.090842
3.0	0.130757	0.081113	0.127125	0.082944	0.123510	0.084568	0.119922	0.085992	0.116375	0.087227
y	x = 2.5		x = 2.6		x = 2.7		x = 2.8		x = 2.9	
0.0	0.001930	0.251723	0.001159	0.239403	0.000682	0.228355	0.000394	0.218399	0.000223	0.209377
0.1	0.014698	0.250050	0.012635	0.238187	0.011037	0.227458	0.009778	0.217722	0.008769	0.208854
0.2	0.026841	0.247092	0.023653	0.235838	0.021057	0.225569	0.018918	0.216181	0.017134	0.207577
0.3	0.038226	0.243042	0.034087	0.232504	0.030626	0.222800	0.027707	0.213858	0.025225	0.205607
0.4	0.048773	0.238092	0.043849	0.228337	0.039656	0.219268	0.036064	0.210843	0.032967	0.203014
0.5	0.058437	0.232420	0.052885	0.223482	0.048090	0.215093	0.043930	0.207232	0.040304	0.199873
0.6	0.067205	0.226190	0.061167	0.218077	0.055890	0.210387	0.051264	0.203119	0.047194	0.196262
0.7	0.075088	0.219546	0.068691	0.212247	0.063043	0.205258	0.058046	0.198594	0.053611	0.192256
0.8	0.082112	0.212614	0.075467	0.206103	0.069548	0.199804	0.064266	0.193741	0.059543	0.187927
0.9	0.088317	0.205504	0.081521	0.199744	0.075416	0.194111	0.069927	0.188638	0.064986	0.183344
1.0	0.093751	0.198307	0.086885	0.193255	0.080670	0.188258	0.075043	0.183354	0.069944	0.178568
1.1	0.098466	0.191099	0.091598	0.186707	0.085338	0.182311	0.079632	0.177950	0.074431	0.173654
1.2	0.102518	0.183943	0.095702	0.180163	0.089451	0.176328	0.083718	0.172480	0.078462	0.168651
1.3	0.105960	0.176889	0.099243	0.173670	0.093044	0.170357	0.087328	0.166990	0.082059	0.163603
1.4	0.108848	0.169977	0.102264	0.167270	0.096155	0.164438	0.090492	0.161519	0.085245	0.158547
1.5	0.111233	0.163237	0.104811	0.160996	0.098820	0.158604	0.093239	0.156099	0.088044	0.153515
1.6	0.113165	0.156692	0.106925	0.154872	0.101076	0.152882	0.095601	0.150758	0.090482	0.148534
1.7	0.114690	0.150359	0.108647	0.148918	0.102957	0.147292	0.097608	0.145518	0.092584	0.143625
1.8	0.115851	0.144249	0.110016	0.143147	0.104498	0.141851	0.099288	0.140395	0.094376	0.138807
1.9	0.116689	0.138368	0.111067	0.137569	0.105730	0.136571	0.100671	0.135403	0.095882	0.134094
2.0	0.117239	0.132720	0.111834	0.132191	0.106683	0.131459	0.101783	0.130553	0.097127	0.129498
2.1	0.117534	0.127305	0.112347	0.127015	0.107386	0.126522	0.102649	0.125851	0.098133	0.125027
2.2	0.117606	0.122121	0.112635	0.122042	0.107864	0.121762	0.103293	0.121303	0.098922	0.120688
2.3	0.117481	0.117164	0.112723	0.117271	0.108140	0.117180	0.103737	0.116911	0.099513	0.116484
2.4	0.117184	0.112428	0.112633	0.112699	0.108238	0.112775	0.104002	0.112676	0.099925	0.112419
2.5	0.116737	0.107909	0.112389	0.108322	0.108177	0.108546	0.104105	0.108597	0.100177	0.108493
2.6	0.116160	0.103597	0.112008	0.104136	0.107975	0.104489	0.104066	0.104674	0.100284	0.104707
2.7	0.115471	0.099487	0.111508	0.100133	0.107648	0.100601	0.103898	0.100905	0.100261	0.101058
2.8	0.114685	0.095570	0.110904	0.096309	0.107213	0.096876	0.103617	0.097284	0.100122	0.097546
2.9	0.113816	0.091838	0.110210	0.092657	0.106682	0.093310	0.103236	0.093810	0.099879	0.094168
3.0	0.112878	0.088283	0.109439	0.089170	0.106067	0.089898	0.102767	0.090479	0.099544	0.090921

See Examples 12–19.

$w(x) = e^{-x^2} + \frac{2i}{\sqrt{\pi}} e^{-x^2}\int_0^x e^{t^2}\,dt$

$w(-x+iy) = \overline{w(x+iy)} \qquad w(x-iy) = 2e^{y^2-x^2}(\cos 2xy + i\sin 2xy) - \overline{w(x+iy)}$

$w(iy) = e^{y^2}\operatorname{erfc} y \qquad w[(1+i)u] = e^{-2iu^2}\left\{1 + (i-1)\left[C\left(\frac{2u}{\sqrt{\pi}}\right) + iS\left(\frac{2u}{\sqrt{\pi}}\right)\right]\right\}$

Table 7.9 ERROR FUNCTION FOR COMPLEX ARGUMENTS

$$w(z) = e^{-z^2}\operatorname{erfc}(-iz) \qquad z = x+iy$$

y	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$	$\Re w(z)$	$\Im w(z)$
	x = 3.0		x = 3.1		x = 3.2		x = 3.3		x = 3.4	
0.0	0.000123	0.201157	0.000067	0.193630	0.000036	0.186704	0.000019	0.180302	0.000010	0.174362
0.1	0.007943	0.200742	0.007254	0.193292	0.006670	0.186421	0.006167	0.180061	0.005728	0.174152
0.2	0.015627	0.199669	0.014338	0.192376	0.013225	0.185630	0.012252	0.179369	0.011394	0.173542
0.3	0.023095	0.197980	0.021250	0.190915	0.019639	0.184354	0.018222	0.178245	0.016966	0.172545
0.4	0.030279	0.195732	0.027929	0.188951	0.025862	0.182626	0.024032	0.176715	0.022403	0.171181
0.5	0.037126	0.192984	0.034328	0.186532	0.031849	0.180484	0.029643	0.174806	0.027670	0.169475
0.6	0.043598	0.189798	0.040407	0.183709	0.037565	0.177970	0.035022	0.172560	0.032738	0.167455
0.7	0.049665	0.186239	0.046141	0.180534	0.042983	0.175128	0.040144	0.170006	0.037582	0.165151
0.8	0.055311	0.182368	0.051509	0.177061	0.048083	0.172003	0.044989	0.167184	0.042185	0.162596
0.9	0.060529	0.178243	0.056501	0.173340	0.052854	0.168637	0.049544	0.164132	0.046532	0.159821
1.0	0.065318	0.173918	0.061114	0.169418	0.057289	0.165072	0.053801	0.160886	0.050615	0.156858
1.1	0.069685	0.169445	0.065350	0.165339	0.061387	0.161349	0.057757	0.157480	0.054428	0.153738
1.2	0.073641	0.164866	0.069216	0.161145	0.065151	0.157502	0.061413	0.153948	0.057971	0.150490
1.3	0.077202	0.160223	0.072722	0.156872	0.068589	0.153567	0.064773	0.150320	0.061245	0.147141
1.4	0.080385	0.155551	0.075883	0.152553	0.071711	0.149572	0.067844	0.146623	0.064258	0.143717
1.5	0.083210	0.150880	0.078712	0.148217	0.074529	0.145545	0.070636	0.142882	0.067012	0.140239
1.6	0.085697	0.146236	0.081229	0.143888	0.077055	0.141510	0.073158	0.139120	0.069518	0.136731
1.7	0.087870	0.141640	0.083450	0.139588	0.079306	0.137488	0.075423	0.135357	0.071785	0.133209
1.8	0.089749	0.137113	0.085394	0.135335	0.081297	0.133495	0.077445	0.131609	0.073823	0.129691
1.9	0.091355	0.132667	0.087080	0.131146	0.083044	0.129548	0.079236	0.127892	0.075646	0.126192
2.0	0.092711	0.128317	0.088525	0.127031	0.084562	0.125660	0.080811	0.124219	0.077263	0.122723
2.1	0.093835	0.124071	0.089749	0.123003	0.085867	0.121840	0.082182	0.120600	0.078687	0.119296
2.2	0.094748	0.119936	0.090767	0.119068	0.086974	0.118099	0.083364	0.117045	0.079930	0.115919
2.3	0.095467	0.115919	0.091597	0.115233	0.087900	0.114442	0.084370	0.113560	0.081004	0.112602
2.4	0.096010	0.112023	0.092255	0.111503	0.088657	0.110875	0.085213	0.110153	0.081921	0.109349
2.5	0.096393	0.108249	0.092754	0.107881	0.089259	0.107403	0.085905	0.106827	0.082690	0.106166
2.6	0.096632	0.104600	0.093110	0.104370	0.089719	0.104027	0.086458	0.103585	0.083324	0.103057
2.7	0.096739	0.101076	0.093336	0.100969	0.090050	0.100751	0.086883	0.100433	0.083832	0.100026
2.8	0.096729	0.097674	0.093442	0.097680	0.090263	0.097575	0.087190	0.097369	0.084225	0.097073
2.9	0.096613	0.094395	0.093442	0.094502	0.090368	0.094499	0.087391	0.094396	0.084511	0.094202
3.0	0.096402	0.091236	0.093345	0.091434	0.090375	0.091523	0.087493	0.091513	0.084700	0.091413
y	x = 3.5		x = 3.6		x = 3.7		x = 3.8		x = 3.9	
0.0	0.000005	0.168830	0.000002	0.163662	0.000001	0.158821	0.000001	0.154273	0.000000	0.149992
0.1	0.005340	0.168645	0.004995	0.163498	0.004685	0.158673	0.004406	0.154140	0.004153	0.149871
0.2	0.010633	0.168102	0.009952	0.163011	0.009339	0.158235	0.008786	0.153743	0.008282	0.149510
0.3	0.015846	0.167212	0.014841	0.162211	0.013935	0.157513	0.013115	0.153088	0.012368	0.148913
0.4	0.020944	0.165990	0.019632	0.161111	0.018446	0.156516	0.017370	0.152183	0.016389	0.148088
0.5	0.025897	0.164456	0.024297	0.159725	0.022847	0.155260	0.021529	0.151040	0.020326	0.147044
0.6	0.030677	0.162633	0.028812	0.158075	0.027118	0.153760	0.025574	0.149672	0.024162	0.145792
0.7	0.035263	0.160548	0.033158	0.156181	0.031239	0.152034	0.029486	0.148094	0.027880	0.144346
0.8	0.039637	0.158227	0.037316	0.154066	0.035195	0.150102	0.033253	0.146324	0.031469	0.142721
0.9	0.043785	0.155698	0.041274	0.151755	0.038974	0.147985	0.036861	0.144380	0.034916	0.140931
1.0	0.047698	0.152988	0.045023	0.149271	0.042565	0.145703	0.040301	0.142279	0.038212	0.138993
1.1	0.051370	0.150124	0.048556	0.146637	0.045962	0.143277	0.043567	0.140039	0.041352	0.136922
1.2	0.054798	0.147132	0.051869	0.143878	0.049161	0.140727	0.046653	0.137680	0.044328	0.134735
1.3	0.057984	0.144038	0.054962	0.141014	0.052159	0.138074	0.049558	0.135218	0.047139	0.132448
1.4	0.060928	0.140862	0.057835	0.138067	0.054958	0.135336	0.052279	0.132671	0.049783	0.130076
1.5	0.063637	0.137628	0.060491	0.135056	0.057557	0.132530	0.054819	0.130054	0.052260	0.127633
1.6	0.066116	0.134354	0.062936	0.131999	0.059962	0.129674	0.057179	0.127384	0.054572	0.125133
1.7	0.068374	0.131058	0.065176	0.128913	0.062177	0.126782	0.059362	0.124673	0.056720	0.122591
1.8	0.070419	0.127755	0.067217	0.125812	0.064206	0.123869	0.061374	0.121935	0.058708	0.120016
1.9	0.072260	0.124460	0.069068	0.122709	0.066058	0.120947	0.063219	0.119182	0.060540	0.117422
2.0	0.073908	0.121185	0.070736	0.119617	0.067738	0.118027	0.064903	0.116425	0.062222	0.114817
2.1	0.075373	0.117940	0.072232	0.116545	0.069254	0.115120	0.066433	0.113673	0.063759	0.112212
2.2	0.076666	0.114735	0.073563	0.113503	0.070615	0.112234	0.067815	0.110935	0.065156	0.109614
2.3	0.077796	0.111578	0.074739	0.110500	0.071829	0.109377	0.069058	0.108218	0.066420	0.107031
2.4	0.078774	0.108474	0.075770	0.107540	0.072902	0.106556	0.070166	0.105530	0.067556	0.104469
2.5	0.079611	0.105431	0.076664	0.104631	0.073845	0.103777	0.071149	0.102875	0.068572	0.101935
2.6	0.080316	0.102451	0.077430	0.101777	0.074663	0.101044	0.072013	0.100260	0.069474	0.099433
2.7	0.080898	0.099538	0.078076	0.098981	0.075366	0.098362	0.072764	0.097688	0.070267	0.096968
2.8	0.081366	0.096696	0.078612	0.096247	0.075961	0.095734	0.073411	0.095163	0.070959	0.094543
2.9	0.081730	0.093927	0.079044	0.093577	0.076455	0.093162	0.073959	0.092688	0.071555	0.092162
3.0	0.081996	0.091230	0.079381	0.090973	0.076855	0.090649	0.074415	0.090265	0.072061	0.089826

If $x > 3.9$ or $y > 3$ $w(z) = iz\left(\dfrac{0.4613135}{z^2-0.1901635} + \dfrac{0.09999216}{z^2-1.7844927} + \dfrac{0.002883894}{z^2-5.5253437}\right) + \epsilon(z)$ $\epsilon(z) < 2 \times 10^{-6}$

If $x > 6$ or $y > 6$ $w(z) = iz\left(\dfrac{0.5124242}{z^2-0.2752551} + \dfrac{0.05176536}{z^2-2.724745}\right) + \eta(z)$ $\eta(z) < 10^{-6}$

COMPLEX ZEROS OF THE ERROR FUNCTION

Table 7.10

$\operatorname{erf} z_n = 0 \qquad z_n = x_n + iy_n$

n	x_n	y_n	n	x_n	y_n
1	1.45061 616	1.88094 300	6	4.15899 840	4.43557 144
2	2.24465 928	2.61657 514	7	4.51631 940	4.78044 764
3	2.83974 105	3.17562 810	8	4.84797 031	5.10158 804
4	3.33546 074	3.64617 438	9	5.15876 791	5.40333 264
5	3.76900 557	4.06069 723	10	5.45219 220	5.68883 744

$$\operatorname{erf} z_n = \operatorname{erf}(-z_n) = \operatorname{erf} \bar{z}_n = \operatorname{erf}(-\bar{z}_n) = 0$$

$$\begin{matrix} x_n \\ y_n \end{matrix} \sim \frac{1}{2}\sqrt{\pi\left(4n-\frac{1}{2}\right)} \pm \frac{\ln\left(\pi\sqrt{2n-\frac{1}{4}}\right)}{2\sqrt{\pi\left(4n-\frac{1}{2}\right)}} \qquad (n \to 0)$$

From H. E. Salzer, Complex zeros of the error function, J. Franklin Inst. **260**, 209–211, 1955 (with permission).

COMPLEX ZEROS OF FRESNEL INTEGRALS

Table 7.11

$C(z_n) = 0 \qquad z_n = x_n + iy_n$

$S(z_n^*) = 0 \qquad z_n^* = x_n^* + iy_n^*$

n	x_n	y_n	x_n^*	y_n^*
0	0.0000	0.0000	0.0000	0.0000
1	1.7437	0.3057	2.0093	0.2886
2	2.6515	0.2529	2.8335	0.2443
3	3.3208	0.2239	3.4675	0.2185
4	3.8759	0.2047	4.0026	0.2008
5	4.3611	0.1909	4.4742	0.1877

$$C(z_n) = C(-z_n) = C(\bar{z}_n) = C(-\bar{z}_n) = C(iz_n) = C(-iz_n) = C(i\bar{z}_n) = C(-i\bar{z}_n) = 0$$

$$x_n \sim \sqrt{4n-1} - \frac{\ln(\pi\sqrt{4n-1})}{\pi^2(4n-1)^{3/2}} \qquad y_n \sim \frac{\ln(\pi\sqrt{4n-1})}{\pi\sqrt{4n-1}} \qquad (n>0)$$

$$x_n^* \sim 2\sqrt{n} - \frac{\ln(2\pi\sqrt{n})}{8\pi^2 n^{3/2}} \qquad y_n^* \sim \frac{\ln(2\pi\sqrt{n})}{2\pi\sqrt{n}}$$

MAXIMA AND MINIMA OF FRESNEL INTEGRALS

Table 7.12

$M_n = C(\sqrt{4n+1}) \qquad m_n = C(\sqrt{4n+3}) \qquad M_n^* = S(\sqrt{4n+2}) \qquad m_n^* = S(\sqrt{4n+4})$

n	M_n	m_n	M_n^*	m_n^*
0	0.779893	0.321056	0.713972	0.343415
1	0.640807	0.380389	0.628940	0.387969
2	0.605721	0.404260	0.600361	0.408301
3	0.588128	0.417922	0.584942	0.420516
4	0.577121	0.427036	0.574957	0.428877
5	0.569413	0.433666	0.567822	0.435059

$$M_n \sim \frac{1}{2} + \frac{\pi^2(4n+1)^2-3}{\pi^3(4n+1)^{5/2}} \qquad m_n \sim \frac{1}{2} - \frac{\pi^2(4n+3)^2-3}{\pi^3(4n+3)^{5/2}} \qquad (n \to \infty)$$

$$M_n^* \sim \frac{1}{2} + \frac{\pi^2(4n+2)^2-3}{\pi^3(4n+2)^{5/2}} \qquad m_n^* \sim \frac{1}{2} - \frac{16\pi^2(n+1)^2-3}{32\pi^3(n+1)^{5/2}}$$

From G. N. Watson, A treatise on the theory of Bessel functions, 2d ed. Cambridge Univ. Press, Cambridge, England, 1958 (with permission).