ESD-TR-77-163

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Technical Note 1977-2

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Astrometry in Small Fields

14 June 1977

Prepared for the Department of the Air Force
under Electronic Systems Division Contract F19628-76-C-0002 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LEXINGTON, MASSACHUSETTS

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract F19628-76-C-0002.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

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ASTROMETRY IN SMALL FIELDS

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Group 94

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ABSTRACT

This report discusses the techniques and analysis for the precise reduction of the topocentric and apparent places of stars and artificial satellites. Included is a full discussion of the method of star constants, independent day numbers, geocentric parallax, parallactic refraction, and errors. Also described, in detail, is the analysis necessary for the real time modeling of the telescope-camera system, its theoretical basis, and differential reduction procedures. In addition, an original method for computing the distance of an artificial satellite from two measurements of position and one of angular velocity is developed.

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I. INTRODUCTION

The purpose of this report is three-fold. One aim was the complete elucidation, through terms of the second-order in small quantities, for reducing the mean place of a star to its apparent place and thence to its topocentric place. Another reason was to develop the basis for the telescope modeling used in the GEODSS Local Astrometric Calibration Procedure and prepare the way for the utilization of charge coupled devices or charge injected devices. These two rely on classical photographic astrometry. In particular, the plate modeling used in both the method of dependences (Schlesinger[1]) and the plate overlap technique (Eichorn[2]) are extended to the real-time problem posed by artificial satellite reductions. Lastly, the analysis necessary for the recovery of an artificial satellite's geocentric position from its measured, topocentric position is presented.

For the most part the material contained herein is a review and summary of the relevant astronomical literature. Original contributions are scattered throughout (principally Eqs. (12), § IIIB, § VI, § VIIA).

The epoch-to-epoch reductions of a star's mean place and its cataloged mean place have already been completely dealt with (Taff[3, 4]). Hence, the initial data are position, proper motion, and annual parallax at the beginning of the nearest Besselian solar year.

II. MEAN PLACE TO APPARENT PLACE TO TOPOCENTRIC PLACE REDUCTIONS

The reduction procedures now used for the mean place to apparent place reductions were developed by Bessel. They represent approximations of the rigorous formulas which retain all second order terms. Their accuracy is further enhanced by restricting their applicability to time intervals of ±0.5 year duration.

A star's position in the equatorial coordinate system is specified by its right ascension and declination (i.e., longitude and latitude). The equatorial coordinate system is specified by its pole (the North Celestial Pole which is the point on the celestial sphere penetrated by the continuation of the Earth's axis through the Earth's north pole), its fundamental circle (the celestial equator which is the great circle on the celestial sphere described by the projection of the Earth's equator as seen from the center of the Earth), its zero vertical (half the great circle through the poles of the celestial equator whose intersection with the celestial equator marks the zero of the longitudinal coordinate, i.e., the vernal equinox or the First Point of Aries), its heliticity (right-handed), and its epoch. The epoch is denoted by a phrase such as "for equator and equinox of 1977.0," etc. One must prescribe the epoch for both the celestial equator and the vernal equinox because the equatorial coordinate system is not fixed relative to an inertial one.

The mean place of a star is denoted by (α_o, δ_o). (The words place and position are used interchangeably herein). For historical reasons the values of right ascension and declination found in catalogs (catalog mean place) include the small effects of elliptic aberration (Taff[4]). As the mean place of a star defines that direction in which an observer situated at the solar system barycenter would view the star,

catalog mean place = mean place + e-terms.

The true place of a star is its solar system barycentric position referred to the true (i.e., actual at date) equator and equinox. It differs from the mean place by the effects of precession, proper motion, and nutation. The apparent place of a star is its geocentric position referred to the true equator and equinox. It differs from the true place by the effects of annual aberration and annual parallax. The topocentric place of a star is its position as determined by an ideal telescope. It differs from the apparent place by the effects of astronomical refraction, diurnal aberration, and geocentric parallax.

The two common methods for performing the mean place to apparent place reduction use either (i) star constants in right ascension (a, b, c, d), star constants in declination (a', b', c', d'), and Besselian day numbers (A, B, C, D, E, J, and J'), or (ii) independent day numbers (f, g, G, h, H, and i). C and D are sometimes referred to as aberrational day numbers since they are used to correct for annual aberration while A, B, and E are used to correct for precession and nutation. In addition, because they represent terms of the second order, J and J' are sometimes referred to as second-order day numbers. All of these quantities are tabulated in the American Ephemeris and Nautical Almanac. In the following A–E, J, J', f, g, h, and i are in seconds of arc and G and H (as well as right ascension and declination) are in radians. The proper motion in right ascension (μ) and the proper motion in declination (μ') are in "/yr.

When all terms of the second order are included the reduction procedures are of comparable accuracy. However, the method employing star constants and Besselian day numbers is to be preferred when simultaneously reducing the positions of several stars to the same instant of time while the method employing independent day numbers is to be preferred when reducing the position of a single star for several different times. Moreover, the latter method is more expeditious when simultaneously reducing the positions of several stars close together on the celestial sphere to the same instant of time.

If (α, δ) denotes the apparent place and τ is the fraction of the year to (from) the beginning of the nearest Besselian solar year, then

$$\alpha = \alpha_o + \sin 1''[\tau\mu + Aa + Bb + Cc + Dd + E + J\tan^2\delta_o + (\pi/\kappa)(Cd\sec\varepsilon - Dc\cos\varepsilon)], \tag{1a}$$

$$= \alpha_o + \sin 1''[\tau\mu + f + g\sin(G + \alpha_o)\tan\delta_o + h\sin(H + \alpha_o)\sec\delta_o + J\tan^2\delta_o], \tag{2a}$$

$$\delta = \delta_o + \sin 1''[\tau\mu' + Aa' + Bb' + Cc' + Dd' + J'\tan\delta_o$$
$$+ (\pi/\kappa)(Cd'\sec\varepsilon - Dc'\cos\varepsilon)], \tag{1b}$$
$$= \delta_o + \sin 1''[\tau\mu' + g\cos(G + \alpha_o) + h\cos(H + \alpha_o)\sin\delta_o$$
$$+ i\cos\delta_o + J'\tan\delta_o], \tag{2b}$$

where

$$a = m/n + \sin\alpha_o\tan\delta_o, \quad a' = \cos\alpha_o, \tag{3a}$$
$$b = \cos\alpha_o\tan\delta_o, \quad b' = -\sin\alpha_o, \tag{3b}$$
$$c = \cos\alpha_o\sec\delta_o, \quad c' = \tan\varepsilon\cos\delta_o - \sin\alpha_o\sin\delta_o, \tag{3c}$$
$$d = \sin\alpha_o\sec\delta_o, \quad d' = \cos\alpha_o\sin\delta_o, \tag{3d}$$

and π (the annual parallax in seconds of arc) has been set equal to zero (for simplicity) in Eqs. (2). The auxiliary quantities are κ = the constant of aberration (= $20\overset{''}{.}4958$), m = the centennial precession in right ascension, n = the centennial precession in declination, and ε = the true obliquity of the eliptic. If T denotes the number of tropical centuries since 1900.0 then,

$$m = 307\overset{s}{.}2337 + 0\overset{s}{.}18630T + 8\overset{s}{.}0 \times 10^{-6}T^2, \tag{4a}$$
$$n = 2004\overset{''}{.}685 - 0\overset{''}{.}8533T - 3\overset{''}{.}7 \times 10^{-4}T^2, \tag{4b}$$
$$\varepsilon = 23^o27'8\overset{''}{.}26 - 46\overset{''}{.}845T - 0\overset{''}{.}0059T^2 + 0\overset{''}{.}00181T^3. \tag{4c}$$

The small difference between the length of a tropical year and a Besselian solar year (the latter is shorter by $0\overset{s}{.}148T$) is negligible.

The independent day numbers are related to the Besselian day number via

$$f = (m/n)A + E, \quad i = C\tan\varepsilon, \tag{5a}$$

$$g\sin G = B, \quad g\cos G = A, \tag{5b}$$

$$h\sin H = C, \quad h\cos H = D. \tag{5c}$$

The second-order day numbers are calculated (using the $\pm$ sign when $\delta_o \lessgtr 0$) by

$$J = \sin 1''[(A \pm D)\sin\alpha_o + (B \pm C)\cos\alpha_o][(A \pm D)\cos\alpha_o - (B \pm C)\sin\alpha_o], \tag{6a}$$

$$= \sin 1''[g\sin(G + \alpha_o) \pm h\sin(H + \alpha_o)][g\cos(G + \alpha_o) \pm h\cos(H + \alpha_o)], \tag{7a}$$

$$J' = -\sin 1''[(A \pm D)\sin\alpha_o + (B \pm C)\cos\alpha_o]^2/2, \tag{6b}$$

$$= -\sin 1''[g\sin(G + \alpha_o) \pm h\sin(H + \alpha_o)]^2/2. \tag{7b}$$

Finally the time of the start of the Besselian solar year (e.g., the instant when the right ascension of the fictitious mean sun is 18^h40^m) can be computed from the expression

$$\text{Jan } 0\overset{d}{.}813516 + T(24\overset{d}{.}219878 - 0\overset{d}{.}000308T) - [25T] + \text{LPYR}, \qquad (4d)$$

where LPYR = 1 if $25T - [25T] = 0$, 0 otherwise. Thus, 1977.0 = Jan $0\overset{d}{.}4626$. This formula is valid if $T \varepsilon [0.25, 0.99]$.

As mentioned above these quantities are tabulated in the American Ephemeris and Nautical Almanac. The dependent variable is (necessarily) ephemeris time but tables in which the mean sidereal time at Greenwich is the argument are also included. Since the future relationship between universal time and ephemeris time can only be deduced from observations not yet made, the latter tables are not exact. The error induced can't exceed $\pm 0\overset{''}{.}01$ unless $|\delta_o| > 84^o$.

In order to calculate the day numbers one needs to know the time. The development of accurate clocks, telescopes, and sophisticated reduction techniques has led to the discovery of irregularities in the Earth's rotation. Hence, there are three different universal times. UT0 is universal time as deduced directly from observations of the stars and the fixed, numerical relationship between an interval of universal time and the corresponding interval of sidereal time (see below). UT1 is UT0 corrected for the motion of the poles (also called polar wandering). UT1 represents the true angular rotation of the earth and is independent of the observer's location. UT2 is UT1 corrected for the average seasonal variations in the Earth's rotation rate (due to polar cap melting, etc.). However, UT2 has not been freed of secular (i.e., tidal friction) or other irregular terms. Both the U. S. Naval Observatory and the

National Bureau of Standards have atomic clocks designed to reproduce UT2. The time they do produce is called coordinated universal time (UTC) and is distributed by radio station WWV. Announced frequency and phase offsets keep UTC within $\pm 0^{s}.1$ of UT2.

The measures of mean universal and mean sidereal times are related via

$$1 \text{ mean sidereal day} = 23^{h}56^{m}4^{s}.09054 \text{ of mean solar time,}$$

$$1 \text{ mean solar day} = 24^{h}3^{m}56^{s}.55536 \text{ of mean sidereal time.}$$

Hence, when both are in the same units

$$1 \text{ mean sidereal day}/1 \text{ mean solar day} = 0.9972695664$$

(= 1/1.0027379093). When the zero point is fixed (usually at 0^{h}UT of Jan 0) these scale factors allow conversion with an accuracy of $\pm 0^{s}.01$. If more precision is required or an apparent time is needed the equation of the equinoxes (= the difference between apparent and mean sidereal time; it's always within $\pm 1^{s}$ of 0) must be considered.

Let (N_D, T_u) represent the time of the observation. Here $N_D \geq 0$ is the number of whole days elapsed since Jan 1.0 = 0^{h}UT Jan 1. T_u is UTC, $T_u \varepsilon [0, 24^{h})$. Then if λ is the east longitude of the observer (in hrs) and $T_s(1.0)$ is the mean sidereal time at 0^{h}UT Jan 1, the mean sidereal time at (N_D, T_u) is given by

$$T_s = T_s(1.0) + \lambda - 24^h + 1.0027379093T_u + 0.0657098222N_D. \qquad (8a)$$

As an example, for 1977 $T_s(1.0) = 6^h42^m7\overset{s}{.}127$ so if $\lambda = T_u = 0$, $N_D = 365$ the formula yields $T_s = 6^h41^m9\overset{s}{.}833$ while the mean sidereal time at 0^hUT Dec 32, 1977 <u>is</u> $6^h41^m9\overset{s}{.}835$.

To compute the fraction of the year elapsed to (from) the beginning of the nearest Besselian solar year we proceed in two steps. Let $\tau(1.0)$ be the fraction of the year to Jan 1.0. Compute τ' via

$$\tau' = \tau(1.0) + (N_D + T_u/24)/365.2422. \qquad (9)$$

If $N_d \leq N_D$(Jul 1), $\tau = \tau'$. If $N_D > N_D$(Jul 1), $\tau = \tau' - 1$. Thus, at $(334^d, 0^h)$ in 1977 $\tau' = [0.537 + (334 + 0/24)]/365.2422 = 0.91593$ so $\tau = -0.08407$ since 1977.0 = Jan $0\overset{d}{.}463$.

Equation (8a) is useful if $T_s(1.0)$ is known for any year. The mean sidereal time at 0^hUT of any calendar date is defined to be

$$T_s = [6^h38^m45\overset{s}{.}836 + 8640184\overset{s}{.}542T + 0\overset{s}{.}0929T^2]\mathrm{mod}(86400^s) \qquad (8b)$$

where T denotes the number of Julian centuries of 36525 days which, at the midnight beginning the day, have elapsed since 12^hUT Jan 0, 1900 at Greenwich. Hence, to obtain T_s at Jan 1.0, 1977 = 0^hUT Jan 1, 1977 we observe that (77 years of length 365 days have elapsed) + (19^d for leap years [1900 was not a leap year]) + ($0\overset{d}{.}5$ from 12^hUT to 0^hUT on Jan 0, 1900) = $28124\overset{d}{.}5$. Straightforward substitution into Eq. (8b) yields T_s = $6^h42^m7\overset{s}{.}127$ as noted above.

Now that we know the instant of time that the observations are to be made at, we can reduce the apparent place to the topocentric place. For stars this includes the effects of geocentric parallax, diurnal aberration, and astronomical refraction. The effects of geocentric parallax are less than those of annual parallax by ≃ (radius of earth)/1 A.U. Hence, they are universally ignored. The computation of diurnal aberration requires the observer's geocentric coordinates.

One's geocentric coordinates are geocentric distance, ρ, in units of the earth's equatorial radius (6378.160km), geocentric latitude, ϕ', and geocentric longitude, Λ. One's geodetic coordinates are height above mean sea level, H, in km, geodetic latitude, ϕ, and geodetic longitude, λ. If H = 0 then

$$\rho\sin\phi' = S\sin\phi, \quad \rho\cos\phi' = C\cos\phi, \tag{10a}$$

$$\tan\phi' = (1 - f)^2\tan\phi, \tag{10b}$$

$$\lambda = \Lambda, \tag{10c}$$

$$\rho^2 = [\cos^2\phi + (1 - f)^4\sin^2\phi]C^2, \tag{10d}$$

where f is the flattening of the earth (1/f = 298.25) and the auxiliary quantities S and C are given by

$$1/C^2 = \cos^2\phi + (1 - f)^2\sin^2\phi, \tag{11a}$$

$$S = (1 - f)^2C. \tag{11b}$$

Equations (11a, 11b, 10b, and 10d) may be expressed directly in terms of f. Through all terms of the fourth order the result is

$$C = 1 + f/2 + 5f^2/16 + 7f^3/32 + 169f^4/1024$$
$$- [f/2 + f^2/2 + 27f^3/64 + 11f^4/32]\cos 2\phi$$
$$+ [3f^2/16 + 9f^3/32 + 77f^4/256]\cos 4\phi$$
$$- [5f^3/64 + 5f^4/32]\cos 6\phi + (35f^4/1024)\cos 8\phi, \tag{12a}$$

$$S = 1 - 3f/2 + 5f^2/16 + 3f^3/32 + 41f^4/1024$$
$$- [f/2 - f^2/2 - 5f^3/64]\cos 2\phi$$
$$+ [3f^2/16 - 3f^3/32 - 19f^4/256]\cos 4\phi$$
$$- (5f^3/64)\cos 6\phi + (35f^4/1024)\cos 8\phi, \tag{12b}$$

$$\phi' = \phi - [f + f^2/2 - f^4/4]\sin 2\phi$$
$$+ [f^2/2 + f^3/2 + f^4/8]\sin 4\phi$$
$$- [f^3/3 + f^4/2]\sin 6\phi + (f^4/4)\sin 8\phi, \tag{12c}$$

$$\rho = 1 - f/3 + 5f^2/16 + 5f^3/32 + 6009f^4/1024$$
$$+ [f/2 - 13f^3/64 - 509f^4/64]\cos 2\phi$$
$$- [5f^2/16 + 5f^3/32 - 517f^4/256]\cos 4\phi$$
$$+ [13f^3/64 + 13f^4/64]\cos 6\phi - (141f^4/1024)\cos 8\phi. \tag{12d}$$

If a correction for H is necessary then it may be approximately included by rewriting Eq. (10a) as

$$\rho\sin\phi' = (S + H/a_e)\sin\phi, \tag{13a}$$

$$\rho\cos\phi' = (C + H/a_e)\cos\phi, \tag{13b}$$

where a_e is the equatorial radius of the earth.

The observer's speed relative to the earth's axis of rotation (due to the rotation of the earth) is

$$v = (1\ \text{revolution/sidereal day})\rho a_e\cos\phi'. \tag{14}$$

As there are 86164.09054 mean solar seconds in one mean sidereal day (see above), $v = 0.4651028\rho\cos\phi'$km/sec. Division of v by the speed of light in vacuo (2.997925×10^5km/sec) yields $v = 0\overset{''}{.}320002\rho\cos\phi'$. The corrected coordinates (α', δ') are given by

$$\alpha' = \alpha + (v\sin 1'')\cos h\sec\delta, \tag{15a}$$

$$\delta' = \delta + (v\sin 1'')\sin h\sin\delta, \tag{15b}$$

where $h = T_s - \alpha$ and $v\sin 1'' = 1.551416 \times 10^{-6}$.

The last correction is for astronomical refraction and is computed by

$$\delta_{obs} = \delta' + R'\cos\eta',$$

$$= \delta' + R'\sec\delta'\csc z'[\sin\phi - \sin\delta'\cos z'], \tag{16a}$$

$$\alpha_{obs} = \alpha' + R'\sec\delta_{obs}\sin\eta',$$

$$= \alpha' + R'\sec\delta_{obs}\csc z'\cos\phi\sin h', \tag{16b}$$

where $h' = T_s - \alpha'$, z' is the zenith distance and η' is the parallactic angle corresponding to (α', δ'). For any right ascension α, declination δ, and geodetic latitude ϕ,

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cosh, \tag{17a}$$

$$\sin z\cos\eta = \mathrm{sgn}(\phi)[\sin\phi\cos\delta - \cos\phi\sin\delta\cosh], \tag{17b}$$

$$\sin z\sin\eta = \cos\phi|\sinh|, \tag{17c}$$

$$h = T_s - \alpha. \tag{17d}$$

The quantity R' (in radians above, seconds of arc below) is the astronomical refraction, R, corrected for the local pressure in mbar, P, and the local temperature in ^{o}F, T_F.

$$R' = (0.5020098P)R/(460 + T_F). \tag{18}$$

The astronomical refraction is the difference between the true and observed zenith distances (assuming refraction to be the sole cause of the difference) and may be approximately expressed by

$$R = z' - z_{obs} = R_1 \tan z_{obs} - R_2 \tan^3 z_{obs}, \tag{19}$$

where R_1 and R_2 are constants. The values used here are $R_1 = 58''.294$, $R_2 = 0''.0668$ and correspond to standard values of $(P, T_F) = (1015.92\text{mb}, 50^{\circ}\text{F})$. Equation (19) is accurate for $z' \leq 75^{\circ}$ and includes effects due to the curvature of the earth.

The fact that only local meteorological conditions affect the refraction can be simply deduced if the atmosphere is plane-parallel. Similarly, the leading term in Eq. (19) may be easily derived for a plane-parallel atmosphere. These results are also valid to first order since (height of the atmosphere/radius of the earth) is a quantity of the first order. That the results are valid to the second order too is known as the theorem of Oriani and Laplace.

We now briefly discuss the accuracy of these procedures. The mean place to apparent place reduction, using J and J', and restricting $|\tau|$ to ≤ 0.5 is $\pm 0''.003$ in each coordinate (Porter and Sadler[5]). The neglect of geocentric parallax is permissable except at the poles (e.g., within $0^{\circ}.5$ of the poles it's still less than $\pm 0''.01$). The astronomical refraction correction here includes all terms through the second order and can systematically affect the reductions (because of a poor choice of R'/R, R_1, and R_2). However, a systematic bias in zenith distance is easily discovered and corrected for. Finally, one should use the apparent sidereal time to compute the refraction correction, not the mean sidereal time. Except for zenith distances so large that Eq. (19) is in doubt, this produces no appreciable error.

III. DIFFERENTIAL REDUCTIONS

It should be clear now that a considerable amount of labor is involved in reducing the position of a star. Even with a modern hand calculator, it takes $20\text{-}30^{m}$ to perform the reduction to $\pm0''.01$. To reduce N stars, all within a few degrees of each other, special procedures have been developed in order to minimize what was once hand labor. These methods have the characteristic that all of the stars are reduced relative to the same point on the celestial sphere and at the same instant of time. Hence, these procedures are referred to as differential.

In the age of electronic computers one might question the necessity or value of analytical devices which shorten routine calculations. After all, the procedures of § II, applied to each of the N stars individually, will unquestionably result in greater accuracy and easier programming. The answer is partly psychological, connected with the esthetics of the astronomer versus those of the arithmatic registers of a central processing unit, and partly practical. The practical side deals with the ultimate accuracy needed, the ability to precisely ascertain truncation and roundoff error, and the development of widely applicable analytical tools.

In the remainder of this section N stars occupy a small ($\lesssim 5^{o}$) area on the sky. They are observed nearly simultaneously. When they are near the celestial poles additional labor is required (cf. § IIIC). The center of the field occupied by the stars has mean coordinates ($<\alpha_o>$, $<\delta_o>$) and proper motions ($<\mu>$, $<\mu'>$) related to those of the stars via

$$\langle\alpha_o\rangle = \Sigma\alpha_o/N, \ \langle\delta_o\rangle = \Sigma\delta_o/N, \tag{20a}$$

$$\langle\mu\rangle = \Sigma\mu/N, \ \langle\mu'\rangle = \Sigma\mu'/N. \tag{20b}$$

The sums in Eqs. (20) extend over all N stars and parallax is neglected. The mean sidereal time of the reduction, T_s, is the midpoint of the observing interval.

A. Star Constants and Besselian Day Numbers

Consider a position with mean equatorial coordinates ($\langle\alpha_o\rangle$, $\langle\delta_o\rangle$) and a second, nearby one, with mean equatorial coordinates (α_o, δ_o). Let the associated proper motions be ($\langle\mu\rangle$, $\langle\mu'\rangle$) and (μ, μ'). Then their apparent right ascensions (declination is treated analogously) from Eq. (1a) are

$$\langle\alpha\rangle = \langle\alpha_o\rangle + \sin 1''[Aa(\langle\alpha_o\rangle, \langle\delta_o\rangle) + Bb(\langle\alpha_o\rangle, \langle\delta_o\rangle) + \ldots + \tau\langle\mu\rangle], \tag{21a}$$

$$\alpha = \alpha_o + \sin 1''[Aa(\alpha_o, \delta_o) + Bb(\alpha_o, \delta_o) + \ldots + \tau\mu], \tag{21b}$$

where the explicit dependence on right ascension and declination of the star constants in right ascension is indicated. If $\Delta\alpha_o = \alpha_o - \langle\alpha_o\rangle$, $\Delta\alpha = \alpha - \langle\alpha\rangle$, etc., then

$$\Delta\alpha = \Delta\alpha_o + \sin 1''\{A[a(\alpha_o, \delta_o) - a(\langle\alpha_o\rangle, \langle\delta_o\rangle)] + \ldots + \tau[\mu - \langle\mu\rangle]\}. \tag{22}$$

However,

$$a(\alpha_o, \delta_o) \simeq a(\langle\alpha_o\rangle, \langle\delta_o\rangle) + \left.\frac{\partial a(x, y)}{\partial x}\right|_{(\langle\alpha_o\rangle, \langle\delta_o\rangle)} \Delta\alpha_o + \left.\frac{\partial a(x, y)}{\partial y}\right|_{(\langle\alpha_o\rangle, \langle\delta_o\rangle)} \Delta\delta_o, \quad (23)$$

so

$$\alpha \simeq \langle\alpha\rangle + \Delta\alpha_o + p\Delta\alpha_o + q\Delta\delta_o + \tau\Delta\mu\sin 1'', \quad (24a)$$

$$\delta \simeq \langle\delta\rangle + \Delta\delta_o + r\Delta\alpha_o + s\Delta\delta_o + \tau\Delta\mu'\sin 1'', \quad (24b)$$

where $\Delta\mu = \mu - \langle\mu\rangle$, $\Delta\mu' = \mu' - \langle\mu'\rangle$, and

$$p = \sin 1''\{[A\cos\langle\alpha_o\rangle - B\sin\langle\alpha_o\rangle]\tan\langle\delta_o\rangle - [C\sin\langle\alpha_o\rangle - D\cos\langle\alpha_o\rangle]\sec\langle\delta_o\rangle\} \quad (25a)$$

$$q = \sin 1''\{[A\sin\langle\alpha_o\rangle + B\cos\langle\alpha_o\rangle]\sec^2\langle\delta_o\rangle + [C\cos\langle\alpha_o\rangle + D\sin\langle\alpha_o\rangle]\sec\langle\delta_o\rangle\tan\langle\delta_o\rangle + 2J\sec^2\langle\delta_o\rangle\tan\langle\delta_o\rangle\}, \quad (25b)$$

$$r = -\sin 1''\{[A\sin\langle\alpha_o\rangle + B\cos\langle\alpha_o\rangle] + [C\cos\langle\alpha_o\rangle + D\sin\langle\alpha_o\rangle]\sin\langle\delta_o\rangle\}, \quad (25c)$$

$$s = -\sin 1''\{C\tan\varepsilon\sin\langle\delta_o\rangle + [C\sin\langle\alpha_o\rangle - D\cos\langle\alpha_o\rangle]\cos\langle\delta_o\rangle + 2J'\sec^2\langle\delta_o\rangle\}. \quad (25d)$$

Hence, as long as $\Delta\alpha_o$, $\Delta\delta_o$ are not too large ($\lesssim 0.1$ rad $= 5^{\circ}\!.7$), the mean place of any nearby star can be reduced to its apparent place once the center of the field has been reduced. As p, q, r, and s only depend on the position of the center of the field, it is little more labor to reduce all N stars simultaneously.

B. Independent Day Numbers

The analytical complexities of the differential reduction can be minimized if independent day numbers are used. This is because, unlike the star constants, the independent day numbers do not depend on the position of the center of the field. Thus, proceeding analogously to the above [and using Eq. (2a) in place of Eq. (1a)] we have

$$\langle\alpha\rangle = \langle\alpha_o\rangle + \sin 1''[f + g\sin(G + \langle\alpha_o\rangle)\tan\langle\delta_o\rangle + \ldots + \tau\langle\mu\rangle], \tag{26a}$$

$$\alpha = \alpha_o + \sin 1''[f + g\sin(G + \alpha_o)\tan\delta_o + \ldots + \tau\mu], \tag{26b}$$

so

$$\Delta\alpha = \Delta\alpha_o + \sin 1''\{g[\sin(G + \alpha_o)\tan\delta_o - \sin(G + \langle\alpha_o\rangle)\tan\langle\delta_o\rangle] + \ldots + \tau[\mu - \langle\mu\rangle]\}, \tag{27}$$

or

$$\alpha \simeq \langle\alpha\rangle + \Delta\alpha_o + P\Delta\alpha_o + Q\Delta\delta_o + \tau\Delta\mu \sin 1'', \tag{28a}$$

$$\delta \simeq \langle\delta\rangle + \Delta\delta_o + R\Delta\alpha_o + S\Delta\delta_o + \tau\Delta\mu' \sin 1'', \tag{28b}$$

where,

$$P = \sin 1''[g\cos(G + \langle\alpha_o\rangle)\tan\langle\delta_o\rangle + h\cos(H + \langle\alpha_o\rangle)\sec\langle\delta_o\rangle], \tag{29a}$$

$$Q = \sin 1''[g\sin(G + \langle\alpha_o\rangle)\sec^2\langle\delta_o\rangle + h\sin(H + \langle\alpha_o\rangle)\sec\langle\delta_o\rangle\tan\langle\delta_o\rangle$$
$$+ 2J\tan\langle\delta_o\rangle\sec^2\langle\delta_o\rangle], \tag{29b}$$

$$R = -\sin 1''[g\sin(G + \langle\alpha_o\rangle) + h\sin(H + \langle\alpha_o\rangle)\sin\langle\delta_o\rangle], \tag{29c}$$

$$S = \sin 1''[h\cos(H + \langle\alpha_o\rangle)\cos\langle\delta_o\rangle - i\sin\langle\delta_o\rangle + J'\sec^2\langle\delta_o\rangle]. \tag{29d}$$

C. The Polar Regions

Right ascension and declination form an orthogonal curvilinear coordinate system, on the celestial sphere, with a non-trivial metric tensor. Hence, we cannot expect approximate procedures to work throughout the right ascension, declination ranges. However, $(\alpha\cos\delta, \delta)$ do form a set of coordinates appropriate everywhere on the celestial sphere. Wherever $\cos\delta \ll 1$ (and $\delta = \pm 81^o$ is traditionally the dividing point in astrometry) it is simpler, and more accurate, to use the direction cosines $(\cos\alpha\cos\delta, \sin\alpha\cos\delta, \sin\delta)$ in place of (α, δ). Thus, pole star tables are frequently given in this form. The above reduction can be extended to the direction cosines (with considerable analytical complexity). Since we never need to observe artificial satellites this close to the poles, the results are not presented here.

As a general rule, any approximation procedure used in spherical astronomy deteriorates systematically with declination. Thus, the position dependence of the second order terms in Eqs. (1) is purely declination, etc. Hence, one must be aware of systematically biasing one's data reduction in this manner.

D. Apparent Place To Topocentric Place

Once $(\langle\alpha\rangle, \langle\delta\rangle)$ have been obtained they must be corrected for diurnal aberration:

$$\langle\alpha'\rangle = \langle\alpha\rangle + (v\sin 1'')\cos\langle h\rangle\sec\langle\delta\rangle, \tag{30a}$$

$$\langle\delta'\rangle = \langle\delta\rangle + (v\sin 1'')\sin\langle h\rangle\sin\langle\delta\rangle, \tag{30b}$$

$$\langle h\rangle = T_s - \langle\alpha\rangle. \tag{30c}$$

The (approximate) differential reduction is obtained from

$$\alpha' \simeq \langle\alpha'\rangle + \Delta\alpha + (v\sin 1'')\sec\langle\delta\rangle[\Delta\alpha\sin\langle h\rangle + \Delta\delta\cos\langle h\rangle\tan\langle\delta\rangle], \tag{31a}$$

$$\delta' \simeq \langle\delta'\rangle + \Delta\delta + (v\sin 1'')[-\Delta\alpha\cos\langle h\rangle\sin\langle\delta\rangle + \Delta\delta\sin\langle h\rangle\cos\langle\delta\rangle], \tag{31b}$$

where, as above, $\Delta\alpha = \alpha - \langle\alpha\rangle$, etc. The last correction is for refraction,

$$\langle\delta_{obs}\rangle = \langle\delta'\rangle + R'\cos\langle\eta'\rangle, \tag{32a}$$

$$\langle\alpha_{obs}\rangle = \langle\alpha'\rangle + R'\sec\langle\delta_{obs}\rangle\sin\langle\eta'\rangle, \tag{32b}$$

and, with $\Delta\alpha' = \alpha' - \langle\alpha'\rangle$, $\Delta\delta' = \delta' - \langle\delta'\rangle$,

$$\alpha_{obs} \simeq \langle\alpha_{obs}\rangle + \Delta\alpha' + I\Delta\alpha' + J\Delta\delta', \tag{33a}$$

$$\delta_{obs} \simeq \langle\delta_{obs}\rangle + \Delta\delta' + K\Delta\alpha' + L\Delta\delta'. \tag{33b}$$

The constants I-L are computed assuming $R = R_o \tan z_{obs}$ [rather than using Eq. (19)], $R_o'/R_o = R'/R$, and $R_o = 58\overset{''}{.}2$ (the error is third-order). The results are

$$I = -R_o'[1 - \tan\langle z'\rangle\cos\langle\eta'\rangle\tan\langle\delta'\rangle + \tan^2\langle z'\rangle\sin^2\langle\eta'\rangle], \tag{34a}$$

$$J = -R_o'\tan\langle z'\rangle\sin\langle\eta'\rangle\sec\langle\delta'\rangle[\tan\langle z'\rangle\cos\langle\eta'\rangle - \tan\langle\delta'\rangle], \tag{34b}$$

$$K = -R_o'\tan\langle z'\rangle\sin\langle\eta'\rangle\cos\langle\delta'\rangle[\tan\langle z'\rangle\cos\langle\eta'\rangle + \tan\langle\delta'\rangle], \tag{34c}$$

and,

$$L = -R_o'[1 + \tan^2\langle z'\rangle\cos^2\langle\eta'\rangle]. \tag{34d}$$

The parallactic angle and zenith distance are computed from Eqs. (17).

Hence, we now have the ability to completely and simultaneously reduce the positions of N stars (relative to any other position) with a minimum of computation. We also see that the correction for astronomical refraction involves the most labor. It is for this reason that, traditionally, this stage of the reduction was left out. It's effects were absorbed in the analysis of the photographic plate (see § V). In addition, because κ is small, the annual and diurnal aberrations were not accounted for. The complete reduction is recommended.

IV. IDEAL ASTRONOMICAL PHOTOGRAPHY

A. Standard Coordinates

Consider the ideal refracting telescope depicted in Fig. 1. OC is the optical axis of the telescope and GH is the focal plane. The focal plane, normal to OC, contains a photographic plate. OC intersects the plate at its center. Produce OC to A where it intersects the celestial sphere. If there were a star at A its light would be focused at O while a nearby star at B (on the celestial sphere) would have its light focused at R (on the plate). To determine the relationship between the linear size of the photographic plate, ℓ, and the angular distance between the corresponding points on the sky, L, imagine R to be at the edge of the plate and the plane RCO to be parallel to the edge of the plate. Then plane trigonometry in ΔRCO yields

$$\ell = 2f\tan L, \tag{35}$$

where f is the focal length of the object glass. Hence, from the focal length and the size of the photographic plate one can compute (in angular measure) the area of the sky which can be recorded. It is also important to know the plate scale ($= 2L/\ell$) so that a linear separation on the plate can be directly transformed into an angular separation on the sky.

Again referring to Fig. 1, the plane tangent to the celestial sphere (also called the plane of the sky) at A is indicated. This plane is (necessarily) perpendicular to OCA and, hence, parallel to both the

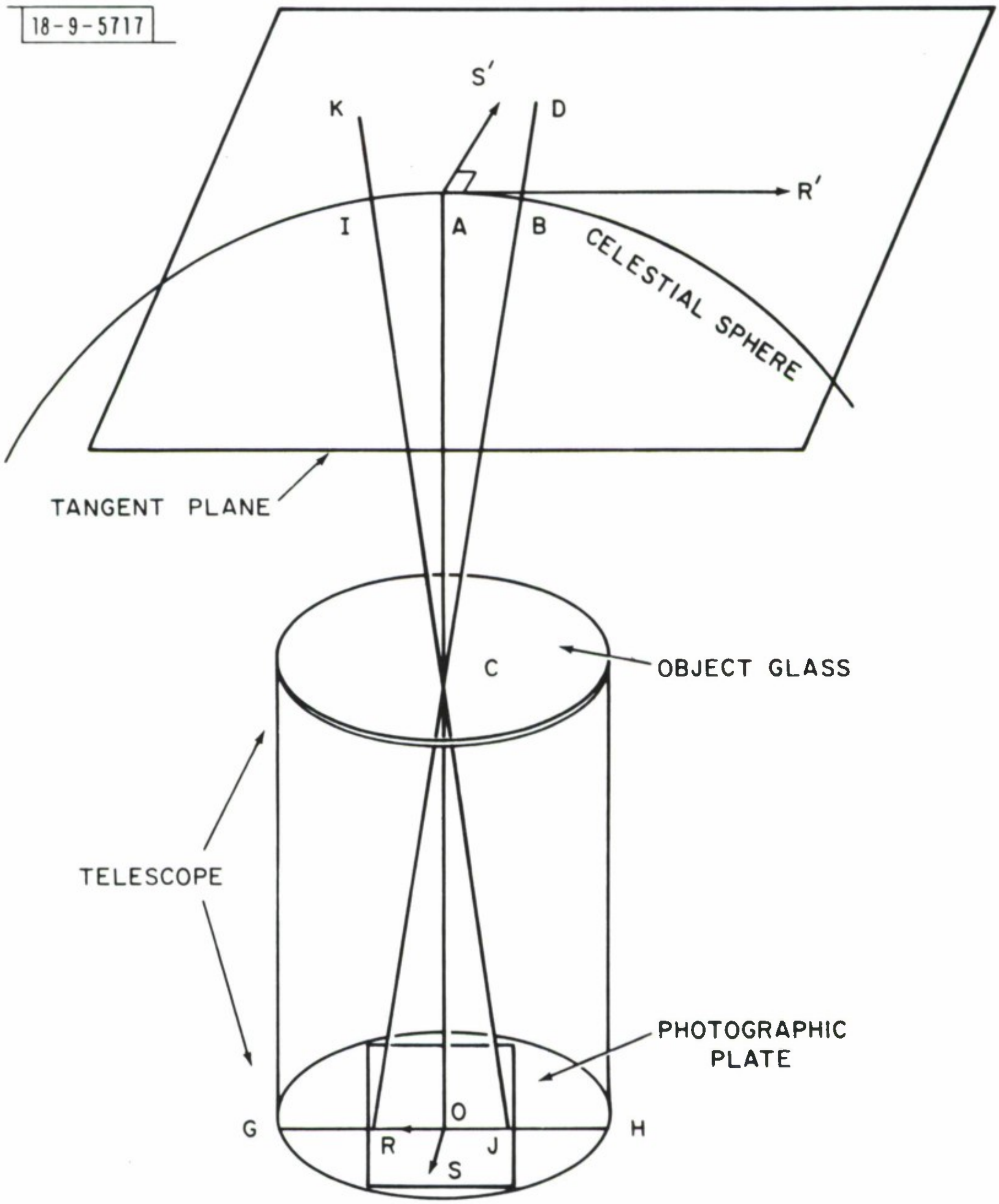

Fig. 1. Refracting telescope diagram for astronomical photography. Not to scale.

focal plane and the photographic plate. Two stars at I and B are shown projected onto the tangent plane at K and D. Let $\sphericalangle$ OCJ = ϕ = $\sphericalangle$ ACI. Then,

$$\tan\phi = OJ/OC = AK/CK, \tag{36}$$

so that there is a similarity between the configurations of the stellar images on the plane of the sky and on the photographic plate. If AS', AR' and OS, OR define the positive directions for Cartesian axes in the two planes (note the direction reversal) then coordinates (ξ', η') on the tangent plane are related to coordinates (ξ, η) on the plate by the scale factor AC/OC, viz.

$$\xi' = (AC/OC)\xi, \quad \eta' = (AC/OC)\eta. \tag{37}$$

The standard coordinates, (ξ, η), (introduced by Turner[6]) are measurable. Just as important, they can be computed from the right ascension and declination of the point A (i.e., the point of tangency) and the point on the celestial sphere, to which they refer. I now demonstrate these statements.

Figure 2 is a portion of Fig. 1 drawn from a different perspective. The arc AB is that portion of the great circle through the point of tangency and the star at B. All radii (of the celestial sphere) connecting C to points on $\overset{\frown}{AB}$ lie in the place of the great circle through A and B

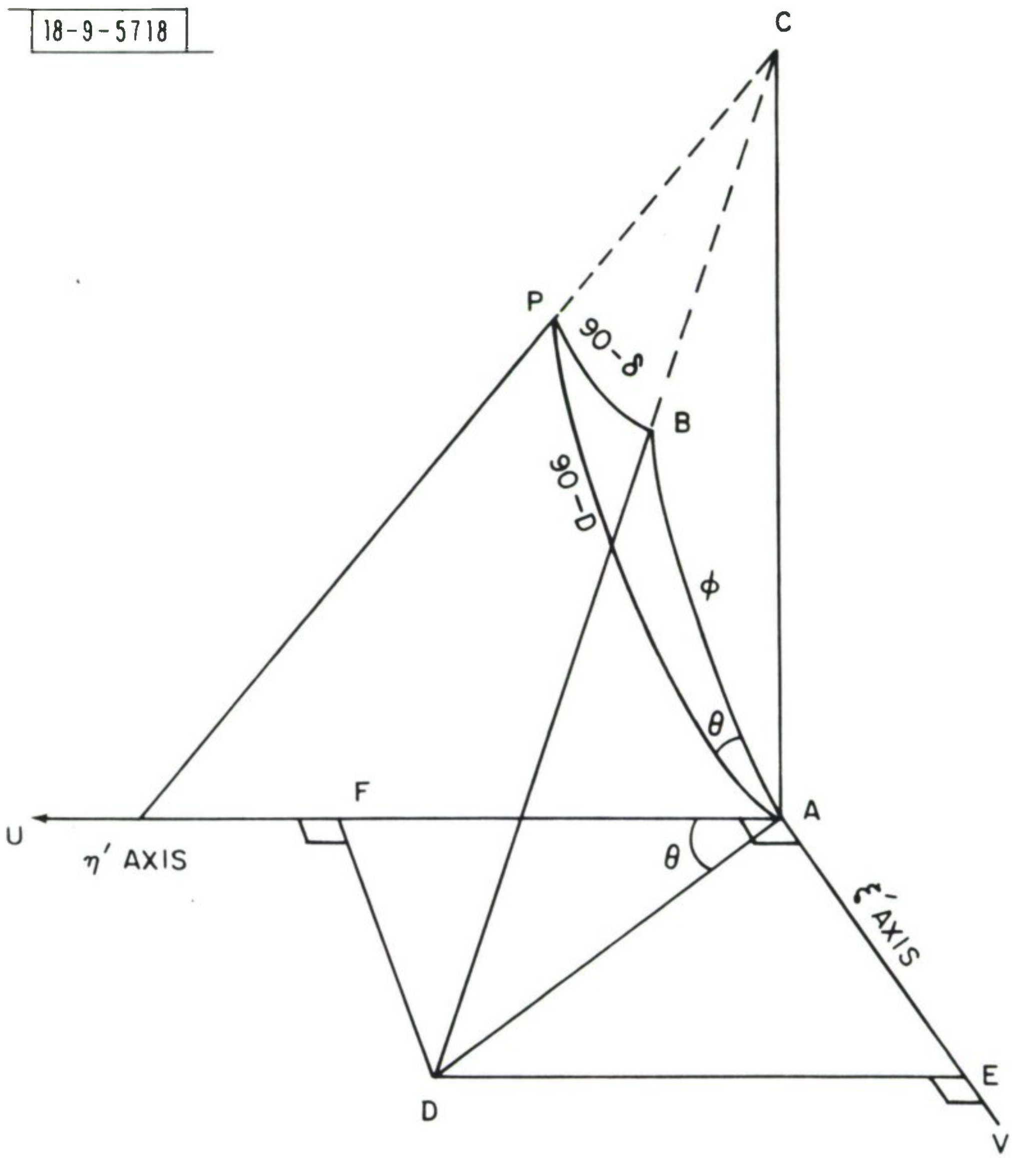

Fig. 2. Same as Fig. 1 seen from a different perspective.

and project onto the straight line segment AD (which lies in the tangent plane). If P is the North Celestial Pole then AP is the meridian for A. Also drawn are ξ', η' axes with the right ascension increasing with increasing ξ'. Hence, in the figure, B is east of A's meridian. Since AD lies in the tangent plane it is perpendicular to AC. Similarly, AU is orthogonal to the great circle arc AP at A. Finally, the dihedral angle UAD is equal to $\sphericalangle$ PAB in the spherical triangle PAB. Therefore, the projection (onto the plane of the sky) of the arcs of great circles preserves angles. Let $\widehat{AB} = \phi$, $\sphericalangle$ BAP $= \theta = \sphericalangle$ UAD. Drop perpendiculars from D to AU (i.e., FD) and from D to AV (i.e., DE). Then, from the plane right triangles FDA, DEA,

$$\xi' = FD = AD\sin\theta, \tag{38a}$$

$$\eta' = ED = AD\cos\theta. \tag{38b}$$

However, AD = AC$\tan\phi$ so [from Eqs. (36, 37, 38)],

$$\xi = f\tan\phi\sin\theta, \tag{39a}$$

$$\eta = f\tan\phi\cos\theta. \tag{39b}$$

Since the focal length of the object glass, f, merely serves to define the linear scale, we may set f = 1 without loss of generality. To complete the demonstration of the assertions made above, let the equatorial coordinates (relative to some equator and equinox) of the point of tangency be (A, Δ) and those of the star at B (relative to the <u>same</u> equator and equinox) be (α, δ). Then from the spherical triangle PAB (in which $\overset{\frown}{AP} = 90^{\circ} - \Delta$, $\overset{\frown}{BP} = 90^{\circ} - \delta$, $\sphericalangle$ APB $= \alpha - A$),

$$\cos\phi = \sin\delta\sin\Delta + \cos\delta\cos\Delta\cos(\alpha - A), \tag{40a}$$

$$\sin\phi\sin\theta = \cos\delta\sin(\alpha - A), \tag{40b}$$

$$\sin\phi\cos\theta = \sin\delta\cos\Delta - \cos\delta\sin\Delta\cos(\alpha - A). \tag{40c}$$

Hence,

$$\begin{aligned} \xi &= \cot\delta\sin(\alpha - A)/[\sin\Delta + \cot\delta\cos\Delta\cos(\alpha - A)], \\ &= \cos q\tan(\alpha - A)\sec(q - \Delta), \end{aligned} \tag{41a}$$

$$\begin{aligned} \eta &= [\cos\Delta - \cot\delta\sin\Delta\cos(\alpha - A)]/[\sin\Delta + \cot\delta\cos\Delta\cos(\alpha - A)], \\ &= \tan(q - \Delta). \end{aligned} \tag{41b}$$

Cotq $= \cot\delta\cos(\alpha - A)$ and q is the declination of that point on the celestial sphere where the great circle arc drawn from B intersects AP in a right angle. The inverse relationships are

$$\tan(\alpha - A) = \xi\sec\Delta/[1 - \eta\tan\Delta], \tag{42a}$$

$$\cot\delta\sin(\alpha - A) = \xi\sec\Delta/[\eta + \tan\Delta], \tag{42b}$$

$$\cot\delta\cos(\alpha - A) = (1 - \eta\tan\Delta)/[\eta + \tan\Delta]. \tag{42c}$$

If $|\alpha - A|$ is small one should use Eq. (42d) in place of Eqs. (42b, c).

$$\sin\delta = [\sin\Delta + \eta\cos\Delta]/[1 + \xi^2 + \eta^2]^{1/2}. \tag{42d}$$

It has now been demonstrated that from the (measurable) standard coordinates plus the equatorial coordinates of the point of tangency (which was chosen by the observer) one can compute the right ascension and declination of any other point on the photographic plate. Conversely, from the equatorial coordinates of the star and the point of tangency one can predict the standard coordinates of the corresponding image point on the photographic plate. Before demonstrating the utility (cf. § IVB) of this result various series expansions derived from the rigorous formulas are listed for reference.

$$\begin{aligned}
\xi &\simeq (\alpha - A)\cos\Delta - (\alpha - A)(\delta - \Delta)\sin\Delta + (\alpha - A)^3\cos\Delta(3\cos^2\Delta - 1)/6 \\
&\quad + \ldots, \\
&\simeq (\alpha - A)\cos\delta + (\alpha - A)^3\cos\delta(3\cos^2\delta - 1)/6 \\
&\quad + (1/2)(\alpha - A)(\delta - \Delta)^2\cos\delta + \ldots, && (43a) \\
\eta &\simeq (\delta - \Delta) + (1/4)(\alpha - A)^2\sin 2\Delta + (1/2)(\alpha - A)^2(\delta - \Delta)\cos 2\Delta \\
&\quad + (\delta - \Delta)^3/3 + \ldots, \\
&\simeq (\delta - \Delta) + (1/4)(\alpha - A)^2\sin 2\delta + (\delta - \Delta)^3/3 + \ldots, && (43b)
\end{aligned}$$

and, inversely,

$$\begin{aligned}
\alpha - A &\simeq \xi\sec\Delta + \xi\eta\sec\Delta\tan\Delta - (\xi^3/3)\sec^3\Delta \\
&\quad + \xi\eta^2\sec\Delta\tan^2\Delta + \ldots, \\
&\simeq \xi\sec\delta + (\xi^3/6)\sec\delta(\sec^2\delta - 3) \\
&\quad - (\xi/2)\eta^2\sec\delta - \ldots && (44a)
\end{aligned}$$

$$\delta - \Delta \simeq \eta - (\xi^2/2)\tan\Delta - (\xi^2/2)\eta\sec^2\Delta - \eta^3/3 + \ldots,$$
$$\simeq \eta - (\xi^2/2)\tan\delta - \eta^3/3 - \ldots \quad . \tag{44b}$$

Finally, note that the convergence of Eqs. (43, 44) deteriorates rapidly as $|\delta| \to 90^o$ and that $|q - \Delta|$ is always small. To exploit this note

$$\tan q = \tan\delta\sec(\alpha - A), \tag{45a}$$

and,

$$\tan(\alpha - A) = \xi\sec q\cos(q - \Delta). \tag{45b}$$

Hence,

$$\tan(q - \Delta) = \tan^2[(\alpha - A)/2]\sin 2q\{[1 - \tan^2[(\alpha - A)/2]\cos 2q\}, \tag{45c}$$

so

$$q \simeq \Delta + \tan^2[(\alpha - A)/2]\sin 2q + (1/2)\tan^4[(\alpha - A)/2]\sin 4q + \ldots, \tag{46}$$

which is a rapidly converging series, the truncation error being $\leq$ 0.''01 if $|\alpha - A| \leq 30^m$.

B. The Method of Dependences

Let us now see how the above analytical development can be used. Since only ideal astronomical photography is discussed here, sources of error will be glossed over (see § V). However, it should not surprise the reader to learn that the coordinates measured on the plate, denoted by (x, y), are not exactly equal to the standard coordinates. Nonetheless, the assumption that the relationship between their differences is linear is usually sufficient. Hence, we write (here)

$$\xi - x = a\xi + b\eta + c, \tag{47a}$$

$$\eta - y = A\xi + B\eta + C. \tag{47b}$$

The constants a, b, c, A, B, and C are called plate constants. In general they should be (and will be) determined by some estimation procedure such as maximum likelihood or least squares.

The problem we face is the following: An artificial satellite and N reference (or comparison) stars have been observed (i.e., photographed). We know the equatorial coordinates of the tangential point (A, Δ), and the equatorial coordinates for all of the stars, $\{(\alpha_j, \delta_j)\}$. Hence, for star j one can compute its standard coordinates, (ξ_j, η_j). We have also measured the coordinates for the stars, $\{(x_j, y_j)\}$, and for the satellite, (X, Y). Using a model for the errors we wish to determine the standard coordinates for the satellite, (Ξ, H), and thence its equatorial coordinates.

A general procedure to solve this problem was developed by Schlesinger[1]. It is called the method of dependences. The minimal case $N = 3$ illustrates the essential features of the problem and can also be (approximately) solved by graphical means. It also leads to the optimal configuration of the comparison stars relative to the satellite. For these reasons we now describe it in some detail.

The situation, on the photographic plate, is illustrated in Fig. 3. The reference stars are at S_1, S_2, and S_3 while the program object (an artifical satellite here) is at C. If one used Eqs. (47) for the plate model and knew the plate constants, then it would be a straightforward matter to compute the satellite's standard coordinates from its measured coordinates. To avoid actually calculating the plate constants let us remember that the least squares equations of condition (for ξ, the treatment of η follows analogously) are

$$\xi_j - x_j = a\xi_j + b\eta_j + c, \; j = 1, 2, 3 \tag{48}$$

and, for the artificial satellite,

$$\Xi - X = a\Xi + bH + c. \tag{49}$$

Multipliers D_j, j = 1, 2, 3, called dependences, are introduced such that

$$\sum_{j=1}^{3} D_j(\xi_j - x_j) - (\Xi - X) = 0. \tag{50a}$$

This yields three equations (equivalent to the normal equations of least squares when the coordinate variances of the reference stars are equal) for the dependences, viz.

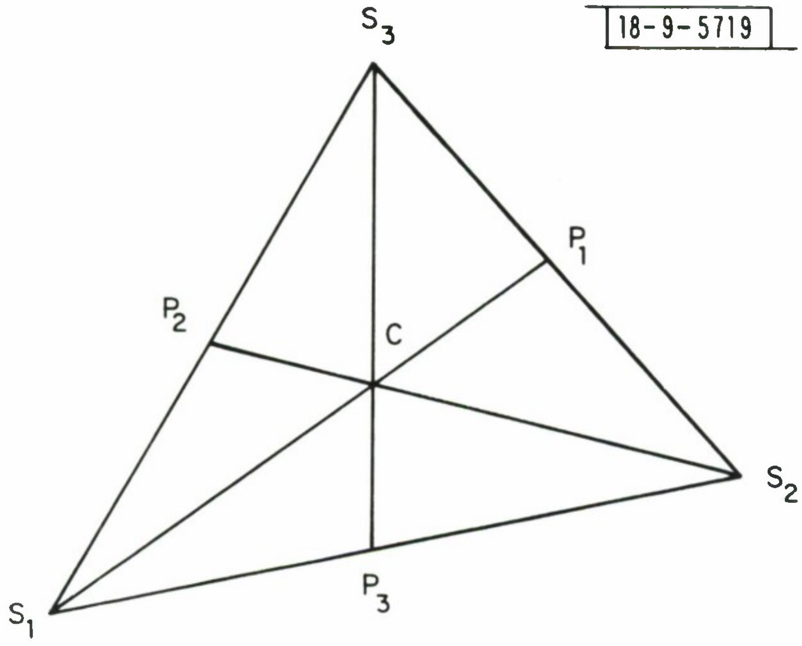

Fig. 3. The exposed photographic plate with reference stars at S_1, S_2, and S_3 and program object at C.

$$\sum_{j=1}^{3} D_j \xi_j = \Xi, \tag{51a}$$

$$\sum_{j=1}^{3} D_j \eta_j = H, \tag{51b}$$

$$\sum_{j=1}^{3} D_j = 1. \tag{51c}$$

It also follows, from the definition of dependences , that

$$\sum_{j=1}^{3} D_j x_j = X, \tag{52}$$

and

$$\Xi = X + \sum_{j=1}^{3} D_j(\xi_j - x_j). \tag{50b}$$

Now, if the linear plate model, Eqs. (47), is sufficient, then since the unmodeled terms are of the second order, the dependences will be determined with sufficient accuracy if the measured coordinates are substituted for the standard coordinates in Eqs. (51).* If this is done then the dependences are given by

$$\frac{D_1}{\begin{vmatrix} X & x_2 & x_3 \\ Y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{D_2}{\begin{vmatrix} X & x_1 & x_3 \\ Y & y_1 & y_3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{D_3}{\begin{vmatrix} X & x_1 & x_2 \\ Y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}}. \tag{53}$$

*This departure from the usual procedures of least squares analysis implies that the original estimation problem was not well-posed. See § VI too.

The last determinant on the right hand side of Eq. (53) is twice the area of the triangle formed by the reference stars. The other determinants have analogous interpretations leading to

$$D_1 = \frac{\text{area } \Delta CS_2S_3}{\text{area } \Delta S_1S_2S_3} = \frac{CP_1}{S_1P_1} , \tag{54a}$$

$$D_2 = \frac{\text{area } \Delta CS_3S_1}{\text{area } \Delta S_1S_2S_3} = \frac{CP_2}{S_2P_2} , \tag{54b}$$

$$D_3 = \frac{\text{area } \Delta CS_1S_2}{\text{area } \Delta S_1S_2S_3} = \frac{CP_3}{S_3P_3} . \tag{54c}$$

Hence, the computation of the dependences has been reduced to counting squares on a piece of graph paper. The satellite's standard coordinates are then calculated from Eqs. (51a, b) and the problem is solved.

In addition to being a neat labor-saving trick, the geometrical significance of the dependences, coupled with the theorem that the center of mass of a plane triangle with a uniform surface density lies at the intersection of its meridians, shows that the satellite should be at the "center of mass" of the reference stars for maximum accuracy. In the general case of $N \geq 3$ one can easily prove this result too (Plummer[7]) as long as the stellar coordinate variances are equal (i.e., the stars have the same "mass").

To summarize, we now know how to go from the photographic plate (charge coupled device, charge injected device, etc.) to the sky and back, and how to arrange matters to achieve the best possible internal accuracy.

V. REAL ASTRONOMICAL PHOTOGRAPHY

The idealized situation considered in § IV will not obtain in practice. Here I discuss the modeling of the most common and important sources of error. These can be divided into two groups; those associated with the measurement of the photographic place (centering error, rotation, non-perpendicularity of the axes) and those associated with the telescope (centering error, tilt, incorrect focal length, radial distortion, decentering distortion, and coma). In addition, the effects caused by not including astronomical refraction and annual aberration in the original reduction process are considered too.

Centering Error (Translation)

This is the error caused by a translation of the photographic plate relative to the measuring device. The differences between the standard coordinates and the measured coordinates will be

$$\xi - x = \text{a constant} = h, \tag{55a}$$

$$\eta - y = \text{a constant} = k. \tag{55b}$$

Rotation

If the photographic plate is rotated relative to the measuring device by an angle ψ ($\psi > 0$ for a counter clockwise rotation) then,

$$\xi - x = (1 - \cos\psi)\xi - \eta\sin\psi, \tag{56a}$$

$$\eta - y = \xi\sin\psi + (1 - \cos\psi)\eta. \tag{56b}$$

Non-Perpendicularity of the Axes

If the x and y axes are not orthogonal then,

$$\xi - x = \eta\tan\psi, \tag{57a}$$

$$\eta - y = (1 - \sec\psi)\eta, \tag{57b}$$

where ψ is the acute angle between the η and y axes (e.g., $x = \xi - \eta\tan\psi$, $y = \eta\sec\psi$).

Centering Error

If the optical axis of the telescope (produced) pierces the sky at ($A + \delta A$, $\Delta + \delta\Delta$) instead of at (A, Δ) then the standard coordinates will be incorrect. From Eqs. (40, 41) the leading terms are

$$\xi - x = \cos\Delta\delta A - \eta\sin\Delta\delta A, \tag{58a}$$

$$\eta - y = \delta\Delta + \xi\sin\Delta\delta A. \tag{58b}$$

If it can be assumed that the net result of all of the other sources of error can be adequately described by a translation, a rotation, and a dilation (so that the plate model is of the form

$$\xi = Ax + By + C, \tag{59a}$$

$$\eta = -Bx + Ay + D, \tag{59b}$$

with $|A| >> |B|$), then one can simply and rigorously correct for decentering via

$$\xi = Ax + By + C + x^2\cos\Delta\delta A + xy\delta\Delta, \tag{60a}$$

$$\eta = -Bx + Ay + D + xy\cos\Delta\delta A + y^2\delta\Delta. \tag{60b}$$

See Eichorn[8].

Tilt

If the photographic plate is tilted by an angle ω relative to the focal plane then the relationship between the true distance (ℓ) of an object from the optical axis and its apparent (measured) distance (ℓ') is

$$\ell/\ell' = \sec\psi\cos(\psi - \omega), \tag{61}$$

where ψ is the angular distance subtended by ℓ at the center of the object glass, $\ell = f\tan\psi$. If the angles are small,

$$\ell - \ell' \simeq \omega\psi^2, \tag{62}$$

so this error is of the second order. In terms of the differences between the standard coordinates and the measured coordinates,

$$\xi - x = (p\xi^2 + q\xi\eta)\tan\omega, \tag{63a}$$

$$\eta - y = (p\xi\eta + q\eta^2)\tan\omega, \tag{63b}$$

where p and q are constants.

Focal Length

If the true focal length is f' (instead of f) then any linear measure, ℓ, on the plate will be in error by

$$\ell/f - \ell/f' = (\ell/f)(1 - f/f'). \tag{64}$$

This error is, therefore, linear in the standard coordinates.

Radial Distortion

A net radial distortion in the imaging process produces third order differences of the form

$$\xi - x = R\xi(\xi^2 + \eta^2), \tag{65a}$$

$$\eta - y = R\eta(\xi^2 + \eta^2), \tag{65b}$$

where R is a constant.

Decentering Distortion

When all of the components of the object glass are not aligned properly the resulting imperfection is called decentering distortion. It results in errors of the second order adequately modeled by

$$\xi - x = 2[P\xi^2 + Q\xi\eta] + P(\xi^2 + \eta^2), \tag{66a}$$

$$\eta - y = 2[P\xi\eta + Q\eta^2] + Q(\xi^2 + \eta^2), \tag{66b}$$

where P and Q are constants.

Coma

Coma is a result of the dependence of the focal length on the apparent magnitude of the star. If m is the apparent magnitude then,

$$\xi - x = Sm\xi, \tag{67a}$$

$$\eta - y = Sm\eta. \tag{67b}$$

The constant S is strongly temperature dependent.

Differential Refraction

When the original reduction is only to the apparent place and not the topocentric place, the plate model absorbs the effects of both astronomical refraction and diurnal aberration. However, unless the field is both small in extent and $\tan z \leq 1$, its approximate treatment requires a full second order expression in both coordinates. When a first order treatment is adequate it can be modeled by

$$\xi - x = R_o'[(1 + X_z^2)\xi + X_zY_z\eta], \tag{68a}$$

$$\eta - y = R_o'[X_zY_z\xi + (1 + Y_z^2)\eta], \tag{68b}$$

where (X_z, Y_z) are the standard coordinates of the zenith. $R_o(R_o')$ was introduced in Eqs. (34).

Differential Aberration

Since diurnal aberration is so small it is (formally) ignored. However, annual aberration may be introduced into the plate model via error terms of the form

$$\xi - x = A\xi, \tag{69a}$$

$$\eta - y = A\eta. \tag{69b}$$

When this is done and the astronomical refraction correction left out too, only general precession need be allowed for.

To summarize, if the full reduction to apparent place is made initially, and the field is not too large, a linear model should suffice.

VI. THE TELESCOPE MODEL

Now that we understand the types of errors to be expected we turn to the problem of constructing, in real time, "plate constants" for the telescope. For the set of reference stars we know the computed equatorial topocentric coordinates, $\{(\alpha_{obs}, \delta_{obs})\}$, and the measured equatorial topocentric coordinates, $\{(\alpha_T, \delta_T)\}$. The latter are subject to error but the former are not. For the artificial satellite we know its measured coordinates, (A_T, Δ_T), and we want to deduce (A_{obs}, Δ_{obs}). The simplest linear model, for declinations within $\pm 81^o$ of the equator, relates $e_\alpha = \alpha_{obs} - \alpha_T$ (e.g., $\xi - x$) to $\varepsilon_\alpha = \alpha_{obs} - \langle\alpha_{obs}\rangle$ (e.g., ξ), $\varepsilon_\delta = \delta_{obs} - \langle\delta_{obs}\rangle$ (e.g., η), $\varepsilon_c = c - \langle c\rangle$, and $\varepsilon_m = m - \langle m\rangle$. Similarly for $e_\delta = \delta_{obs} - \delta_T$. The quantities c and m refer to the color and apparent magnitude of the reference stars (observed at a <u>constant</u> camera gain setting). The average values are denoted by $\langle c\rangle$, $\langle m\rangle$. If necessary, one can obviously extend the model to include more than one color. Furthermore, if the camera gain is adjusted during each individual observation such that the size of the disc visible to the observer is always the same (and small), then the magnitude term (but <u>not</u> the color term) may be eliminated.

Only actual observations and data analysis can determine the relative importance of the various terms as a function of the size of the field, spread in apparent magnitude, spread in color, etc. In order to illustrate the type of analysis necessary the linear model is assumed. Thus,

$$e_\alpha = u + v\varepsilon_\alpha + w\varepsilon_\delta + x\varepsilon_c + y\varepsilon_m, \tag{70a}$$

$$e_\delta = U + V\varepsilon_\alpha + W\varepsilon_\delta + X\varepsilon_c + Y\varepsilon_m. \tag{70b}$$

We obtain estimates for u, v, ..., Y by the procedures of least squares using equal weights. More accurate weighting cannot be used for two reasons. First, the Smithsonian Astrophysical Observatory Catalog contains incorrectly computed coordinate variances. Second, insufficient measurements of m and c will be obtained to accurately estimate their variances. After the plate constants are obtained one computes (A_{obs}, Δ_{obs}) from

$$A_{obs} - A_T = u + v(A_{obs} - \langle\alpha_{obs}\rangle) + w(\Delta_{obs} - \langle\delta_{obs}\rangle) + x(C - \langle c\rangle) + y(M - \langle m\rangle), \tag{71a}$$

$$\Delta_{obs} - \Delta_T = U + V(A_{obs} - \langle\alpha_{obs}\rangle) + W(\Delta_{obs} - \langle\delta_{obs}\rangle) + X(C - \langle c\rangle) + Y(M - \langle m\rangle), \tag{71b}$$

where M, C are the apparent magnitude and color of the artificial satellite.

Equations (71), regarded as two, linear, simultaneous, inhomogeneous equations in the two unknowns A_{obs}, Δ_{obs} have a determinant = 1 + (small quantity). Hence, the number of truly significant figures remaining at this stage of the calculation is critical. Thus, one might want to introduce α_T, δ_T on the right hand sides of Eqs. (70) initially via $\varepsilon_\alpha \to E_\alpha = \alpha_T - \langle\alpha_{obs}\rangle$, $\varepsilon_\delta \to E_\delta = \delta_T - \langle\delta_{obs}\rangle$. This would be analogous to replacing (ξ, η) by (x, y) in the N = 3 version of the method of dependences [cf. below Eq. (50b)]. Then, the corresponding version of Eqs. (71) would still represent two, linear, simultaneous, inhomogeneous equations in the two unknowns A_{obs}, Δ_{obs} but the determinant of the system would be exactly unity. Hence, the need for high precision throughout the entire process would be alleviated. Unfortunately, the estimation problem posed by this substitution (e.g., $\varepsilon_\alpha \rightarrow E_\alpha$, $\varepsilon_\delta \rightarrow E_\delta$) requires a priori knowledge of the variances and covariances of e_α, e_δ, E_α, and E_δ. The covariances can be expressed in terms of the variances (given a weak assumption about the perpendicularity of the telescope's axes) but no estimate for the variances is possible. Indeed, the telescope model is needed precisely because the variances are unknown. Thus, the substitution will not be used.

There are still three possible sources of systematic error. One is that the linear model may not suffice. In order to decide this point an external estimate of the ultimate accuracy is needed. This is discussed below. Another source of error is due to the fact that (typically) the reference stars will be several magnitudes brighter than the artificial satellite. Hence, if the camera gain setting is kept constant, one will be extrapolating the magnitude term instead of interpolating. This can, and should be, avoided. The last point concerns an error introduced by the reduction procedure itself (depending on the color).

The refractivity of air varies by about two percent over the wavelength range 4000-7000Å. When the seeing is good the stars are not seen as points but as vertical spectra with the blue end up. The effect is small (see Fig. 4), the angular distance between the red and blue-green ends being 0$''$.35, 0$''$.60, 1$''$.04, 2$''$.24 for zenith distances of 30(15)75°. Therefore, if all of the reference stars are the same color, the inclusion of a color term in Eqs. (70) will not only incorrectly model this effect, but will also systematically bias the results. Ideally one should tabulate R_0, R_1, and R_2 [cf. Eqs. (19, 34)] as a function of c and differentially correct (from the beginning) for color. This would be an extremely difficult thing to do. Nonetheless, one can estimate this error (see below).

One can obtain an estimate for the <u>internal</u> error in the artificial satellite's topocentric position as follows: Solve Eqs. (71) analytically for A_{obs}, Δ_{obs}, say $A_{obs} = f(u, v, \ldots, Y)$. The variance of A_{obs} is given by

$$\sigma^2_{A_{obs}} = (\sigma_u \frac{\partial f}{\partial u})^2 + (\sigma_v \frac{\partial f}{\partial v})^2 + \ldots + (\sigma Y \frac{\partial f}{\partial Y})^2$$

$$+ 2[\Sigma_{uv} \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \ldots + \Sigma_{xy} \frac{\partial f}{\partial x}\frac{\partial f}{\partial Y}], \qquad (72)$$

where Σ_{uv} is the covariance of u, v, etc. The solution of the normal equations provides, coupled with the actual unbiased residual, estimates for the variances and covariances appearing in Eq. (72). Thus, an estimate for the internal variance of A_{obs} (the situation is similar for the declination) is available. As a formal result this is fine, but

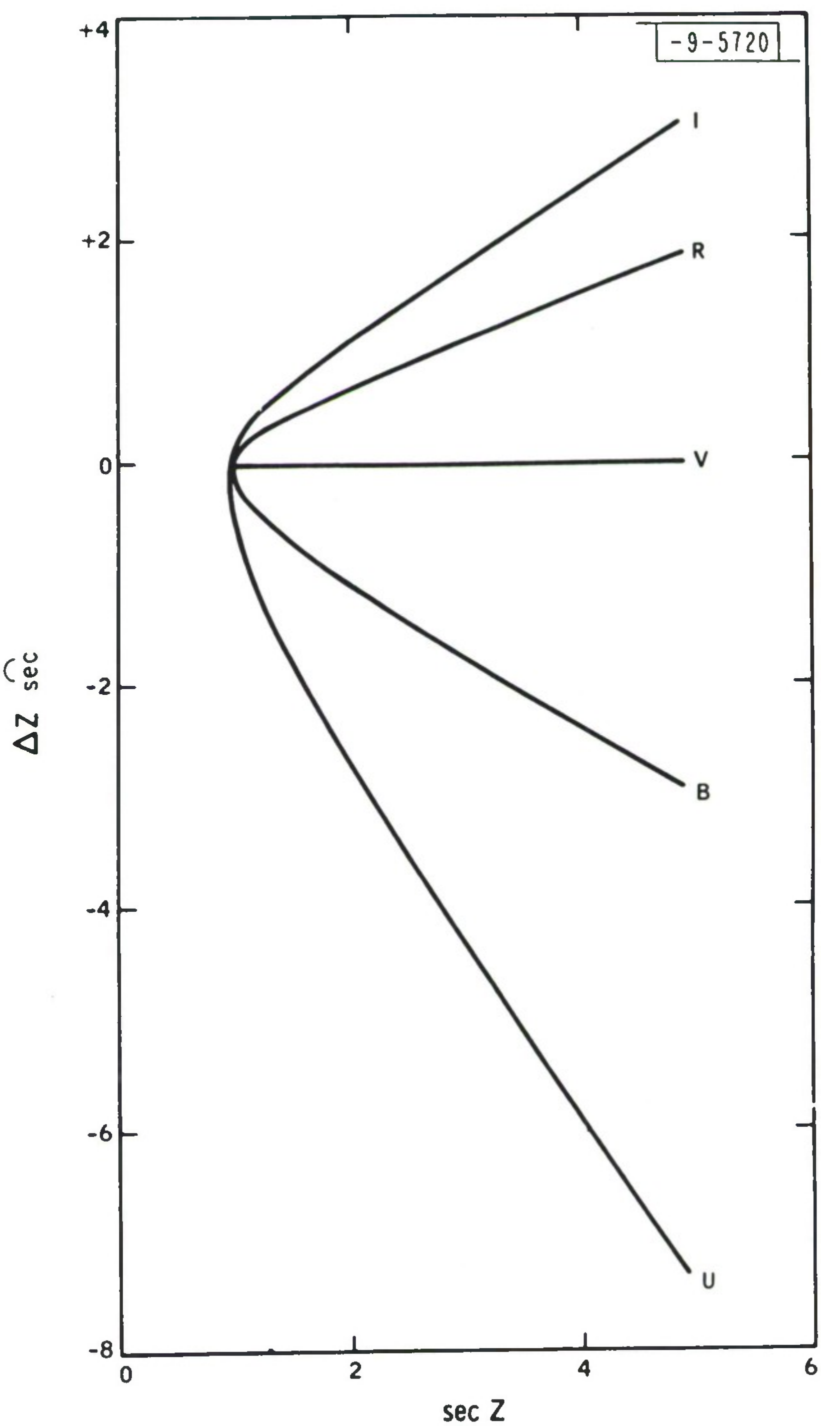

Fig. 4. Dispersion caused by atmosphere in the UBVRI system. After Taylor.[9]

probably useless. It is the external error (e.g., relative to the reference system of the FK4) that is of interest. A better way to estimate this is to have observed and reduced N + 1 stars, allowing N to contribute to the plate model. The extra star, which most closely matches the artificial satellite in color, apparent magnitude, and position, serves as a control on the entire procedure.

VII. ARTIFICIAL SATELLITE REDUCTIONS

At this stage of the analysis I assume that the topocentric equatorial coordinates of the artificial satellite, (A_{obs}, Δ_{obs}) are known and one wants to compute geocentric equatorial coordinates (A, Δ). To do this one needs to correct for refraction, planetary aberration, diurnal aberration, and geocentric parallax. Three of these involve the distance to the artificial satellite. If this is not known it must be estimated. An estimation procedure is described in § VIIA and the rest of § VII discusses the four corrections.

A. Distance Estimation

In this subsection only lower case variables refer to geocentric values (α, δ, h = hour angle, ω = angular speed, d = distance) and upper case variables refer to topocentric values. Also, in this subsection only, the difference between the two sets is due solely to geocentric parallax.

The geocentric location of the observer is given by $\underline{s} = (s = \rho a_e, \phi', T_s)$. The fundamental relationship between the geocentric location and the topocentric location is

$$\underline{d} = \underline{D} + \underline{s}. \tag{73}$$

In component form,

$$d\cos\delta\cos\alpha = D\cos\Delta\cos A + s\cos\phi'\cos T_s, \tag{74a}$$

$$d\cos\delta\sin\alpha = D\cos\Delta\sin A + s\cos\phi'\sin T_s, \tag{74b}$$

$$d\sin\delta = D\sin\Delta + s\sin\phi'. \tag{74c}$$

These equations may be manipulated to yield

$$\sin(\alpha - A) = (s/d)\cos\phi'\sec\Delta\sin H, \tag{75a}$$

$$\sin(\delta - \Delta) = (s/d)\sin\phi\csc\Gamma\sin(\Gamma - \Delta), \tag{75b}$$

$$\tan\Gamma = \tan\phi'\cos[(A - \alpha)/2]\sec[H + (A - \alpha)/2], \tag{75c}$$

$$d = D\sin(\Delta - \Gamma)\csc(\delta - \Gamma), \tag{75d}$$

$$d^2 = D^2 + s^2 + 2Ds(\sin\phi'\sin\Delta + \cos\phi'\cos\Delta\cos H), \tag*{(75c)}$$

or, inversely,

$$\tan(\alpha - A) = a\sin h/(1 - a\cos h), \tag{76a}$$

$$\tan(\delta - \Delta) = b\sin(\gamma - \delta)/[1 - b\cos(\gamma - \delta)], \tag{76b}$$

$$a = (s/d)\cos\phi'\sec\delta, \tag{76c}$$

$$b = (s/d)\sin\phi'\csc\gamma, \tag{76d}$$

$$\tan\gamma \equiv \tan\Gamma = \tan\phi'\cos[(\alpha - A)/2]\sec[h + (\alpha - A)/2], \tag{76e}$$

$$D = d\sin(\delta - \gamma)\csc(\Delta - \gamma), \tag{76f}$$

$$D^2 = d^2 + s^2 - 2ds(\sin\phi'\sin\delta + \cos\phi'\cos\delta\cos h). \tag{76g}$$

These relationships should be used rather than (74) since they will always yield more accurate results.

In order to estimate a distance one needs three positions or two positions and one angular velocity. Utilizing the first data set is a standard problem in celestial mechanics and I shall not treat it here. Instead, I will develop the second approach.

Expand $\alpha - A$, $\delta - \Delta$ in powers of s/d. The first-order result is

$$\alpha - A = (s/d)\cos\phi'\sin H\sec\Delta, \tag{77a}$$

$$\delta - \Delta = (s/d)[\sin\phi'\cos\Delta - \cos\phi'\sin\Delta\cos H]. \tag{77b}$$

Differentiate Eqs. (77) with respect to time (rigorously Ephemeris time but mean solar time is sufficient) and form $\omega^2 = \dot{\alpha}^2\cos^2\delta + \dot{\delta}^2$ where $\dot{z} = dz/dt \ \forall\ z$. The leading terms are

$$\begin{aligned}\omega^2 = \Omega^2 + 2(s/d)\ \{&-\Omega^2[\sin\phi'\sin\Delta + \cos\phi'\cos\Delta\cos H]\\ &+ [\dot{A}\cos\Delta\cos H + \dot{\Delta}\sin\Delta\sin H]\dot{T}_s\cos\phi' - (\dot{d}/d)[\Delta(\sin\phi'\cos\Delta\\ &- \cos\phi'\sin\Delta\cos H) + \dot{A}\cos\phi'\cos\Delta\sin H]\},\end{aligned} \tag{78}$$

so that not only d but $\dot{d}/d$ are required to estimate ω^2. It turns out that $\dot{d}/d$ can be approximately computed without knowledge of d. I now show this.

Differentiate Eqs. (75a, 75b) with respect to time. The <u>exact</u> result is

$$\dot{d}/d + \dot{\alpha}\cot(\alpha - A) - \dot{\delta}\tan\delta = \dot{A}\cot(\alpha - A) + \dot{H}\cot H, \tag{79a}$$

$$\dot{d}/d + \dot{\alpha}EF + \dot{\delta}\cot(\delta - \Delta) = \dot{A}EF + \dot{\Delta}[\cot(\delta - \Delta) - \cot(\Gamma - \Delta)]$$
$$+ \dot{H}E\tan[H - (\alpha - A)/2], \tag{79b}$$

where

$$E = \sin\Gamma\cos\Gamma[\cot(\Gamma - \Delta) - \cot\Gamma], \tag{80a}$$

$$2F = \tan[(\alpha - A)/2] + \tan[H - (\alpha - A)/2]. \tag{80b}$$

In addition, by differentiating the geocentric relationship which expresses the constancy of the direction of the total orbital angular momentum vector, one finds,

$$\dot{\alpha}\sin\delta\cos\delta\cot(\alpha - \Omega') - \dot{\delta} = 0, \tag{79c}$$

where Ω' is the longitude of the ascending node. The only approximation involved in determining $\dot{d}/d$ from Eqs. (79) concerns the computation of Ω'. One calculates Ω' from the two topocentric (instead of geocentric) positions, viz.

$$\tan\Omega' \simeq \frac{\tan\Delta_1\sin A_2 - \tan\Delta_2\sin A_1}{\tan\Delta_1\cos A_2 - \tan\Delta_2\cos A_1} . \tag{80c}$$

Thus, the observational data, the geometry, and the physics provides three, linear, simultaneous, inhomogeneous equations in the three unknowns

$\dot{\alpha}$, $\dot{\delta}$, $\dot{d}/d$. We now continue the estimation problem for d regarding $\dot{d}/d$ as known.

If the orbit were circular and if $\omega = \Omega$, then d would be given by

$$d = d_1 \equiv (GM_{\oplus}/\Omega^2)^{1/3}, \tag{81}$$

where G is the gravitational constant and $M_{\oplus}$ is the mass of the earth; $GM_{\oplus} = 3.98603 \times 10^{14} m^3/sec^2$. A better guess for d corrects d_1 for the effects of an eccentric orbit. I write this as

$$d_2 = F(e) d_1. \tag{82}$$

F(e) is discussed below. A new estimate for the distance, d_3, is then calculated using $d = d_2$ in Eq. (78), and then putting ω^2 into Eq. (81) in place of Ω^2. Finally, the last estimate for d is computed from Eq. (82) with d_3 replacing d_1.

F(e) is the average, over an orbital period, of the ratio of the geocentric distance in the elliptic orbit to the geocentric distance in a (fictitious) circular orbit with the same instantaneous angular speed. Using standard relationships F(e) can be written as (e = eccentricity, E = eccentric anomaly)

$$\begin{aligned} F(e) &= [(1 - e^2)^{1/3}/2\pi] \int_0^{2\pi} (1 + e\cos E)^{2/3} dE \\ &= P_{2/3}(\varepsilon)/\varepsilon^{4/3};\ \varepsilon^2 = 1/(1 - e^2), \end{aligned} \tag{83}$$

where $P_{2/3}$ is the Legendre function of order 2/3. Table 1 lists F(e) for e = 0(0.05)1. If the eccentricity can not be estimated to ±0.1 (from other information) then an average value of F(e) should be used.

$$\int_0^1 F(e)de = 0.8263. \tag{84}$$

Experience with this procedure indicates that $\dot{d}/d$ is poorly determined from Eqs. (79) because of the dominance of the cotangents in Eqs. (79a, 79b). Since it is small, it may be better to set it equal to zero in Eq. (78) if one doubts its sign. The overall accuracy is then ±10 percent in d for $d \geq 3s$, increasing with d.

B. Refraction

The light rays from an extra-terrestrial object must pass through both space and the earth's atmosphere before reaching an earthbound telescope. I assume that the extra-atmospheric path is a straight line. When the light enters the atmosphere the path becomes curved, with a continually changing curvature. The light ray's path is bent towards the geocentric radius vector to the source so that refraction can alter zenith distance but not azimuth. The difference in direction between the final tangent vector to the light ray's path (e.g., at the telescope) and the initial tangent vector to the light ray's path (e.g., at the top of the atmosphere) is called the astronomical refraction. In Fig. 5 $\sphericalangle$ S'OZ (the telescope is at O, Z denotes the zenith of O) is the apparent zenith distance of the source and $\sphericalangle$ SO'Z is the true zenith distance of

TABLE I.

ECCENTRICITY CORRECTION FACTOR

e	F(e)
0	1
0.05	0.999027
0.10	0.996101
0.15	0.991197
0.20	0.984275
0.25	0.975276
0.30	0.964119
0.35	0.950697
0.40	0.934873
0.45	0.916469
0.50	0.895257
0.55	0.870938
0.60	0.843115
0.65	0.811248
0.70	0.774575
0.75	0.731972
0.80	0.681663
0.85	0.620560
0.90	0.542382
0.95	0.429811
1	0

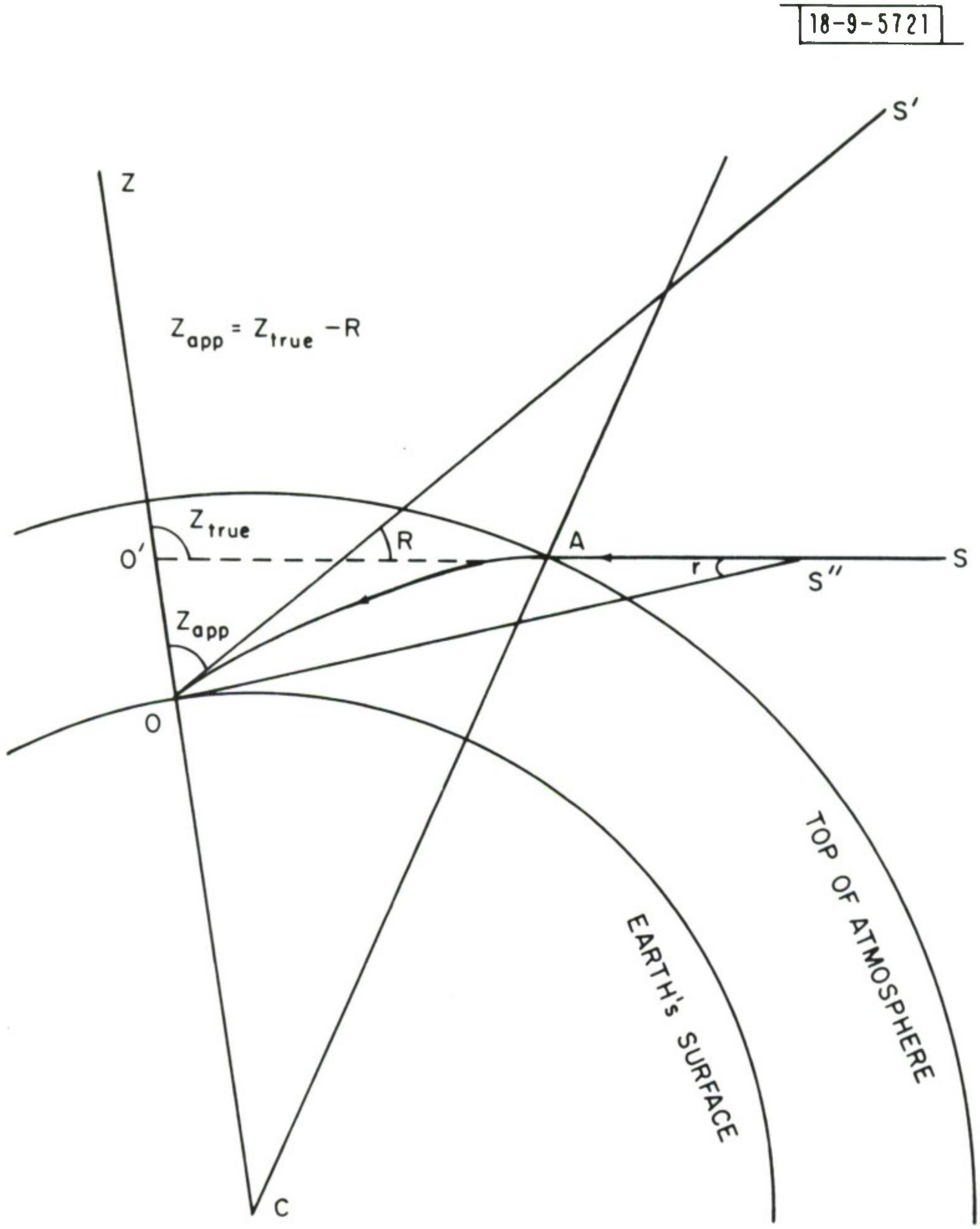

Fig. 5. Diagram for refraction calculation. C is the center of a spherical earth, the observer is at O, and light from S enters the top of the atmosphere at A. Not to scale.

the source.* The difference between these angles is the parallax of the source as seen from 0' relative to 0. To determine the height of 0' ($\overline{00'}$= L), regard the earth as a perfect sphere with radius a_e. Then, for a spherical atmosphere,

$$1 + L/a_e = \mu_o \sin z_{app} \csc z_{true} = \mu_o \sin(z_{true} - R) \csc z_{true}, \qquad (85)$$

where μ_o is the index of refraction of the atmosphere at the telescope and $R = z_{true} - z_{app}$ is called the astronomical refraction. The net refraction is R - r where r (the parallactic refraction) measures the angle subtended by L along the original light path (e.g., O'S" = OS" in Fig. 5). The parallactic refraction for the natural celestial objects can only be appreciable for the Moon (and then only at large zenith distances). Hence, it is universally ignored except for the precise predictions of solar eclipses. However, r increases as the geocentric distance decreases, reaches one minute of arc for an object 500km above the earth's surface. Hence, its universal neglect for artificial satellites will result in a systematic bias.

An approximate expression for r (D = topocentric distance in km, H = height above earth's surface in km) is,

$$\begin{aligned} r \cot z_{app} \cos z_{app} = & -(482''.9/D)[1 - 0.00130(2\sec^2 z_{app} \\ & + \tan^2 z_{app}][1 - \exp(-0.1205H)] \\ & - 0''.0757(\sec^2 z_{app} + \tan^2 z_{app})H\exp(-0.1205H). \qquad (86) \end{aligned}$$

*In this subsection only the difference between true and apparent angles is due solely to refraction.

Thus, the equations analogous to Eqs. (16) for artificial satellites are

$$\delta_{app} = \delta_{true} + (R' - r')\sec\delta_{true}\csc z_{true}[\sin\phi - \sin\delta_{true}\cos z_{true}], \tag{87a}$$

$$\alpha_{app} = \alpha_{true} + (R' - r')\sec\delta_{app}\csc z_{true}\cos\phi\sinh_{true}. \tag{87b}$$

These equations must be solved iteratively for (α_{true}, δ_{true}) and $r'/r = R'/R$.

C. Planetary Aberration

Planetary aberration refers to the aberration of light due explicitly to the motion of the source. Thus, since the artificial satellite is D km away, it required D/c (c = speed of light = 2.997925 x 10^5 km/sec) seconds for the light to reach the observer. Neglecting this correction ($\simeq$ 2" for a geostationary satellite) will also systematically bias the results. If (A"', Δ"') are the satellite's topocentric coordinates corrected for refraction and (A", Δ") are its topocentric coordinates corrected for both refraction and planetary aberration then, (approximately)

$$A'' = A''' - \dot{A}D/c, \tag{88a}$$

$$\Delta'' = \Delta''' - \dot{\Delta}D/c. \tag{88b}$$

D. Diurnal Aberration

This is exactly as for the stars. Hence, (A', Δ') is related to (A", Δ") via [cf. Eqs. (15)],

$$A'' = A' + (v\sin 1'')\cos H'\sec\Delta', \tag{89a}$$

$$\Delta'' = \Delta' + (v\sin 1'')\sin H'\sin\Delta' \tag{89b}$$

$$H' = T_s - A'. \tag{89c}$$

E. Geocentric Parallax

The last correction one must apply to (A', Δ') to yield geocentric coordinates, (A, Δ), is the one for geocentric parallax. Adapting Eqs. (75) to the notation here we calculate successively

$$d^2 = D^2 + s^2 + 2Ds(\sin\phi'\sin\Delta' + \cos\phi'\cos\Delta'\cos H'), \tag{90a}$$

$$A = A' + \arcsin[(s/d)\cos\phi'\sec\Delta'\sin H'], \tag{90b}$$

$$\Delta = \Delta' + \arcsin[(s/d)\sin\phi'\csc\Gamma\sin(\Gamma' - \Delta)], \tag{90c}$$

with

$$\tan\Gamma' = \tan\phi'\cos[(A' - A)/2]\sec[H' + (A' - A)/2]. \tag{90d}$$

REFERENCES

1. F. Schlesinger, Astrophys. J. 33, 161 (1911).

2. H. Eichorn, Astron. Nach. 285, 233 (1960).

3. L. G. Taff, "The Handling of Star Catalogs: The Transformations of Positions and Proper Motions," Technical Note 1976-12, Lincoln Laboratory, M.I.T. (17 February 1976), DDC AD-A023248/8.

4. L. G. Taff, "The Handling and Uses of the SAO Catalog," Project Report ETS-6, Lincoln Laboratory, M.I.T. (5 November 1976), DDC AD-A035871.

5. J. G. Porter and D. H. Sadler, Mon. Not. R. Astr. Soc. 113, 455 (1953).

6. H. H. Turner, Mon. Not. R. Astr. Soc. 54, 13 (1893).

7. H. C. Plummer, Mon. Not. R. Astr. Soc. 92, 892 (1932).

8. H. Eichorn, Astron. Nach 293, 127 (1971).

9. A. J. Taylor, Methods of Experimental Physics 12A, 95 (1974).

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ESD-TR-77-163	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Astrometry in Small Fields		5. TYPE OF REPORT & PERIOD COVERED Technical Note
		6. PERFORMING ORG. REPORT NUMBER Technical Note 1977-2
7. AUTHOR(s) Laurence G. Taff		8. CONTRACT OR GRANT NUMBER(s) F19628-76-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lincoln Laboratory, M.I.T. P.O. Box 73 Lexington, MA 02173		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element No. 63428F Project No. 2128
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Systems Command, USAF Andrews AFB Washington, DC 20331		12. REPORT DATE 14 June 1977
		13. NUMBER OF PAGES 66
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Electronic Systems Division Hanscom AFB Bedford, MA 01731		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

astrometry	satellite detection	astronomical photography
artificial satellites	GEODSS	telescope-camera system

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report discusses the techniques and analysis for the precise reduction of the topocentric and apparent places of stars and artificial satellites. Included is a full discussion of the method of star constants, independent day numbers, geocentric parallax, parallactic refraction, and errors. Also described, in detail, is the analysis necessary for the real time modeling of the telescope-camera system, its theoretical basis, and differential reduction procedures. In addition, an original method for computing the distance of an artificial satellite from two measurements of position and one of angular velocity is developed.

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