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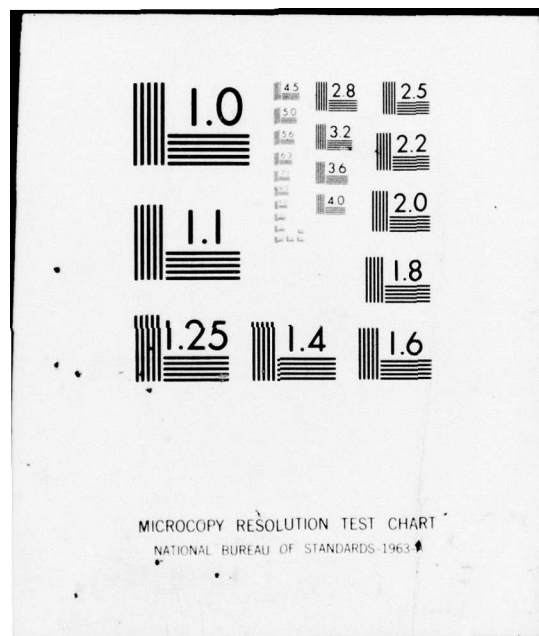
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Abdalla T. El-Helbawy

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Preface: This technical report is a short manuscript prepared for the proceedings of the 41st Session of the International Statistical Institute meeting in New Delhi, December, 1977 and to be presented at that meeting. Results summarized have ~~have~~ been developed in detail in ONR Technical Reports, No.'s 99, 100 and 102 submitted earlier.

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SOME OPTIMAL DESIGN RESULTS IN PAIRED COMPARISONS

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INTRODUCTION

The authors, Bradley and El-Helbawy (1976), El-Helbawy and Bradley (1977a,b), have developed the methodology for consideration of specified treatment contrasts in paired comparisons. The procedures developed give much new flexibility to the use of paired comparisons and, in particular, to the use of factorial treatment combinations in such experiments.

The probability model developed by Bradley and Terry (1952), originally proposed by Zermelo (1929), is used. Many additional references are given by Davidson and Farquhar (1976) in their bibliography and Bradley (1976) reviews various approaches to the model and its extensions.

In this short presentation, we summarize important results on treatment contrasts and indicate how they may be used to consider optimal design questions. Some simple optimal design results are given.

SUMMARY OF METHODOLOGY

Suppose that the paired comparisons experiment has t treatments, $T_1, \dots, T_t$, with n_{ij} comparisons of T_i and T_j ,

$n_{ii} = 0$, $n_{ij} = n_{ji}$, $i, j = 1, \dots, t$. A parameter π_i is associated with T_i , $\pi_i > 0$, such that the probability of selection of T_i when compared with T_j is

$$\text{pr}(T_i > T_j) = \pi_i / (\pi_i + \pi_j), \quad i \neq j. \quad (1)$$

The convenient scale-determining constraint is

$$\sum_{i=1}^t \gamma_i = 0, \quad \gamma_i = \log \pi_i, \quad i = 1, \dots, t, \quad (2)$$

different from that used by Bradley and Terry. On the assumption of independence of selection judgments, the likelihood function is

$$L(\underline{\pi}) = \prod_i \pi_i^{a_i} / \prod_{i < j} (\pi_i + \pi_j)^{n_{ij}}, \quad (3)$$

where a_i is the total number of selections of T_i , $\sum_i a_i = N = \sum_{i < j} n_{ij}$.

$\underline{\pi}$ is the column vector with typical element π_i and other vectors below are defined similarly.

Treatment contrasts are specified as linear, orthonormal contrasts on the γ_i . The typical estimation problem is to maximize L subject to (2) and

$$\underline{B}_m \underline{\gamma}(\underline{\pi}) = \underline{0}_m \quad (4)$$

where $\underline{0}_m$ is a column vector of zeros and $\underline{B}_m$ consists of m , zero-sum, orthonormal rows. The resulting likelihood equations are

$$\sum_j [a_j - \sum_k n_{jk} p_j / (p_j + p_k)] D_{ij} = 0, \quad i=1, \dots, t, \quad (5)$$

$$\sum_i \gamma_i(\underline{\pi}) = 0, \quad \text{and} \quad \underline{B}_m \underline{\gamma}(\underline{p}) = \underline{0}_m,$$

where D_{ij} is the (i,j) -element of $\underline{I}_t - \underline{B}'_m \underline{B}_m / t$, $\underline{I}_t$ the t -square identity matrix, $\underline{p}$ is the estimator of $\underline{\pi}$, and $\underline{\gamma}(\underline{p})$ of $\underline{\gamma}(\underline{\pi})$. El-Helbawy and Bradley (1977b) show that $\sqrt{N}[\underline{\gamma}(\underline{p}) - \underline{\gamma}(\underline{\pi})]$ has the singular, t -variate normal limiting distribution function in $(t-m-1)$ dimensions with zero mean vector and dispersion matrix $\underline{\Sigma}_m$ given in the references. El-Helbawy and Bradley (1977a) examine the solution of (5) and convergence properties of a suggested iterative scheme.

The typical testing situation assumes (4) and specifies $H_0: \underline{B}_n \underline{\gamma}(\underline{\pi}) = \underline{0}_n$ against the alternative, $H_a: \underline{B}_n \underline{\gamma}(\underline{\pi}) \neq \underline{0}_n$ and uses the likelihood ratio statistic, $\lambda_N(H_0, H_a)$. It is shown in the (1977b) paper that $-2\log \lambda_N(H_0, H_a)$ has the chi-square limiting distribution with n degrees of freedom, central under H_0 and non-central under H_a with non-centrality parameter,

$$\lambda^2 = \underline{\delta}' \underline{\Sigma}_0^{-1} \underline{\delta}, \quad (6)$$

where $\underline{\Sigma}_0$ is a dispersion matrix dependent on H_0 given in the references and $\underline{\delta}_1 = \lim_{N \rightarrow \infty} \underline{\delta}_1^N$, $\underline{B}_n \underline{\gamma}(\underline{\pi}^N) = N^{-1/2} \underline{\delta}_1^N$ and $\{\underline{\pi}^N\}$ is a sequence of local alternatives to H_0 satisfying (2) and (4).

Bradley and El-Helbawy (1976, 1977b) show how the contrasts described by $\underline{B}_m$ and $\underline{B}_n$ may be related to factorial effects when the treatments are factorial treatment combinations and, indeed, give a reparameterization of the problem for factorials.

SOME OPTIMAL DESIGN RESULTS

The results summarized above for the first time provide means of considering asymptotically optimal design of paired comparisons experiments. We limit consideration to two examples with $t = 8$ and a 2^3 -factorial. T_1 is associated with $T_{\underline{\alpha}}$, $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_s = 0, 1$, $s = 1, 2, 3$, α_s designating the level of Factor s in the treatment combination.

Consider a test of no interaction between Factors 1 and 2; $\underline{B}_m$ in (4) does not exist and $\underline{B}_n$ describes the usual analysis of variance contrast for the specified treatment contrast, now in terms of the $\underline{\gamma}_1$. The objective is to maximize asymptotic power, that is, to maximize λ^2 in (6) for the desired test. $\underline{\Sigma}_0$ in (6) depends on $\underline{\pi}_0$ and $\lambda_{ij} = \lim_{N \rightarrow \infty} n_{ij}/N$, $i \neq j$. The maximization is with respect to the λ_{ij} 's and $\underline{\pi}_0$ is taken to be $\underline{1}_8$, a column vector of unities, consistent with H_0 and the concept that any other effects present are of the same order of magnitude relative to N as the contrast under test. The experiment is assumed to be as balanced as possible but to

permit optimality consideration; we take $\lambda_{ij} = a$ or b respectively as T_i and T_j represent factorial treatment combinations with factor levels α_1 and α_2 such that $(-1)^{\alpha_1 + \alpha_2}$ does or does not have the same sign for the two treatments, $12a + 16b = 1$. Maximization of λ^2 with respect to a and b , $12a + 16b = 1$, yields $a = 0$, $b = 1/16$; no observations are taken on comparisons that yield no information on the two-factor interaction under test. The same result occurs, for example, for the same test with $\underline{B}_m$ chosen to assume that the three-factor and other two-factor interactions are null.

Suppose that all factorial effects are assumed null except the three interactions involving Factor 1. Then $\underline{B}_m$ has four rows. We take $\underline{\pi} = \underline{1}_8$, a central value satisfying (2) and (4) and make the simplifying assumption of as much balance in the experiment as possible but permitting optimality considerations. We are concerned with the dispersion matrix $\underline{\Sigma}_m$ and show that one should take all $\lambda_{ij} = 0$ except for those treatment comparisons yielding information on all of the $F_1F_2^-$, $F_1F_3^-$ and $F_1F_2F_3^-$ interactions for A-, D- and E- optimality minimizing respectively $t_r \underline{\Sigma}_m$, $|\underline{\Sigma}_m|$ and the largest variance of $\underline{\Sigma}_m$.

Some other examples are given by El-Helbawy and Bradley (1977b). While the results noted are consistent with intuition, formal demonstration is given for the first time and the way is open for more general consideration of optimal design in paired comparisons.

SUMMARY

The authors have shown (Biometrika, 1976) how to consider specified treatment contrasts in paired comparisons and given applications to factorials. In a subsequent paper, pending publication, they consider asymptotic theory and applications to optimal design when the treatments are factorial treatment combinations. This paper is a summary of some of the main results.

SOMMAIRE

Les auteurs ont démontré (Biometrika, 1976) comment à considérer des contrastes spécifiés entre traitements en comparaisons par paires et donné des applications pour traitements factoriels. Dans des subséquentes recherches, ne pas encore publiées, ils ont considéré la théorie asymptotique et les applications à dessein optimal quand les traitements sont des combinaisons factoriels. Ce papier est un sommaire des résultats principaux.

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