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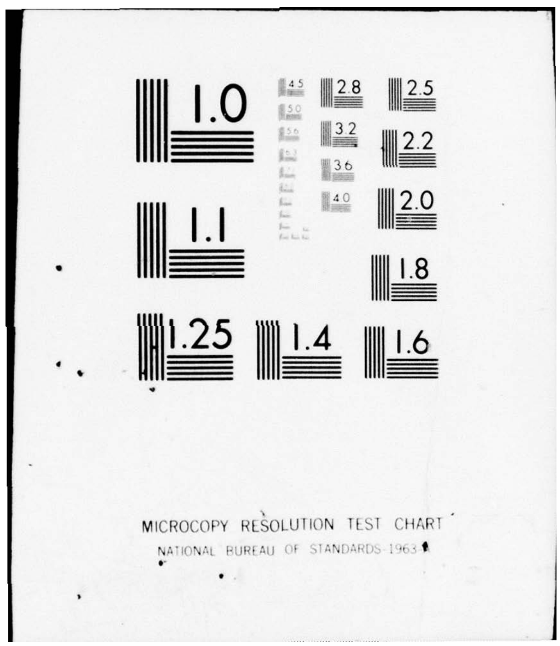
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AN ALGORITHM FOR DETERMINING
OPTIMUM STOCK LEVELS IN A
MULTI-ECHELON INVENTORY SYSTEM

Captain John A. Muckstadt

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Many mathematical models for controlling inventory were developed during the past decade. One of the models, called METRIC, was designed specifically for Air Force use in determining appropriate base and depot inventory levels for a class of items called recoverable items. METRIC can be used in three ways in determining stock levels for these items.

First, the model can be used to assist in the initial procurement-allocation problem. The objective is to determine both base and depot stock levels so that the sum of the expected backorders is minimized over all recoverable items at all bases having a particular weapon system, where the optimization is performed subject to a constraint on system investment.

Second, METRIC can be used to redistribute stocks in an optimal manner. The objective in the redistribution problem is to find base and depot stock levels for each item that minimizes the expected total base backorders, given the available system stock.

Last, the model can be used as an analysis tool. It can be used to assess system performance and investment cost for any distribution of assets.

The objective of this report is to present an algorithm that can be used to obtain the solution to the redistribution problem.

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AN ALGORITHM FOR DETERMINING OPTIMUM STOCK LEVELS IN A MULTI-ECHELON INVENTORY SYSTEM

BY

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April 1971

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SUMMARY

Many mathematical models for controlling inventory were developed during the past decade. One of the models, called METRIC, was designed specifically for Air Force use in determining appropriate base and depot inventory levels for a class of items called recoverable items. METRIC can be used in three ways in determining stock levels for these items.

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Last, the model can be used as an analysis tool. It can be used to assess system performance and investment cost for any distribution of assets.

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I. INTRODUCTION

Many mathematical models for controlling inventory were developed during the past decade. One of these models, called METRIC, was designed specifically for Air Force use in determining appropriate base and depot inventory levels for a class of items called recoverable items.[3] These recoverable items typically are expensive and experience low demand rates; however, their proper management is imperative since approximately 80 percent of the Air Force's total investment in spares is concentrated in these items. METRIC can be used in three ways to assist in determining stock levels for recoverable items.[3]

The first purpose of the model is optimization. The objective of this optimization is to determine base and depot stock levels so that the sum of the expected backorders is minimized over all recoverable items at all bases having a particular weapon system. This problem is called the procurement-allocation problem. The minimization is performed subject to a constraint on system investment. Thus this optimization is of prime importance in the acquisition phase of a weapon system.

METRIC can also be used to allocate in an optimal manner the existing amounts of stock between the bases and depot. Thus the second way this model can be used is in redistributing assets. The objective in the redistribution problem is to find the base and depot stock levels for each item that minimizes the expected total base backorders, given the available amount of system stock of that item. This redistribution is of particular importance as demand patterns change over time thereby causing items to be in either long or short supply at various locations.

Lastly, the model can be used to perform analysis. It is a simple matter to assess system performance and investment cost for any specific distribution of stocks between the bases and depot. This type of analysis can be performed at any stage of the life cycle of a weapon system.

In addition to the model a computational procedure was given in the original papers for finding the optimum base-depot stock levels.[3,4] This technique was aimed at solving the procurement-allocation problem described above. The algorithm as stated requires a large amount of computer time to find the optimum stock levels. A much more effective technique was subsequently developed for solving this problem.[1] However, it is also possible to construct a better solution method for solving the redistribution problem.

The objective of this Report is to present an algorithm that can be used to obtain the solution to the redistribution problem. A brief discussion of the backorder function and demand distribution is also included. In addition, a computer program is included which was developed for allocating spare aircraft engines. By making some minor modifications to certain input-output format statements, this program could be used to obtain the optimal redistribution for any type of item.

II. BACKGROUND

2.1 Introduction

In this chapter a variety of background material is given. Much of what is said here is a summary of results given in other papers.[2,3,4] However, for the sake of completeness it is included in the Report. In particular, is chapter contains a discussion of the backorder function and its properties. In addition, a section is included on the probability model used as the basis for making the calculations in the backorder function.

2.2 The Backorder Function

In the previous section it was stated that the objective in the redistribution problem is to minimize the expected number of total base backorders for any specific item. It should be mentioned that we are not explicitly interested in depot backorders. They are of importance only in how they affect base backorders.

Let us now define what is meant by the backorder criterion. A backorder exists at a specific point in time if there is an unsatisfied demand at base level for a specific item. Backorders are then accumulated as follows. For a fixed period of time calculate the duration, in days, corresponding to each backorder. Add the lengths of these individual backorders at all pertinent bases and divide by the length of the period. The quantity to be minimized is the expected value of this ratio. Note that this performance measure is linear, that is, 5 backorders each lasting two days is as undesirable as 10 backorders each lasting one day. The dimension of this measure is backorder days per day.

Mathematically speaking the expected number of backorders in a time period of length T is expressed as

$$B(s) = \sum_{x=s+1}^{\infty} (x-s)\, p(x|\lambda T) \; , \qquad \text{Eq 1}$$

where s is the stock level, x is the quantity demanded, λ is mean demand rate, and $p(x|\lambda T)$ is the probability of observing x demands during the time period. Equation 1 will be subsequently referred to as the backorder function. More will be said about $p(x|\lambda T)$ in the next section. Thus $x-s$, for $x>s$, measures the number of requests in excess of supply, or backorders, and $B(s)$ is the expected number of backorders.

In the algorithm for computing the optimal allocation of spares described below, certain calculations must be performed. First, for each base it is necessary to compute the expected time between placing a request for an item from the depot and the base receipt of the item. In addition, this time could also include build-up time at the base. (For aircraft engines this total time is designated ARBUT--Automatic Resupply and Buildup Time). This time will be called the depot response time. Clearly the depot response time is dependent on the depot stock level. For example, if the depot has a very large quantity of stock on hand, then the average depot response time is dependent only on the average administrative and pipeline times. Designate this time for base j as R_j . On the other hand, if there is no stock on hand at the depot, then the average depot response time at base j is R_j+D , where D is the average depot repair time. Thus if the depot stock is any positive finite quantity, the average depot response time will be between R_j and $R_j + D$.

Let s represent the depot spare stock. Then the expected number of units delayed at the depot at some arbitrary point in time is given by

$$B(s|\lambda D) = \sum_{x>s} (x-s)p(x|\lambda D) ,$$

where $\lambda = \sum \lambda_j (1-r_j)$, λ_j = monthly demand rate at base j, and r_j = percentage of units that are base reparable. By dividing $B(s|\lambda D)$ by λ we get the average delay per demand measured in months. Next, define

$$d(s) = \frac{B(x|\lambda D)}{\lambda D}$$

Hence $d(s) \cdot D$ is the average delay per demand.

Now let us compute the expected response time at base j given that the depot stock is s. Let

$$T = r_j W_j + (1-r_j)(R_j + d(s) \cdot D),$$

where W_j is the average base repair time. Then T represents the average response time at base j. If s_j denotes the stock level of base j, then the expected number of backorders for base j can then be found from Equation 1 using the above definition of T and substituting s_j for s.

2.3 The Demand Distribution

The probability distribution for demand, $p(x|\lambda T)$, for each item is assumed to be a compound Poisson distribution. By using this type of distribution it is possible to consider a wide variety of shapes for the demand distributions. Another reason for using a compound Poisson distribution rather than a simple Poisson distribution is that it has a variance-to-mean ratio which is greater than or equal to 1 which is the ratio value of the Poisson. In practice, the variance displayed by actual data for the vast majority of items is greater than the mean thereby necessitating the use of a probability model other than the Poisson.

A compound Poisson probability model results when any one of a number of assumptions is made. Two sets of assumptions which lead to a particular type of compound Poisson model are described below.

A compound Poisson distribution is obtained whenever the individual demand requests follow a Poisson process and the number of demands per request is also a random variable.[2] In the METRIC model it is assumed that the order size distribution is a logarithmic distribution.[2] It has been shown that if the number of customer orders is a Poisson process and the number of demands per request are independent and identically distributed random variables following a logarithmic distribution, then the probability distribution for the total number of demands (for some specified time period) has a negative binomial distribution. The negative binomial is a particularly useful distribution to work with. First, it adequately describes many observed demand patterns; second, mathematically it is a simple distribution to work with.

There is another way in which a compound Poisson distribution can be obtained. For certain items the order size will almost always equal one. An example of such an item is an aircraft engine. Since demands occur one at a time, one might assume that a simple Poisson distribution would be an appropriate probability model to use in this situation. Unfortunately, this may not be the case. To choose the appropriate probability model to use in a decision problem we should actually determine the distribution of the demand about the forecasted value; that is, what is of real interest is the distribution of forecast errors. To test demand data to see if it follows a Poisson distribution, in practice, retrospective analysis is used. For example, a Chi-square goodness of fit test can be made on past data using the sample mean as an estimate of the true mean. If this statistical test is passed we may be willing to consider demand to have a Poisson distribution whose mean equals the sample mean. If the underlying process is stationary, the Poisson model could probably be used in making future decisions. On the other hand, if the process is really non-stationary, then this retrospective information must be used with great caution.

Subsequent decisions cannot be solely based on the results of the above statistical analysis. What is really needed in this case is to determine the distribution of demand around the forecasted value. Only if the distribution of forecast errors follows a Poisson distribution can a Poisson model be used in decision making. Often a Poisson model does not describe the distribution of forecast errors since the variance exceeds the mean. In these cases a compound Poisson distribution is often of value. Since the method used to forecast demand is of such obvious importance in choosing the correct model, let us now discuss it in greater depth.

Instead of using a point estimate for forecasting future demand, it is preferable to use a distribution of estimates. This is the case because if only a single number is used to estimate future mean demand, unless we are certain that this value equals the true mean demand, we will always underestimate the expected number of backorders. In other words, if we estimate the true mean demand as 3 when in fact it is felt that it is equally likely to be either 2, 3, or 4, then by using 3 as a point estimate of true mean demand we will under state the expected number of backorders. The reason for this is that the backorder function is a convex function of true mean demand. Thus a single point estimate of demand does not provide sufficient information for decision making purposes. To include a distribution of forecast values in the model rather than a single estimate we will use a Bayesian procedure. However, to use a Bayesian model for the true mean demand it is necessary to have a prior estimate of the probability distribution of the true mean demand. There are several ways that this prior distribution can be determined. One way is to choose the best distribution from a particular class of distributions. This is the method employed here. The family of distributions chosen for use is the gamma distribution. The best member of the family is selected based on the initial estimate of the distributions' parameter values.

The choice of the gamma is made for two reasons. First, the gamma distribution is a two parameter, unimodal, right-skewed, continuous distribution which is positive for only non-negative values. There are a wide variety of shapes this type of distribution can assume. The exact shape depends on the values taken on by the two parameters. The decision maker can reflect his degree of confidence in the initial estimate of true mean demand by choosing the parameters so that the distribution either has a large or small variance. If α and β are the parameters of the distribution, then its mean is $\alpha\beta$ and its variance is $\alpha\beta^2$. It should be pointed out that the assumption that the true mean demand is gamma distributed was really made assuming the analysis was made for a fixed time period. However, it turns out that this implies that the true mean demand has a gamma distribution over a time period of any duration.

The second reason for choosing the gamma distribution is that mathematically it is easy to work with. Furthermore, if we assume that the conditional demand distribution is Poisson over some time period, given the value of the mean, then the unconditional probability distribution of total demand over that time period is negative binomial. In what follows, the conditional demand distribution will be assumed to be Poisson and the distribution of true mean demand will be assumed to be gamma.

The Bayesian procedure for revising the probability distribution of the true mean demand is performed as follows. Suppose that in a time period of length Q there are k observed demands. Then combine the prior distribution of the true mean demand over that time period, call it f(u) , and the probability of observing those k demands, using Bayes formula as follows:

$$g(u|k) = \frac{p(k|u) \cdot f(u)}{\int_0^\infty p(k|u) \cdot f(u) du} ,$$

where $p(k|u)$ is the Poisson probability for observing k demands when the mean demand rate is u, and $g(u|k)$ is the posterior density for the demand rate u given k demands in the time period of length Q. Then

$$g(u|k) = \frac{(1 + 1/\beta)^{k+\alpha}}{(k + \alpha - 1)!} \quad u^{k+\alpha-1} \quad \exp[-u(1+1/\beta)] ,$$

where α, β are the two parameters of the gamma prior. Thus the posterior distribution of true mean demand is also gamma with parameters $k+\alpha$ and $\frac{\beta+1}{\beta}$.

It must be assumed that the variance to mean ratio in the prior density functions of true mean demand is the same for each base, that is, β is the same for all the bases. This assumption is made so that the prior density of true mean depot demand is also gamma with $\alpha = \sum(1-r_j)\alpha_j$ and β as the two parameters, where α_j is the first gamma parameter at base j and r_j is the portion of the failures that are base reparable.

2.4 The Backorder Function Revisited

We are interested in computing the expected number of backorders over a period of length T, the average response time discussed earlier. But $g(u|k)$ was calculated for a time period of length Q. As mentioned above, the same posterior densities are appropriate for any period of time. The Poisson probability of observing demands in T is

$$p(x|\frac{uT}{Q}) = \frac{(\frac{uT}{Q})^x \quad e^{-uT/Q}}{x!}$$

. Thus the expected number of

backorders in T at base j given k demands during the data period of length Q when base stock is s_j is

$$B(s_j|k) = \int_0^\infty g(u|k) \sum_{x>s_j} (x-s_j)\cdot p(x|\tfrac{uT}{Q})\, du$$

$$= \sum_{x>s_j} (x-s_j)\cdot h(x|s) ,$$

where $h(x|s) = \binom{k+\alpha+x-1}{x} \left(\frac{\beta Q+Q}{\beta Q+Q+\beta T}\right)^{k+\alpha} \left(\frac{\beta T}{\beta Q+Q+\beta T}\right)^{x}$, Eq 2

s is the depot stock level, and $T = r_j W_j + (1-r_j)(R_j + d(s)D)$.

Thus $h(x|s)$ is a negative binomial probability distribution in which p equals $(\beta Q+Q)/(\beta Q+Q+\beta T)$ and second parameter, r , equals $k+\alpha$.

Let us now state some properties associated with the backorder function.

1) The backorder function for base j is a convex function of its stock level. This follows since $B(s_j+1)-B(s_j) = -\sum_{x\geq s_j+1} p(x|\lambda T)$.

2) $d(s) = \frac{B(s|\lambda D)}{\lambda D} = \frac{\sum (x-s)\cdot p(x|\lambda D)}{\lambda D}$ is a convex, strictly decreasing function of s , where $p(x|\lambda D)$ is the probability of x units being in depot resupply given a demand rate λ and an average depot repair time of D . This result is obvious since the backorder function is convex and strictly decreasing.

3) Since $d(s)$ is convex, T is a strictly decreasing convex function of s .

4) $\lim_{s\to\infty} T = r_j W_j + (1-r_j)R_j = \overline{T}$.

5) In the steady state negative binomial distribution for the number of units in resupply at any base, $h(x|s)$, the value of p is a convex strictly decreasing function of T .

6) p is a strictly increasing function of depot stock s .

7) $$\lim_{s\to\infty} p = \frac{\beta Q+Q}{\beta Q+Q+\beta(r_j W_j+(1-r_j)R_j)}$$

8) The mean of the negative binomial distribution, $\frac{(k+\alpha)(1-p)}{p}$, is a strictly decreasing function of p and hence of s .

9) Thus $\lim_{s\to\infty} \frac{(k+\alpha)(1-p)}{p} = (k+\alpha) \frac{\beta\overline{T}}{\beta Q+Q}$.

10) The variance, $\frac{(k+\alpha)(1-p)}{p^2}$, is a strictly decreasing function of p and s .

11) The backorder function need not be a convex function of depot stock.

Property 11) unfortunately makes the computational procedure for finding the optimum depot stock and base stocks somewhat more difficult. If the function were convex, then some simple search procedure could be employed to find the optimum solution. Although the backorder function is not always convex, in a very substantial number of cases it will be. In those cases where it is not convex, experience indicates that it is only "slightly" non-convex, that is, the function may have small oscillations near the optimum, but at a significant distance from the optimum the function is strictly monotone increasing. These properties are exploited in the algorithm described in the next section for obtaining a local optimum solution to the problem.

III. THE ALGORITHM

3.1 Introduction

In this section an algorithm is presented for finding a local optimal solution to the redistribution problem. The algorithm is based on the marginal analysis procedure described by Sherbrooke.[3,4] Based on certain properties of the backorder function, it is shown how to alter Sherbrooke's method in order to reduce the number of required calculations at the expense of finding only a local optimum.

3.2 Background

Recall that the objective of the redistribution problem is to allocate an existing spare stock of an item between the bases and depot such that the total number of expected backorders is minimized. For each given level of depot stock it is possible to calculate the resulting number of expected backorders at the bases. In Table 1, each row corresponds to a particular level of depot spare stock,

DEPOT STOCK	EXPECTED BACKORDERS TOTAL SPARE STOCK AT BASES		
	0	1	2
0	x	xx	xxx
1	xx	xxx	
2	xxx		

Table 1

and each column corresponds to a given amount of spare stock allocated to all the bases. In the table the number found at the intersection of a row and a column indicates the expected number of backorders given the total base and depot spare stocks. For example, the number in the intersection of the second row and second solumn -- denoted by xxx -- is the expected number of backorders given that 1 unit of stock is allocated to the depot and 1 unit to the bases. Clearly if there are 2 units of spare stock in the system the expected number of backorders that result from the various possible distributions can be found in the table on the appropriate diagonal. Thus the three xxx symbols correspond to the expected backorders resulting from the different possible allocations of the 2 units of stock. If there are M units of stock currently in the system, then by calculating the diagonal corresponding to the possible distributions of this stock the optimum solution can be obtained by simply locating the smallest number on the diagonal.

Instead of calculating the whole table as Sherbrooke does, it is possible to substantially limit the search. This is of particular importance when there are many hundreds or even thousands of units of a particular item in spare stock. If the search can be limited to a small percentage of the table, a significant reduction in computational effort will occur. When this calculation is performed for thousands of items the benefits of the proposed procedure are great.

3.2 The Optimum Base Allocation for a Fixed Depot Stock

Suppose M units of an item are in stock and s units are allocated to the depot. Then M-s units are available to be distributed between the bases. Let us now describe a method for determining the optimum distribution of the M-s units of stock.

For each base j calculate $h(x_j|s)$. The best allocation of the $M-s$ units occurs when $\sum_j \sum_{x_j>s_j} (x_j-s_j)\cdot h(x_j|s)$

is minimized, where s_j is the amount of stock located at base $j(\Sigma s_j = M-s)$. If $M=s$, That is, all the spare stock is located at the depot, then the only solution is to have $s_j=0$ for each base j . If $M-s = 1$, then there is one unit of stock that is available for distribution to the bases. The problem is to determine which base gets the unit. For each base calculate

$$B(1) - B(0) = -\sum_{x_j=1}^{\infty} h(x_j|s) .$$

The base that gets the unit of stock is the one for which this quantity is smallest. This corresponds to the base which marginally reduces the expected number of backorders by the largest amount. Let base m be the one which receives the unit of stock.

If there are two units to allocate to the bases, then the first unit again goes to base m . This is due to the fact that by allocating this unit to base m expected backorders are reduced by as large an amount as possible. The second unit goes to the base which can now reduce expected backorders by the maximum amount. This base is found by calculating $B(1) - B(0)$ for all bases except base m ; for base m calculate $B(2) - B(1) = -\sum_{x_m>1} h(x_m|s)$. Again determine the smallest of these numbers. The corresponding base gets the second unit of stock.

In general, the computation is made as follows: If there are n bases and $s_1, s_2, \ldots, s_n$ units of stock are currently being allocated to bases

1, 2, ..., n respectively, then to allocate the next unit of stock, find the minimum of the numbers

$$B(s_1+1) - B(s_1) = -\sum_{x_1>s_1} h(x_1|s)$$

$$B(s_2+1) - B(s_2) = -\sum_{x_2>s_2} h(x_2|s)$$

$$\vdots$$

$$B(s_n+1) - B(s_n) = -\sum_{x_n>s_n} h(x_n|s) \quad .$$

The base corresponding to the smallest number gets the additional unit of stock. Note that this calculation is simple to make. To add one unit it is necessary to calculate only the difference corresponding to the base which received the previous unit of stock since all the other numbers s_j remain unchanged.

Furthermore, $B(s_j+2) - B(s_j+1) = B(s_j+1) - B(s_j) - h(s_j+1|s)$, or the new difference is equal to old difference minus the value of the probability $h(s_j+1|s)$. Thus the calculation to determine which base gets the additional unit of stock can be found by subtracting only one number and finding the minimum of the resultant quantities. It is also a simple task to calculate the negative binomial probabilities, $h(x|s)$, using the following recursion relation

$$h(x+1|s) = \frac{(x+k+\alpha)}{(x+1)} (1-p)\, h(x|s) \; ,$$

where $h(x|s)$ is defined in Equation 2.

Thus for a given depot stock s and base stock $M-s$, the optimum allocation of the stock between the bases can be found using the technique described above. This technique is especially easy to implement on a computer.

3.3 The Solution Procedure

In the last section a method was given for determining the optimum stock level for each base given the depot stock level. Now let us develop a technique for finding the proper depot stock level. As mentioned previously, the optimum depot stock level can be found by performing the calculations described in the previous section for all possible values of depot stock (0 to M). For even moderately sized values of M this procedure is inefficient.

The expected number of backorders for base j given a depot stock level of s and base stock level s_j is

$$\sum_{x_j > s_j} (x_j - s_j) \cdot h(x_j | s)$$

If $s+1$ units are in depot stock, then the only difference in the calculation is that $h(x_j|s+1)$ is used in place of $h(x_j|s)$. In Section 2.4, various properties were developed associated with $h(x|s)$. In particular it was shown that as s increases the negative binomial probability distribution approaches a limiting distribution since p approaches a limiting value. It is easy to see that as s exceeds the average demand in a time period of length D, λD, the impact on the probability values becomes negligible. This occurs because $d(s) \to 0$ quite rapidly after s exceeds λD, and, therefore, T quickly approaches $\overline{T}$. To illustrate this fact consider the following example.

Suppose we have six bases at which $W_1 = \ldots = W_6 = 20$, $R_1 = \ldots = R_6 = 20$, $\lambda_1 = \ldots = \lambda_6 = .1$, $r_1 = \ldots = r_6 = .9$ and $D = 40$. To keep the calculations simple, let us assume we know the true mean demand at each base, and therefore the demand distribution at each base follows a Poisson distribution. The values found in Table 2 indicate how $d(s)$ and T change as a function of s .

CHANGES IN d(r) AND T
AS A FUNCTION OF r
($\lambda D = 2.4$)

Depot Stock	$d(s) = \frac{\sum(x-s)\ p(x\mid\lambda D)}{\lambda D}$	T
0	1.000	24
1	0.621	22.84
2	0.33	21.32
3	0.154	20.616
4	0.061	20.244
5	0.022	20.088
6	0.0067	20.0268
7	0.0012	20.0048
8	0.0004	20.0016
9	0.0	20.000

Table 2

Since the difference between $h(x|s)$ and $h(x|s+1)$ is so small as s becomes large, there is, for all practical purposes, no difference in the expected number of backorders for like values of total base stock when depot stock is either s or $s+1$. Hence there is no advantage in investigating system performance for values of depot stock larger than some number. Designate this number by $\overline{M}$. The value of $\overline{M}$ can be based on some criteria such as d(s) is less than some prescribed value, T is within some specified distance of $\overline{T}$, p is as close to its limiting value as desired, or the mean of the distribution is within some distance of its limiting value. Thus it is not necessary to compute all the entries in Table I; no more than the first $\overline{M}+1$ rows need be calculated.

In most situations the total system stock is large enough to permit the depot stock level to be at least λD. Since significant reductions can be made to the value of d(s) when $s \leq \lambda D$, it appears likely that the optimum value of s will not be very much less than λD. In test problems this was always the case when the system stock was quite large compared to total demand. On the other hand, only when system stick became relatively small as compared to expected demand did the optimal value of depot stock drop much below λD. Thus the algorithm, for all practical purposes, limits the search for the best value of depot stock to a limited range aground $[\lambda D]$, ($[\lambda D]$ is the greatest integer value less than or equal to λD).

The steps of the computational procedure are given below and a flow chart is given in Appendix A.

Step 0) Determine the values of the parameters used in the probability distributions. Set Z, the run length used to terminate calculations, $\overline{M}$, and the starting value for depot stock, b. Calculate the expected number of base backorders when depot stock equals b, that is, B(b). Call this value v. Set a=b+1, d=b, q=0, and c=1.

Step 1) Calculate $B(a)$. If $B(a) \geq B(a-1)$ go to Step 2. Otherwise go to Step 4.

Step 2) Replace the value of q with $q+1$ and check the value of q and a. If either $a=\overline{M}$, $a=o$ or $q=Z$, go to Step d. Otherwise replace the value of a with $a+c$ and return to Step 1.

Step 3) If $c=1$, then reset the value of c to -1, set $a=b-1$, $q=0$, and return to Step 1; otherwise terminate computations with the solution being to stock d units at the depot and the corresponding expected number of backorders being equal to $B(d)$.

Step 4) Set $q=0$. If $B(a)<v$, set $v=B(a)$ and $d=a$, and go to Step 5. If $B(a) \geq v$, go to Step 5.

Step 5) If either $a=\overline{M}$ or $a=0$, go to Step 3. Otherwise replace the value of a with $a+c$ and return to Step 1.

3.4 Some Final Comments

The computational procedure stated above requires that a starting point and a range value be specified. If no better starting point is available, then begin with depot stock equal to $[\lambda D]$. However, once some experience is available, it is best to start the procedure with the initial value of depot stock equal to its previously calculated optimal value. Since the parameter values do not change very rapidly, the previous optimal value of depot stock if not still optimal should be quite close to new optimum.

The range value, Z , which is used to terminate the computational procedure, must be chosen with some care. Once the value of the backorder function increases or stays the same for Z consecutive values of depot stock the procedure stops searching for a better solution in that direction. Hence, if Z is too small there is a possibility of missing the optimum. On the other hand, if Z is too large then an excessive amount of computer time will be used to find the optimum. In the cases where system stock is 100, a value of $Z=5$ seems to work well. However, if the system stock is several thousand it is probably necessary to increase the value of Z. At this time this is an open question and is therefore an area requiring further investigation.

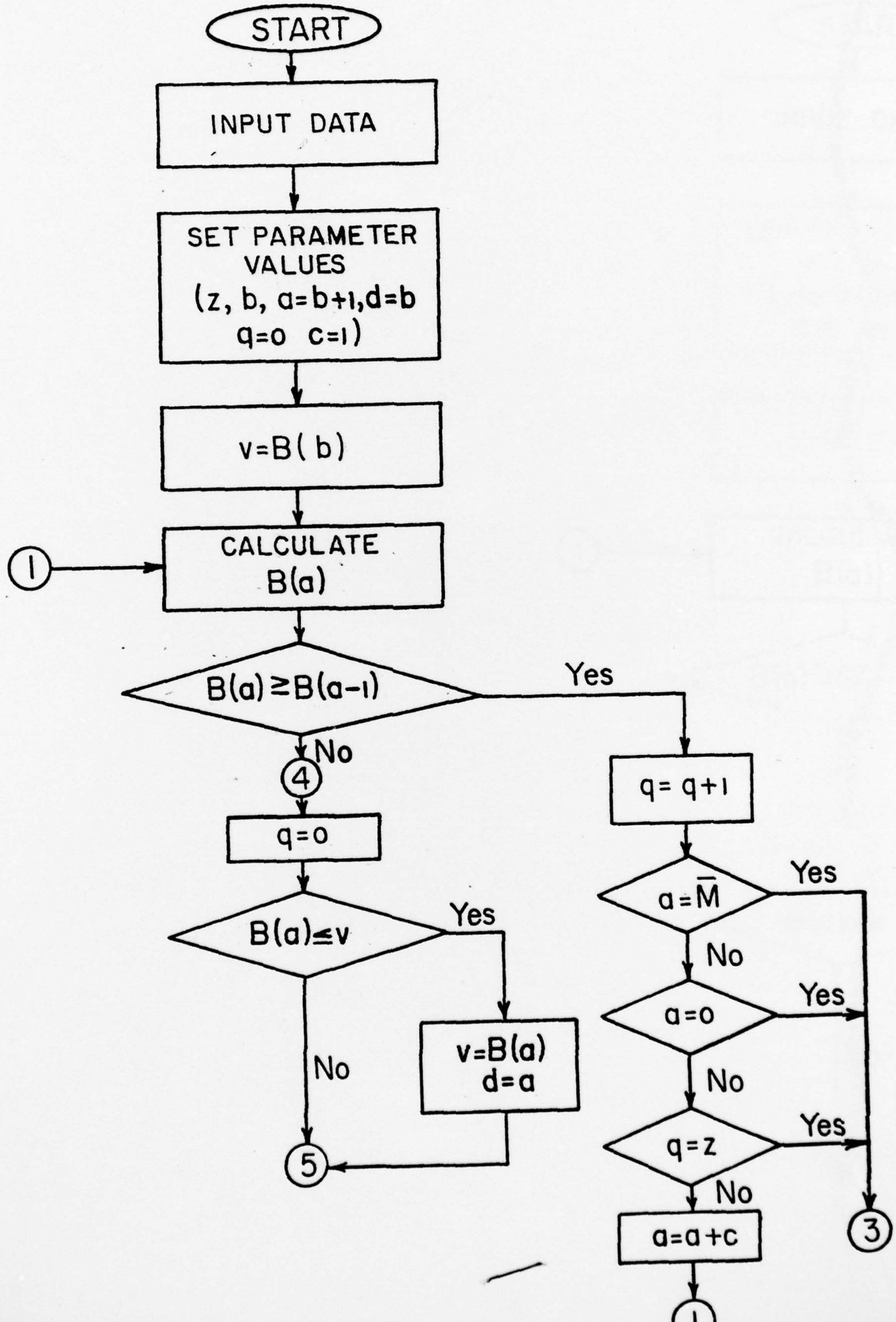
In the vast majority of the cases tested the backorder function has been convex although there is no guarantee that it will be. In these cases, the local minimum obtained by the algorithm will also be the global minimum. Even when the backorder function is not convex, the algorithm will most likely produce the optimal solution. In fact, the optimal solution was found for all the test problems.

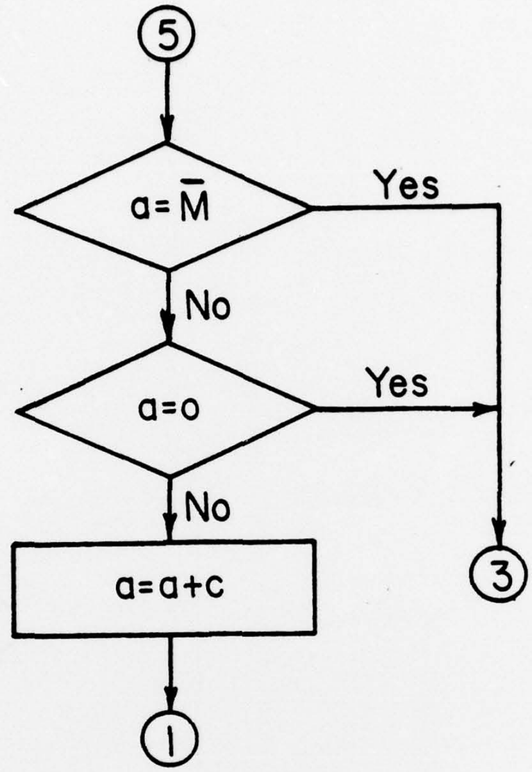
It could be argued that by using a more sophisticated search method the computational requirements could be reduced even further. See, for example, Item 5 in the Bibliography. This may be the case. It is felt, however, that in the algorithm by using the past optimal value as the starting point for depot stock that very little improvement will be made in computer run times by using a more complicated search procedure. This is another area that requires additional testing. In its present form the proposed procedure should reduce computational times to a small fraction of their current values. This has been the case in each of the test problems.

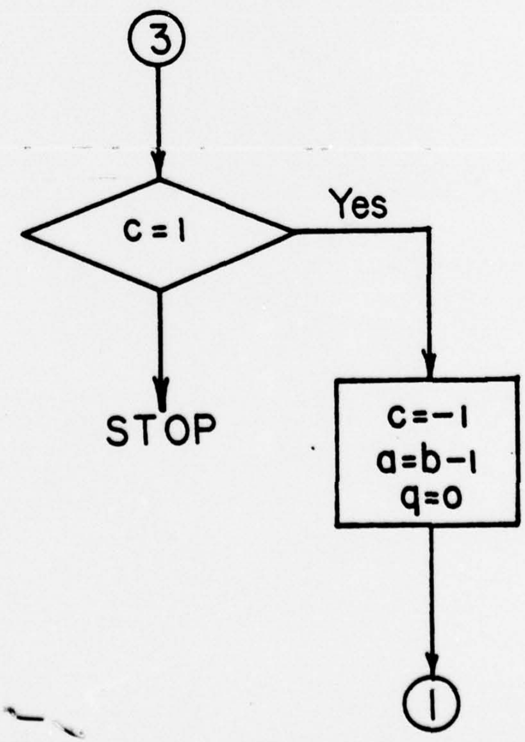
APPENDIX A

FLOW CHART OF THE

ALGORITHM


START
INPUT DATA
SET PARAMETER VALUES (z, b, a=b+1, d=b q=0 c=1)
v=B(b)
CALCULATE B(a)
1
B(a)≥B(a−1)
Yes
No
4
q=0
B(a)≤v
Yes
No
v=B(a) d=a
5
q=q+1
a=M̄
Yes
No
a=0
Yes
No
q=z
Yes
No
a=a+c
1
3


5
a = M̄
Yes
No
a=o
Yes
No
a=a+c
3
1


3
c = 1
Yes
STOP
c=-1
a=b-1
q=0
1

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