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OHIO STATE UNIV COLUMBUS DEPT OF MATHEMATICS
CHARACTERIZATION OF PROJECTIVE INCIDENCE STRUCTURES, (U)
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6 Characterization of Projective Incidence Structures,

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1. Introduction and statement of theorems

For a finite set X , $|X|$ will denote the number of elements of X . An incidence structure is an ordered triple (P, L, I) where P and L are disjoint sets and $I \subseteq P \times L$. Elements of P will be called points or vertices and elements of L lines. A line ℓ and a point p are called incident iff $(p, \ell) \in I$. We also say in this case that ℓ contains p or p lies on ℓ . Two lines ℓ and m are said to intersect iff they have a common incident point. With any incidence structure (P, L, I) is associated its dual incidence structure (L, P, I^*) where $I^* = \{(\ell, p) : (p, \ell) \in I\}$. If L is a set of subsets of P and $(p, \ell) \in I$ iff $p \in \ell$, we will refer to (P, L, I) as $(P, L, \in)$ or (P, L) . The dual of $(P, L, \in)$ will be written as $(L, P, \supseteq)$. If each element of L and P is a set and $(p, \ell) \in I$ iff $p \subseteq \ell$, we write (P, L, I) as $(P, L, \subseteq)$ and its dual as $(L, P, \supseteq)$. For a line ℓ , P_ℓ will denote the set of points incident with line ℓ . If P_ℓ is a finite set, we write $k(\ell)$ for the cardinality of P_ℓ . Similarly, for a point p , L_p denotes the set of lines ℓ incident with the point p and we write $r(p)$ for $|L_p|$. An incidence structure is said to be simple iff for any two distinct lines ℓ and ℓ' , $P_\ell \neq P_{\ell'}$. Incidence structures (P, L, I) and (P', L', I') will be called isomorphic iff there exist bijections $\sigma: P \rightarrow P'$ and $\tau: L \rightarrow L'$ such that $(p, \ell) \in I$ iff $(\sigma(p), \tau(\ell)) \in I'$.

An incidence structure $\pi = (P, L, I)$ is said to be finite iff

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both P and L are finite sets. All incidence structures in this paper are finite. For a finite incidence structure, we will set

$r(\pi) = \min\{r(p): p \in P\}$ and $k(\pi) = \min\{k(l): l \in L\}$. Let q be a

positive integer. If $q = 1$, we define $s(\pi, q)$ to be equal to

$k(\pi)$. If $q \geq 2$, we define $s(\pi, q)$ to be the unique real number

s which satisfies $q^s - 1 = k(\pi)(q - 1)$. If $q = 1$, we define

$d(\pi, q)$ to be equal to $r(\pi) + s(\pi, q) - 1$. If $q \geq 2$, we define

$d(\pi, q)$ to be the unique real number d which satisfies $q^{d-s(\pi, q) + 1} - 1$

$= (q - 1)r(\pi)$. We normally write $s(\pi, q)$ as $s(\pi)$ and $d(\pi, q)$ as $d(\pi)$.

The incidence structure π is said to be semilinear iff $\forall p, p' \in P$,

$p \neq p'$, $\exists$ at most one line l incident with both p and p' . Let r and

k be positive integers. A semilinear incidence structure π is said to be an

(r, k) incidence structure iff for every point p , $r(p) = r$ and every line l ,

$k(l) = k$. Let π be a semilinear incidence structure and l and

m be two lines. A line n will be called a transversal of l and

m iff n intersects both l and m and $P_n \cap P_l \neq P_n \cap P_m$. A

semilinear incidence structure π is said to satisfy Pasch's axiom

iff for any pair of intersecting lines m_1 and m_2 and any pair

of transversals n_1 and n_2 of m_1 and m_2 , n_1 intersects n_2 .

A subset $F \subseteq P$ is called a flat iff $\forall l \in L$, $|P_l \cap F| \geq 2$ implies

$P_l \subseteq F$. Clearly, any intersection of flats is a flat. For $S \subseteq P$,

the flat $\langle S \rangle = \bigcap_{F \supseteq S} F$ is said to be the flat generated by S . For

any flat F , rank F is the smallest integer n such that there

exists a set $S \subseteq P$, $|S| = n$ and $\langle S \rangle = F$. The rank of the flat

P is called rank π .

A simple graph is a simple incidence structure in which every line is incident with exactly two points. Points and lines of a simple graph will usually be called vertices and edges, respectively. Two vertices p and p' will be called adjacent iff there exists an edge l incident with p and p' . Adjacency is a symmetric relation on the set of vertices of a graph and determines a simple graph completely. All graphs considered in this paper will be finite and simple. Let G be a simple graph with vertex set V and edge set E . Let n be a nonnegative integer. A path of length n from u to v is a sequence $(u = v_0, l_1, v_1, l_2, v_2, \dots, l_n, v_n = v)$ where l_i is an edge incident with v_{i-1} and v_i , $i = 1, 2, \dots, n$. If the vertices $v_0, v_1, \dots, v_n$ are all distinct, then the path is said to be simple. If for any two vertices u and v there exists a path from u to v , then the graph G is said to be connected. In a connected graph G the distance $d(u, v)$ between two vertices u and v is the smallest nonnegative integer n such that a path of length n from u to v in G exists.

Let $\pi = (P, L, I)$ be an incidence structure. The adjacency graph $G(\pi)$ of π is a graph having vertex set P and two vertices adjacent iff some line of π contains both. The graph $G(\pi^*)$ of the dual incidence structure π^* will be called the line graph of π .

Distance between two points p and p' of π will be same as the distance between them in $G(\pi)$. For $S \subseteq P$ and $l, m \in L$, we will set $d(l, S) = \min \{d(p, p') : p' \in S, p \text{ incident with } l\}$ and $d(l, m) = \min \{d(l, p) : p \text{ incident with } m\}$ where $d(p, p')$ is the distance between the vertices p and p' in $G(\pi)$. Sometimes the points of π will be called vertices.

Let $q \geq 2$ be a prime power and $1 \leq s \leq d$ be integers. Let V be a d -dimensional vector space over a finite field of order q . Let W_i be the set of i -dimensional subspaces of V , $1 \leq i \leq d$. Let $(W_{s-1}, W_s, \subseteq)$ be the incidence structure whose points are $(s-1)$ -dimensional subspaces, lines are s -dimensional subspaces and incidence is set inclusion. Any incidence structure π isomorphic to $(W_{s-1}, W_s, \subseteq)$ will be called an (s, q, d) projective incidence structure (p.i.s.). For $q = 1$, also we define an $(s, 1, d)$ -projective incidence structure. Let Y be a finite set with $|Y| = d$. A subset $Y' \subseteq Y$ is called an i -subset of Y iff $|Y'| = i$. Let Z_i be the set of i -subsets of Y . Any incidence structure isomorphic to $(Z_{s-1}, Z_s, \subseteq)$ will be called an $(s, 1, d)$ -projective incidence structure. The incidence structure $(W_{d-s+1}, W_{d-s}, \supseteq)$ is dual to $(W_{s-1}, W_s, \subseteq)$. Also, $(Z_{d-s+1}, Z_{d-s}, \supseteq)$ is dual to $(Z_{s-1}, Z_s, \subseteq)$.

The following classical theorem about finite projective spaces characterizes $(2, q, d)$ -projective incidence structures for $d \geq 4$.

Theorem Let π be a finite incidence structure satisfying

- (p1) There exists exactly one line joining two distinct points.
- (p2) Every line contains at least three points.
- (p3) Pasch's axiom.
- (p4) Rank of $\pi \geq 4$.

Then there exists a prime power $q \geq 2$ and an integer $d \geq 4$ such that π is a $(2, q, d)$ -projective incidence structure. Conversely, any $(2, q, d)$ -projective incidence structure with $d \geq 4$, $q \geq 2$ satisfies (p1) - (p4).

Extending this classical theorem, we prove a characterization

of (s,q,d) -projective incidence structures when $3 \leq s < d-1$.

Theorem 1. Let $q \geq 1$ be an integer and π be a finite incidence structure satisfying

- (f1) $3 \leq s(\pi, q) < d(\pi, q) - 1$.
- (f2) There exists at most one line joining two distinct points.
- (f3) If p is a point and ℓ is a line such that $d(p, \ell) = 1$, then there are exactly $(q + 1)$ lines which pass through p and intersect ℓ .
- (f4) If p and p' are two distinct points such that $d(p, p') = 2$, then there are exactly $(q + 1)$ lines ℓ such that ℓ passes through p' and $d(p, \ell) = 1$.
- (f5) $G(\pi)$ is connected.

Then $s = s(\pi, q)$ and $d = d(\pi, q)$ are integers, $q = 1$ or a prime power and π is an (s, q, d) -projective incidence structure. Conversely, for $3 \leq s < d - 1$, any (s, q, d) -projective incidence structure satisfies (f1) - (f5).

We also show that the axioms (f1) - (f5) are minimal for the purpose of characterizing (s, q, d) -p.i.s., $3 \leq s < d - 1$. For any choice of $j \in \{1, 2, 3, 4, 5\}$, there exists incidence structures π' which satisfy the four axioms other than (f_j) and is not an (s, q, d) -p.i.s. with $3 \leq s < d - 1$. A finite incidence structure π satisfying (f2) - (f5) is called an (s, q, d) -pseudo projective incidence structure where $s(\pi, q) = s$ and $d(\pi, q) = d$. The axiom (f5) in the statement of Theorem 1 is not an essential axiom. Let $\pi_i = (P_i, L_i, I_i)$, $i = 1, 2$ be two incidence structures such that $P_1 \cap P_2 = L_1 \cap L_2 = \emptyset$. We define the direct sum $\pi = \pi_1 + \pi_2$ by

$\pi = (P_1 \cup P_2, L_1 \cup L_2, I)$ where $(p, \ell) \in I$ iff $\exists i, 1 \leq i \leq 2$, $p \in P_i$, $\ell \in L_i$, and $(p, \ell) \in I_i$.

Theorem 2. Let $q \geq 1$ be an integer and π be a finite incidence structure satisfying the axioms (f1) - (f4). Then $q = 1$ or a prime power and π is isomorphic to the direct sum of one or more projective incidence structures. Conversely, if $q = 1$ or a prime power and $3 \leq s < d - 1$ and π is the direct sum of several (s_i, q, d_i) -p.i.s. where $3 \leq s_i < d_i - 1$, then π satisfies axioms (f1) - (f4).

Outline of the Proof. Let π be an (s, q, d) -pseudo projective incidence structure. Let m and n be two lines containing a common point O . A line ℓ is said to be a transversal of m and n iff ℓ intersects both m and n and does not contain O . Let $C(m, n)$ be the set of lines containing the transversals of m and n and all lines ℓ which contain O and intersect at least one transversal of m and n . $C(m, n)$ is called the plane generated by m and n . Let $\mathcal{C}$ be the set of all planes. One of the important steps in the proof is to show that the incidence structure $(L, \mathcal{C}, \epsilon)$ is an $(s+1, q, d)$ -pseudo projective incidence structure. One starts with an (s, q, d) -pseudo p.i.s. and finally obtains an $(d-1, q, d)$ -pseudo p.i.s. which is then shown to be the dual of a projective space.

2. Preliminary propositions

Lemma 1. Let $q \geq 1$ be an integer, π be a finite incidence structure such that $r(p)$ and $k(l)$ are positive for all points p and line l .

Let π satisfy the axioms (f2), (f3) and (f5) and $r = r(\pi)$, $k = k(\pi)$.

Then π is an (r, k) -incidence structure.

Proof: Let $\pi = (P, L, I)$. To show that $\forall l \in L, k(l) = k$, it is sufficient to show that $\forall l' \in L, k(l) = k(l')$. Let l and l' be two intersecting lines and z be the common point. We calculate

$$b = |\{(p, p') : (p, l) \in I, (p', l') \in I, p, p' \neq z, d(p, p') = 1\}|$$

For every point $p \neq z$ of l , $d(p, l') = 1$. So there are q points p' of l' such that $d(p, p') = 1$ and $p' \neq z$. Hence, $b = (k(l)-1)(q)$. By symmetry $b = (k(l')-1)(q)$. Since $q \geq 1$, $k(l) = k(l')$. Let l and l' be any two lines. Since $G(\pi)$ is connected, we can find a sequence $l_0 = l, l_1, l_2, \dots, l_i = l'$ such that l_{j-1} and l_j intersect for $j = 1, 2, \dots, i$. Since $k(l_{j-1}) = k(l_j)$ for $j = 1, 2, \dots, i$, it follows that $k(l') = k(l)$. It is easily checked that the dual incidence structure π^* satisfies (f2), (f3) and (f5). Therefore, we get $r(p) = r(p'), \forall p, p' \in P$ and hence, $r(p) = r, \forall p \in P$.

Lemma 2. Let $q = 1$ or a prime power and $3 \leq s < d-1$ be integers. Then any (s, q, d) -projective incidence structure is an (s, q, d) -pseudo projective incidence structure.

Proof: First we consider the case q a prime power, $q \geq 2$. Let $\pi = (W_{s-1}, W_s, \subset)$ be an (s, q, d) -projective incidence structure

where $3 \leq s \leq d-2$ and W_i is the set of i -dimensional subspaces of a vector space V of dimension d over $GF(q)$, $0 \leq i \leq d$.

The number of $(s-1)$ -dimensional subspaces contained in an s -dimensional subspace is $\frac{q^s-1}{q-1}$ and hence, $k(\pi) = \frac{q^s-1}{q-1}$ and $s(\pi) = s$.

Similarly, the number of s -dimensional subspaces containing a given

$(s-1)$ -dimensional subspace is $\frac{q^{d-s+1}-1}{q-1}$. Therefore, $r(\pi) = \frac{q^{d-s+1}-1}{q-1}$

and $d(\pi) = d$. The axiom (f1) holds since $3 \leq s \leq d-2$. Let p and p' be two $(s-1)$ -dimensional subspaces and ℓ be an s -dimensional subspace such that $p, p' \subseteq \ell$. Then ℓ is the subspace spanned by p and p' . Hence, there exists at most one line joining p and p'

and π is semilinear. Let p and p' be two $(s-1)$ -dimensional

subspaces such that $\{u_1, u_2, \dots, u_i, v_1, \dots, v_{s-i}\}$ and

$\{u_1, u_2, \dots, u_i, w_1, w_2, \dots, w_{s-i}\}$ are respectively bases of p

and p' , $0 \leq i \leq s-1$. Let p_j be the subspace spanned by

$\{u_1, u_2, \dots, u_i, w_1, w_2, \dots, w_j, v_{j+1}, \dots, v_{s-i}\}$, $j=0, 1, \dots, s-i$.

Then $p_0 = p$ and $p_{s-i} = p'$ and p_j and p_{j+1} are adjacent in $G(\pi)$.

Hence, there exists a path joining p and p' in $G(\pi)$. This

establishes that $G(\pi)$ is connected. Let $p \in W_{s-1}$ and $\ell \in W_s$ such

that $d(p, \ell) = 1$. Then $p \not\subseteq \ell$ and there exists an $\ell' \in W_s$ such

that $p \subseteq \ell'$ and $\ell \cap \ell' \in W_{s-1}$. It follows that $p \cap \ell = u$ is an

$(s-2)$ -dimensional subspace. There are $(q+1)$ $(s-1)$ -dimensional

subspaces p_i , $1 \leq i \leq q+1$ such that $u \subseteq p_i \subseteq \ell$. Let $\ell_i = \langle p, p_i \rangle$.

Then ℓ_i , $1 \leq i \leq q+1$ are the only lines of π which contain p and

intersect ℓ in a point. It follows that π satisfies (f3). Let

$p, p' \in W_{s-1}$ such that $d(p, p') = 2$. This implies that $p \cap p' = v \in W_{s-3}$.

Let $u_1, u_2, \dots, u_{q+1}$ be the $(s-2)$ -dimensional subspaces such that $v \subseteq u_i \subseteq p$, $1 \leq i \leq q+1$. Let $\ell_i = \langle u_i, p' \rangle$, $1 \leq i \leq q+1$. Then $\ell_1, \ell_2, \dots, \ell_{q+1}$ are the only lines of π which pass through p' and have distance 1 from p . Therefore, π satisfies (f4). This establishes the lemma when $q \geq 2$. For $q = 1$, we take $\pi = (Z_{s-1}, Z_s, \subseteq)$ where Z_i is the set of i -element subsets of a d -set Y , $0 \leq i \leq d$. It is easily checked that π satisfies the axioms (f1) - (f5).

In the sequel we will assume without loss of generality (wlog) that lines are subsets of points. We assume that q is a fixed positive integer and s and d real numbers satisfying $3 \leq s < d-1$ and π is a pseudo projective incidence structure and $s(\pi) = s$, $d(\pi) = d$, $r(\pi) = r$, $k(\pi) = k$.

Lemma 3. Let p and p' be two distinct points of π such that $d(p, p') = 2$. Let L_1 be the set of lines containing p and at distance 1 from p' and let L_2 be the set of lines containing p' and at distance 1 from p . Then each line of L_1 intersects each line of L_2 .

Proof: Let $n \in L_2$ and $n^* = \{z \in n : d(z, p) = 1\}$. Then $|n^*|$ equals the number of lines of L_1 which intersect n . By (f3), $|n^*| = (q+1)$ and by (f4), $|L_1| = q+1$. Hence, each line of L_1 intersects n .

For a pair of lines m and n , $T(m, n)$ denotes the set of transversals of m and n .

Lemma 4. Let m and n be two distinct lines of π such that $d(m, n) = 1$. Then (i), $d(p, m) = 1$ for exactly $(q+1)$ points p of n and (ii), $|T(m, n)| \leq (q+1)^2$.

Proof: Since $d(m,n) = 1$, there exists points x and y such that $x \in m$, $y \in n$ and $d(x,y) = 1$. Since $d(x,n) = 1$, by (f3) there exists $(q+1)$ points $y_0 = y, y_1, \dots, y_q$ such that $d(x,y_i) = 1$, $y_i \in n$ and $d(y_i, m) = 1$, $0 \leq i \leq q$. If possible, let $y \in n$, $y \neq y_i$, $0 \leq i \leq q$ and $d(y,m) = 1$. Then $d(x,y) = 2$, $d(x,n) = d(y,m) = 1$ and $x \in m$, $y \in n$. By Lemma 4, m and n must intersect whence $d(m,n) \neq 1$. This completes the proof of (i) and (ii) follows easily.

Let m and n be intersecting lines and x be the point of intersection. We let

$$C(m,n) = T(m,n) \cup \{h: h \in L, x \in h, h \cap n' \neq \emptyset \\ \text{for some } n' \in T(m,n)\}$$

Lemma 5. Pasch's axiom is valid in (P,L) .

For any pair of intersecting lines m_1 and m_2 , $|T(m_1, m_2)| = (k-1)q$.

Proof: Let $\{x\} = m_1 \cap m_2$. For each $y \in m_1 - x$, y is adjacent to q vertices of $m_2 - x$. So, q transversals of m_1 and m_2 contain y . Therefore, $|T(m_1, m_2)| = (k-1)q$. Let $n \in T(m_1, m_2)$. Let $a \in n \cap m_1$, $b \in n \cap m_2$, S_1 be the set of $(q-1)$ vertices of $m_1 - \{x, a\}$ adjacent to b and S_2 be the set of $(q-1)$ vertices of $m_2 - \{x, b\}$ adjacent to a . Let $h \in T(m_1, m_2)$ such that $\{c\} = h \cap m_1 \notin S_1$. Then b and c are not adjacent. We get $d(b,c) = 2$, $b \in n$, $d(c,n) = 1$, $c \in h$, $d(b,h) = 1$. By Lemma 4, n and h intersect. It follows that if $h \in T(m_1, m_2)$ and h and n do not intersect, then $h \cap m_1 \in S_1$. Similarly, $h \cap m_2 \in S_2$. Therefore the number of lines of $T(m_1, m_2)$ not intersecting n is at most $(q-1)^2 = |S_1| |S_2|$. If $q = 1$, $(q-1)^2 = 0$. Then all lines of $T(m_1, m_2)$ intersect n , so Pasch's axiom is valid.

Let $q \geq 2$. If possible, let n and h be two non-intersecting lines of $T(m_1, m_2)$. There are at least $|T(m_1, m_2)| - 2(q-1)^2 = (k-1)q - 2(q-1)^2$ lines of $T(m_1, m_2)$ which intersect both n and h . Also, m_1 and m_2 intersect both n and h . Hence, the number of lines intersecting both n and h is at least $kq - 2q^2 + 3q$. On the other hand, since $d(n, h) = 1$, by Lemma 5 there are at most $(q+1)^2$ lines intersecting both n and h . This gives us $(q+1)^2 \geq kq - 2q^2 + 3q$. Since $s \geq 3$, $k \geq q^2 + q + 1 \geq 3q$ and $(q+1)^2 \geq 3q^2 - 2q^2 + 3q$. Simplifying the inequality we get $1 \geq q$ which contradicts the assumption.

If S is a set of lines such that any two lines of S intersect each other, then S is a clique in the line graph of (P, L) ; we refer to such a set S as a clique of lines.

Lemma 6. Let m_1 and m_2 be intersecting lines. Then $C(m_1, m_2)$ is a maximal clique of lines.

Proof: We denote $T(m_1, m_2)$ by T and $C(m_1, m_2)$ by C . Let $\{x\} = m_1 \cap m_2$. T is a clique of lines, and so is $C-T$ since x belongs to each line of $C-T$. It is sufficient to show that if $h \in C-T$ and $n' \in T$, then h intersects n' . Since $h \in C-T$, $x \in h$ and h intersects n for some transversal n of m_1 and m_2 . We may assume (by exchanging m_1 and m_2 if necessary) that $n \cap m_2 \neq n' \cap m_2$. Then $h, n' \in T(n, m_2)$. So, h and n' intersect. Hence C is a clique of lines. It is clear from the definition of C that no proper superset of C is a clique of lines.

We call each $C(m_1, m_2)$ a plane, and let $\mathcal{C}$ be the set of planes.

Corollary 1. Each plane contains $qk + 1$ lines.

Proof: Let m and n be lines which intersect at x . We show that $|C(m, n)| = qk + 1$. Let $h \in T(m, n)$. By Lemma 6 every line

of $C(m,n)-T(m,n)$ intersects h , so $C(m,n)-T(m,n)$ is the set of $q+1$ lines which contain x and intersect h . $|T(m,n)| = (k-1)q$.

Lemma 7. Let K be a clique of lines, $m,n \in K$ ($m \neq n$), and $\{x\} = m \cap n$. Then either all lines of K contains x or $K \subseteq C(m,n)$.

Proof: We assume that some line n' of K does not contain x and show that $K \subseteq C(m,n)$. Let $h \in K$. Then h intersects m , n , and n' . If $x \notin h$, then $h \in T(m,n)$. So $h \in C(m,n)$. Next suppose $x \in h$. Since h intersects n' , $h \in C(m,n)$. Therefore, $K \subseteq C(m,n)$.

Lemma 8. (i) Each pair of intersecting lines is in a unique plane. (ii) If the plane C contains at least 1 line containing x , then C contains exactly $q+1$ lines containing x . (iii) Each line is contained in $(r-1)/q$ planes.

Proof: (i) Let m and n be intersecting lines and the plane C contain m and n . By Lemma 7 $C \subseteq C(m,n)$. But all planes have the same cardinality, so $C = C(m,n)$. (ii) Let $x \in m \in C$. Let $n \in C$ so that $x \notin n$. Then $C = C(m,n)$. Every line of C which contains x also intersects n . There are $q+1$ lines which contain x and intersect n . One of these lines is m , and the remaining q lines are transversals of m and n , so $q+1$ lines of C contain x . (iii) Let m be a line. Choose $x \in m$ and let $m_2, m_3, \dots, m_r$ be the lines containing x which are distinct from m . Each plane which contains m contains exactly q lines among $m_2, m_3, \dots, m_r$. By part (i), each line m_i is contained in a unique plane containing m . Hence exactly $(r-1)/q$ planes contain m .

From Lemma 8 and Corollary 1, the following statement is immediate.

Corollary 2. $(L, \mathcal{C})$ is a semilinear $((r - \mathcal{C}/q, qk + 1) -$
incidence structure.

For any plane C we define $\bar{C} = \bigcup_{m \in C} m$.

Lemma 9. Let $m \in L$ and $C \in \mathcal{C}$. If $|m \cap \bar{C}| \geq 2$, then $m \in C$.

Proof: Let $x, y \in m \cap \bar{C}$. Then for some n_1 and n_2 ($n_1, n_2 \neq m$) $x \in n_1 \in C$ and $y \in n_2 \in C$. Lines n_1 and n_2 intersect since all lines of C intersect, so $C = C(n_1, n_2)$. Since m is a transversal of n_1 and n_2 , $m \in C$.

Since each pair of intersecting lines is contained in a plane, and each plane is a clique of lines, two lines contain a point in common iff they are both contained in some plane. Therefore the adjacency graph of $(L, \mathcal{C})$ is identical to the line graph of (P, L) . Let H be the adjacency graph of $(L, \mathcal{C})$.

Lemma 10. If m and n are distinct lines then $d_H(m, n) = d_G(m, n) + 1$.
If the line m is not contained in the plane C then $d_H(m, C) = d_G(m, \bar{C}) + 1$.

Proof: Let $d_G(m, n) = i - 1$ where $m \neq n$. Denote m by m_0 and n by m_i . Let $(m_0, x_1, m_1, x_2, \dots, x_i, m_i)$ be a sequence of points and lines such that x_j is contained in m_{j-1} and m_j ($1 \leq j \leq i$). Let $C_j = C(m_{j-1}, m_j)$ for $1 \leq j \leq i$. Then $(m_0, C_1, m_1, C_2, \dots, C_i, m_i)$ is a sequence of lines and planes so that C_j contains m_{j-1} and m_j ($1 \leq j \leq i$), so $d_H(m, n) \leq i$. Since the direction of this argument is reversible, we may conclude that $d_H(m, n) = d_G(m, n) + 1$.

Let $m \notin C$. Now $d_G(m, \bar{C}) = \min\{d_G(m, n) : n \in C\}$ and $d_H(m, C) = \min\{d_H(m, n) : n \in C\}$. Since $d_H(m, n) = d_G(m, n) + 1$ for distinct lines m and n , $d_H(m, C) = d_G(m, \bar{C}) + 1$.

Lemma 11. $(L, \mathcal{C})$ is an $(s+1, q, d)$ -pseudo projective incidence structure.

Proof: We have already established that $(L, \mathcal{C})$ is a semilinear (r^*, k^*) -incidence structure where $r^* = (r-1)/q = (q^{d-s}-1)/(q-1)$ and $k^* = qk+1 = (q^{s+1}-1)/(q-1)$. (If $q = 1$ then $r^* = (r-1)/q = d-s$ and $k^* = qk+1 = s+1$.) The graph H is connected since G is. We prove (f4). Let m and n be lines and $d_H(m, n) = 2$ (so $d_G(m, n) = 1$). Let $S = \{C : C \in \mathcal{C}, n \in C, d_H(m, C) = 1\}$. We are to show $|S| = q + 1$. Now $S = \{C : C \in \mathcal{C}, n \in C, m \cap \bar{C} \neq \emptyset\}$.

If h is a line and z a vertex so that $d_G(z, h) = 1$ then there exists at least two lines h_1 and h_2 so that $z \in h_1$ and h_1 intersects h ($i = 1, 2$). The plane $C(h_1, h_2)$ contains both h and z . For any plane C containing both z and h we have $|h_1 \cap \bar{C}| \geq 2$ so $h_i \in C$ ($i = 1, 2$), and consequently $C = C(h_1, h_2)$. Therefore for any line h and vertex z so that $d_G(z, h) = 1$, a unique plane contains both z and h .

Lines m and n do not intersect. So, no plane contains both. Every plane S contains n and at least one point of m . Let $x_0, x_1, \dots, x_q$ be the points of m satisfying $d_G(x_i, n) = 1$ ($0 \leq i \leq q$). Let C_i be the unique plane containing x_i and n ($0 \leq i \leq q$). If for some i and j ($i \neq j$) $C_i = C_j$ then $|m \cap \bar{C}_i| \geq 2$. By Lemma 9 this would imply that $m \in C_i$, which is false. Then $S = \{C_0, C_1, \dots, C_q\}$, so $|S| = q + 1$.

To prove (f3), let $d_H(m, C) = 1$. Then $d_G(m, C) = 0$. So, $m \cap \bar{C} \neq \emptyset$. By Lemma 9, $|m \cap \bar{C}| = 1$. Let $\{x\} = m \cap \bar{C}$.

We are to show that $d_H(m, n) = 1$ for exactly $q + 1$ lines n of C . In other words $d_G(m, n) = 0$ for exactly $q + 1$ lines n of C . But this is clear, since exactly $q + 1$ lines of C contain x .

Lemma 12. Pasch's axiom is valid in $(C, L, \ni)$.

Proof: We first state Pasch's axiom for $(C, L, \ni)$, recalling that two lines intersect (i.e. contain a vertex in common) iff they are both incident with some plane. Pasch's axiom for $(C, L, \ni)$ states that if lines m and n intersect, and lines h_1 and h_2 intersect both m and n but no plane contains h_1, m , and n and no plane contains h_2, m , and n , then h_1 and h_2 intersect.

Let $\{x\} = m \cap n$. Now $h_1 \notin C(m, n)$, so $h_1 \notin T(m, n)$. Since $h_1 \notin T(m, n)$ but h_1 intersects both m and n , $x \in h_1$. Similarly $x \in h_2$. Therefore h_1 and h_2 intersect, and Pasch's axiom is valid.

Let $\bar{P} = \{\bar{P} : p \in P\}$ where $\bar{p} = \{p : p \in L, p \in m\}$.

Lemma 13. The mapping $\alpha : P \rightarrow \bar{P}$ defined by $\alpha(p) = \bar{p}$ is a bijection.

Proof: The mapping α is clearly surjective. We show that α is injective. (P, L) is a semilinear (r, k) -incidence structure, therefore $|\bar{x}| = r > 1$ and $|\bar{x} \cap \bar{y}| \leq 1$ for all $x, y \in P$. It follows that $\bar{x} \neq \bar{y}$ for all distinct $x, y \in P$.

Lemma 14. $\bar{P} \cup \mathcal{C}$ is a partition of the set of maximal cliques of H .

Proof: It is clear from Lemma 7 that every maximal clique of lines is contained in $\bar{P} \cup \mathcal{C}$. We have shown that every plane is a maximal clique of lines. Therefore it is sufficient to show that $\bar{x}$ is a maximal clique of lines for every $x \in P$, and that $\bar{X}$ and $\mathcal{C}$ are disjoint. $\bar{X}$ and $\mathcal{C}$ are disjoint because the lines of a plane are not concurrent.

Let $x \in P$. Clearly $\bar{x}$ is a clique of lines. Let K be a maximal clique containing $\bar{x}$. If possible, let $K \supsetneq \bar{x}$. Let $m \in K - \bar{x}$. Then $x \notin m$. By (f3), the number of lines of $\bar{x}$ intersecting m is at most $q + 1$. Since K is a clique of lines, every line of $\bar{x}$ intersects m . Therefore $q + 1 \geq |\bar{x}| = r > q + 1$ which is a contradiction.

In Lemmas 15 - 17 we examine (s, q, d) -pseudo projective incidence structures where $s = d - 1$.

Lemma 15. Let (P, L) be a $(d-1, q, d)$ -pseudo projective incidence structure. Then any two lines intersect and if $q = 1$, $|L| = d$.

Proof: (P, L) is an (r, k) -incidence structure where $r = q + 1$ and $k = (q^{d-1} - 1)/(q - 1)$ (if $q = 1$ then $k = d - 1$).

Let G be the adjacency graph of (P, L) . Since G is connected, the distance between any two lines is finite. If not all lines intersect then there are lines m and n so that $d(m, n) = 1$. Assume that $d(m, n) = 1$. Then for some $x \in m, d(x, n) = 1$. By (f3) $q + 1$ lines contain x and intersect n . Then these lines together with m constitute $q + 2$ lines containing x , which violates the condition $r = q + 1$. Therefore any two lines intersect.

Let $m \in L$. Since $k(r-1)$ lines intersect m and all lines intersect, $|L| = k(r-1) + 1$. If $q = 1$, $k = d-1$ and $|L| = d$.

Lemma 16. Let (P, L) be a $(d-1, q, d)$ -pseudo projective incidence structure where $d > 3$ and $q \geq 2$, and let the incidence structure dual to (P, L) satisfy Pasch's axiom. Then

- (i) q is a prime power and d is an integer,
- (ii) the incidence structure dual to (P, L) is a $(2, q, d)$ -

projective incidence structure,

and (iii) (P, L) is a $(d-1, q, d)$ -projective incidence structure.

Proof: For $r = q + 1$ and some k , (P, L) is an (r, k) -incidence structure.

We show that $(L, P, \mathfrak{D})$ satisfies the axioms $(p_1) - (p_4)$ of section 1. Now elements of L will be called points and elements of P will be called lines. By Lemma 15 any two points are incident with some line. Therefore $(L, P, \mathfrak{D})$ satisfies (p_1) . By hypothesis (p_3) is satisfied. Every element of P is incident with $q + 1 \geq 3$ elements of L . Since $d > 3$ every element of L is incident with more than $q + 1$ elements of P . It easily follows that rank of (P, L) is at least 4. Therefore by the theorem about finite projective spaces $(L, P, \mathfrak{E})$ is a $(2, q', d')$ -projective incidence structure. Clearly we must have $q' = q$ and $d' = d$. This establishes (ii) and (i). Since a $(d-1, q, d)$ -projective incidence structure is dual to a $(2, q, d)$ -projective incidence structure (iii) follows.

Lemma 17. Let (P, L) be a $(d-1, 1, d)$ -pseudo projective incidence structure where $d > 2$. Then d is an integer and (P, L) is a $(d-1, 1, d)$ -projective incidence structure.

Proof: (P, L) is an (r, k) -incidence structure with $r = 2$ and $k = d-1$. Since k is an integer, d is an integer. We examine the dual incidence structure $(L, P, \mathfrak{E})$. Elements of L will be called dual points and elements of P dual lines. Each dual line is incident with exactly 2 dual points. Therefore dual lines are equivalent to the edges of the adjacency graph of $(L, P, \mathfrak{E})$. By Lemma 15, each pair of dual points is incident with some dual line.

So, the adjacency graph of (L, P, ϵ) is the complete graph on $|L| = d$ vertices. Let Y be a d -set and Z_i be the set of i -subsets of Y , $1 \leq i \leq d-1$. We have proved that (L, P, ϵ) is isomorphic to (Y, Z_2) . Therefore (P, L) is isomorphic to (Z_2, Y, ϵ) and hence to $(Z_{d-2}, Z_{d-1}, \epsilon)$.

Lemma 18. There is no (s, q, d) -pseudo projective incidence structure where $3 \leq s$ and $d-2 < s < d-1$.

Proof: Assume $\pi = (P, L)$ is an (s, q, d) -pseudo projective incidence structure where $3 \leq s$ and $d-2 < s < d-1$. If $q = 1$ then $r(\pi) = d-s+1$ is not an integer. Therefore $q > 1$. Define $\mathcal{C}$ as in Lemmas 6 - 11. By Lemma 11 $\pi^* = (L, \mathcal{C})$ is an $(s+1, q, d)$ -pseudo projective incidence structure. $r(\pi^*) = (q^{d-s}-1)/(q-1)$ so $1 < r(\pi^*) < q+1$. Since $r(\pi^*) \geq 2$ and $k(\pi^*) \geq 2$ there exist $m \in L$ and $C \in \mathcal{C}$ so that in the adjacency graph of π^* $d(m, C) = 1$. By (f3), $r(m) \geq q+1$. Since $r(m) = r(\pi^*)$ the impossibility of the assumed incidence structure is established.

3. Proof of the Theorems.

The heart of the inductive procedure for Theorem 1 is contained in the next lemma.

Lemma 19. For $j = 1, 2$ let

- (i) B_j be a set,
- (ii) A_j and C_j be sets of subsets of B_j ,
- (iii) the incidence structures (B_j, A_j) and (B_j, C_j) have the same adjacency graph H_j ,
- (iv) $A_j \cup C_j$ be the set of maximal cliques of H_j ,
- (v) $A_j \cap C_j = \emptyset$.

Let (B_1, C_1) and (B_2, C_2) be isomorphic. Then $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

Proof. By hypothesis (B_1, C_1) and (B_2, C_2) are isomorphic; let $\sigma: B_1 \rightarrow B_2$ and $\tau: C_1 \rightarrow C_2$ be bijections which preserve incidence. For any $B' \subseteq B_1$ we let $\sigma(B') = \{\sigma(b): b \in B'\}$; in particular, for $c \in C_1$, $\sigma(c) = \{\sigma(b): b \in c\}$. Then $\sigma(c) = \tau(c)$ for all $c \in C_1$.

σ is an isomorphism between the adjacency graph H_1 of (B_1, C_1) and the adjacency graph H_2 of (B_2, C_2) . Therefore σ induces a bijection between the maximal cliques of H_1 and the maximal cliques of H_2 . The set of maximal cliques of H_1 is $A_1 \cup C_1$ and the set of

maximal cliques of H_2 is $A_2 \cup C_2$. Since $A_1 \cap C_1 = \emptyset = A_2 \cap C_2$ and σ induces a bijection from C_1 to C_2 , σ induces a bijection from A_1 to A_2 . Then the bijection $\sigma: B_1 \rightarrow B_2$ and the bijection from A_1 to A_2 induced by σ show that the incidence structures (B_1, A_1) and (B_2, A_2) are isomorphic, and also that $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

In order to shorten the proof of Theorem 1, we introduce some terminology. For $q = 1$ and a positive integer d , $V_{d,q}$ will denote a finite d -element set. For q a prime power $V_{d,q}$ will denote a d -dimensional vectorspace over $GF(q)$. For $q = 1$ an i dimensional object of $V_{d,q}$ will mean an i -element subset of $V_{d,q}$. For q a prime power, an i -dimensional object of $V_{d,q}$ will mean an i -dimensional subspace of $V_{d,q}$. For $0 \leq i \leq d$, W_i will denote the set of i -dimensional objects of $V_{d,q}$.

Proof of Theorem 1. Assume that there exists a counter example to the statement of Theorem 1. Among all such counter examples we choose an incidence structure $\pi = (P, L, I)$ for which $r(\pi)$ is as small as possible. Wlog we assume that lines are subsets of points. We write s for $s(\pi)$ and d for $d(\pi)$. Let $\mathcal{C}$ be as in Section 2. By Lemma 11, $\pi^* = (L, \mathcal{C})$ is an $(s+1, q, d)$ - pseudo projective incidence structure. Note that $r(\pi^*) < r(\pi)$ and that the dual of π^* satisfies Pasch's axiom by Lemma 12. By Lemma 18, $s \leq d - 2$. If $s < d - 2$, then π^* satisfies

the hypotheses of the theorem and $r(\pi^*) < r(\pi)$. Therefore π^* is an $(s+1, q, d)$ - projective incidence structure. If $s = d-2$, then by Lemmas 16 and 17 π^* is an $(d-1, q, d)$ - projective incidence structure. So in either case d is an integer, $q = 1$ or is a prime power and π^* is isomorphic to $(W_s, W_{s+1}, \subseteq)$ where W_i is the class of i dimensional objects of a $V_{d,q}$ where $i = s, s+1$. For $w \in W_{s+1}$, let $\bar{w} = \{u: u \in W_s \text{ and } u \subseteq w\}$ and $\bar{W}_{s+1} = \{\bar{w}: w \in W_{s+1}\}$. For $w \in W_{s-1}$, let $w' = \{u: u \in W_s, u \supset w\}$ and $W'_{s-1} = \{w': w \in W_{s-1}\}$. It is easily seen that $(W_s, W_{s+1}, \subseteq)$ is isomorphic to $(W_s, \bar{W}_{s+1})$ and $(W_{s-1}, W_s, \subseteq)$ is isomorphic to $(W_{s-1}, W'_s, \supseteq)$. We now apply Lemma 19 with $B_1 = W_s$, $C_1 = \bar{W}_{s+1}$ and $A_1 = W'_{s-1}$, $B_2 = L$, $C_2 = \mathcal{C}$ and $A_2 = \bar{P}$. $(W_s, \bar{W}_{s+1})$ and (W_s, W'_{s-1}) have the same adjacency graph H_1 . $W'_{s-1} \cup \bar{W}_{s+1}$ is a partition of the set of maximal cliques of H_1 . By the remark after Lemma 9, $(L, \mathcal{C})$ and $(L, \bar{P})$ have the same adjacency graph H_2 . By Lemma 14, $\bar{P} \cup \mathcal{C}$ is a partition of the set of maximal cliques of H_2 . Finally $(L, \mathcal{C})$ and $(W_s, \bar{W}_{s+1})$ are isomorphic. Therefore by Lemma 19 $(\bar{P}, L, \supseteq)$ and $(W'_{s-1}, W_s, \supseteq)$ are isomorphic and hence $(P, L, \supseteq)$ and $(W_{s-1}, W_s, \subseteq)$ are isomorphic. Hence there is no counter example to the statement of Theorem 1.

Proof of Theorem 2. Wlog assume that lines of π are subsets of points. Consider the connected components of $G(\pi)$. Let P_1 be

the vertex set of the i th component, $1 \leq i \leq t$. Let L_i be the set of lines of π which contain at least one point of P_i , $1 \leq i \leq t$. Then $P = P_1 \cup P_2 \cup \dots \cup P_t$ and $L = L_1 \cup L_2 \cup \dots \cup L_t$ are partitions and each line of L_i is a subset of P_i , $1 \leq i \leq t$. It is easily checked that for $1 \leq i \leq t$, (P_i, L_i) satisfy the axioms (f1) - (f5) with respect to the integer q . Therefore for some integers s_i and d_i (P_i, L_i) is an (s_i, q, d_i) -projective incidence structure and π is the direct sum of these incidence structure. The converse follows from **Lemma 2**.

4. Minimality of the Axioms.

Let $\mathcal{P}$ be the class of (s, q, d) - projective incidence structure with $3 \leq s \leq d - 2$. The axioms (f1) - (f5) form a minimal set of axioms for the purpose of characterization of the class $\mathcal{P}$. We now demonstrate the minimality of the axiom set (f1) - (f5). For $j \in \{1, 2, 3, 4, 5\}$ we choose q and construct an incidence structure π' which satisfies the four axioms other than (fj) and is not a member of $\mathcal{P}$. For $j = 5$, we saw that the direct sum of two (s, q, d) - projective incidence structures ($3 \leq s \leq d - 2$) satisfy the four axioms other than (f5). For $j = 1$, our example is a nonDesarguesian finite projective plane π of order q . It is easy to see that π satisfies the axioms (f2) - (f5) and that $s(\pi) = 2$. Since π is non Desarguesian, π is not an (s, q, d) - projective incidence structure. For $j = 2$, we construct an example as follows. Let q be a given prime power. We choose positive integers s and d satisfying $3 \leq s \leq d - 2$ and $(2q+1)^2 + (2q+1) + 1 \leq \text{Min}(\frac{q^s-1}{q-1}, 2\frac{(q^{d-s+1}-1)}{q-1})$. Let π be an (s, q, d) - projective incidence structure. The point set of the incidence structure π' will be same as that of π and for each line ℓ of π , π' will have two lines ℓ and ℓ' with $P_\ell = P_{\ell'}$. It is easily checked $k(\pi') = \frac{q^s-1}{q-1}$ and $r(\pi') = 2\frac{(q^{d-s+1}-1)}{q-1}$. Therefore with respect to $(2q+1)$, $3 \leq s(\pi') \leq d(\pi') - 2$ and also π' satisfies (f3), (f4) and (f5) w.r.t. $2q+1$. Clearly π' is not a member of $\mathcal{P}$ and is not an (s, q, d) - projective incidence structure. For $j = 3$, we proceed to construct an example as follows. Consider an affine space

$\text{Aff}(n, q)$ where q is a prime power, and $q^2 + q + 1 \leq \frac{q^{n-2}-1}{q-1}$. Let π' be an incidence structure whose points are the planes of the affine space and lines are the 3-spaces of the affine space and incidence is containment. Points and lines of π will be respectively called ideal points and ideal lines. It will be helpful to view the affine space as a projective space $\text{PG}(n, q)$ minus a hyperplane Σ . The number of planes contained in an affine 3-space is $q^3 + q^2 + q$. Therefore $k(\pi) = q^3 + q^2 + q$ and $r(\pi) = \frac{q^{n-2}-1}{q-1}$. For an ideal point p and an ideal line ℓ , p' and ℓ' will respectively denote the corresponding projective plane and projective 3-space. The axiom (f1) is satisfied by π' . Clearly (f2) and (f5) hold for π' . Let p_1 and p_2 be two ideal points such that $d(p_1, p_2) = 2$. Then p_1 and p_2 are affine planes (Figure 1). Since $d(p_1, p_2) = 2$, there exist an ideal point p_3 such that $d(p_i, p_3) = 1$, $i = 1, 2$. Hence $\langle p'_1, p'_3 \rangle$ is a 3-space for $i = 1, 2$. Therefore $p'_1 \cap p'_3$ is a line for $i = 1, 2$. Therefore $p'_1 \cap p'_2$ is a point O . Let ℓ be an ideal line such that p_1 is incident with ℓ and $d(p_2, \ell) = 1$. Then ℓ' is a projective 3-space such that $p'_1 \subseteq \ell'$ and $p'_2 \cap \ell'$ is a projective line passing through O . Let x_i ($i = 0, 1, \dots, q$) be the projective lines of p'_2 passing through O . Then letting $\ell'_i = \langle p'_1, x_0 \rangle$, $0 \leq i \leq q$, ℓ'_i $0 \leq i \leq q$ are all the projective 3-spaces satisfying $p'_1 \subseteq \ell'_i$, $p'_2 \cap \ell'_i =$ a projective line. The corresponding affine 3-spaces ℓ_i , $0 \leq i \leq q$ are all the ideal lines satisfying $d(p_1, \ell) = 0$ and $d(p_2, \ell) = 1$. We proved that π' satisfies (f4) w.r.t. q .

$\text{Aff}(n, q)$ where q is a prime power, and $q^2 + q + 1 \leq \frac{q^{n-2}-1}{q-1}$. Let π' be an incidence structure whose points are the planes of the affine space and lines are the 3-spaces of the affine space and incidence is containment. Points and lines of π will be respectively called ideal points and ideal lines. It will be helpful to view the affine space as a projective space $\text{PG}(n, q)$ minus a hyperplane Σ . The number of planes contained in an affine 3-space is $q^3 + q^2 + q$. Therefore $k(\pi) = q^3 + q^2 + q$ and $r(\pi) = \frac{q^{n-2}-1}{q-1}$. For an ideal point p and an ideal line ℓ , p' and ℓ' will respectively denote the corresponding projective plane and projective 3-space. The axiom (f1) is satisfied by π' . Clearly (f2) and (f5) hold for π' . Let p_1 and p_2 be two ideal points such that $d(p_1, p_2) = 2$. Then p_1 and p_2 are affine planes (Figure 1). Since $d(p_1, p_2) = 2$, there exist an ideal point p_3 such that $d(p_i, p_3) = 1$, $i = 1, 2$. Hence $\langle p'_i, p'_3 \rangle$ is a 3-space for $i = 1, 2$. Therefore $p'_1 \cap p'_3$ is a line for $i = 1, 2$. Therefore $p'_1 \cap p'_2$ is a point O . Let ℓ be an ideal line such that p_1 is incident with ℓ and $d(p_2, \ell) = 1$. Then ℓ' is a projective 3-space such that $p'_1 \subseteq \ell'$ and $p'_2 \cap \ell'$ is a projective line passing through O . Let x_i ($i = 0, 1, \dots, q$) be the projective lines of p'_2 passing through O . Then letting $\ell'_i = \langle p'_1, x_0 \rangle$, $0 \leq i \leq q$, ℓ'_i $0 \leq i \leq q$ are all the projective 3-spaces satisfying $p'_1 \subseteq \ell'_i$, $p'_2 \cap \ell'_i =$ a projective line. The corresponding affine 3-spaces ℓ_i , $0 \leq i \leq q$ are all the ideal lines satisfying $d(p_1, \ell) = 0$ and $d(p_2, \ell) = 1$. We proved that π' satisfies (f4) w.r.t. q .

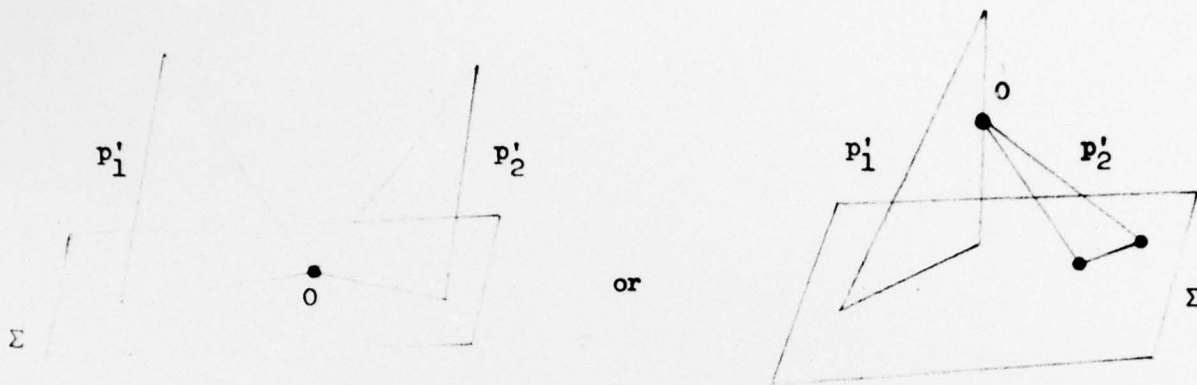


Figure 1

We now show that (f3) does not hold in π' . Let ℓ be an ideal line and p be an ideal point such that $d(p, \ell) = 1$. Let p' and ℓ' be the corresponding projective plane and 3-space respectively. Since $d(p, \ell) = 1$, $p' \cap \ell'$ must be a projective line. Case 1. (Figure 2) $p' \cap \ell' = y$ is a projective line contained in Σ . There are $(q + 1)$ planes of ℓ' which contain y . Of these one is $\ell' \cap \Sigma$ which does not correspond to an ideal point of π . Therefore in case 1 there are q ideal points p_i such that p_i is incident with ℓ and $d(p, p_i) = 1$, $1 \leq i \leq q$.

Case 2. (Figure 3) $p' \cap \ell' = y$ is a projective line not contained in Σ . In this case there will be $(q + 1)$ ideal points p_i such that p_i is incident with ℓ and $d(p, p_i) = 1$, $0 \leq i \leq q$. With respect to q , π' satisfies all the four axioms except (f3) and $\pi' \notin \mathcal{P}$.



Figure 2

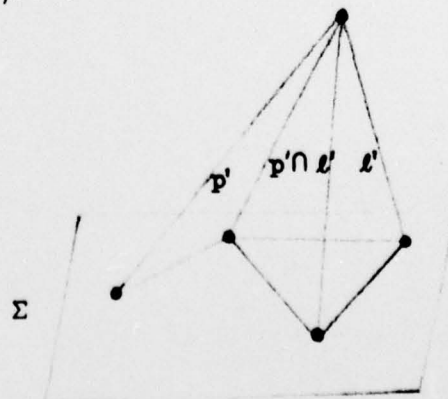


Figure 3

We now consider $j = 4$. Let q be a prime power, $n \geq 4$, $PG(n, q)$ be an n -dimensional projective space over $GF(q)$ and Σ_{n-3} be an $(n-3)$ -flat of $PG(n, q)$. Let π' be an incidence structure whose points are the lines of $PG(n, q)$ not intersecting Σ_{n-3} and lines are the planes of $PG(n, q)$ not intersecting Σ_{n-3} . As before points and lines of π will be referred to as ideal points and ideal lines respectively. Lines and planes of $PG(n, q)$ will be called projective lines and projective planes. Clearly every ideal line is incident with $q^2 + q + 1$ ideal point and hence $k(\pi) = q^2 + q + 1$. The number of projective planes of $PG(n, q)$ containing a given projective line is $\frac{q^{n-1}-1}{q-1}$. Of these projective planes $\frac{q^{n-2}-1}{q-1}$ will intersect Σ_{n-3} . Hence the number of ideal lines passing through a given ideal point is q^{n-2} . Since $n \geq 4$, the axiom (f1) holds for π' with respect to $(q-1)$.

Clearly (f2) and (f5) hold for π' . We now check (f3) for π . Let p and ℓ respectively be an ideal point and an ideal line such that $d(p, \ell) = 1$. Then the projective line p intersects the projective plane ℓ in a projective point O (Figure 4). Let $\Sigma_{n-1} = \langle p, \Sigma_{n-3} \rangle$ be the span of p and Σ_{n-3} and $p_0 = \ell \cap \Sigma_{n-1}$. There are $q+1$ projective lines of ℓ passing through O . Let $p_0, p_1, \dots, p_q$ be these lines. The projective plane $\langle p, p_0 \rangle$ is contained in Σ_{n-1} , intersects Σ_{n-3} and hence is not an ideal line of π . Therefore the distance between the ideal points p and p_0 is greater than 1. The ideal points

$p_1, p_2, \dots, p_q$ are the only ideal points p satisfying $p_i \subset \ell$ and $d(p, p_i) = 1$, $1 \leq i \leq q$. Therefore π satisfies (f3) with respect to $(q - 1)$.

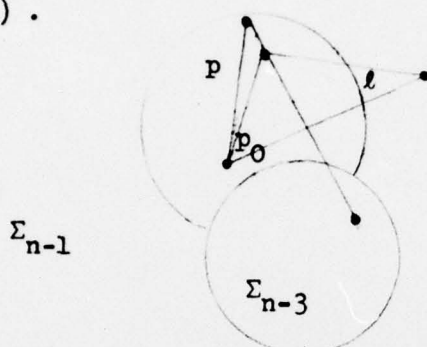


Figure 4

We now show that (f4) does not hold in π' . Let p_1 be an ideal point and $\langle \Sigma_{n-3}, p_1 \rangle = \Sigma_{n-1}$. Let p_2 be an ideal point such that $d(p_1, p_2) = 2$. Case 1. p_2 is a projective line not intersecting p_1 and Σ_{n-3} and not contained in Σ_{n-1} (Figure 5). Let $x_i, 0 \leq i \leq q$ be the $q + 1$ points of p_2 where $x_0 \in \Sigma_{n-1}$, and $\ell_i = \langle p_1, x_i \rangle$, $0 \leq i \leq q$. The projective plane ℓ_0 intersects Σ_{n-3} . In this case $\ell_1, \dots, \ell_q$ are the only ideal lines which contain p_1 and have distance one from p_2 .

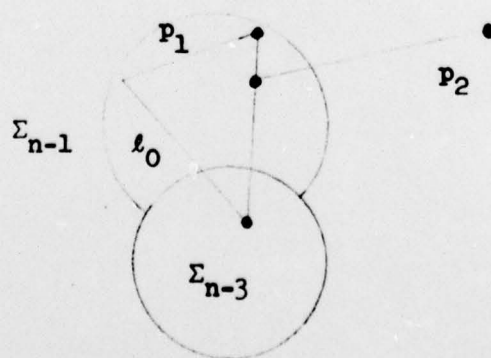


Figure 5

Case 2. The projective line p_2 intersects p_1 and is contained in Σ_{n-1} (Figure 6). The projective plane $\langle p_1, p_2 \rangle$ intersects Σ_{n-3} and hence is not an ideal line. Therefore $d(p_1, p_2) = 2$. Let ℓ_1 be any projective plane which contains p_1

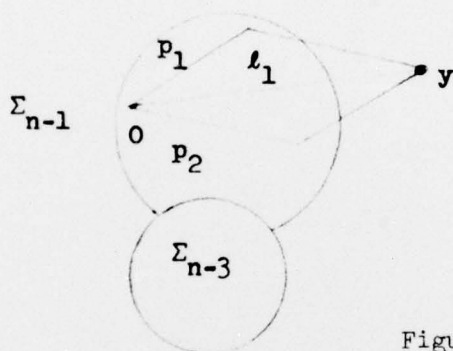


Figure 6

and is not contained in Σ_{n-1} . Then it is easily seen that $p_1 \subseteq \ell_1$ and $d(p_2, \ell_1) = 1$. In case 2 the number of ideal lines ℓ satisfying $p_1 \subseteq \ell$, $d(p_2, \ell) = 1$ is q^{n-2} . Therefore (f4) does not hold in π with respect to $(q-1)$. Obviously π' is not a $(s, q-1, d)$ -projective incidence structure for any choice of s and d .

This completes the proof of the minimality of the system of axioms (f1) - (f5).

Concluding Remarks. Consider a simple graph whose vertices are s -dimensional subspaces of a d -dimensional vector space V over $GF(q)$. Two vertices in this graph are adjacent iff the corresponding s -dimensional subspaces intersect in an $(s-1)$ -dimensional subspace. This graph will be called $\rightarrow$ next page

an (s, q, d) -projective graph. The Theorem 1 of this paper can be used to obtain a characterization of the (s, q, d) -projective graphs provided d is larger than some function of s and q . We are also considering characterization problems of Affine spaces and Polar spaces ^{are also concerned} in terms of flats of higher dimensions. These results will be communicated in a subsequent communication. 

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