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*Icebreaker simulation*

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*Cover: United States Coast Guard Cutter Burton  
Island under way in the Weddell Sea in  
February 1977. The thickness of the sea  
ice is 2 to 3 meters. (Photograph by  
Stephen F. Ackley.)*

# CRREL Report 77-16

## *Icebreaker simulation*

Donald E. Nevel

July 1977

Prepared for

UNITED STATES COAST GUARD

By

CORPS OF ENGINEERS, U.S. ARMY

**COLD REGIONS RESEARCH AND ENGINEERING LABORATORY**

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A brief discussion is given of the ways an icebreaker breaks ice. Since the icebreaking process is so complex, the solution of a mathematical model does not appear to be feasible. As an alternative, it is suggested that physical models be used to design icebreakers. The appropriate scaling laws for physical models are developed and their practical limitation discussed.		

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### PREFACE

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## ICEBREAKER SIMULATION

by

Donald E. Nevel

In designing an icebreaker, we attempt to choose the size, shape, weight and thrust that will efficiently break ice. In order to make this selection rationally, we must first look at the ways in which an ice sheet can break. Any combination of these failures can occur during actual icebreaking.

When a vertical load is applied to the edge of an ice sheet, the ice bends. With sufficient load a crack develops perpendicular to the edge of the ice sheet. Then a circumferential crack forms. Secondary circumferential cracks may also develop closer to the load than the first one. In general, breaking the ice by bending is very efficient; the broken pieces are relatively large.

When a horizontal load is applied to the edge of an ice sheet, the ice enters a state of plane stress. With sufficient load the ice cracks perpendicular to the edge of the ice sheet. If the two pieces of ice are not constrained, they are wedged apart. This type of icebreaking has been observed in thin ice.

If horizontal and vertical loads are applied simultaneously, the horizontal load helps bend the ice sheet. On the other hand, we usually assume that the vertical load does not affect the horizontal plane stresses. The secondary circumferential cracks in bending may be significantly influenced by these horizontal stresses. These cracks are important since they produce smaller, more reasonably sized pieces of ice.

Another way in which a horizontal load can break the ice is by crushing. The process of crushing is not really understood and is probably a combination of tensile and shear failures. Crushing occurs when the ship first hits the ice. Crushing does no useful work, and hence should be kept to a minimum. However, due to the shape of the ship's bow, some crushing always occurs.

The effect of a vertical dynamic load is to cause a local failure. An example of this is the vertical oscillating icebreaker.

Ideally, we would like to formulate the problem mathematically and solve it. The nearest problem that has been solved is a static vertical load on a semi-infinite plate on an elastic foundation, which was solved first by Westergaard<sup>5</sup> and later by Shapiro<sup>4</sup> and Nevel.<sup>3</sup>

The problem of a static horizontal and vertical load on a semi-infinite plate on an elastic foundation has not been solved. For this case, with a concentrated horizontal force  $P_H$ , the differential equation in Cartesian coordinates is

$$\xi^4 \nabla^4 w + w = \frac{q}{\rho g} - \frac{2P_H}{\rho g \pi} \frac{y}{(x^2 + y^2)^2} \left( x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} \right) \quad (1)$$

and in polar coordinates is

$$\xi^4 \nabla^4 w + w = \frac{q}{\rho g} - \frac{P_H}{\rho g \pi} \frac{\sin \theta}{r} \frac{\partial^2 w}{\partial r^2} \quad (2)$$

where  $w$  = deflection

$x, y$  = Cartesian coordinates

$r, \theta$  = polar coordinates

$\rho$  = mass density of water

$g$  = gravitational constant

$q$  = applied vertical stress

$\zeta^4 = E h^3 / 12 \rho g (1 - \nu^2)$

$E$  = Young's modulus of ice

$\nu$  = Poisson's ratio of ice

$h$  = thickness of ice.

Solving either of these differential equations is still insufficient since the boundary conditions during icebreaking may be considerably different than a semi-infinite plate. In addition, dynamic effects have not been considered in the above formulation.

Another closely related problem is the infinite wedge on an elastic foundation. This problem was solved by Nevel,<sup>2</sup> assuming beam theory. Milano<sup>1</sup> has applied the results to the icebreaker problem. Here again the boundary conditions of the wedge are considerably different than those occurring during icebreaking.

Rather than considering the force required to break the ice, White<sup>6</sup> has considered the force that a ship applies to the ice. In this analysis the ship hits the ice, crushes it, slides up on it and comes to rest. The ice sheet is assumed not to deflect. The equations of motion for the ship are derived and numerically evaluated by means of a computer. The results are discussed in terms of the sustained vertical force and the extracting thrust. This method has the advantage of showing the influence of the ship's parameters. However, the limitation is neglecting the deflection and the breaking of the ice.

In contemporary icebreakers, the thrust of the propellers and the momentum of the ship cause the ship to ride up on the ice. After the ship has come to a halt, the vertical force on the ice is due to raising of the center of gravity of the ship and the vertical component of the thrust. This vertical thrust is due to the trim of the ship. The horizontal force on the ice is just the horizontal component of the thrust. Although this horizontal force helps bend the ice, a vertical force is more efficient. Hence, in order to have a more efficient icebreaker we intuitively say that the thrust and momentum should be increased, the stem angle lowered, the coefficient of friction decreased, and the crushing reduced. These intuitive concepts agree with White's results.

After the ship has come to rest on the ice, the thrust is essentially transmitted as a horizontal force to the ice. This could be transmitted as a vertical force if the power were switched from the horizontal thrusting propeller to a vertical one. If the vertical thrusting propeller were at the stern, it is estimated that one half of the thrust could be exerted vertically on the ice.

In the past, bow propellers have been effective in breaking the ice. As far as I can determine, they are effective because of Bernoulli's principle. That is to say, with an increase in the water velocity there is a decrease in pressure. This causes an additional downward force on the ice. In polar ice, bow propellers are not used because the ice does not break before it strikes the propellers. A possible solution to this problem is to place the propellers inside a tunnel going through the ship. This would pull the water into the tunnel near the bow and discharge it farther back along the side of the ship. The use of jets rather than propellers may be another consideration.

The icebreaking process is a complex one which has not been adequately analyzed. One method of solution is to simulate the process by means of a model. If all the important parameters can be scaled, then the size, shape, weight and thrust of the model ship can be varied to increase the effectiveness of icebreaking. In particular, one should determine the most effective shape of the bow. A large variety of bow shapes, such as straight, convex and concave bows (Fig. 1) should be investigated.

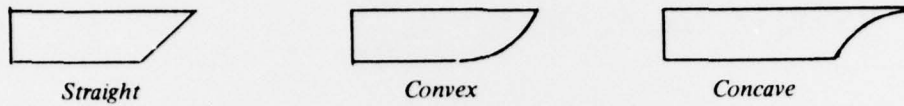
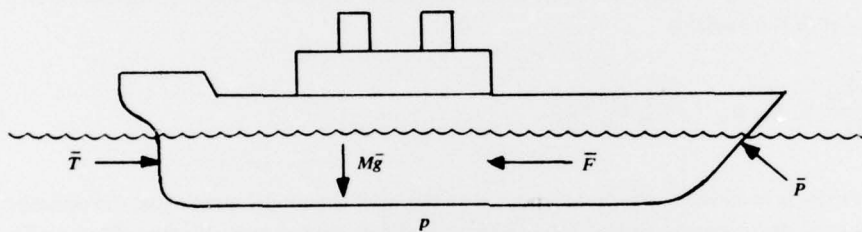


Figure 1. Bow shapes.



$\bar{T}$  = propeller thrust  
 $\bar{P}$  = force from the ice  
 $M$  = mass of the ship  
 $\bar{g}$  = gravitational vector  
 $p$  = water pressure  
 $\bar{F}$  = water friction force per unit area

Figure 2. Forces on an icebreaker.

Since the primary purpose of an icebreaker is to break ice, it seems reasonable to design the ship to do just this efficiently. Other factors such as stability, stern swamping, and extraction thrust should be considered after the efficient bow shape has been found. One can design for the ramming condition in which the ship comes to a halt, or for the continuous condition in which the velocity lost during deceleration just equals the velocity gained during acceleration.

For the model of the ship one requires that all the linear dimensions of the prototype (full scale ship) be scaled by the same factor. This factor  $\lambda$  is defined as

$$L = \lambda L_m \quad (3)$$

where  $L$  is a prototype linear distance and  $L_m$  is a model linear distance. In this paper a parameter with a subscript  $m$  means the parameter is referred to the model. A parameter with no subscript means the parameter is referred to the prototype.

In order to simulate the ship's motion we will consider the equation describing this. Figure 2 shows the forces on the ship. For the prototype the equation for the conservation of linear momentum is

$$M\bar{g} + \bar{T} + \bar{P} + \int p d\bar{s} + \int \bar{F} ds = M \frac{d\bar{V}}{dt} \quad (4)$$

where  $\bar{V}$  = velocity

$t$  = time

$ds$  = ship's differential surface area.

The linear momentum equation for the model ship is

$$M_m \bar{g} + \bar{T}_m + \bar{P}_m + \int p_m d\bar{s}_m + \int \bar{F}_m ds_m = M_m \frac{d\bar{V}_m}{dt_m} \quad (5)$$

For the water, the equation for the conservation of momentum is

$$\rho \frac{D\bar{V}}{Dt} = \nabla p + \mu \nabla^2 \bar{V} + \rho \bar{g} \quad (6)$$

where  $\mu$  is the viscosity of water and  $D/Dt$  is the substantial derivative. The momentum equation for the water of the model is

$$\rho \frac{D\bar{V}_m}{Dt_m} = \nabla_m p_m + \mu \nabla_m^2 \bar{V}_m + \rho \bar{g} \quad (7)$$

If the model is to simulate the prototype, the equation of the model must equal the corresponding equation of the prototype except for a constant multiplication factor. We find this factor to be unity between eq 6 and 7 by comparing the last terms. Comparison of the second term says  $p/L = p_m/L_m$ . Using  $L = \lambda L_m$  this gives

$$p = \lambda p_m \quad (8)$$

Comparison of the first term says  $\bar{V}/t = \bar{V}_m/t_m$ . Using  $L = \lambda L_m$  it yields

$$t = \lambda^{1/2} t_m \quad (9)$$

Hence the velocity is scaled by

$$\bar{V} = \lambda^{1/2} \bar{V}_m \quad (10)$$

Comparison of the third term shows that it cannot be scaled. This term represents the friction of the water and is relatively unimportant; hence we neglect it.

Now we compare the fourth term and eq 4 and 5. Using eq 3 and 8 we find that if eq 5 is multiplied by  $\lambda^3$  each term is equal to the corresponding term of eq 4. Hence we get

$$M = \lambda^3 M_m \quad (11)$$

$$\bar{T} = \lambda^3 \bar{T}_m \quad (12)$$

$$\bar{P} = \lambda^3 \bar{P}_m \quad (13)$$

$$\bar{F} = \lambda \bar{F}_m \quad (14)$$

$$t = \sqrt{\lambda} t_m \quad (15)$$

Since the friction of the water has not been scaled correctly, we cannot guarantee that eq 14 will be satisfied.

The force  $\bar{P}$  from the ice can be resolved into two components and the components must obey the same scaling as  $\bar{P}$  of eq 13. We resolve  $\bar{P}$  into a force  $P_N$  normal to the ship's surface and a force  $P_t$  tangential to the ship's surface. When the ship is sliding onto the ice we have

$$P_t = f P_N \quad (16)$$

where  $f$  is the coefficient of friction between the ship and the ice. For the model we have

$$(P_t)_m = f_m (P_N)_m \quad (17)$$

Using  $\bar{P} = \lambda^3 \bar{P}_m$  we obtain

$$f = f_m \quad (18)$$

We note that if the conservation of linear momentum of the ship is modeled, the conservation of angular momentum will also be modeled. This follows since the angular momentum equals length times linear momentum. Of course the model is assumed to have the same weight distribution as the prototype.

We now turn our attention from the ship to the ice. The force  $\bar{P}$  from the ship on the ice may be resolved into a vertical force  $P_v$  and a horizontal force  $P_H$ . First we shall consider the static case of the vertical force only. The maximum force that can be exerted on the ice depends upon the bending stress  $\sigma$  that will cause the ice to break. The general solution to the bending of an ice sheet under a vertical load for a wide variety of boundary conditions can be expressed as

$$\frac{\sigma h^2}{P_v} = G\left(v, \frac{x}{\ell}, \frac{y}{\ell}, \frac{a}{\ell}, \frac{b}{\ell}, \frac{h}{\ell}\right) \quad (19)$$

where  $a$  and  $b$  are the lengths of the load distribution. For the model ice sheet we have

$$\frac{\sigma_m h_m^2}{(P_v)_m} = G\left(v_m, \frac{x_m}{\ell_m}, \frac{y_m}{\ell_m}, \frac{a_m}{\ell_m}, \frac{b_m}{\ell_m}, \frac{h_m}{\ell_m}\right) \quad (20)$$

Since the function  $G$  is the same function in eq 19 and 20, their values will be equal if their arguments are the same. Thus we obtain

$$\frac{\sigma h^2}{P_v} = \frac{\sigma_m h_m^2}{(P_v)_m} \quad (21)$$

Utilizing eq 13 we obtain

$$h = \lambda^{3/2} (\sigma_m / \sigma)^{1/2} h_m \quad (22)$$

In order for eq 22 to be valid we need

$$v = v_m \quad (23)$$

$$x/\ell = x_m/\ell_m, \quad y/\ell = y_m/\ell_m \quad (24)$$

$$a/\ell = a_m/\ell_m, \quad b/\ell = b_m/\ell_m \quad (25)$$

$$h/\ell = h_m/\ell_m. \quad (26)$$

Equation 23 is reasonably satisfied since Poisson's ratio does not vary greatly for most materials, and previous solutions show that the function  $G$  does not depend greatly upon  $v$ .

Let us assume that a horizontal distance  $x$  in the ice sheet is scaled by

$$x = \theta x_m. \quad (27)$$

It seems desirable to have  $\theta = \lambda$ , since the ship is in contact with the ice. However at this time we will not make them equal. They can be made equal at a later time if this proves to be convenient. Now eq 24 and 25 will be satisfied if

$$\ell = \theta \ell_m. \quad (28)^*$$

Using eq 22 this becomes

$$\theta = \lambda^{9/8} \left( \frac{E}{E_m} \right)^{1/4} \left( \frac{1-v_m^2}{1-v^2} \right)^{1/4} \left( \frac{\sigma_m}{\sigma} \right)^{3/8} \quad (29)$$

Using eq 28, we see that eq 26 cannot be scaled. From previous solutions, it is known that  $h/\ell$  has an influence on the value of  $G$  only in the vicinity of concentrated loads on thick ice sheets. Although the icebreaker may approximate this for initial cracking, the ultimate failure does not occur in the vicinity of the load, and hence should not depend significantly upon  $h/\ell$ . Therefore eq 26 need not be scaled.

We note that from previous solutions,<sup>3</sup> the deflection  $w$  of an ice sheet can be expressed as

$$\frac{w \rho g \ell^2}{P_v} = H\left(v, \frac{x}{\ell}, \frac{y}{\ell}, \frac{a}{\ell}, \frac{b}{\ell}\right). \quad (30)$$

Since the arguments of the function  $H$  have already been scaled we obtain

$$\frac{w \ell^2}{P_v} = \frac{w_m \ell_m^2}{(P_v)_m}. \quad (31)$$

Using eq 13 and 28 this becomes

$$w = \frac{\lambda^3}{\theta^2} w_m. \quad (32)$$

In order to consider the effects of dynamic forces and the forces in the plane of the plate, we consider the complete differential equation, which is

$$\ell^4 \nabla^4 w + \frac{p}{\rho g} + \frac{\gamma h}{\rho g} \frac{\partial^2 w}{\partial t^2} = \frac{q}{\rho g} + \frac{N_x}{\rho g} \frac{\partial^2 w}{\partial x^2} + 2 \frac{N_{xy}}{\rho g} \frac{\partial^2 w}{\partial x \partial y} + \frac{N_y}{\rho g} \frac{\partial^2 w}{\partial y^2} \quad (33)$$

where  $\gamma$  = density of ice

$N_x$  = force per unit length in the plane of the plate in the  $x$  direction

$N_y$  = same but in the  $y$  direction

$N_{xy}$  = shear force per unit length in the plane of the plate.

There is a similar equation with subscripts  $m$  describing the modeled ice sheet.

Comparison of the first terms shows that if eq 33 is multiplied by  $\theta^2/\lambda^3$  it is equal to the modeled equation. The second terms yield

$$p = (\lambda^3/\theta^2)p_m. \quad (34)$$

Comparing the third terms, which represent the vertical acceleration force of the ice sheet, and using eq 9, 22 and 32, we obtain

$$\gamma = (\sigma/\sigma_m)^{1/2} \lambda^{-1/2} \gamma_m. \quad (35)$$

The fourth term is satisfied and the remaining terms, which represent the forces in the plane of the plate, yield

$$P = \theta^3 P_m. \quad (36)$$

Therefore the forces in the plane of the plate can only be scaled correctly when  $\theta = \lambda$ .

If  $\theta \neq \lambda$ , then the horizontal distances in the ice and the forces in the plane of the plate will not be scaled correctly. The effect of the incorrect horizontal length of contact area of the ship will be insignificant on ultimate failure. But the effect of incorrect horizontal scaling on the size of the broken ice pieces will be significant. If the size of the broken ice pieces is not important, such as in the ramming condition, it is not necessary for  $\theta = \lambda$ . However, under the continuous condition, where the size of the broken ice pieces determines the ship's acceleration distance, it is necessary that  $\theta = \lambda$ . In order that  $\theta = \lambda$ , we must choose the model material for ice such that the following equation gives a reasonable value of  $\lambda$ :

$$\lambda = \left(\frac{\sigma}{\sigma_m}\right)^3 \left(\frac{E_m}{E}\right)^2 \left(\frac{1-v^2}{1-v_m^2}\right)^2. \quad (37)$$

We note that if the model material is the same as sea ice, then  $\theta > \lambda$ . For  $\theta > \lambda$ , the model will simulate a prototype that will break the ice into pieces proportional to  $\theta x_m$  while a real icebreaker would break pieces proportional to  $\lambda x_m$ . This means the model will have a greater acceleration distance than the real icebreaker. Furthermore the model will simulate a prototype that has horizontal forces of  $\theta^3 P_m$  while the real forces are  $\lambda^3 P_m$ . Both of these observations say that the model icebreaker will do better than the corresponding real icebreaker. By similar reasoning the reverse argument could be made when  $\theta < \lambda$ .

Saunders-Roe used a wax for the model ice sheet. The properties of this wax were  $\sigma = 8.66 \text{ kg/cm}^2$  and  $E = 3.17 \times 10^3 \text{ kg/cm}^2$ . The U.S. Coast Guard used freshwater ice for tests at the Navy Electronics Laboratory in March 1967. The properties of this ice were  $\sigma = 11.27 \text{ kg/cm}^2$  and  $E/(1-v^2) = 100.5 \times 10^3 \text{ kg/cm}^2$ . For freshwater ice  $v \approx 1/3$  which gives  $E = 89.3 \times 10^3 \text{ kg/cm}^2$ . Low salinity ice was also tried at N.E.L. This ice showed a greater variation in thickness than the freshwater ice. The properties of this low salinity ice were  $\sigma = 11.72 \text{ kg/cm}^2$  and  $E/(1-v^2) = 91.0 \times 10^3 \text{ kg/cm}^2$ . Using  $v = 1/3$  we get  $E = 80.9 \times 10^3 \text{ kg/cm}^2$ . The properties for sea ice are approximately  $\sigma = 5 \text{ kg/cm}^2$  and  $E = 20 \times 10^3 \text{ kg/cm}^2$ .

In order to determine which material is better we compute  $\theta$  as determined by eq 29 for the three model materials. Since Poisson's ratio is not accurately known for these materials, we assume the same Poisson's ratio for all. For  $\lambda = 36$  we get for wax, freshwater ice and low salinity ice a  $\theta$  of 110, 52.5 and 54.7 respectively. For  $\lambda = 44$  we get for wax, freshwater ice and low salinity ice a  $\theta$  of 138, 65.8 and 68.6 respectively. Hence it appears that freshwater ice will give the best simulation of the process.

There is a possibility that various additives to the water may produce an ice which will give a  $\theta$  value closer to  $\lambda$ . However this remains for future research.

In summary the relationships between a prototype and model icebreaker are determined by the following equations:

$$L = \lambda L_m \quad (38)$$

$$M = \lambda^3 M_m \quad (39)$$

$$\bar{T} = \lambda^3 \bar{T}_m \quad (40)$$

$$\bar{P} = \lambda^3 \bar{P}_m \quad (41)$$

$$\bar{V} = \lambda^{1/2} \bar{V}_m \quad (42)$$

$$t = \lambda^{1/2} t_m \quad (43)$$

$$f = f_m \quad (44)$$

$$h = \lambda^{3/2} (\sigma_m / \sigma)^{1/2} h_m \quad (45)$$

$$x = \theta x_m \quad (46)$$

$$\ell = \theta \ell_m \quad (47)$$

$$w = (\lambda^3 / \theta^2) w_m \quad (48)$$

where

$$\theta = \lambda^{9/8} \left( \frac{E}{E_m} \right)^{1/4} \left( \frac{1 - \nu_m^2}{1 - \nu^2} \right)^{1/4} \left( \frac{\sigma_m}{\sigma} \right)^{3/8} \quad (49)$$

In order to simulate the vertical acceleration of the ice sheet, we need

$$\gamma = (\sigma / \sigma_m)^{1/2} \lambda^{-1/2} \gamma_m \quad (50)$$

In order to simulate the forces in the plane of the ice sheet and the size of the broken ice pieces, we need  $\theta = \lambda$ , which means

$$\lambda = \left( \frac{\sigma}{\sigma_m} \right)^3 \left( \frac{E_m}{E} \right)^2 \left( \frac{1 - \nu^2}{1 - \nu_m^2} \right)^2 \quad (51)$$

### Literature cited

1. Milano, V.R. Notes on preliminary design of ice breakers. Thesis at Webb Institute of Naval Architecture.
2. Nevel, D.E. (1961) The narrow free infinite wedge on an elastic foundation. U.S. Army Cold Regions Research and Engineering Laboratory (CRREL) Research Report 79. AD 277538.
3. Nevel, D.E. (1965) A semi-infinite plate on an elastic foundation. CRREL Research Report 136. AD 616313.
4. Shapiro, G.S. (1955) Deflection of a semi-infinite plate on an elastic foundation. *Prikladnaia Matematika i Mekhanika*, Institute Mekhaniki, Akad. Nauk SSSR, tom VII. See also CRREL Translation 48, 1955.
5. Westergaard, H.M. (1923) Om beregning af plader paa elastisk underlag med saerligt henblik paa sporgsmaalet om spaendinger i betonveje. *Ingenioren*, vol. 32, no. 42, p. 513-524.
6. White, R.M. (1965) Dynamically developed force at the bow of an ice breaker. Thesis, Massachusetts Institute of Technology.