AD-A042 733 WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
ON INVERSES OF VANDERMONDE AND CONFLUENT VANDERMONDE MATRICES I--ETC(U)
JUN 77 W GAUTSCHI DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1759 NL

| OF |

ADA042 733

END
DATE
FILMED
9-77
DDC

MRC Technical Summary Report #1759

ON INVERSES OF VANDERMONDE AND CONFLUENT VANDERMONDE MATRICES III

Walter Gautschi

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

June 1977

(Received May 27, 1977)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C.
20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ON INVERSES OF VANDERMONDE AND CONFLUENT VANDERMONDE MATRICES III

Walter Gautschi

Technical Summary Report #1759
June 1977

ABSTRACT

We derive lower bounds for the norm of the inverse Vandermonde matrix and the norm of certain inverse confluent Vandermonde matrices. They supplement upper bounds which were obtained in previous papers.

AMS (MOS) Subject Classifications: 15A12, 65F35

Key Words: matrix norms, norm estimates, inverse Vandermonde matrix, inverse *confluent* Vandermonde matrix

Work Unit Number 7 (Numerical Analysis)

EXPLANATION

The sensitivity of a system of linear algebraic equations to small perturbations in the data depends in large measure on the magnitude of the inverse of the coefficient matrix. It is therefore of interest to estimate the norm (i.e., the magnitude) of the inverse of a matrix. We do this here for the Vandermonde matrix and certain related matrices, which occur frequently in problems of numerical analysis, providing lower bounds for the norms in question. Upper bounds, and exact formulas in special cases, have been given previously.

Sponsored in part by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under grant MCS 76-00842A01.

ON INVERSES OF VANDERMONDE AND CONFLUENT VANDERMONDE MATRICES III

Walter Gautschi

1. Introduction. Norm estimates for the inverse of a Vandermonde matrix, or the inverse of confluent Vandermonde matrices, have been the subject of several previous papers [1], [2], [4]. The emphasis there was on upper bounds in the case of general complex nodes, or identities when the nodes are positive [1], [2] or real and symmetric with respect to the origin [4]. We now wish to supplement these results by providing lower bounds in the case of arbitrary complex nodes. We obtain these bounds by applying Jensen's formula in the theory of analytic functions to appropriate polynomials.

2. Jensen's formula for polynomials. Given a polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0 , \tag{2.1}$$

with complex coefficients a_μ, let $\zeta_1, \zeta_2, \ldots, \zeta_n$ denote its zeros ordered such that

$$|\zeta_1| \le |\zeta_2| \le \cdots \le |\zeta_r| \le 1 < |\zeta_{r+1}| \le |\zeta_{r+2}| \le \cdots \le |\zeta_n| .$$

Jensen's formula, applied to (2.1) on the unit circle, then gives [6]

$$|a_n \zeta_{r+1} \zeta_{r+2} \cdots \zeta_n| = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|p(e^{i\theta})|\,d\theta\right) ,$$

hence, letting $M = \max\limits_{0\le\theta\le 2\pi} |p(e^{i\theta})|$,

$$|a_n \zeta_{r+1} \zeta_{r+2} \cdots \zeta_n| \le M \le \sum_{\mu=0}^{n} |a_\mu| . \tag{2.2}$$

Thus,

$$\sum_{\mu=0}^{n} |a_\mu| \ge |a_n| \prod_{\nu=1}^{n} \max(1, |\zeta_\nu|) . \tag{2.3}$$

Equality in (2.3) holds if and only if $a_0 = a_1 = \cdots = a_{n-1} = 0$, i.e., $p(z) = a_n z^n$. Indeed, if $p(z) = a_n z^n$, then (2.3) (with equality) is trivial. Conversely, if we have

Sponsored in part by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under grant MCS 76-00842A01.

equality in (2.3), we must have equality in (2.2), hence, by Jensen's formula, $|p(e^{i\theta})| \equiv M$ for $0 \leq \theta \leq 2\pi$. Since

$$|p(e^{i\theta})|^2 = \sum_{k,\ell=0}^{n} a_k \bar{a}_\ell e^{i(k-\ell)\theta} = \sum_{\lambda=-n}^{n} c_\lambda e^{i\lambda\theta}$$

is a trigonometric polynomial, with coefficients

$$c_\lambda = \sum_{k=-\infty}^{\infty} a_k \bar{a}_{k-\lambda}, \quad c_{-\lambda} = \bar{c}_\lambda$$

(the convention $a_\mu = 0$ if $\mu < 0$ or $\mu > n$ is used here), it can be constant equal to M^2 only if $c_n = c_{n-1} = \cdots = c_1 = 0$ and $c_0 = M^2$. The first condition, $c_n = 0$, implies $a_n \bar{a}_0 = 0$, hence $a_0 = 0$ (since $a_n \neq 0$). The second condition, $a_n \bar{a}_1 + a_{n-1} \bar{a}_0 = 0$, then gives $a_1 = 0$, and continuing in this manner, we find recursively $a_0 = a_1 = \cdots = a_{n-1} = 0$.

3. Inverse Vandermonde matrix. We denote the Vandermonde matrix of order n by

$$V_n(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{bmatrix}, \tag{3.1}$$

where $z^T = [z_1, z_2, \ldots, z_n]$ is a vector of n complex numbers, called "nodes". If the nodes are mutually distinct, then $V_n(z)$ has an inverse, which we denote by

$$V_n^{-1}(z) = [u_{\lambda\mu}]_{\lambda,\mu=1}^{n}. \tag{3.2}$$

We are interested in the ℓ_∞-norm of (3.2),

$$\|V_n^{-1}(z)\|_\infty = \max_{1\leq\lambda\leq n} \sum_{\mu=1}^{n} |u_{\lambda\mu}| .$$

Theorem 3.1. *If $z_1, z_2, \ldots, z_n$ are mutually distinct complex numbers, and $n > 1$, then*

$$\|V_n^{-1}(z)\|_\infty > \max_{1\leq\lambda\leq n} \prod_{\substack{\nu=1 \\ \nu\neq\lambda}}^{n} \frac{\max(1,|z_\nu|)}{|z_\lambda - z_\nu|} . \tag{3.3}$$

Proof. We recall [4] that the elements $u_{\lambda\mu}$ in (3.2) are the coefficients of the fundamental Lagrange interpolation polynomials associated with the nodes z_ν,

$$\prod_{\substack{\nu=1\\ \nu\neq\lambda}}^{n} \frac{z - z_\nu}{z_\lambda - z_\nu} = u_{\lambda 1} + u_{\lambda 2} z + \cdots + u_{\lambda n} z^{n-1} . \tag{3.4}$$

Applying (2.3) and the remark following (2.3) to the polynomial of degree $n - 1$ in (3.4), we find

$$\sum_{\mu=1}^{n} |u_{\lambda\mu}| > \left(\prod_{\nu\neq\lambda} \frac{1}{|z_\lambda - z_\nu|} \right) \prod_{\nu\neq\lambda} \max(1, |z_\nu|) . \tag{3.5}$$

If λ_0 is the index λ for which the right-hand expression in (3.5) attains its maximum, then that maximum is less than $\sum_{\mu=1}^{n} |u_{\lambda_0 \mu}|$, hence less or equal than $\max_{1\leq\lambda\leq n} \sum_{\mu=1}^{n} |u_{\lambda\mu}|$. This establishes (3.3) and proves Theorem 3.1.

The lower bound in (3.3) supplements the upper (attainable) bound in [1], which is of the same form as (3.3) except that the ℓ_∞-norm of the 2-vectors $[1, z_\nu]$ in the numerator factors is replaced by the ℓ_1-norm.

4. Inverse confluent Vandermonde matrices. The technique used in the proof of Theorem 3.1 can be adapted to confluent Vandermonde matrices. We illustrate this with the particular matrix

$$U_{2n}(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ z_1 & z_2 & \cdots & z_n & 1 & 1 & \cdots & 1 \\ z_1^2 & z_2^2 & \cdots & z_n^2 & 2z_1 & 2z_2 & \cdots & 2z_n \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ z_1^{2n-1} & z_2^{2n-1} & \cdots & z_n^{2n-1} & (2n-1)z_1^{2n-2} & (2n-1)z_2^{2n-2} & \cdots & (2n-1)z_n^{2n-2} \end{bmatrix} \tag{4.1}$$

considered previously in [1], [2].

Theorem 4.1. *If $z_1, z_2, \ldots, z_n$ are mutually distinct complex numbers, and $n > 1$, then*

$$\|U_{2n}^{-1}(z)\|_\infty > \max_{1\leq\lambda\leq n} b_\lambda \prod_{\substack{\nu=1\\ \nu\neq\lambda}}^{n} \left(\frac{\max(1, |z_\nu|)}{|z_\lambda - z_\nu|} \right)^2 , \tag{4.2}$$

where b_λ is the larger of the two quantities

$$(4.3)\qquad b_\lambda^{(1)} = \max(1,|z_\lambda|),\ b_\lambda^{(2)} = \max\left(2\left|\sum_{\nu\neq\lambda} 1/(z_\lambda - z_\nu)\right|,\ \left|1 + 2z_\lambda \sum_{\nu\neq\lambda} 1/(z_\lambda - z_\nu)\right|\right).$$

Proof. We have [2]

$$U_{2n}^{-1} = \begin{bmatrix} V \\ W \end{bmatrix},\quad V = [v_{\lambda\mu}],\quad W = [w_{\lambda\mu}],$$

where

$$(4.4)\qquad \begin{aligned} \ell_\lambda^2(z)[1 - 2\ell_\lambda'(z_\lambda)(z - z_\lambda)] &= \sum_{\mu=1}^{2n} v_{\lambda\mu} z^{\mu-1} \\ \ell_\lambda^2(z)(z - z_\lambda) &= \sum_{\mu=1}^{2n} w_{\lambda\mu} z^{\mu-1} \end{aligned} \qquad 1 \le \lambda \le n,$$

and $\ell_\lambda(z)$ denotes the fundamental Lagrange interpolation polynomial in (3.4). Applying (2.3) to the polynomials in (4.4), and taking note of the remark following (2.3), one finds

$$\sum_{\mu=1}^{2n} |v_{\lambda\mu}| > b_\lambda^{(2)} \prod_{\nu\neq\lambda} \left(\frac{\max(1,|z_\nu|)}{|z_\lambda - z_\nu|}\right)^2,$$

$$\sum_{\mu=1}^{2n} |w_{\lambda\mu}| > b_\lambda^{(1)} \prod_{\nu\neq\lambda} \left(\frac{\max(1,|z_\nu|)}{|z_\lambda - z_\nu|}\right)^2,$$

where $b_\lambda^{(1)}$, $b_\lambda^{(2)}$ are as defined in (4.3). Denoting the products $\prod_{\nu\neq\lambda}$ on the right by π_λ, and observing that $\|U_{2n}^{-1}\|_\infty = \max(\max_\lambda \sum_{\mu=1}^{2n} |v_{\lambda\mu}|, \max_\lambda \sum_{\mu=1}^{2n} |w_{\lambda\mu}|)$, an argument similar to the one after (3.5) will show that $b_\lambda^{(1)}\pi_\lambda < \|U_{2n}^{-1}\|_\infty$, $b_\lambda^{(2)}\pi_\lambda < \|U_{2n}^{-1}\|_\infty$ for all $\lambda = 1,2,\dots,n$, hence $\max(b_\lambda^{(1)},b_\lambda^{(2)})\pi_\lambda < \|U_{2n}^{-1}\|_\infty$ for all $\lambda = 1,2,\dots,n$. This proves Theorem 4.1.

The lower bound in (4.2) supplements the (attainable) upper bound in [2], which is of the same form as (4.2) except that the ℓ_∞-norm of the 2-vectors $[1,z_\nu]$ in the numerator factors, and the ℓ_∞-norms defining $b_\lambda^{(1)}$ and $b_\lambda^{(2)}$ are all replaced by the respective ℓ_1-norms. In the case of positive nodes z_ν another (usually sharper) lower bound can be found in [3, Theorem 2.1].

5. Examples.

Example 5.1 (roots of unity). $z_\nu = e^{2\pi i(\nu-1)/n}$, $\nu = 1,2,\ldots,n$.

In view of

$$\ell_\lambda(z) = \frac{1}{n}\sum_{\mu=1}^{n}\left(\frac{z}{z_\lambda}\right)^{\mu-1}, \quad \lambda = 1,2,\ldots,n,$$

we obtain from (3.4), and from (4.4) after a little computation,

$$\|V_n^{-1}(z)\|_\infty = 1, \quad \|U_{2n}^{-1}(z)\|_\infty = 2 - \frac{1}{n}. \tag{5.1}$$

The lower bounds in (3.3) and (4.2) both evaluate to $1/n$, while the upper bounds in [1], [2] are $2^{n-1}/n$ and $(2n-1)4^{n-1}/n^2$, respectively.

Example 5.2 (roots of unity on half-circle). $z_\nu = e^{2\pi i(\nu-1)/N}$, $\nu = 1,2,\ldots,n$, where $n = [N/2] + 1$.

The true norms of V_n^{-1} and U_{2n}^{-1}, as well as the lower bounds of Theorems 3.1 and 4.1 and the upper bounds in [1], [2] are shown in Table 5.1 for $N = 5(5)20$ (2). It is interesting to note how deletion of the roots of unity on a half-circle results in substantially larger values of $\|V_n^{-1}\|_\infty$ and $\|U_{2n}^{-1}\|_\infty$.

		$\|V_n^{-1}\|_\infty$			$\|U_{2n}^{-1}\|_\infty$		
N	n	lower	true	upper	lower	true	upper
5	3	7.24(-1)	1.89	2.89	1.57	1.79(1)	4.19(1)
10	6	1.17	1.47(1)	3.75(1)	8.29	2.36(3)	1.56(4)
15	8	4.25	2.03(2)	5.45(2)	1.46(2)	6.18(5)	4.48(6)
20	11	1.17(1)	2.76(3)	1.20(4)	1.52(3)	1.59(8)	3.03(9)

Table 5.1. Norm estimates for Example 5.2.

Example 5.3. $e_n(z_\nu) = 0$, $\nu = 1,2,\ldots,n$, where $e_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$.

Using the zeros of e_n, tabulated in [5], we obtain the results in Table 5.2.

---

(2) The integers in parentheses indicate exponents of 10.

	$\lVert V_n^{-1} \rVert_\infty$			$\lVert U_{2n}^{-1} \rVert_\infty$		
n	lower	true	upper	lower	true	upper
5	1.12	2.08	3.93	4.76	2.21(1)	7.90(1)
10	2.44	5.22	1.45(1)	4.45(1)	2.52(2)	1.96(3)
15	7.45	1.69(1)	5.71(1)	6.08(2)	3.69(3)	4.22(4)
20	2.27(1)	5.36(1)	2.02(2)	7.54(3)	4.79(4)	6.84(5)

Table 5.2. Norm estimates for Example 5.3.

Example 5.4. $e_N(z_\nu) = 0$, $\mathrm{Im}\, z_\nu \geq 0$, $\nu = 1,2,\ldots,n$, where $n = \left[\frac{N+1}{2}\right]$.

		$\lVert V_n^{-1} \rVert_\infty$			$\lVert U_{2n}^{-1} \rVert_\infty$		
N	n	lower	true	upper	lower	true	upper
5	3	1.62	2.74	3.13	6.69	2.51(1)	3.29(1)
10	5	5.71	1.09(1)	1.27(1)	1.38(2)	6.15(2)	8.35(2)
15	8	3.07(1)	6.82(1)	8.63(1)	5.71(3)	3.24(4)	5.19(4)
20	10	1.60(2)	3.60(2)	4.49(2)	1.88(5)	1.07(6)	1.66(6)

Table 5.3. Norm estimates for Example 5.4.

Similarly as in Example 5.2, deletion of the zeros in the lower half-plane has the effect of increasing the norms of V_n^{-1} and U_{2n}^{-1}.

REFERENCES

1. Gautschi, W.: On inverses of Vandermonde and confluent Vandermonde matrices. Numer. Math. 4, 117-123 (1962).

2. Gautschi, W.: On inverses of Vandermonde and confluent Vandermonde matrices II. Numer. Math. 5, 425-430 (1963).

3. Gautschi, W.: Construction of Gauss-Christoffel quadrature formulas. Math. Comp. 22, 251-270 (1968).

4. Gautschi, W.: Norm estimates for inverses of Vandermonde matrices. Numer. Math. 23, 337-347 (1975).

5. Iverson, K. E.: The zeros of the partial sums of e^z. Math. Tables Aids Comput. 7, 165-168 (1953).

6. Mahler, K.: An application of Jensen's formula to polynomials. Mathematika 7, 98-100 (1960).

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) MRC-TSR-1759	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: (9) Technical	
4. TITLE (and Subtitle): (6) ON INVERSES OF VANDERMONDE AND CONFLUENT VANDERMONDE MATRICES III.	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) Walter Gautschi	8. CONTRACT OR GRANT NUMBER(s): (15) DAAG29-75-C-0024, VNSF-MCS-76-00842A01
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 7 (Numerical Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below. (12) 10p.	12. REPORT DATE: (11) June 1977
	13. NUMBER OF PAGES: 7
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709	National Science Foundation Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

matrix norms
norm estimates
inverse Vandermonde matrix
inverse confluent Vandermonde matrix

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We derive lower bounds for the norm of the inverse Vandermonde matrix and the norm of certain inverse confluent Vandermonde matrices are derived. They supplement upper bounds which were obtained in previous papers.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

221 200