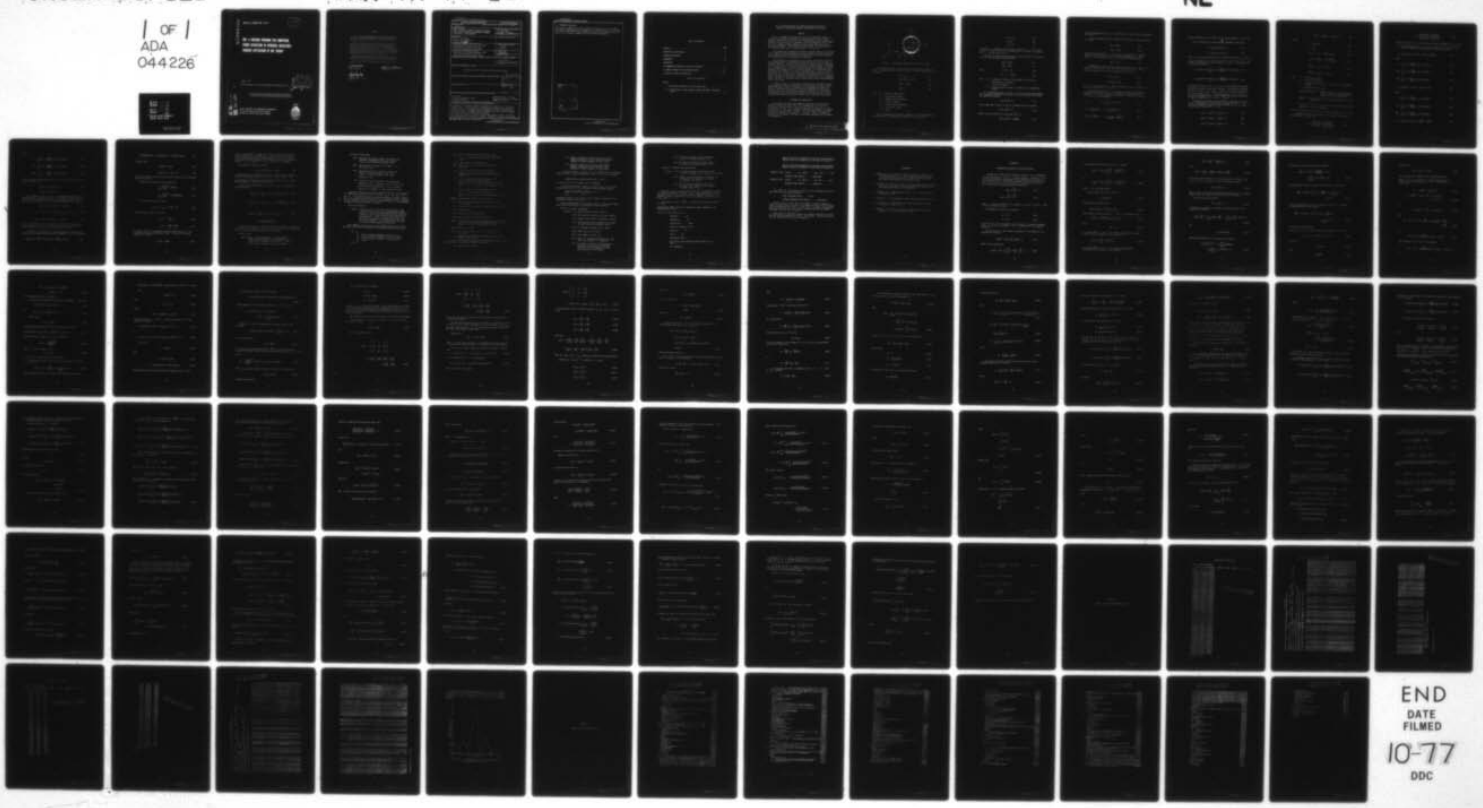
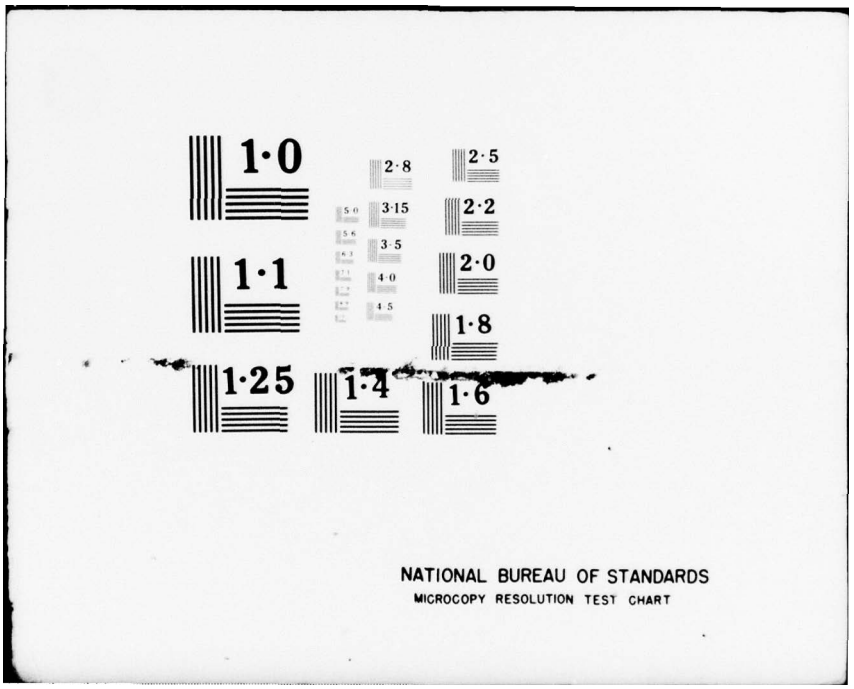
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MIE: A FORTRAN PROGRAM FOR COMPUTING POWER DEPOSITION IN SPHERICAL DIELECTRICS THROUGH APPLICATION OF MIE THEORY

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Interim Report for Period September 1976-January 1977

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USAF SCHOOL OF AEROSPACE MEDICINE
Aerospace Medical Division (AFSC)
Brooks Air Force Base, Texas 78235

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This technical report has been reviewed and is approved for publication.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A homogeneous, nonmagnetic spherical dielectric is immersed in an electromagnetic field. The problem is to determine the absorbed power density at internal points, average absorbed power density, and total absorbed power inside the sphere. A FORTRAN computer program called MIE has been designed to compute the required numerical quantities.
This report describes the program in sufficient depth to permit job setups and implementation on any modern computer. Mathematical theory and formulae,

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20. ABSTRACT (Continued)

structure and sequence of control parameter and data cards, output format, and function subprograms and subroutines are covered in detail. An extensive discussion of the Mie solution, sample problems with associated computer results, and a listing of program MIE are included in appendixes.



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MIE: A FORTRAN PROGRAM FOR COMPUTING POWER DEPOSITION IN SPHERICAL DIELECTRICS THROUGH APPLICATION OF MIE THEORY

PURPOSE

MIE is a FORTRAN IV program written for the IBM 360/65 system, to compute the absorbed power density at internal points, the average absorbed power density, and the total absorbed power inside a homogeneous spherical dielectric that is immersed in an electromagnetic field. Results are obtained through applying the Mie theory and using double-precision arithmetic. Here double-precision numbers have approximately 16.8 decimal digits and an exponent range of -78 to +75.

This program was originally produced to provide special test results for a computer program designed to solve the problem of ascertaining the power deposition inside arbitrarily shaped, finitely conducting biological bodies exposed to electromagnetic radiation.

The knowledge to be gleaned from this effort is directly related to the research effort of the Radiation Sciences Division at the School of Aerospace Medicine. Briefly, here studies are being currently conducted (1) to determine the radiofrequency radiation-induced effects in biological specimens, (2) to seek out possible hazards to personnel in a radiofrequency environment, (3) to accurately measure and to determine the distribution of energy in the whole biological body or just in a particular organ, (4) to find ways to reduce any potentially adverse action between RF emitters and cardiac pacemaker and prosthetics, (5) to extrapolate response to radiation from the test animal to man in a meaningful manner, and (6) to contribute in the design of realistic safety standards with a solid basis.

To benefit users of this report, program MIE is described in sufficient depth to facilitate implementation on any modern computer and job setups. Detailed coverage is provided of the mathematical theory and formulas, structure and sequence of control parameter and data cards, output format, and function subprograms and subroutines. Included in the appendixes are an extensive discussion of the Mie solution, sample problems with associated computer results, and a listing of the program.

MATHEMATICAL DESCRIPTION

We consider the plane electromagnetic wave that is irradiating a homogeneous spherical dielectric to be propagating in the positive z-direction, and the electric field $\underline{E}$ to be linearly polarized in the x-direction (cf. Fig. 1). A system of Cartesian coordinates with origin at the center of the sphere is used. Also, the medium which surrounds the sphere is taken as free space (or vacuum). Thus our embedding medium is a nonconductor, and both the surrounding medium and the sphere are nonmagnetic.

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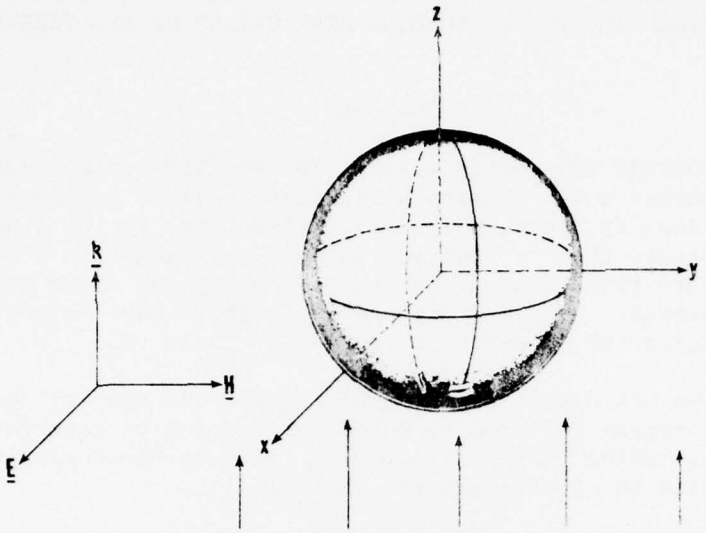


Figure 1. Directional approach of the incident wave.

The Gaussian system of units (nonrationalized c.g.s. units) is used. In this system the macroscopic form of Maxwell's equations may be written as

$$\underline{\nabla}\cdot\underline{D} = 4\pi\rho, \tag{1}$$

$$\underline{\nabla}\times\underline{H} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{D}}{\partial t}, \tag{2}$$

$$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial \underline{H}}{\partial t}, \tag{3}$$

$$\underline{\nabla}\cdot\underline{H} = 0, \tag{4}$$

with $\underline{D}$ = electric displacement,

ρ = free-charge density,

$\underline{H}$ = magnetic-field intensity,

$\underline{J}$ = current density,

$\underline{E}$ = electric-field intensity,

c = velocity of light,

t = time.

For our homogeneous, isotropic, permeable, conducting dielectric, and linear case, the constitutive (material) relations are

$$\underline{D} = \varepsilon \underline{E} , \tag{5}$$

$$\underline{B} = \underline{H} , \tag{6}$$

$$\underline{J} = \sigma \underline{E} . \tag{7}$$

The symbol ε represents the dielectric constant, and σ the conductivity. In these relations and in equations 1-4, the magnetic permeability, μ, was set equal to one.

Since we are considering time-harmonic fields, Maxwell's equations 2 and 3 take the simpler time-free form

$$\nabla \underline{x} \underline{H} = im^2 k \underline{E} , \tag{8}$$

$$\nabla \underline{x} \underline{E} = -ik \underline{H} , \tag{9}$$

where

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} , \tag{10}$$

$$m^2 = \varepsilon - i \frac{4\pi\sigma}{\omega} , \tag{11}$$

with k = propagation constant or wave number in free space,

λ = wavelength in free space,

ω = circular frequency,

m = complex refractive index, a constant for a homogeneous medium.

In a homogeneous medium, relative to a Cartesian coordinate system, each rectangular component of vectors $\underline{E}$ and $\underline{H}$ satisfies the scalar-wave Helmholtz equation

$$\nabla^2 \psi + m^2 k^2 \psi = 0. \tag{12}$$

At the same time, vectors $\underline{E}$ and $\underline{H}$ satisfy the vector equation

$$\nabla^2 \underline{A} + m^2 k^2 \underline{A} = 0. \tag{13}$$

Recall that the formula for evaluating $\nabla^2 \underline{A}$ is

$$\nabla^2 \underline{A} = \nabla(\nabla \cdot \underline{A}) - \nabla \underline{x} (\nabla \underline{x} \underline{A}). \tag{14}$$

(Details showing that vector $\underline{E}$ satisfies eq. 13 can be found in Appendix A, pp. 19-21.)

Now two independent vector solutions of equation 13 can be expressed as vectors

$$\underline{M}_{\psi} = \underline{\nabla}\times(\underline{r}\psi), \tag{15}$$

$$mk\underline{N}_{\psi} = \underline{\nabla}\times\underline{M}_{\psi}, \tag{16}$$

where the scalar function ψ is a solution of equation 12. (Details pertinent to eqs. 15 and 16 can be found in Appendix A, pp. 27-30.) Since $\underline{r}$ is a constant vector, the following relationship holds:

$$mk\underline{M}_{\psi} = \underline{\nabla}\times\underline{N}_{\psi} \tag{17}$$

Consequently, if we choose two independent solutions, u and v, of equation 12 and construct the field vectors M_u, M_v, N_u, N_v, we will find that equations 8 and 9 are satisfied by the vectors

$$\underline{E} = \underline{M}_{v} + i\underline{N}_{u}, \tag{18}$$

$$\underline{H} = m(-\underline{M}_{u} + i\underline{N}_{v}). \tag{19}$$

Here i is the complex unit. (Details pertinent to eqs. 18 and 19 can be found in Appendix A, pp. 30-31.)

Convenience dictates that our problem be embedded in a spherical system of coordinates r(radial), θ(scattering or colatitude), and ϕ(azimuthal). In this system, the components of vectors $\underline{M}_{\psi}$ and $\underline{N}_{\psi}$ (ψ = u or v) can be expressed as

$$M_r = 0, \qquad N_r = \frac{1}{mk}\frac{\partial^2}{\partial r^2}(r\psi) + mkr\psi, \tag{20}$$

$$M_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\psi), \quad N_\theta = \frac{1}{mkr}\frac{\partial^2}{\partial r\partial\theta}(r\psi), \tag{21}$$

$$M_\phi = -\frac{1}{r}\frac{\partial}{\partial\theta}(r\psi), \qquad N_\phi = \frac{1}{mkr\sin\phi}\frac{\partial^2}{\partial r\partial\phi}(r\psi). \tag{22}$$

(Details pertinent to eqs. 20-22 can be found in Appendix A, pp. 32-34.)

For our incident plane-wave radiation expressed in the form

$$\underline{E} = \underline{a}_x E_o \exp[-i(kz-\omega t)], \tag{23}$$

$$\underline{H} = \underline{a}_y E_o \exp[-i(kz-\omega t)], \tag{24}$$

where E_o is the wave amplitude, and polarization vectors $\underline{a}_x$ and $\underline{a}_y$ are unit vectors directed along the x,y-axes, van de Hulst (5, p. 122) chooses the independent scalar solutions, u and v, of equation 12 for the induced wave as

$$u = E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} mc_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr), \tag{25}$$

$$v = E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} md_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr), \tag{26}$$

The expansion coefficients c_n and d_n are determined by using the appropriate boundary conditions; $p_n^1(\cos\theta)$, the associated Legendre function of order 1; and $j_n(mkr)$, the spherical Bessel function of the first kind and order n. (Details pertinent to eqs. 25 and 26 and the corresponding forms for the incident and scattered waves can be found in Appendix A, pp. 34, 36, 37, 49-61.)

The boundary conditions on the tangential components of vectors $\underline{E}$ and $\underline{H}$, when applied to the components in equations 20-22, yield a system of algebraic equations in the unknowns a_n, b_n, c_n, and d_n (van de Hulst's notation [5, p. 123]):

$$\psi_n(x) - a_n\zeta_n(x) - mc_n\psi_n(y) = 0, \tag{27}$$

$$\psi_n'(x) - a_n\zeta_n(x) - c_n\psi_n'(y) = 0, \tag{28}$$

$$\psi_n(x) - b_n\zeta_n(x) - d_n\psi_n(y) = 0, \tag{29}$$

$$\psi_n'(x) - b_n \zeta_n'(x) - m\, d_n \psi_n'(y) = 0, \tag{30}$$

where

$$x = \frac{2\pi a}{\lambda} = ka, \tag{31}$$

$$y = mka, \tag{32}$$

$$\psi_n(z) = z j_n(z) = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z), \tag{33}$$

$$\zeta_n(z) = z h_n^{(2)}(z) = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(z), \tag{34}$$

$$h_n^{(2)}(z) = j_n(z) - i\eta_n(z), \tag{35}$$

$$H_n^{(2)}(z) = J_{n+\frac{1}{2}}(z) - iN_{n+\frac{1}{2}}(z). \tag{36}$$

Here x = MIE-size parameter,

a = radius of the sphere,

m = complex refractive index, a constant for a homogeneous medium,

λ = wavelength in free space,

$J_{n+\frac{1}{2}}(z)$ and $N_{n+\frac{1}{2}}(z)$ = Bessel functions of half-odd-integer order, of the first and second kind (Neumann function) respectively,

$H_{n+\frac{1}{2}}^{(2)}(z)$ = Hankel function of half-odd-integer order, of the second kind.

The prime on $\psi_n(z)$ or $J_n(z)$ denotes first derivative with respect to the function argument.

(Details pertinent to eqs. 27-36 can be found in Appendix A, pp. 35-45.)

Elementary algebraic manipulations of equations 27-30 result in the solution of c_n and d_n:

$$c_n = \frac{\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)}{\psi_n'(y)\zeta_n(x) - m\psi_n(y)\zeta_n'(x)}, \tag{37}$$

$$d_n = \frac{\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)}{m\psi_n'(y)\zeta_n(x) - \psi_n(y)\zeta_n'(x)} . \tag{38}$$

(Details pertinent to eqs. 37 and 38 can be found in Appendix A, pp. 42-49.)

Utilizing equations 18, 20, 21, 22, 25, and 26, we can derive the r, θ, and φ components for the induced electric-field vector:

$$E_r = iE_o\exp(i\omega t)\cos\phi(m^2krU + 2mUR + m^2krURR), \tag{39}$$

where

$$U = \sum_{n=1}^{\infty} c_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr), \tag{40}$$

$$UR = \sum_{n=1}^{\infty} c_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n'(mkr), \tag{41}$$

$$URR = \sum_{n=1}^{\infty} c_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n''(mkr). \tag{42}$$

$$E_\theta = E_o\exp(i\omega t)\cos\phi[mU + i(\frac{1}{kr}UR + mURR)] \tag{43}$$

where

$$U = \sum_{n=1}^{\infty} d_n(-i)^n \frac{2n+1}{n(n+1)} \pi_n(\cos\theta) j_n(mkr), \tag{44}$$

$$UR = \sum_{n=1}^{\infty} c_n(-i)^n \frac{2n+1}{n(n+1)} \tau_n(\cos\theta) j_n(mkr), \tag{45}$$

$$URR = \sum_{n=1}^{\infty} c_n(-i)^n \frac{2n+1}{n(n+1)} \tau_n(\cos\theta) j_n'(mkr). \tag{46}$$

$$E_\phi = -E_o\exp(i\omega t)\sin\phi[mU + i(\frac{1}{kr}UR + mURR)] , \tag{47}$$

where

$$U = \sum_{n=1}^{\infty} d_n (-i)^n \frac{2n+1}{n(n+1)} \tau_n(\cos\theta) j_n(mkr), \tag{48}$$

$$UR = \sum_{n=1}^{\infty} cn (-i)^n \frac{2n+1}{n(n+1)} \pi_n(\cos\theta) j_n(mkr), \tag{49}$$

$$URR = \sum_{n=1}^{\infty} c_n (-i)^n \frac{2n+1}{n(n+1)} \pi_n(\cos\theta) j_n'(mkr). \tag{50}$$

Here a single or double prime denotes a first or second derivative with respect to mkr and

$$\pi_n(\cos\theta) = \frac{1}{\sin\theta} P_n^1(\cos\theta), \tag{51}$$

$$\tau_n(\cos\theta) = \frac{d}{d\theta} P_n^1(\cos\theta). \tag{52}$$

For a sphere of radius a (cm), the absorbed power density at an internal point, the average absorbed power density, and the total absorbed power are computed by means of the following formulas (with σ and $\underline{E}$ in electrostatic units of the nonrationalized c.g.s. system):

$$P_d(r,\theta,\phi) = 0.05\sigma \, \underline{E}\cdot\underline{E}^* \quad \text{watts/m}^3, \tag{53}$$

$$P_{tot} = 10^{-6} \int_0^{\pi} \int_0^{a} \int_0^{2\pi} P_d r^2 \sin\theta \, d_\phi \, d_r \, d_\theta \quad \text{watts}, \tag{54}$$

$$P_{avg} = 10^6 P_{tot} / [(4/3)\pi a^3] \quad \text{watts/m}^3. \tag{55}$$

Here, * denotes the complex conjugate. In program MIE a numerical integration product rule, based on three m-point Gauss-Legendre quadrature formulas, is used to evaluate the triple integral.

To complete our discussion, it seems appropriate to consider the formulas used in generating the values of certain functions. The formulas

$$P_{n+1}^1(\cos\theta) = \frac{2n+1}{n} \cos\theta \, P_n^1(\cos\theta) - \frac{n+1}{n} P_{n-1}^1(\cos\theta), \tag{56}$$

$$\sin\theta \frac{d}{d\theta} P_n^1(\cos\theta) = n\cos\theta P_n^1(\cos\theta) - (n+1)P_{n-1}^1(\cos\theta), \tag{57}$$

together with

$$P_1^1(\cos\theta) = \sin\theta, \tag{58}$$

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta \tag{59}$$

are used to generate function and derivative values of the associated Legendre functions.

Special limit values are also obtained by

$$\lim_{\theta\to 0} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{n(n+1)}{2}, \tag{60}$$

$$\lim_{\theta\to \pi} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{(-1)^{n+1} n(n+1)}{2}. \tag{61}$$

The forward recurrence relation

$$\eta_{n+1}(z) + \eta_{n-1}(z) = \frac{2n+1}{z} \eta_n(z) \tag{62}$$

is used together with the relations

$$\eta_0(z) = -\frac{\cos z}{z}, \tag{63}$$

$$\eta_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \tag{64}$$

to generate values of the Neumann spherical Bessel functions. The generating process is terminated at order N when the following termination criterion

$$|\eta_n(z)| \geq \text{STOPR} \tag{65}$$

is met. Here STOPR is a number, say 1.0D25. The user's needs will determine whether or not STOPR should retain its presently assigned value. Our own demands were satisfactorily met for both real ($2\pi r/f$) and complex ($2\pi mr/f$) arguments of the Neumann functions for parameter ranges: $7 \leq |m| \leq 100$, $5 \leq r \leq 25$ cm, and $20 \leq f \leq 1000$ MHz.

The backward recurrence relation

$$j_{n-1}(z) = \frac{2n+1}{z} j_n(z) - j_{n+1}(z) \tag{66}$$

in combination with an appropriate starting value is used to generate values of the spherical Bessel functions of the first kind, $j_n(z)$.

This technique of using the backward relation in place of the forward relation helps to avoid stability problems.

Values of the derivatives of $j_n(z)$ and $n_n(z)$ are obtained by using the formulas:

$$\frac{d}{dz} j_n(z) = \frac{1}{2n+1} [n j_{n-1}(z) - (n+1) j_{n+1}(z)], \tag{67}$$

$$\frac{d^2}{dz^2} j_n(z) = -\frac{2}{z} \frac{d}{dz} j_n(z) - [1 - \frac{n(n+1)}{z^2}] j_n(z), \tag{68}$$

$$\frac{d}{dz} n_n(z) = \frac{1}{2n+1} [n n_{n-1}(z) - (n+1) n_{n+1}(z)]. \tag{69}$$

PROGRAM DESCRIPTION

Program MIE consists of a driver routine, four subroutine subprograms, and one function subprogram. A list of the driver routine and subprograms, along with a brief explanation of their use, follows.

Driver routine:

MAIN - used to input/output data, to compute spherical Bessel function values for a real argument, to compute series expansion coefficients c_n and d_n, and to direct the course of the calculations.

Subroutine subprograms:

EVEC - generates the complex radial, colatitude, and azimuthal components of the electric-field vector, $\underline{E}$, and the scalar product $\underline{E}\cdot\underline{E}^*$.

TERM - generates the nth power of -i, where i is the complex unit.

BESS - generates an array of values of each of the spherical Bessel functions $j_n(mx)$ and $\eta_n(mx)$ for complex argument mx. Array dimension = 100.

PL - generates an array of values of the associated Legendre function, $P_n^1(\cos\theta)$, and an array for its first derivative, $\frac{d}{d\theta} P_n^1(\cos\theta)$, for n varying from one to a maximum N. Array dimension = 100.

Subroutines BESS and PL, plus the algorithm (in-line coded in MAIN) for computing the spherical Bessel functions $j_n(x)$ and $\eta_n(x)$ for real x, are modified versions of subroutines found in program SUP[1]--one of two allied programs developed by D. S. Drumheller and D. E. Setzer (1) as a research tool in the study of problems related to electromagnetic transmission through atmospheric aerosols.

Function subprogram:

GAUSS3 - a product rule for the numerical evaluation of the triple integral for the total absorbed power within the sphere. A basic m-point Gauss-Legendre quadrature formula is used. User's option is available for the selection of m, number of weighting coefficients, and associated points for Gaussian quadrature, from the set of integers {2,3,4,5,6,8,10,12,14} for each of the three Gauss quadrature rules used.

Blank COMMON is used by driver routine (MAIN) and subroutine EVEC. The list of the arrays, variables, and constants stored in this area is

CN -
DN - } array of complex expansion coefficients for coefficients for components of electric-field vector $\underline{E}$ inside the sphere. Array dimension = 100.

AJR - array of spherical Bessel functions $j_n(x)$, $n = 1,\ldots,N$ (complex x) values. Array dimension = 100.

AM - complex index of refraction (m).

AMK - complex index of refraction (m) times wave number (k).

CEX - complex time-variation factor [exp(iωt)].

Z - complex index of refraction (m) times Mie-size parameter (x).

DP - array of spherical Bessel functions $\eta_n(x)$, $n = 1,\ldots,N$ (real x) values or first derivative values of associated Legendre functions $p_n^1(\theta)$, $n = 1,\ldots,N$ (real θ). Array dimension = 100.

P - array of spherical Bessel functions $j_n(x)$, $n = 1,\ldots,N$ (real x) values or values of $p_n^1(\theta)$, $n = 1,\ldots,N$ (real θ). Array dimension = 100.

ALAMDA - wavelength of incident electric wave.

D1 - twice the radial distance to an interior point of the sphere.

EO - intensity (field strength) of incident electric field.

PHI - azimuthal angle.

PIE - the number 3.14159265358973DØ.

RKR - the reciprocal of Mie-size parameter.

STOPR - a test quantity for the termination of the recurrence relation for determining the spherical Bessel functions $\eta_n(mx)$. It is equal in magnitude to 1.0D25.

SINTH - the value of $\sin\theta$.

THETA - the colatitude angle (θ) of an interior point of the sphere.

NC - maximum order of spherical Bessel functions $\eta_n(mx)$ (complex mx) less two.

A single labeled common area, GAUSS, is used by the driver routine, MAIN, and function subprogram, GAUSS3, for values of

D - diameter of the sphere.

M1 - number of points to be used in an m-point Gauss-Legendre quadrature rule to evaluate innermost integral of triple integral for total power.

M2 - number of points to be used in m-point Gauss-Legendre quadrature rule to evaluate middle integral of triple integral for total power.

In subroutine BESS, if variable N, the maximum order of spherical Bessel function $\eta_n(mx)$ (complex mx), tests ≤ 2, the error message

PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z = ...

is printed out and the computer run is terminated.

In function subprogram, GAUSS3, an invalid value of one of the parameters M1, M2, or M3 institutes an error-message printout:

ERROR IN INTEGRATION CONTROLS. M1 = . . .

M2 = . . . M3 = . . .

A numerical value of zero for the triple integral, equation 54, is returned to driver routine, MAIN.

Input to program MIE is by keypunched cards. There are two basic input cards with structure and sequential order as follows:

Card No. 1 (control parameters)

Columns: 1-10 FREQ. Frequency in MHz. (E10.3)

11-20 EPS. Relative dielectric constant. (E10.3)

21-30 SIGMA1. Conductivity in mho/meter. (E10.3)

31-40 EØ. Intensity (field strength) of incident electric field in volt/meter. (E10.3)

41-50 D. Diameter of sphere in cm. (E10.3)

51-60 TIME. Time in sec. (E10.3)

61-65 NOC. Number of cases. (I5)

66-70 IOPT = 0: Average power density and total power not computed; = 1: otherwise.

71-73 M1. Number of points for Gauss-Legendre quadrature rule applied to innermost integral of triple integral for total power deposited in the sphere.

74-76 M2. Same definition as M1 except application is to middle integral. (I3)

77-79 M3. Same definition as M1 except application is to outermost integral. (I3)

Cards Nos. 2-(NOC+1) (coordinate data)

Columns: 1-10 R. Radial spherical coordinate of interior point of sphere in cm. Range: R > 0. (E10.3)

11-20 THETAD. Colatitude spherical coordinate of interior point of sphere in deg. Range: $0 \leq$ THETAD $\leq 180.$ (E10.3)

21-30 PHID. Azimuthal spherical coordinate of interior point of sphere in deg. Range: $0 \leq$ PHID $\leq 360.$ (E10.3)

The last card of a single data set must be a termination card with the symbols /* punched in columns 1 and 2. Also program MIE can handle multiple data sets. Each data set [Cards 1-(NOC+1)] is stacked one behind the other, with the last card in the complete data deck a termination card.

Under option control IOPT=0, program output printouts consist of the title

DEPOSITION OF POWER INSIDE OF A HOMOGENEOUS SPHERE IMMERSED IN AN ELECTROMAGNETIC FIELD

followed by such information as

FREQUENCY = . . . MHZ

WAVELENGTH = . . . CM

CONDUCTIVITY = . . . MHO/M

RELATIVE DIELECTRIC CONST = . . .

DIAMETER = . . . CM

TIME = . . . SEC

REFRACTIVE INDEX = . . .

THE HIGHEST ORDER NEUMANN FUNCTION COMPUTED IS OF ORDER = . . .

SIZE PARAMETER = . . .

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SPHERICAL BESSEL FUNCTION OF ARGUMENT X TO THE VALUE OF SIN(X)/X = . . .

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SPHERICAL BESSEL FUNCTION OF ARGUMENT Z TO THE VALUE OF SIN(Z)/Z = . . .

INTERNAL POINT: RADIUS = . . . CM THETA = . . . DEG PHI = . . . DEG

ABSORBED POWER DENSITY = . . . WATTS/M**3

INTERNAL POINT: RADIUS = . . . CM THETA = . . . DEG PHI = . . . DEG

ABSORBED POWER DENSITY = . . . WATTS/M**3

. . .

For IOPT = 1, the printouts consist of the information set forth above plus two additional statements:

TOTAL ABSORBED POWER = . . . WATTS

AVERAGE ABSORBED POWER DENSITY = . . . WATTS/M**3

Option IOPT = 0 with NOC = 0 produces a printout similar in format to that presented on page 16 but without the data on the internal points; whereas, option IOPT = 1 with NOC = 0 yields a like printout plus total absorbed power and average absorbed power density results formatted as in the above paragraph.

When using the IBM 360/65 system, the combined compilation and link editing times of program MIE is about 0.30 minute. Execution time for specific problems will be found in Appendix B.

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APPENDIX A

MATHEMATICAL ANALYSIS OF THE MIE SOLUTION

Consider four vector-valued functions ($\underline{E}$, $\underline{H}$, $\underline{D}$, and $\underline{B}$) defined on the Cartesian product $(R^3 \times R)$ of space and time. Functions $\underline{E}$ and $\underline{H}$ denote the electric- and magnetic-field intensities; functions $\underline{D}$ and $\underline{B}$ denote electric displacement and magnetic induction respectively. Vector $\underline{D}$ is also called electric induction by O'Rahilly (3, p. 35). These functions are related by Maxwell's equations

$$\underline{\nabla} \times \underline{E} + \frac{1}{c} \frac{\partial \underline{B}}{\partial t} = 0, \qquad \text{(A-1)}$$

$$\underline{\nabla} \times \underline{H} - \frac{1}{c} \frac{\partial \underline{D}}{\partial t} = \frac{4\pi\sigma \underline{E}}{c}, \qquad \text{(A-2)}$$

where σ is the conductivity of the medium at (x,y,z) at time t, and c is the velocity of light.

We assume the constitutive relations

$$\underline{B} = \mu \underline{H}, \qquad \text{(A-3)}$$

$$\underline{D} = \varepsilon \underline{E}, \qquad \text{(A-4)}$$

and divide R^3 into two regions in each of which μ (magnetic permeability), ε (dielectric constant), and σ (conductivity) are constant functions.

We take the curl of both sides of equation A-1 and substitute into equation A-2 and have

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \underline{\nabla}^2 \underline{E} + \frac{1}{c} \frac{\partial}{\partial t} (\underline{\nabla} \times \underline{B}) = 0, \qquad \text{(A-5)}$$

which in turn implies that

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \underline{\nabla}^2 \underline{E} + \frac{\mu}{c^2} \frac{\partial^2 \underline{D}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2} \frac{\partial \underline{E}}{\partial t} = 0. \qquad \text{(A-6)}$$

From equation A-6 we see that $\underline{\nabla}\cdot\underline{E} = \rho/\varepsilon$ implies

$$\underline{\nabla}(\rho/\varepsilon) - \underline{\nabla}^2\underline{E} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial\underline{E}}{\partial t} = 0 \qquad \text{(A-7)}$$

or

$$\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) - \underline{\nabla}^2\underline{E} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial\underline{E}}{\partial t} = 0, \qquad \text{(A-8)}$$

where ρ is the charge density.

Let us assume that the $\underline{E}$ field is

$$\underline{E} = \underline{E}_o\exp(i\omega t). \qquad \text{(A-9)}$$

Then equation A-2 implies that

$$\underline{\nabla}\times\underline{H} = \left(\frac{i\omega\varepsilon}{c}\underline{E}_o + \frac{4\pi\sigma}{c}\underline{E}_o\right)\exp(i\omega t). \qquad \text{(A-10)}$$

From equation A-10 we see that if ε and σ are constant, then

$$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{H}) = \left(\frac{i\omega\varepsilon+4\pi\sigma}{c}\right)\exp(i\omega t)\underline{\nabla}\cdot\underline{E}_o = 0. \qquad \text{(A-11)}$$

Thus

$$\underline{\nabla}\cdot\underline{E} = 0 \qquad \text{(A-12)}$$

in a region where ε and σ are constant. Therefore, for time-harmonic waves, equation A-8 may be replaced by

$$-\underline{\nabla}^2\underline{E} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2}\frac{\partial\underline{E}}{\partial t} = 0, \qquad \text{(A-13)}$$

in a region where ε and σ are constant. Substituting equation A-9 into equation A-13 we find that

$$\underline{\nabla}^2 \underline{E} + \left(\frac{\mu\varepsilon\omega^2}{c^2} - i\,\frac{4\pi\mu\sigma\omega}{c^2}\right)\underline{E} = 0. \qquad \text{(A-14)}$$

Let

$$m^2k^2 = \omega^2\left[\frac{\mu\varepsilon}{c^2} - i\,\frac{4\pi\mu\sigma}{\omega c^2}\right] = \left(\frac{\omega}{c}\right)^2\left(\mu\varepsilon - i\,\frac{4\pi\mu\sigma}{\omega}\right). \qquad \text{(A-15)}$$

Then substituting equation A-15 into equation A-14, we have, with $k=\omega/c$ and $m^2 = \mu\varepsilon - i4\pi\mu\sigma/\omega$, the vector Helmholtz equation

$$\underline{\nabla}^2 \underline{E} + m^2k^2\underline{E} = 0, \qquad \text{(A-16)}$$

where k and m are the wave number and complex index of refraction, respectively, and remain as such in the rest of the paper.

Now the scalar-wave equation

$$\nabla^2\psi + m^2k^2\psi = 0 \qquad \text{(A-17)}$$

in spherical coordinates, a separable coordinate system for the equation, has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2\psi}{\partial\phi^2} + m^2k^2\psi = 0. \qquad \text{(A-18)}$$

Let

$$\psi = R(r)\Theta(\theta)\Phi(\phi). \qquad \text{(A-19)}$$

Substituting equation A-19 into equation A-18 yields

$$\Theta\Phi\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2R')\right] + \frac{\Phi R}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\Theta') + \frac{\Theta R}{r^2\sin^2\theta}\Phi'' + m^2k^2R\Theta\Phi = 0. \qquad \text{(A-20)}$$

Expansion of terms in equation A-20 results in

$$\Theta\Phi[R'' + (2/r)R'] + \frac{\Phi R}{r^2}\left(\frac{\cos\theta}{\sin\theta}\,\Theta' + \Theta''\right)$$

$$+ \frac{\Theta R}{r^2\sin^2\theta}\,\Phi'' + m^2k^2R\Theta\Phi = 0, \qquad \text{(A-21)}$$

and dividing all terms in equation A-21 by $R\Theta\Phi$, we infer that

$$\frac{1}{R}[R'' + (2/r)R'] + \left[\frac{1}{\theta r^2}\right]\left(\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta'\right) + \frac{1}{r^2\sin^2\theta\Phi}\,\Phi''$$

$$+ m^2k^2 = 0. \qquad \text{(A-22)}$$

Finally, multiplying all terms of equation A-22 by r^2, we obtain the separated form of the equation

$$\left\{\frac{r^2}{R}[R'' + (2/r)R'] + m^2k^2r^2\right\} + \frac{1}{\Theta}\left(\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta'\right)$$

$$+ \frac{1}{\sin^2\theta\Phi}\,\Phi'' = 0. \qquad \text{(A-23)}$$

We use the following lemma:

Lemma A-1. *Let* y *be a Bessel function of order* $(n + \frac{1}{2})$, *then*

$$x^2y'' + x\,y' = [n(n+1) + \tfrac{1}{4} - x^2]y \qquad \text{(A-24)}$$

along with

$$\nu^2 = n(n+1) \qquad \text{(A-25)}$$

and

$$u = y/\sqrt{x} \qquad \text{(A-26)}$$

implies that

$$x^2u'' + 2xu' = (\nu^2 - x^2)u\,. \tag{A-27}$$

Proof of Lemma A-1. Equation A-24 is satisfied because $(n+\frac{1}{2})^2 = n^2 + n + \frac{1}{4} = n(n+1) + \frac{1}{4}$ and by the definition of Bessel's equation (Whittaker [6, p. 38]). Differentiating u we find that

$$u' = y'/\sqrt{x} - (1/2)y/x^{3/2}, \tag{A-28}$$

$$u'' = y''/\sqrt{x} - (1/2)y'/x^{3/2} - (1/2)y'/x^{3/2} + (3/4)y/x^{5/2}, \tag{A-29}$$

or

$$u'' = y''/\sqrt{x} - y'/x^{3/2} + (3/4)y/x^{5/2}. \tag{A-30}$$

Now

$$x^2u'' + 2x\,u' = x^{3/2}y'' - \sqrt{x}y' + (3/4)y/\sqrt{x} + 2\sqrt{x}y' - y/\sqrt{x} \tag{A-31}$$

or

$$x^2u'' + 2xu' = (1/\sqrt{x})[x^2y'' + xy' - (1/4)y]. \tag{A-32}$$

From equations A-24 and A-32 it follows that

$$x^2u'' + 2xu' = (1/\sqrt{x})\{[n(n+1) + (1/4)]y - (1/4)y - x^2y\},$$

or

$$x^2u'' + 2xu' = [n(n+1) - x^2](y/\sqrt{x}),$$

$$= [n(n+1) - x^2]u. \tag{A-33}$$

This completes the proof of Lemma A-1.

Now equation A-23 implies that there is a constant ν^2 such that

$$(1/R)(r^2R'' + 2rR') + m^2k^2r^2 = \nu^2 \tag{A-34}$$

or

$$r^2R'' + 2rR' + (m^2k^2r^2 - \nu^2)R = 0. \tag{A-35}$$

Let us write

$$R(r) = z(mkr). \tag{A-36}$$

On substituting equation A-36 into equation A-35, we obtain

$$m^2k^2r^2z''(mkr) + 2mkrz'(mkr) + (m^2k^2r^2 - \nu^2)\, z\,(mkr) = 0, \tag{A-37}$$

which according to Lemma A-1 is true provided that

$$z(mkr) = C\,\frac{Z_{n+\frac{1}{2}}(mkr)}{\sqrt{mkr}}\,, \tag{A-38}$$

where C is an arbitrary constant

$$\nu^2 = n(n+1), \tag{A-39}$$

and $Z_{n+\frac{1}{2}}$ is a Bessel function of order $n+\frac{1}{2}$.

Substituting equations A-36 and A-38 into equation A-23 results in the relation

$$n(n+1) + \frac{1}{\Theta}\left(\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta'\right) + \frac{1}{\sin^2\theta\Phi}\,\Phi'' = 0. \tag{A-40}$$

Now multiplying all terms of equation A-40 by $\sin^2\theta$, we find that

$$[n(n+1)\sin^2\theta + (1/\Theta)(\sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta')] + (1/\Phi)\Phi'' = 0. \quad \text{(A-41)}$$

Let

$$(1/\Phi)\Phi'' = -\ell^2. \quad \text{(A-42)}$$

Then

$$\Phi'' + \ell^2\Phi = 0 \quad \text{(A-43)}$$

implies

$$\Phi(\phi) = c_1\sin(\ell\phi) + c_2\cos(\ell\phi) \quad \text{(A-44)}$$

for some constants, c_1 and c_2. Substituting equation A-42 into equation A-41 yields

$$n(n+1)\sin^2\theta\Theta + \sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta' - \ell^2\Theta = 0 \quad \text{(A-45)}$$

or

$$(1-\cos^2\theta)\Theta'' + \sin\theta\cos\theta\Theta' + [n(n+1)(1-\cos^2\theta)-\ell^2]\Theta = 0. \quad \text{(A-46)}$$

Let us set

$$\Theta = w(\cos\theta). \quad \text{(A-47)}$$

Then

$$\Theta' = -\sin\theta\, w'(\cos\theta), \quad \text{(A-48)}$$

$$\Theta'' = -\cos\theta\, w'(\cos\theta) + \sin^2\theta\, w''(\cos\theta). \quad \text{(A-49)}$$

Substituting equations A-48 and A-49 into equation A-46, we obtain

$$(1-\cos^2\theta)[-\cos\theta\, w'(\cos\theta) + \sin^2\theta\, w''(\cos\theta)]$$

$$+ \cos\theta \sin^2\theta\, w'(\cos\theta) + [n(n+1)(1-\cos^2\theta)-\ell^2]w(\cos\theta) = 0 \tag{A-50}$$

and dividing all terms of equation A-50 by $(1-\cos^2\theta)$ yields

$$(1-\cos^2\theta)w''(\cos\theta) - 2\cos\theta\, w'(\cos\theta)$$

$$+ \left[n(n+1) - \frac{\ell^2}{1-\cos^2\theta}\right] w(\cos\theta) = 0. \tag{A-51}$$

Now set $x = \cos\theta$ in equation A-51 and discover that $w(x)$ satisfies

$$(1-x^2)w''(x)-2xw'(x) + \left[n(n+1) - \frac{\ell^2}{1-x^2}\right]w(x) = 0. \tag{A-52}$$

It can be shown that

$$w(x) = P_n^{\ell}(x) \tag{A-53}$$

is a solution of equation A-52, where $P_n^{\ell}(x)$ is a solution of the associated Legendre ordinary differential equation (Whittaker [6, p. 324]).

Hence

$$R\Theta\Phi = \frac{Z_{n+\frac{1}{2}}(mkr)}{\sqrt{mkr}}\, P_n^{\ell}(\cos\theta)[c_1\cos(\ell\phi) + c_2\sin(\ell\phi)], \tag{A-54}$$

where $Z_{n+\frac{1}{2}}$ denotes a Bessel function of order $n+\frac{1}{2}$. Observe that

$$\psi = R(r)\Theta(\theta)\Phi(\phi) \tag{A-55}$$

satisfies equation A-17.

The following vector functions:

$$\underline{L} = \underline{\nabla}\psi, \tag{A-56}$$

$$\underline{M} = \underline{\nabla} x (\underline{a}\psi) = \underline{\nabla}\psi x \underline{a}, \tag{A-57}$$

$$\underline{N} = (1/mk)\underline{\nabla} x \underline{M}, \tag{A-58}$$

in which $\underline{a}$ is a constant vector and ψ is a spatial function, satisfy the same vector-wave equation since the components of these vectors are linear combinations of derivatives of solutions of the basic scalar Helmholtz wave equation--a linear partial differential equation with constant coefficients.

The validity of equation A-57 is based on the following proposition:

Proposition A-1. *For every* C^1 *function* ψ *and every constant vector* $\underline{a}$, *we have*

$$\underline{\nabla} x (\underline{a}\psi) = \underline{\nabla}\psi x \underline{a}. \tag{A-59}$$

Proof of proposition A-1.

$$\underline{\nabla} x \underline{a}\psi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1\psi & a_2\psi & a_3\psi \end{vmatrix}$$

$$= \underline{i}\left(a_3\frac{\partial\psi}{\partial y} - a_2\frac{\partial\psi}{\partial z}\right) - \underline{j}\left(a_3\frac{\partial\psi}{\partial x} - a_1\frac{\partial\psi}{\partial z}\right)$$

$$+ \underline{k}\left(a_2\frac{\partial\psi}{\partial x} - a_1\frac{\partial\psi}{\partial y}\right). \tag{A-60}$$

$$\underline{\nabla}\psi x\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \underline{i}(a_3\frac{\partial\psi}{\partial y} - a_2\frac{\partial\psi}{\partial z}) - \underline{j}(a_3\frac{\partial\psi}{\partial x} - a_1\frac{\partial\psi}{\partial z})$$

$$+ \underline{k}(a_2\frac{\partial\psi}{\partial x} - a_1\frac{\partial\psi}{\partial y}). \qquad \text{(A-61)}$$

Since the right sides of equations A-60 and A-61 are identical, the proposition is proven.

The approach of Stratton (4, pp. 392-423) is to express the vector potential $\underline{A}$ as a linear combination of vector functions $\underline{L}_n$, $\underline{M}_n$ and $\underline{N}_n$ where these depend on ψ_n, the nth spherical harmonic associated with the scalar Helmholtz wave equation.

Observe that

$$\underline{\nabla}x(\underline{r}\psi) = \underline{\nabla}\psi x\underline{r} + \psi\underline{\nabla}x\underline{r}, \qquad \text{(A-62)}$$

where $\underline{r}$ is the position vector. To simplify equation A-62, we need to represent the basic operators in an orthogonal coordinate system.

Let (x,y,z) denote a point in Cartesian coordinates. Suppose that

$$x = x(u^1,u^2,u^3), \quad y = y(u^1,u^2,u^3), \quad z = z(u^1,u^2,u^3). \qquad \text{(A-63)}$$

If the position vector, $\underline{r}$, is expressed as

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \quad (\underline{i},\underline{j},\underline{k} \text{ unit base vectors}), \qquad \text{(A-64)}$$

then in Cartesian coordinates

$$\underline{\nabla}\psi x \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \psi_x & \psi_y & \psi_z \\ x & y & z \end{vmatrix}$$

$$= \underline{i}(z\psi_y - y\psi_z) - \underline{j}(z\psi_x - x\psi_z) + \underline{k}(y\psi_x - x\psi_y). \qquad \text{(A-65)}$$

It is therefore natural to define operators L_x, L_y, and L_z by the rules

$$L_x\psi = z\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial z}, \qquad \text{(A-66)}$$

$$L_y\psi = x\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial x}, \qquad \text{(A-67)}$$

$$L_z\psi = y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}. \qquad \text{(A-68)}$$

Note that

$$L_x\nabla^2\psi = z\left(\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial z^2\partial y}\right) - y\left(\frac{\partial^3\psi}{\partial z\partial x^2} + \frac{\partial^3\psi}{\partial z\partial y^2} + \frac{\partial^3\psi}{\partial z^3}\right)$$

$$= \left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2\right]\left(z\frac{\partial\psi}{\partial y} - x\frac{\partial\psi}{\partial y}\right). \qquad \text{(A-69)}$$

That is to say, $L_x\nabla^2 = \nabla^2 L_x$. Moreover, we have the following theorem:

Theorem A-1. *For all* C^3 *functions* ψ, *we have*

$$\nabla^2 L_x\psi = L_x\nabla^2\psi, \qquad \text{(A-70)}$$

$$\nabla^2 L_y\psi = L_y\nabla^2\psi, \qquad \text{(A-71)}$$

$$\nabla^2 L_z\psi = L_z\nabla^2\psi. \qquad \text{(A-72)}$$

Hence, if

$$\nabla^2\psi = -m^2k^2\psi, \tag{A-73}$$

the vector function

$$\begin{aligned}\underline{V} &= \underline{i}L_x\psi + \underline{j}L_y\psi + \underline{k}L_z\psi \\ &= \nabla\psi x\underline{r}\end{aligned} \tag{A-74}$$

satisfies

$$\nabla^2\underline{V} = -m^2k^2\underline{V}. \tag{A-75}$$

Proof of Theorem A-1. If $\underline{V}$ is defined by equation A-74, then equations A-70 through A-73 imply that

$$\begin{aligned}\nabla^2\underline{V} &= \underline{i}\nabla^2L_x\psi + \underline{j}\nabla^2L_y\psi + \underline{k}\nabla^2L_z\psi \\ &= \underline{i}L_x\nabla^2\psi + \underline{j}L_y\nabla^2\psi + \underline{k}L_z\nabla^2\psi \\ &= \underline{i}L_x(-m^2k^2\psi) + \underline{j}L_y(-m^2k^2\psi) + \underline{k}L_z(-m^2k^2\psi) \\ &= -m^2k^2\underline{V}.\end{aligned} \tag{A-76}$$

This establishes Theorem A-1.

If u and v are two solutions of the scalar-wave equation, then Maxwell's equations are satisfied by

$$\underline{E} = \underline{M}_v + i\underline{N}_u \quad (i \text{ is the complex unit}). \tag{A-77}$$

Equation A-1 implies

$$\nabla x\underline{E} + \frac{i\omega}{c}\underline{B} = 0. \tag{A-78}$$

Thus

$$\underline{H} = -\frac{c}{i\mu\omega}\,\underline{\nabla}x\underline{E} = i\left(\frac{c}{\mu\omega}\right)\underline{\nabla}x\underline{E}\ . \tag{A-79}$$

Furthermore, in view of equations A-58 and A-77

$$\underline{H} = \left(\frac{ic}{\mu\omega}\right)mk\underline{N}_v + \left(\frac{ic}{\mu\omega}\right)[\underline{\nabla}x(\underline{\nabla}x\underline{M}_u)/mk]i \tag{A-80}$$

or, equivalently,

$$\underline{H} = \frac{imkc}{\mu\omega}\,\underline{N}_v + \left(\frac{ic}{\mu\omega mk}\right)[\underline{\nabla}(\underline{\nabla}\cdot\underline{M}_u) - \underline{\nabla}^2\underline{M}_u]i\ . \tag{A-81}$$

The definition of $\underline{M}_u$ by the rule

$$\underline{M}_u = \underline{\nabla}x\underline{r}u, \tag{A-82}$$

the vector identity (the divergence of a curl is zero), and equations A-75 or A-76, imply that

$$\underline{H} = \frac{imkc}{\mu\omega}\,\underline{N}_v + \frac{imkc}{\mu\omega}\,\underline{M}_u i \tag{A-83}$$

or

$$\underline{H} = \frac{mkc}{\mu\omega}\,(-\underline{M}_u + i\underline{N}_v). \tag{A-84}$$

In the Gaussian units for a nonmagnetic body, $\mu = 1$. Thus, since $k = \omega/c$, we have

$$\underline{H} = m(-\underline{M}_u + i\underline{N}_v)\ . \tag{A-85}$$

In the spherical coordinate system with unit base vectors, $\underline{e}_r$, $\underline{e}_\theta$, and $\underline{e}_\phi$, vector $\underline{A}$ can be represented as

$$\underline{A} = A_r \underline{e}_r + A_\theta \underline{e}_\theta + A_\phi \underline{e}_\phi \ . \tag{A-85.1}$$

Then

$$\begin{aligned} \underline{\nabla x A} = \frac{1}{r^2 \sin\theta} \{ & [\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)]\underline{e}_r \\ & + [\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r \sin\theta A_\phi)]r\underline{e}_\theta \\ & + [\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}]r \sin\theta \, \underline{e}_\phi \} \ . \end{aligned} \tag{A-86}$$

We take $A_r = r\psi$, $A_\theta = 0$, and $A_\phi = 0$, and observe that

$$\underline{M}_\psi = M_r \underline{e}_r + M_\theta \underline{e}_\theta + M_\phi \underline{e}_\phi \tag{A-87}$$

implies that

$$M_r = 0, \tag{A-88}$$

$$M_\theta = \frac{1}{r \sin\theta} \frac{\partial}{\partial\phi}(r\psi), \tag{A-89}$$

$$M_\phi = -\frac{1}{r} \frac{\partial}{\partial\theta}(r\psi). \tag{A-90}$$

Furthermore, the functions N_ψ satisfy the relation

$$\underline{N}_\psi = \frac{1}{mk} \underline{\nabla x M}_\psi \ , \tag{A-91}$$

which implies that if

$$\underline{N}_{\psi} = N_r \underline{e}_r + N_\theta \underline{e}_\theta + N_\phi \underline{e}_\phi \ , \tag{A-92}$$

then

$$N_r = \frac{1}{mk} \{ - \left(\frac{1}{r^2 \sin\theta}\right) \frac{\partial}{\partial\theta} [\sin\theta \frac{\partial}{\partial\theta} (r\psi)] - \left(\frac{1}{r^2 \sin^2\theta}\right) \frac{\partial^2}{\partial\phi^2} (r\psi) \}, \tag{A-93}$$

$$N_\theta = \frac{1}{mk} \{ - \frac{\partial}{\partial r} [r \sin\theta \, (- \frac{1}{r} \frac{\partial}{\partial\theta}(r\psi))] r \} \left(\frac{1}{r^2 \sin\theta}\right)$$

$$= \frac{1}{mk} [\frac{1}{r} \frac{\partial^2}{\partial r \partial\theta}(r\psi)] \ , \tag{A-94}$$

$$N_\phi = \frac{1}{mkr^2 \sin\theta} \{ \frac{\partial}{\partial r} [\frac{r}{r\sin\theta} \frac{\partial}{\partial\phi}(r\psi)] \} r\sin\theta, \tag{A-95}$$

Hence

$$N_\phi = \frac{1}{mkr \sin\theta} \frac{\partial^2}{\partial r \partial\phi}(r\psi) \ . \tag{A-96}$$

Now equation A-93 may be simplified by using the fact that ψ satisfies equation A-18. We have

$$N_r = \frac{1}{mk} [\frac{1}{r} \frac{\partial}{\partial r} (r^2 \frac{\partial\psi}{\partial r}) + m^2 k^2 r\psi]. \tag{A-97}$$

Since

$$\frac{\partial}{\partial r}(r\psi) = r \frac{\partial\psi}{\partial r} + \psi, \tag{A-98}$$

partial differentiation with respect to r yields

$$\frac{\partial^2}{\partial r^2}(r\psi) = r\frac{\partial^2\psi}{\partial r^2} + 2\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) . \qquad \text{(A-99)}$$

Substituting equation A-99 into equation A-97, we see that

$$N_r = \frac{1}{mk}\left[\frac{\partial^2}{\partial r^2}(r\psi) + m^2k^2r\psi\right] . \qquad \text{(A-100)}$$

The incident wave is described by the vectors

$$\underline{E} = \underline{a}_x E_o \exp[-i(kz-\omega t)] , \qquad \text{(A-101)}$$

$$\underline{H} = \underline{a}_y E_o \exp[-i(kz-\omega t)] , \qquad \text{(A-102)}$$

where vectors $\underline{a}_x$ and $\underline{a}_y$ are unit vectors directed along the x,y-axes. Vectors $\underline{E}$ and $\underline{H}$ can be expressed also in the form of equations A-77 and A-85 with scalar functions

$$u = E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(kr) , \qquad \text{(A-103)}$$

$$v = E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(kr) . \qquad \text{(A-104)}$$

The representation of vector $\underline{E}$ is given by

$$\underline{E} = \underline{M}_v + i\underline{N}_u . \qquad \text{(A-105)}$$

The terms

$$(\underline{M}_v)_\theta = \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} = F_\theta(\theta,\phi)v, \qquad \text{(A-106)}$$

$$(\underline{N}_u)_\theta = \frac{1}{mk}\frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}(ru) = G_\theta(\theta,\phi)\frac{\partial}{\partial r}(ru) \tag{A-107}$$

make up E_θ. Similarly, for E_ϕ

$$(\underline{M}_v)_\phi = -\frac{\partial v}{\partial \theta} = F_\phi(\theta,\phi)v, \tag{A-108}$$

$$(\underline{N}_u)_\phi = \frac{1}{mkr\sin\theta}\frac{\partial^2}{\partial r \partial \phi}(ru) = G_\phi(\theta,\phi)\frac{\partial}{\partial r}(ru) \tag{A-109}$$

This does not remain true when in equations A-103 and A-104 the $(-i)^n$ is replaced by $(-i)^n a_n$ and $(-i)^n b_n$, respectively, where the $(a_1, a_2, \ldots, a_n, \ldots)$ and $(b_1, b_2, \ldots, b_n, \ldots)$ are members of the sequence space appropriate to the functions being represented. We see by differentiation in the tangential directions that if $\frac{1}{m}\frac{\partial}{\partial r}(ru)$ and v are continuous in r for each θ and ϕ, then E_θ and E_ϕ are continuous across the boundary of the sphere.

Now we repeat this argument for the $\underline{H}$ vector. The $\underline{H}$ vector is defined by

$$\underline{H} = m(-\underline{M}_u + i\underline{N}_v). \tag{A-110}$$

To get the tangential components of $\underline{H}$, simply interchange u and v in the preceding argument. Thus, the continuity of tangential $\underline{H}$ is assured if $\frac{\partial(rv)}{\partial r}$ and mu are continuous in r for each θ and ϕ.

We introduce a new set of functions which differ from spherical Bessel functions by an additional factor, z. These functions are defined by

$$\psi_n(z) = zj_n(z) = (\pi z/2)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z), \tag{A-111}$$

$$\chi_n(z) = -z\eta_n(z) = -(\pi z/2)^{\frac{1}{2}} N_{n+\frac{1}{2}}(z), \tag{A-112}$$

$$\zeta_n(z) = z h_n^{(2)}(z) = (\pi z/2)^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(z), \qquad \text{(A-113)}$$

where

$$J_\nu(z) = \left[\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}\right](z/2)^\nu \qquad \text{(A-114)}$$

is holomorphic for integral ν and converges for all ν if the appropriate definition of $(z/2)^\nu$ is given and

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad \text{(A-115)}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\, Y_\nu(z)$$

$$= \frac{J_{-\nu}(z) - \exp(i\nu\pi) J_\nu(z)}{-i \sin(\nu\pi)}, \qquad \text{(A-116)}$$

$$N_\nu(z) = Y_\nu(z). \qquad \text{(A-117)}$$

(The functions N_ν are called Bessel functions of the second kind, Neumann functions, or Weber functions.)

An outgoing scattered wave can be obtained by applying a linear operator to the ordered pair (u,v), where

$$u = E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} -a_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) h_n^{(2)}(kr), \qquad \text{(A-118)}$$

$$v = E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} -b_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) h_n^{(2)}(kr). \qquad \text{(A-119)}$$

Similarly, an induced wave can be obtained by applying a linear operator to (u,v), where

$$u = E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} mc_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr), \qquad \text{(A-120)}$$

$$v = E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} md_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr). \qquad \text{(A-121)}$$

The boundary conditions are defined by

$$\underline{E}^{\text{incident}}_{\text{tangent}} + \underline{E}^{\text{scattered}}_{\text{tangent}} = \underline{E}^{\text{inside}}_{\text{tangent}} \qquad \text{(A-122)}$$

and

$$\underline{H}^{\text{incident}}_{\text{tangent}} + \underline{H}^{\text{scattered}}_{\text{tangent}} = \underline{H}^{\text{inside}}_{\text{tangent}} . \qquad \text{(A-123)}$$

Since the tangent space of the sphere is two-dimensional, we have four equations in four unknowns. These are assured by the continuity of mu, $\frac{1}{m}\frac{\partial(ru)}{\partial r}$, v, and $\frac{\partial(rv)}{\partial r}$. Thus, equations A-103 and A-104, equations A-118 through A-123, and the continuity relations imply that the relations that must be satisfied are

$$(mu)_{(A-103)} + (mu)_{(A-118)} = (mu)_{(A-120)}, \qquad \text{(A-124)}$$

$$\left(\frac{1}{m}\frac{\partial(ru)}{\partial r}\right)_{(A-103)} + \left(\frac{1}{m}\frac{\partial(ru)}{\partial r}\right)_{(A-118)} = \left(\frac{1}{m}\frac{\partial(ru)}{\partial r}\right)_{(A-120)}, \qquad \text{(A-125)}$$

$$v_{(A-104)} + v_{(A-119)} = v_{(A-121)}, \qquad \text{(A-126)}$$

$$\left(\frac{\partial(rv)}{\partial r}\right)_{(A-104)} + \left(\frac{\partial(rv)}{\partial r}\right)_{(A-119)} = \left(\frac{\partial(rv)}{\partial r}\right)_{(A-121)} . \qquad \text{(A-127)}$$

In equations A-124 and A-125, the m denotes the refractive index of the propagating medium. Outside the sphere, $m = m_o = 1$.

Equation A-124 with $r = a$ yields

$$m_o E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(ka)$$

$$+ m_o E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} -a_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) h_n^{(2)}(ka)$$

$$= m E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} m c_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mka). \qquad \text{(A-128)}$$

Multiplying all terms of equation A-128 by

$$x = ka, \qquad \text{(A-129)}$$

setting

$$y = mka = mx, \qquad \text{(A-130)}$$

and using the fact that

$$m_o = 1, \qquad \text{(A-131)}$$

we finally arrive at

$$\begin{aligned} x j_n(x) - a_n x h_n^{(2)}(x) &= m^2 c_n x j_n(mka) \\ &= m c_n y j_n(y) \\ &= m c_n \psi_n(y). \end{aligned} \qquad \text{(A-132)}$$

Equations A-111 through A-113 and A-132 yield

$$\psi_n(x) - a_n \zeta_n(x) = m c_n \psi_n(y). \qquad \text{(A-133)}$$

Next we make use of the continuity of $\frac{1}{m}\frac{\partial(ru)}{\partial r}$. Using equations A-103, A-118, A-120, and A-125 results in

$$\frac{1}{m_o k} E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) \frac{\partial}{\partial r}(krj_n(kr))$$

$$+ \frac{1}{m_o k} E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-a_n(-i)^n) \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) \frac{\partial}{\partial r}(krh_n^{(2)}(kr))$$

$$= \frac{1}{mk} E_o \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} mc_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) \frac{\partial}{\partial r}(mkrj_n(mkr)). \qquad \text{(A-134)}$$

Thus, using equations A-111 through A-113 and the orthogonality of the $P_n^1(\cos\theta)$, we arrive at

$$\psi_n'(x) - a_n \zeta_n'(x) = c_n \psi_n'(y) . \qquad \text{(A-135)}$$

Here we have used the fact that $y = mkr$ and that

$$\frac{\partial}{\partial r}[mkrj_n(mkr)] = mk \frac{\partial}{\partial y}[yj_n(y)]. \qquad \text{(A-136)}$$

The continuity of v is expressed in equation A-126, and use of A-104, A-119, and A-121 shows that

$$E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(kr)$$

$$+ E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} -b_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) h_n^{(2)}(kr)$$

$$= E_o \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} md_n (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(mkr). \qquad \text{(A-137)}$$

Thus, multiplying both sides of equation A-137 by kr and using equations A-111 through A-113, A-129, and A-130, we have

$$\psi_n(x) - b_n\zeta_n(x) = d_n\psi_n(y). \tag{A-138}$$

The continuity of $\frac{\partial(rv)}{\partial r}$ is expressed by the equation A-127 and this, in view of equations A-104, A-119, and A-121, implies that

$$E_o\exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\frac{\partial}{\partial r}[rj_n(kr)]$$

$$+ E_o\exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} -b_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\frac{\partial}{\partial r}[rh_n^{(2)}(kr)]$$

$$= E_o\exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} md_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\frac{\partial}{\partial r}[rj_n(mkr)]. \tag{A-139}$$

Using equations A-129, A-130, A-136, and A-139, and the orthogonality of the $P_n^1(\cos\theta)$, we deduce that

$$\psi_n'(x) - b_n\zeta_n'(x) = md_n\psi_n'(y). \tag{A-140}$$

Dividing the right side of equation A-133 by the right side of equation A-135, we obtain

$$\frac{\psi_n(x) - a_n\zeta_n(x)}{\psi_n'(x) - a_n\zeta_n'(x)} = \frac{m\psi_n(y)}{\psi_n'(y)} \tag{A-141}$$

which implies that

$$\frac{\psi_n(x)\psi_n'(y) - m\psi_n(y)\psi_n'(x)}{\psi_n'(y)\zeta_n(x) - m\psi_n(y)\zeta_n'(x)} = a_n . \tag{A-142}$$

Similarly, equations A-138 and A-140 imply that

$$\frac{m\psi_n'(y)\psi_n(x) - \psi_n(y)\psi_n'(x)}{m\psi_n'(y)\zeta_n(x) - \psi_n(y)\zeta_n'(x)} = b_n \ . \tag{A-143}$$

Observe that

$$\frac{d}{dx}[\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)] = \psi_n''(x)\zeta_n(x) - \psi_n(x)\zeta_n''(x). \tag{A-144}$$

Now

$$\psi_n''(x) = xj_n''(x) + 2j_n'(x) \tag{A-145}$$

implies that

$$\begin{aligned} x\psi_n''(x) &= x^2 j_n''(x) + 2xj_n'(x) \\ &= [n(n+1) - x^2]j_n(x). \end{aligned} \tag{A-146}$$

Similarly,

$$x\zeta_n''(x) = [n(n+1) - x^2]h_n^{(2)}(x) \ . \tag{A-147}$$

Thus, in view of definitions A-111 and A-113

$$x\frac{d}{dx}[\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)] = 0, \tag{A-148}$$

which implies that

$$\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x) = c, \tag{A-149}$$

where c is independent of x.

Lemma A-2. *For all complex* z, *we have*

$$\psi_n'(z)\zeta_n(z) - \psi_n(z)\zeta_n'(z) = i\ . \tag{A-150}$$

Proposition A-2. *For every positive integer* n *and for* x *and* y *defined by equations* A-129 *and* A-130, *we have*

$$c_n = \frac{i}{\psi_n'(y)\zeta_n(x) - m\psi_n(y)\zeta_n'(x)}\ , \tag{A-151}$$

$$d_n = \frac{i}{m\psi_n'(y) - \psi_n(y)\zeta_n'(x)}\ . \tag{A-152}$$

Proof of Proposition A-2. From equation A-133 it follows that

$$\psi_n(x) - mc_n\psi_n(y) = a_n\zeta_n(x)\ , \tag{A-153}$$

and from equation A-135 it follows that

$$\psi_n'(x) - c_n\psi_n'(y) = a_n\zeta_n'(x). \tag{A-154}$$

Dividing the right side of equation A-153 by the right side of equation A-154, we deduce that

$$\frac{\psi_n(x) - mc_n\psi_n(y)}{\psi_n'(x) - c_n\psi_n'(y)} = \frac{\zeta_n(x)}{\zeta_n'(x)}\ , \tag{A-155}$$

or, equivalently,

$$\psi_n(x)\zeta_n'(x) - mc_n\psi_n(y)\zeta_n'(x)$$

$$= \zeta_n(x)\psi_n'(x) - c_n\psi_n'(y)\zeta_n(x) \ . \qquad \text{(A-156)}$$

Thus

$$c_n = \frac{\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)}{\psi_n'(y)\zeta_n(x) - m\psi_n(y)\zeta_n'(x)} \ . \qquad \text{(A-157)}$$

By Lemma A-2, equation A-157 implies equation A-151.

Equation A-138 leads to

$$\psi_n(x) - d_n\psi_n(y) = b_n\zeta_n(x) , \qquad \text{(A-158)}$$

and equation A-140 leads to

$$\psi_n'(x) - md_n\psi_n'(y) = b_n\zeta_n'(x) \ . \qquad \text{(A-159)}$$

Division of the right side of equation A-158 by the right side of equation A-159 results in the expression

$$\frac{\psi_n(x) - d_n\psi_n(y)}{\psi_n'(x) - md_n\psi_n'(y)} = \frac{\zeta_n(x)}{\zeta_n'(x)} \ . \qquad \text{(A-160)}$$

Thus

$$d_n = \frac{\psi_n'(x)\zeta_n(x) - \psi_n(x)\zeta_n'(x)}{m\psi_n'(y)\zeta_n(x) - \psi_n(y)\zeta_n'(x)} \ . \qquad \text{(A-161)}$$

We obtain equation A-152 by using equation A-161 and Lemma A-2. This completes the proof of Proposition A-2.

Proof of Lemma A-2. Observe that

$$J_{\nu}(z) = [\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}](z/2)^{\nu} . \tag{A-162}$$

Thus, definition A-111 implies that

$$\psi_n(z) = (\pi z/2)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{\Gamma(k+1)\Gamma(k+n+\frac{1}{2}+1)}(z/2)^n (z/2)^{\frac{1}{2}}$$

$$= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{\Gamma(k+1)\Gamma(k+n+3/2)}(z/2)^{n+1} , \tag{A-163}$$

and

$$\psi_n'(z) = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k+1)(z/2)^k z^n / 2^{n+1}}{\Gamma(k+1)\Gamma(k+n+3/2)} . \tag{A-164}$$

In addition, definitions A-113 and A-116 imply that

$$\zeta_n(z) = (\pi z/2)^{\frac{1}{2}} \{ J_{n+\frac{1}{2}}(z) - i[\frac{J_{n+\frac{1}{2}}(z)\cos((n+\frac{1}{2})\pi) - J_{-(n+\frac{1}{2})}(z)}{\sin((n+\frac{1}{2})\pi)}] \} \tag{A-165}$$

or

$$\zeta_n(z) = (\pi z/2)^{\frac{1}{2}} [J_{n+\frac{1}{2}}(z) - i(-1)^{n+1} J_{-(n+\frac{1}{2})}(z)] . \tag{A-166}$$

Hence, equation A-166 implies that

$$\zeta_n(z) = \sqrt{\pi} \sum_{k=o}^{\infty} \frac{(-1)^k (z/2)^k}{\Gamma(k+1)\Gamma(k+n+3/2)} (z/2)^{n+1}$$

$$+ i(-1)^n \sqrt{\pi} \sum_{k=o}^{\infty} \frac{(-1)^k (z/2)^k}{\Gamma(k+1)\Gamma[k-(n+\frac{1}{2})+1]} (z/2)^{-n}, \qquad \text{(A-167)}$$

$$\zeta_n'(z) = \sqrt{\pi} \sum_{k=o}^{\infty} \frac{(-1)^k (n+k+1)(z/2)^k z^n / 2^{n+1}}{\Gamma(k+1)\Gamma(k+n+3/2)}$$

$$+ i(-1)^n \sqrt{\pi} \sum_{k=o}^{\infty} \frac{(-1)^k (k-n) 2^n (z/2)^k z^{-n-1}}{\Gamma(k+1)\Gamma[k-(n+\frac{1}{2})+1]} . \qquad \text{(A-168)}$$

The nonzero terms are

$$\psi_n(z)\zeta_n'(z)\big|_{z=o} = \frac{i(-1)^n \pi 2^n (-n)(\frac{1}{2})^{n+1}}{\Gamma(1)\Gamma(\frac{1}{2}-n)\Gamma(1)\Gamma(n+3/2)} , \qquad \text{(A-169)}$$

$$\psi_n'(z)\zeta_n(z)\big|_{z=o} = \frac{i(-1)^n \pi (\frac{1}{2})(n+1)}{\Gamma(1)\Gamma(n+3/2)\Gamma(\frac{1}{2}-n)\Gamma(1)} . \qquad \text{(A-170)}$$

Finally, we observe that

$$\zeta_n(z)\psi_n'(z) - \psi_n(z)\zeta_n'(z) = A_n$$

$$= \frac{i(-1)^n \pi (n+\frac{1}{2})}{\Gamma^2(1)\Gamma(n+3/2)\Gamma(\frac{1}{2}-n)} . \qquad \text{(A-171)}$$

To complete the computation, we observe that

$$\Gamma(-\tfrac{1}{2}) = -2\,\Gamma(\tfrac{1}{2}) \tag{A-172}$$

since

$$\Gamma(-\tfrac{1}{2}+1) = (-\tfrac{1}{2})\Gamma(-\tfrac{1}{2}) \tag{A-173}$$

follows from the known relation

$$\Gamma(z+1) = z\,\Gamma(z)\,. \tag{A-174}$$

Furthermore, $n = 1$ and equation A-171 implies that

$$A_1 = \frac{i(-1)(3/2)}{\Gamma^2(1)\Gamma(1+3/2)\Gamma(-\tfrac{1}{2})}\,. \tag{A-175}$$

Application of equation A-174 to equation A-175 results in

$$A_1 = \frac{-i(3/2)\pi}{(3/2)(\tfrac{1}{2})\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2})(-2)}$$

$$= \frac{i\pi}{\Gamma^2(\tfrac{1}{2})}\,. \tag{A-176}$$

Now we see that for $Re z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \tag{A-177}$$

since

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$$

$$= -\int_0^\infty t^z de^{-t}$$

$$= -t^z e^{-t}\Big|_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt$$

$$= z\,\Gamma(z)\,. \qquad \text{(A-178)}$$

Observe that

$$\Gamma(1) = \int_0^\infty e^{-t} dt$$

$$= 1, \qquad \text{(A-179)}$$

and

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt. \qquad \text{(A-180)}$$

Substituting $t = s^2$ in equation A-180, we see that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty (1/s) e^{-s^2} 2s\,ds$$

$$= 2\int_0^\infty e^{-s^2} ds$$

$$= \sqrt{\pi} \qquad \text{(A-181)}$$

since

$$I = \int_0^\infty e^{-s^2} ds \tag{A-182}$$

implies that

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

$$= \pi/4 \tag{A-183}$$

or

$$I = \sqrt{\pi}/2 \ . \tag{A-184}$$

Thus, equations A-176 and A-181 through A-183 imply that

$$A_1 = i \ . \tag{A-185}$$

We claim that $A_n = i$ for all n. Our claim is established by mathematical induction on n. Suppose that $n > 1$ and that $A_{n-1} = i$. Then

$$\Gamma(\tfrac{1}{2}-n) = \frac{\Gamma(3/2-n)}{(\tfrac{1}{2}-n)} \tag{A-186}$$

and

$$\Gamma(n+3/2) = (n+\tfrac{1}{2})\Gamma(n+\tfrac{1}{2}) \tag{A-187}$$

imply that

$$A_n = \frac{i(-1)^n \pi (n+\frac{1}{2})}{(n+\frac{1}{2})\Gamma(n+\frac{1}{2})\ \frac{\Gamma(3/2-n)}{(\frac{1}{2}-n)}} \quad . \tag{A-188}$$

Equation A-188 and the original definition in equation A-171 imply that

$$A_n = A_{n-1} = \frac{i(-1)^{n-1} \pi (n-1+\frac{1}{2})}{\Gamma(n-1+3/2)\Gamma(\frac{1}{2}-(n-1))} \tag{A-189}$$

This completes the proof of Lemma A-2.

Previous results were developed on the assumption of a particular form for the expansion of a plane electromagnetic wave. We now verify this expansion. Each component ψ of a plane wave is a solution of the scalar Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0. \tag{A-190}$$

In spherical coordinates, equation A-190 takes the form

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} + k^2 \psi = 0. \tag{A-191}$$

Let us write

$$\psi = R(r)\Theta(\theta)\Phi(\phi), \tag{A-192}$$

$$\psi(r,\theta,\phi) = \sum a_{m,n,p} R_m(r)\Theta_n(\theta)\Phi_p(\phi), \qquad \text{(A-193)}$$

and observe that $R(r) = R_m(r)$, $\Theta(\theta) = \Theta_n(\theta)$, and $\Phi(\phi) = \Phi_p(\phi)$ imply

$$\frac{1}{R(r)}\left[\frac{1}{r^2}\frac{d}{dr}(r^2R'(r))\right] + \frac{1}{\Theta(\theta)}\left[\frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\Theta'(\theta))\right]$$

$$+ \frac{1}{\Phi(\phi)}\left[\frac{1}{r^2\sin^2\theta}\Phi''(\phi)\right] + k^2 = 0. \qquad \text{(A-194)}$$

Suppose that electric vector $\underline{E}$ is given by

$$\underline{E} = a_x \exp(-ikz)\exp(i\omega t), \qquad \text{(A-195)}$$

where $\underline{a}_x$ is the unit base vector directed along the x-axis of a Cartesian coordinate system. The expression for $\underline{a}_x$ in terms of the unit base vectors $\underline{e}_r$, $\underline{e}_\theta$, and $\underline{e}_\phi$ in the spherical coordinate system is

$$\underline{a}_x = (\underline{a}_x \cdot \underline{e}_r)\underline{e}_r + (\underline{a}_x \cdot \underline{e}_\theta)\underline{e}_\theta + (\underline{a}_x \cdot \underline{e}_\phi)\underline{e}_\phi \qquad \text{(A-196)}$$

which, upon replacement of the inner products, becomes

$$\underline{a}_x = \sin\theta\cos\phi\ \underline{e}_r + \cos\theta\cos\phi\ \underline{e}_\theta - \sin\phi\ \underline{e}_\phi . \qquad \text{(A-197)}$$

Substituting equation A-197 into equation A-195, with z replaced by $r\cos\theta$, yields

$$\underline{E} = \sin\theta\cos\phi\exp(-ikr\cos\theta)\exp(i\omega t)\underline{e}_r$$

$$+ \cos\theta\cos\phi\exp(-ikr\cos\theta)\exp(i\omega t)\underline{e}_\theta$$

$$- \sin\phi\exp(-ir\cos\theta)\exp(i\omega t)\underline{e}_\phi . \qquad \text{(A-198)}$$

Through use of $\underline{E} = \underline{M}_v + i\underline{N}_u$ and equations A-87 through A-90 and A-93 through A-96, the vector $\underline{E}$ can be expressed as

$$\underline{E} = [0\underline{e}_r + (\frac{1}{\sin\theta})\frac{\partial v}{\partial \phi}\underline{e}_\theta - \frac{\partial v}{\partial \theta}\underline{e}_\phi]$$

$$+ i\{\frac{1}{mk}[\frac{1}{r}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + m^2k^2ru]\underline{e}_r$$

$$+ \frac{1}{mkr}\frac{\partial^2(ru)}{\partial r\partial\theta}\underline{e}_\theta + \frac{1}{mkr\sin\theta}\frac{\partial^2(ru)}{\partial r\partial\phi}\underline{e}_\phi\}. \qquad \text{(A-199)}$$

Our starting point in finding expressions for u and v is to write down the spherical harmonic expansion of $\exp(-ikr\cos\theta)$. We set

$$\exp(-ikr\cos\theta) = \sum_{n=o}^{\infty} a_n P_n(\cos\theta) j_n(kr), \qquad \text{(A-200)}$$

where $P_n(\cos\theta)$ and $j_n(kr)$ are the Legendre and spherical Bessel functions respectively. Now $j_n(s)$ has an infinite series expansion

$$j_n(s) = 2^n s^n \sum_{m=o}^{\infty} \frac{(-1)^m(n+m)!}{m!(2n+2m+1)!}s^{2m} \qquad \text{(A-201)}$$

from which we obtain

$$\frac{d^n}{ds^n} j_n(s)\Big|_{s=o} = \frac{2^n(n!)^2}{(2n+1)!}. \qquad \text{(A-202)}$$

Multiplying both sides of equation A-200 by $P_n(\cos\theta)\sin\theta$, we obtain, upon integrating from 0 to π with respect to θ, the result

$$a_n j_n(kr) \int_0^{\pi} P_n^2(\cos\theta)\sin\theta d\theta = \int_0^{\pi} \exp(-ikr\cos\theta) P_n(\cos\theta)\sin\theta d\theta. \quad \text{(A-203)}$$

Thus, the relation

$$\int_0^{\pi} P_n^2(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1} \quad \text{(A-204)}$$

implies that

$$a_n j_n(kr)\left(\frac{2}{2n+1}\right) = \int_0^{\pi} \exp(-ikr\cos\theta) P_n(\cos\theta)\sin\theta d\theta. \quad \text{(A-205)}$$

Letting $s = kr$ in equation A-205, we find that

$$a_n j_n(s)\left(\frac{2}{2n+1}\right) = \int_0^{\pi} \exp(-is\cos\theta) P_n(\cos\theta)\sin\theta d\theta. \quad \text{(A-206)}$$

Differentiating both sides of equation A-206 with respect to s n times, setting $s = 0$, and using equation A-202, we obtain

$$a_n \frac{2^n (n!)^2}{(2n+1)!}\left(\frac{2}{2n+1}\right) = \int_0^{\pi} (-i\cos\theta)^n P_n(\cos\theta)\sin\theta d\theta. \quad \text{(A-207)}$$

Consequently,

$$a_n \frac{2^n (n!)^2}{(2n+1)!}\left(\frac{2}{2n+1}\right) = (-i)^n \int_0^{\pi} \cos^n\theta P_n(\cos\theta)\sin\theta d\theta. \quad \text{(A-208)}$$

By equation A-208 and the fact that

$$\int_0^{\pi} \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{2^{n+1}(n!)^2}{(2n+1)}, \quad \text{(A-209)}$$

we thus obtain

$$a_n = (-1)^n(2n+1). \tag{A-210}$$

Now the functions u and v were previously expanded in terms of products of the associated Legendre functions of the first kind and order one and spherical Bessel functions, $P_n^1(\cos\theta)j_n(kr)$. It thus behooves us to represent $\exp(-ikr\cos\theta)$ in terms of these functions. Differentiation of both sides of equation A-200 with respect to θ results in the relation

$$ikr\sin\theta\exp(-ikr\cos\theta) = -\sum_{n=o}^{\infty} a_n P_n^{(1)}(\cos\theta)\sin\theta j_n(kr). \tag{A-211}$$

Using equation A-211 and the fact that

$$P_n^1(x) = \sqrt{1-x^2}\ P_n^{(1)}(x), \tag{A-212}$$

we deduce that

$$-ikr\sin\theta\exp(-ikr\cos\theta) = \sum_{n=o}^{\infty} a_n P_n^1(\cos\theta)j_n(kr). \tag{A-213}$$

Noting that

$$r^2\frac{d^2}{dr^2}j_n(kr) + 2r\frac{d}{dr}j_n(kr) + (k^2r^2-n(n+1))j_n(kr) = 0 \tag{A-214}$$

and defining u by

$$u = \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\zeta_n'(kr), \qquad \text{(A-215)}$$

shows that the r component of $\underline{E}$ is, by equations A-198 through A-200, and A-213, given by

$$E_r = \cos\phi\sin\theta\exp(-ikr\cos\theta)\exp(i\omega t)$$

$$= \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n (2n+1) P_n^1(\cos\theta)\left(\frac{j_n(kr)}{-ikr}\right), \qquad \text{(A-216)}$$

since by equations A-199 and A-215, we deduce that E_r is given by

$$E_r = \frac{i}{kr}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + k^2 r^2 u\right]$$

$$= i\exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\frac{n(n+1)}{kr} j_n(kr)$$

$$= \exp(i\omega t)\cos\phi \sum_{n=1}^{\infty} (-i)^n (2n+1) P_n^1(\cos\theta)\left(\frac{j_n(kr)}{-ikr}\right) \qquad \text{(A-217)}$$

which is exactly the right side of equation A-216.

Now an analysis of the coefficient of $\underline{e}_\theta$ in equation A-198 is in order. An essential step toward this goal is to expand the function

$$F(r,\theta) = \exp(-ikr\cos\theta)\cos\theta. \qquad \text{(A-218)}$$

Differentiating both sides of equation A-200 with respect to r, we find that

$$-i\cos\theta\exp(-ikr\cos\theta) = \sum_{n=o}^{\infty} a_n P_n(\cos\theta) j_n'(kr). \qquad \text{(A-219)}$$

Since $\underline{E} = \underline{M}_v + i\underline{N}_u$ and setting $m = 1$, the coefficient of $\underline{e}_\theta$ equation A-199 is given by

$$\frac{1}{\sin\theta}\frac{\partial v}{\partial \phi} + i\left(\frac{1}{k}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{kr}\frac{\partial u}{\partial \theta}\right) . \qquad \text{(A-220)}$$

One might be tempted to set

$$\frac{\partial v}{\partial \phi} = u \qquad \text{(A-221)}$$

and use equation A-215 to deduce that

$$v = \exp(i\omega t)\sin\phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta) j_n(kr). \qquad \text{(A-222)}$$

Also one might attempt to prove that

$$\frac{1}{\sin\theta}\frac{\partial v}{\partial \phi} + i\left(\frac{1}{k}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{kr}\frac{\partial u}{\partial \theta}\right) = \exp(i\omega t)\sin\theta\cos\phi \cdot \exp(-ikr\cos\theta) \qquad \text{(A-223)}$$

by substitution of series. However, there is a much easier method. This involves seeking an expression for u in the form

$$u = \frac{B(\theta,\phi)\exp(-ikr\cos\theta)}{r} . \qquad \text{(A-224)}$$

Then

$$\frac{\partial u}{\partial r} = B(\theta,\phi)\exp(-ikr\cos\theta)\left(-\frac{1}{r^2} - \frac{ik\cos\theta}{r}\right), \qquad \text{(A-225)}$$

$$r^2\frac{\partial u}{\partial r} = B(\theta,\phi)\exp(-ikr\cos\theta)(-1-ikr\cos\theta), \qquad \text{(A-226)}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = B(\theta,\phi)\exp(-ikr\cos\theta)[(-ik\cos\theta)(-1-ikr\cos\theta) - ik\cos\theta]. \qquad \text{(A-227)}$$

Equations A-217, A-224, and A-227 lead to

$$E_r = \frac{i}{kr}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + k^2 r^2 u\right]$$

$$= B(\theta,\phi)\exp(-ikr\cos\theta)\cdot\left\{\frac{\cos\theta}{r}\left[(-1-ikr\cos\theta) + 1\right]\right\}$$

$$+ ikr\left[B(\theta,\phi)\exp(-ikr\cos\theta)/r\right]$$

$$= ik(1-\cos^2\theta)B(\theta,\phi)\exp(-ikr\cos\theta)$$

$$= ik\sin^2\theta B(\theta,\phi)\exp(-ikr\cos\theta). \qquad \text{(A-228)}$$

Now we desire to find $B(\theta,\phi)$ so that equation A-228 becomes

$$\frac{i}{kr}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} + k^2 r^2 u\right)\right] = \exp(i\omega t)\sin\theta\cos\phi\exp(-kr\cos\theta). \qquad \text{(A-229)}$$

The choice

$$B(\theta,\phi) = \frac{\cos\phi}{ik\sin\theta}\exp(i\omega t) \qquad \text{(A-230)}$$

will serve our purpose. Thus, we must attempt to expand

$$u = \exp(i\omega t)\cos\phi\,\frac{\exp(-ikr\cos\theta)}{ikr\sin\theta}. \qquad \text{(A-231)}$$

The method of undetermined coefficients will enable us to obtain equation A-215 by using equation A-229.

Let

$$v = \exp(i\omega t)\sin\phi\,\frac{\exp(-ikr\cos\theta)}{ikr\sin\theta}, \qquad \text{(A-232)}$$

Let $a(t) = \exp(i\omega t)$, and form the derivatives

$$\frac{\partial v}{\partial \phi} = a(t)\exp(-ikr\cos\theta)\frac{\cos\phi}{ikr\sin\theta}, \tag{A-233}$$

$$\frac{\partial u}{\partial \theta} = a(t)\exp(-ikr\cos\theta)\cos\phi\left(\frac{-\cos\theta}{ikr\sin^2\theta} + 1\right), \tag{A-234}$$

$$\frac{\partial^2 u}{\partial r\partial\theta} = a(t)\exp(-ikr\cos\theta)\cos\phi\left(\frac{ik\cos^2\theta}{irk\sin^2\theta} - ik\cos\theta + \frac{\cos\theta}{ikr^2\sin^2\theta}\right). \tag{A-235}$$

Construct the left member of equation A-223 by using equations A-233 through A-235 and find that

$$\frac{1}{\sin\theta}\frac{\partial v}{\partial \phi} + i\left(\frac{1}{k}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{kr}\frac{\partial u}{\partial\theta}\right)$$

$$= a(t)\exp(-ikr\cos\theta)\cos\phi\left(\frac{1}{ikr\sin^2\theta} + \frac{i\cos^2\theta}{kr\sin^2\theta}\right.$$

$$\left. + \cos\theta + \frac{\cos\theta}{k^2r^2\sin^2\theta} - \frac{\cos\theta}{k^2r^2\sin^2\theta} + \frac{i}{kr}\right)$$

$$= a(t)\exp(-ikr\cos\theta)\cos\phi\left(\frac{1}{ikr\sin^2\theta} + \frac{i\cos^2\theta}{kr\sin^2\theta}\right.$$

$$\left. + \frac{i\sin^2\theta}{kr\sin^2\theta} + \cos\theta\right)$$

$$= a(t)\cos\theta\cos\phi\exp(-ikr\cos\theta) = E_\theta . \tag{A-236}$$

This establishes the validity of equation A-223. Finally, to complete the argument, we must show that

$$-\frac{\partial v}{\partial \phi} + \frac{i}{krsin\theta}\frac{\partial^2}{\partial r \partial \phi}(ru) = -exp(i\omega t)sin\phi exp(-ikrcos\theta), \qquad (A-237)$$

We have from equation A-232 that

$$\frac{\partial v}{\partial \theta} = exp(i\omega t)sin\phi exp(-ikrcos\theta)\left(\frac{-cos\theta}{ikrsin^2\theta} + 1\right) . \qquad (A-238)$$

Also by equation A-231,

$$\frac{\partial^2}{\partial r \partial \phi}(ru) = exp(i\omega t)sin\phi exp(-ikrcos\theta)\left(\frac{cos\theta}{sin\theta}\right) . \qquad (A-239)$$

Thus, equation A-239 implies that

$$\frac{1}{krsin\theta}\frac{\partial^2}{\partial r \partial \phi}(ru) = exp(i\omega t)sin\phi exp(-ikrcos\theta)\left(\frac{cos\theta}{krsin^2\theta}\right) . \qquad (A-240)$$

Consequently, combining equations A-238 and A-240, we deduce that

$$-\frac{\partial v}{\partial \theta} + \frac{i}{krsin\theta}\frac{\partial^2}{\partial r \partial \phi}(ru) = -exp(i\omega t)sin\phi exp(-ikrcos\theta)$$

$$\cdot \left(1 + \frac{cos\theta}{ikrsin^2\theta} + \frac{icos\theta}{krsin^2\theta}\right)$$

$$= -exp(i\omega t)sin\phi exp(-ikrcos\theta) = E_\phi . \qquad (A-241)$$

This completes the derivation. The following lemma has been proved:

Lemma A-3. *If* u *and* v *are defined by equations* A-231 *and* A-232, *respectively, then* u *satisfies equation* A-215, v *satisfies equation* A-222, *and* u *and* v *satisfy equation* A-199, *where vector* $\underline{E}$ *is defined by equation* A-198.

In proving that the u defined by equation A-231 has the representation A-215, we used some properties of Legendre polynomials which are consequences of the following lemma:

Lemma A-4. *For every positive integer* n,

$$\int_0^\pi \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{2^{n+1}(n!)^2}{(2n+1)!} \tag{A-242}$$

and

$$\int_0^\pi [P_n(\cos\theta)]^2 \sin\theta d\theta = \frac{2}{2n+1} \quad . \tag{A-243}$$

Proof of Lemma A-4. We use Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{A-244}$$

and observe that the substitution $x = \cos\theta$ implies that

$$\int_0^\pi \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{1}{2^n n!}\int_{-1}^1 x^n \frac{d^n}{dx^n}(x^2-1)^n dx$$

$$\int_0^\pi \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{(-1)^n}{2^n n!}\int_{-1}^1 \left(\frac{d^n}{dx^n}x^n\right)(x^2-1)^n$$

$$= \frac{(-1)^n}{2^n}\int_{-1}^1 (x-1)^n(x+1)^n dx \tag{A-245}$$

Continuing the integration by parts of the right side of equation A-245, we conclude that

$$\int_0^{\pi} \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{(-1)^{2n}}{2^n(n+1)(n+2)\ldots(n+n)} \int_{-1}^{1} [\left(\frac{d^n}{dx^n}\right)(x-1)^n](x+1)^{2n} dx$$

$$= \frac{2^{2n+1}(n!)^2}{2^n(2n+1)!}$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!} \quad . \qquad \text{(A-246)}$$

To complete the proof of the lemma, observe that

$$\int_0^{\pi} [P_n(\cos\theta)]^2 \sin\theta d\theta = \int_{-1}^{1} [P_n(x)]^2 dx$$

$$= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} [\frac{d^n}{dx^n}(x^2-1)^n][\frac{d^n}{dx^n}(x^2-1)^n]dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} [\frac{d^{2n}}{dx^{2n}}(x^2-1)^n](x^2-1)^n dx. \qquad \text{(A-247)}$$

Since

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)! \qquad \text{(A-248)}$$

equation A-245 implies that

$$\int_0^{\pi} [P_n(\cos\theta)]^2 \sin\theta d\theta = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n dx. \qquad \text{(A-249)}$$

But another integration by parts shows that

$$\int_{-1}^{1} (x^2-1)^n dx = \int_{-1}^{1} (x-1)^n (x+1)^n dx$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!} \quad . \qquad \text{(A-250)}$$

Substituting equation A-250 into equation A-249, we obtain equation A-243.

APPENDIX B

SAMPLE PROBLEMS WITH COMPUTER RESULTS

SAMPLE PROBLEM 1 DECK SETUP

CARD 1 (CONTROL PARAMETERS)

```
   1.000E3   5.986E1 1.00152E0   1.000E0   1.000E1   0.000E0   66    1  8  8  8
```

CARDS 2 - (NOC+1) (DATA CARDS)

```
 8.660E-01 5.474E+01 4.500E+01
 1.658E+00 2.524E+01 4.500E+01
 2.598E+00 1.579E+01 4.500E+01
 3.571E+00 1.142E+01 4.500E+01
 4.555E+00 8.930E+00 4.500E+01
 1.658E+00 7.245E+01 7.157E+01
 2.179E+00 4.651E+01 7.157E+01
 2.958E+00 3.231E+01 7.157E+01
 3.841E+00 2.431E+01 7.157E+01
 4.770E+00 1.936E+01 7.157E+01
 2.598E+00 7.890E+01 7.869E+01
 2.958E+00 5.953E+01 7.869E+01
 3.571E+00 4.556E+01 7.869E+01
 4.330E+00 3.607E+01 7.869E+01
 3.571E+00 8.195E+01 8.187E+01
 3.841E+00 6.701E+01 8.187E+01
 4.330E+00 5.474E+01 8.187E+01
 4.975E+00 4.529E+01 8.187E+01
 4.555E+00 8.370E+01 8.366E+01
 4.770E+00 7.167E+01 8.366E+01
 1.658E+00 7.245E+01 1.843E+01
 2.179E+00 4.651E+01 1.843E+01
 2.958E+00 3.231E+01 1.843E+01
 3.841E+00 2.431E+01 1.843E+01
 4.770E+00 1.936E+01 1.843E+01
 2.179E+00 7.674E+01 4.500E+01
 2.598E+00 5.474E+01 4.500E+01
 3.279E+00 4.032E+01 4.500E+01
 4.093E+00 3.122E+01 4.500E+01
 4.975E+00 2.524E+01 4.500E+01
 2.958E+00 8.027E+01 5.904E+01
 3.279E+00 6.277E+01 5.904E+01
 3.841E+00 4.939E+01 5.904E+01
 4.555E+00 3.979E+01 5.904E+01
 3.841E+00 8.252E+01 6.680E+01
 4.093E+00 6.850E+01 6.680E+01
 4.555E+00 5.671E+01 6.680E+01
 4.770E+00 8.398E+01 7.157E+01
 4.975E+00 7.245E+01 7.157E+01
 2.598E+00 7.890E+01 1.131E+01
 2.958E+00 5.953E+01 1.131E+01
 3.571E+00 4.556E+01 1.131E+01
 4.330E+00 3.607E+01 1.131E+01
 2.958E+00 8.027E+01 3.096E+01
 3.279E+00 6.277E+01 3.096E+01
 3.841E+00 4.939E+01 3.096E+01
 4.555E+00 3.979E+01 3.096E+01
 3.571E+00 8.195E+01 4.500E+01
 3.841E+00 6.701E+01 4.500E+01
 4.330E+00 5.474E+01 4.500E+01
 4.330E+00 8.337E+01 5.446E+01
 4.555E+00 7.077E+01 5.446E+01
 3.571E+00 8.195E+01 8.130E+00
 3.841E+00 6.701E+01 8.130E+00
 4.330E+00 5.474E+01 8.130E+00
 4.975E+00 4.529E+01 8.130E+00
 3.841E+00 8.252E+01 2.320E+01
 4.093E+00 6.850E+01 2.320E+01
 4.555E+00 5.671E+01 2.320E+01
 4.330E+00 8.337E+01 3.554E+01
 4.555E+00 7.077E+01 3.554E+01
 4.975E+00 8.423E+01 4.500E+01
 4.555E+00 8.370E+01 6.340E+00
 4.770E+00 7.167E+01 6.340E+00
 4.770E+00 8.398E+01 1.843E+01
 4.975E+00 7.245E+01 1.843E+01
```

TERMINATION CARD

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BEST AVAILABLE COPY

SAMPLE PROBLEM 1

DEPOSITION OF POWER INSIDE A HOMOGENEOUS SPHERE IMMERSED IN AN ELECTROMAGNETIC FIELD

FREQUENCY = 1000.00 MHZ WAVELENGTH = 29.97925 CM FIELD STRENGTH = 1.00 V/M

CONDUCTIVITY = 1.001520 MHO/M RELATIVE DIELECTRIC CONSTANT = 59.86 DIAMETER = 10.00 CM TIME = 0.0 SEC

REFRACTIVE INDEX = 7.82226D+00 -1.15231D+00

THE HIGHEST ORDER NEUMANN FUNCTION COMPUTED IS OF ORDER = 21 SIZE PARAMETER = 1.04792D+00

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SHERICAL BESSEL FUNCTION OF ARGUMENT X
TO THE VALUE OF SIN(X)/X = 1.00000D+00

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SHERICAL BESSEL FUNCTION OF ARGUMENT Z
TO THE VALUE OF SIN(Z)/Z = 1.00000D+00 7.13511D-16

INTERNAL POINT:	RADIUS =	0.866 CM	THETA =	54.74 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.09375857	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.658 CM	THETA =	25.24 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.03009165	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.598 CM	THETA =	15.79 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.05211915	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.571 CM	THETA =	11.42 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.00718193	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.555 CM	THETA =	8.93 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.02330504	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.658 CM	THETA =	72.45 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.01844696	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.179 CM	THETA =	46.51 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.01423773	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.958 CM	THETA =	32.31 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.01821150	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.841 CM	THETA =	24.31 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.00658279	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.770 CM	THETA =	19.36 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.01939612	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.598 CM	THETA =	78.90 DEG	PHI =	78.69 DEG	ABSORBED POWER DENSITY =	0.01421862	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.958 CM	THETA =	59.53 DEG	PHI =	78.69 DEG	ABSORBED POWER DENSITY =	0.00863334	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.571 CM	THETA =	45.56 DEG	PHI =	78.69 DEG	ABSORBED POWER DENSITY =	0.00203966	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.330 CM	THETA =	36.07 DEG	PHI =	78.69 DEG	ABSORBED POWER DENSITY =	0.01071924	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.571 CM	THETA =	81.95 DEG	PHI =	81.87 DEG	ABSORBED POWER DENSITY =	0.00685367	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.841 CM	THETA =	67.01 DEG	PHI =	81.87 DEG	ABSORBED POWER DENSITY =	0.00286858	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.330 CM	THETA =	54.74 DEG	PHI =	81.87 DEG	ABSORBED POWER DENSITY =	0.00525806	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.975 CM	THETA =	45.29 DEG	PHI =	81.87 DEG	ABSORBED POWER DENSITY =	0.00698834	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.555 CM	THETA =	83.70 DEG	PHI =	83.66 DEG	ABSORBED POWER DENSITY =	0.00902432	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.770 CM	THETA =	71.67 DEG	PHI =	83.66 DEG	ABSORBED POWER DENSITY =	0.00647169	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.658 CM	THETA =	72.45 DEG	PHI =	18.43 DEG	ABSORBED POWER DENSITY =	0.07920415	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.179 CM	THETA =	46.51 DEG	PHI =	18.43 DEG	ABSORBED POWER DENSITY =	0.05024135	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.958 CM	THETA =	32.31 DEG	PHI =	18.43 DEG	ABSORBED POWER DENSITY =	0.02015498	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.841 CM	THETA =	24.31 DEG	PHI =	18.43 DEG	ABSORBED POWER DENSITY =	0.00823060	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.770 CM	THETA =	19.36 DEG	PHI =	18.43 DEG	ABSORBED POWER DENSITY =	0.02027197	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.179 CM	THETA =	76.74 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.02137977	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.598 CM	THETA =	54.74 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.01572690	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.279 CM	THETA =	40.32 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.00521008	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.093 CM	THETA =	31.22 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.01125208	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.975 CM	THETA =	25.29 DEG	PHI =	45.00 DEG	ABSORBED POWER DENSITY =	0.01501013	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.958 CM	THETA =	80.27 DEG	PHI =	59.04 DEG	ABSORBED POWER DENSITY =	0.01176989	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.279 CM	THETA =	62.77 DEG	PHI =	59.04 DEG	ABSORBED POWER DENSITY =	0.00532103	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.841 CM	THETA =	49.39 DEG	PHI =	59.04 DEG	ABSORBED POWER DENSITY =	0.00428980	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.555 CM	THETA =	39.79 DEG	PHI =	59.04 DEG	ABSORBED POWER DENSITY =	0.01136860	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.841 CM	THETA =	82.52 DEG	PHI =	66.80 DEG	ABSORBED POWER DENSITY =	0.00819485	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.093 CM	THETA =	68.50 DEG	PHI =	66.80 DEG	ABSORBED POWER DENSITY =	0.00535788	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.555 CM	THETA =	56.71 DEG	PHI =	66.80 DEG	ABSORBED POWER DENSITY =	0.00632642	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.770 CM	THETA =	83.98 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.00936273	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.975 CM	THETA =	72.45 DEG	PHI =	71.57 DEG	ABSORBED POWER DENSITY =	0.00625014	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.598 CM	THETA =	78.90 DEG	PHI =	11.31 DEG	ABSORBED POWER DENSITY =	0.00915834	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.958 CM	THETA =	59.53 DEG	PHI =	11.31 DEG	ABSORBED POWER DENSITY =	0.00502638	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.571 CM	THETA =	45.56 DEG	PHI =	11.31 DEG	ABSORBED POWER DENSITY =	0.00922500	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.330 CM	THETA =	36.07 DEG	PHI =	11.31 DEG	ABSORBED POWER DENSITY =	0.01559046	WATTS/M**3

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INTERNAL POINT:  RADIUS =   2.958 CM    THETA =  80.27 DEG    PHI =  30.96 DEG    ABSORBED POWER DENSITY =  0.01024710 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.279 CM    THETA =  62.77 DEG    PHI =  30.96 DEG    ABSORBED POWER DENSITY =  0.00588853 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.841 CM    THETA =  49.39 DEG    PHI =  30.96 DEG    ABSORBED POWER DENSITY =  0.00808898 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.555 CM    THETA =  39.79 DEG    PHI =  30.96 DEG    ABSORBED POWER DENSITY =  0.01323507 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.571 CM    THETA =  81.95 DEG    PHI =  45.00 DEG    ABSORBED POWER DENSITY =  0.01331412 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.841 CM    THETA =  67.01 DEG    PHI =  45.00 DEG    ABSORBED POWER DENSITY =  0.00864617 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.330 CM    THETA =  54.74 DEG    PHI =  45.00 DEG    ABSORBED POWER DENSITY =  0.00852068 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.330 CM    THETA =  83.37 DEG    PHI =  54.46 DEG    ABSORBED POWER DENSITY =  0.00978453 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.555 CM    THETA =  70.77 DEG    PHI =  54.46 DEG    ABSORBED POWER DENSITY =  0.00690478 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.571 CM    THETA =  81.95 DEG    PHI =   8.13 DEG    ABSORBED POWER DENSITY =  0.01976837 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.841 CM    THETA =  67.01 DEG    PHI =   8.13 DEG    ABSORBED POWER DENSITY =  0.01442377 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.330 CM    THETA =  54.74 DEG    PHI =   8.13 DEG    ABSORBED POWER DENSITY =  0.01178249 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.975 CM    THETA =  45.29 DEG    PHI =   8.13 DEG    ABSORBED POWER DENSITY =  0.00748877 WATTS/M**3
INTERNAL POINT:  RADIUS =   3.881 CM    THETA =  82.52 DEG    PHI =  23.20 DEG    ABSORBED POWER DENSITY =  0.01738899 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.093 CM    THETA =  68.50 DEG    PHI =  23.20 DEG    ABSORBED POWER DENSITY =  0.01229063 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.555 CM    THETA =  56.71 DEG    PHI =  23.20 DEG    ABSORBED POWER DENSITY =  0.00864552 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.330 CM    THETA =  83.37 DEG    PHI =  35.54 DEG    ABSORBED POWER DENSITY =  0.01132381 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.555 CM    THETA =  70.77 DEG    PHI =  35.54 DEG    ABSORBED POWER DENSITY =  0.00757644 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.975 CM    THETA =  88.23 DEG    PHI =  45.00 DEG    ABSORBED POWER DENSITY =  0.01062630 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.555 CM    THETA =  83.70 DEG    PHI =   6.34 DEG    ABSORBED POWER DENSITY =  0.01101610 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.770 CM    THETA =  71.67 DEG    PHI =   6.34 DEG    ABSORBED POWER DENSITY =  0.00690943 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.770 CM    THETA =  83.98 DEG    PHI =  18.43 DEG    ABSORBED POWER DENSITY =  0.01083597 WATTS/M**3
INTERNAL POINT:  RADIUS =   4.975 CM    THETA =  72.45 DEG    PHI =  18.43 DEG    ABSORBED POWER DENSITY =  0.00709315 WATTS/M**3

        TOTAL ABSORBED POWER=  1.08523D-05 WATTS

        AVERAGE ABSORBED POWER DENSITY=  2.07264D-02 WATTS/M**3


APPROXIMATE EXECUTION TIME = 0.17 CPU MINUTE
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SAMPLE PROBLEM 2 DECK SETUP

CARD 1 (CONTROL PARAMETERS)

1.000E3 5.986E1 1.00152E0 1.000E0 1.000E1 0.000E0 101 0 0 0 0

CARDS 2 - (NOC+1) (DATA CARDS)

0.1E-3	0.0E0	0.0E0
0.1E0	0.0E0	0.0E0
0.2E0	0.0E0	0.0E0
0.3E0	0.0E0	0.0E0
0.4E0	0.0E0	0.0E0
0.5E0	0.0E0	0.0E0
0.6E0	0.0E0	0.0E0
0.7E0	0.0E0	0.0E0
0.8E0	0.0E0	0.0E0
0.9E0	0.0E0	0.0E0
1.0E0	0.0E0	0.0E0
1.1E0	0.0E0	0.0E0
1.2E0	0.0E0	0.0E0
1.3E0	0.0E0	0.0E0
1.4E0	0.0E0	0.0E0
1.5E0	0.0E0	0.0E0
1.6E0	0.0E0	0.0E0
1.7E0	0.0E0	0.0E0
1.8E0	0.0E0	0.0E0
1.9E0	0.0E0	0.0E0
2.0E0	0.0E0	0.0E0
2.1E0	0.0E0	0.0E0
2.2E0	0.0E0	0.0E0
2.3E0	0.0E0	0.0E0
2.4E0	0.0E0	0.0E0
2.5E0	0.0E0	0.0E0
2.6E0	0.0E0	0.0E0
2.7E0	0.0E0	0.0E0
2.8E0	0.0E0	0.0E0
2.9E0	0.0E0	0.0E0
3.0E0	0.0E0	0.0E0
3.1E0	0.0E0	0.0E0
3.2E0	0.0E0	0.0E0
3.3E0	0.0E0	0.0E0
3.4E0	0.0E0	0.0E0
3.5E0	0.0E0	0.0E0
3.6E0	0.0E0	0.0E0
3.7E0	0.0E0	0.0E0
3.8E0	0.0E0	0.0E0
3.9E0	0.0E0	0.0E0
4.0E0	0.0E0	0.0E0
4.1E0	0.0E0	0.0E0
4.2E0	0.0E0	0.0E0
4.3E0	0.0E0	0.0E0
4.4E0	0.0E0	0.0E0
4.5E0	0.0E0	0.0E0
4.6E0	0.0E0	0.0E0
4.7E0	0.0E0	0.0E0
4.8E0	0.0E0	0.0E0
4.9E0	0.0E0	0.0E0

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5.0E0      0.0E0      0.0E0
0.1E0    180.0E0      0.0E0
0.2E0    180.0E0      0.0E0
0.3E0    180.0E0      0.0E0
0.4E0    180.0E0      0.0E0
0.5E0    180.0E0      0.0E0
0.6E0    180.0E0      0.0E0
0.7E0    180.0E0      0.0E0
0.8E0    180.0E0      0.0E0
0.9E0    180.0E0      0.0E0
1.0E0    180.0E0      0.0E0
1.1E0    180.0E0      0.0E0
1.2E0    180.0E0      0.0E0
1.3E0    180.0E0      0.0E0
1.4E0    180.0E0      0.0E0
1.5E0    180.0E0      0.0E0
1.6E0    180.0E0      0.0E0
1.7E0    180.0E0      0.0E0
1.8E0    180.0E0      0.0E0
1.9E0    180.0E0      0.0E0
2.0E0    180.0E0      0.0E0
2.1E0    180.0E0      0.0E0
2.2E0    180.0E0      0.0E0
2.3E0    180.0E0      0.0E0
2.4E0    180.0E0      0.0E0
2.5E0    180.0E0      0.0E0
2.6E0    180.0E0      0.0E0
2.7E0    180.0E0      0.0E0
2.8E0    180.0E0      0.0E0
2.9E0    180.0E0      0.0E0
3.0E0    180.0E0      0.0E0
3.1E0    180.0E0      0.0E0
3.2E0    180.0E0      0.0E0
3.3E0    180.0E0      0.0E0
3.4E0    180.0E0      0.0E0
3.5E0    180.0E0      0.0E0
3.6E0    180.0E0      0.0E0
3.7E0    180.0E0      0.0E0
3.8E0    180.0E0      0.0E0
3.9E0    180.0E0      0.0E0
4.0E0    180.0E0      0.0E0
4.1E0    180.0E0      0.0E0
4.2E0    180.0E0      0.0E0
4.3E0    180.0E0      0.0E0
4.4E0    180.0E0      0.0E0
4.5E0    180.0E0      0.0E0
4.6E0    180.0E0      0.0E0
4.7E0    180.0E0      0.0E0
4.8E0    180.0E0      0.0E0
4.9E0    180.0E0      0.0E0
5.0E0    180.0E0      0.0E0
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TERMINATION CARD

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SAMPLE PROBLEM 2

DEPOSITION OF POWER INSIDE A HOMOGENEOUS SPHERE IMMERSED IN AN ELECTROMAGNETIC FIELD

FREQUENCY = 1000.00 MHZ WAVELENGTH = 29.97925 CM FIELD STRENGTH = 1.00 V/M

CONDUCTIVITY = 1.001520 MHO/M RELATIVE DIELECTRIC CONSTANT = 59.86 DIAMETER = 10.00 CM TIME = 0.0 SEC

REFRACTIVE INDEX = 7.82226D+00 -1.15231D+00

THE HIGHEST ORDER NEUMANN FUNCTION COMPUTED IS OF ORDER = 21 SIZE PARAMETER = 1.04792D+00

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SPHERICAL BESSEL FUNCTION OF ARGUMENT X
TO THE VALUE OF SIN(X)/X = 1.00000D+00

RATIO OF THE FIRST CALCULATION OF THE ZERO ORDER SPHERICAL BESSEL FUNCTION OF ARGUMENT Z
TO THE VALUE OF SIN(Z)/Z = 1.00000D+00 7.13511D-16

INTERNAL POINT:	RADIUS =	0.0001CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.16492808 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.100 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.15847236 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.200 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.14861959 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.300 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.13587876 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.400 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.12083088 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.500 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.10447731 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.600 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.08745981 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.700 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.07071352 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.800 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05586628 WATTS/M**3
INTERNAL POINT:	RADIUS =	0.900 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04325135 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.000 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02986406 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.100 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02132636 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.200 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01586434 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.300 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01349977 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.400 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01405594 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.500 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01717725 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.600 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02236057 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.700 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02899605 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.800 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03641389 WATTS/M**3
INTERNAL POINT:	RADIUS =	1.900 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04393368 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.000 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05091286 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.100 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05678608 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.200 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.06111481 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.300 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.06358676 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.400 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.06406177 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.500 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.06254305 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.600 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05916914 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.700 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05428331 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.800 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04820560 WATTS/M**3
INTERNAL POINT:	RADIUS =	2.900 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04140312 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.000 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03434283 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.100 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02748238 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.200 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02123859 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.300 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01593694 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.400 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01184127 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.500 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.00905259 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.600 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.00762578 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.700 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.00787185 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.800 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.00942258 WATTS/M**3
INTERNAL POINT:	RADIUS =	3.900 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01228003 WATTS/M**3
INTERNAL POINT:	RADIUS =	4.000 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01568218 WATTS/M**3
INTERNAL POINT:	RADIUS =	4.100 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.01932912 WATTS/M**3
INTERNAL POINT:	RADIUS =	4.200 CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02300426 WATTS/M**3

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INTERNAL POINT:	RADIUS =	4.300	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02042313	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.400	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02236238	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.500	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02368345	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.600	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02431844	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.700	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02425531	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.800	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02357193	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.900	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02238461	WATTS/M**3
INTERNAL POINT:	RADIUS =	5.000	CM	THETA =	0.0	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02085281	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.100	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.16769784	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.200	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.16666903	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.300	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.16196792	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.400	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.15391656	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.500	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.14301543	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.600	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.12990664	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.700	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.11532887	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.800	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.10006705	WATTS/M**3
INTERNAL POINT:	RADIUS =	0.900	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.08490052	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.000	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.07055386	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.100	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.05765102	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.200	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04668423	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.300	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03798579	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.400	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03171812	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.500	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02787413	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.600	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02628987	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.700	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02666781	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.800	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02860828	WATTS/M**3
INTERNAL POINT:	RADIUS =	1.900	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03164648	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.000	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03529164	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.100	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03906601	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.200	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04253939	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.300	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04535811	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.400	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04726578	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.500	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04811481	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.600	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04786852	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.700	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04659816	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.800	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04444792	WATTS/M**3
INTERNAL POINT:	RADIUS =	2.900	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.04165390	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.000	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03847898	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.100	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03520598	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.200	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03210761	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.300	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02942315	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.400	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02733984	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.500	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02598017	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.600	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02539577	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.700	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02556803	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.800	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02641406	WATTS/M**3
INTERNAL POINT:	RADIUS =	3.900	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02780315	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.000	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.02956362	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.100	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03150973	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.200	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03345547	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.300	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03523249	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.400	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03670430	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.500	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03777670	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.600	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03840348	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.700	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03858740	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.800	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03837658	WATTS/M**3
INTERNAL POINT:	RADIUS =	4.900	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03785656	WATTS/M**3
INTERNAL POINT:	RADIUS =	5.000	CM	THETA =	180.00	DEG	PHI =	0.0	DEG	ABSORBED POWER DENSITY =	0.03714009	WATTS/M**3

APPPOXIMATE EXECUTION TIME = 0.17 CPU MINUTE

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The plot of the computed absorbed power densities at internal points of the sphere, Figure B-1, is similar to that in Kritikos (2,p.58) for a sphere of radius 5 cm and 1000 MHz. The electrical properties ε and σ are those of biological tissue of high water content.

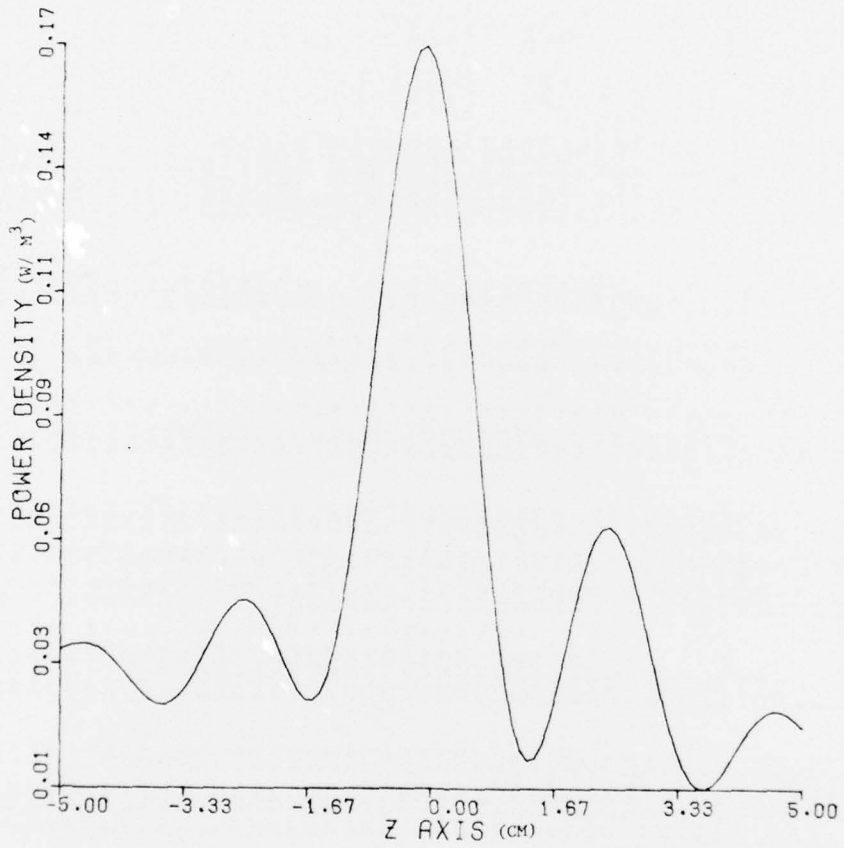


Figure B-1. Distribution of the power density inside the sphere along the z axis.

APPENDIX C

SOURCE LISTING OF PROGRAM MIE

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C                              PROGRAM MIE                                      MIE0001
C          DEPOSITION OF POWER INSIDE OF A HOMOGENEOUS SPHERE                   MIE0002
C          IMMERSED IN AN ELECTROMAGNETIC FIELD                                 MIE0003
C                                                                               MIE0004
      IMPLICIT REAL*8 (A-H,O-Z)                                                 MIE0005
      COMMON /GAUSS/D,M1,M2,M                                                   MIE0006
      COMMON CM(100),DM(100),AJR(100),AM,AMM,CEX,Z,DP(100),P(101),ALAMDAMIE0007
     1,D1,PO,PHI,PIE,SKE,STOPR,SIGTH,CHITA,MC,NM2                               MIE0008
      COMPLEX*16 CM,DM,AJR,AM,AMM,CEX,Z,CQ,RQ,CPHI,CPHIP,EPSZ,EPSIP            MIE0009
      PIE=3.14159265358979D0                                                    MIE0010
      RAD=180.D0/PIE                                                            MIE0011
      STOPR = 1.D05                                                             MIE0012
      OLDTH=999.D0                                                              MIE0013
      OLDD1=1.D-70                                                              MIE0014
    5 READ (5,10, END=80) FREQ,EPS,SIGM1,EO,D,TIME,MOC,IOPT,M1,M2,M3            MIE0015
   10 FORMAT (6E10.0,2I5,3I3)                                                   MIE0016
      OMEGA=2.D6*PIE*FREQ                                                       MIE0017
      SIGMA=SIGM1*9.D9                                                          MIE0018
      AM=CDSQRT(DCMPLX(EPS,(-4.DO*PIE*SIGMA)/OMEGA))                            MIE0019
      ARG=OMEGA*TIME                                                            MIE0020
      CEX=DCMPLX(DCOS(ARG),DSIN(ARG))                                           MIE0021
      ALAMDA=2.997924562D4/FREQ                                                 MIE0022
      AMK=2.DO*AM*PIE/ALAMDA                                                    MIE0023
      EO=EO*.3333333333333333D-4                                                MIE0024
      X=PIE*D/ALAMDA                                                            MIE0025
      Z=AM*X                                                                    MIE0026
C ***         GENERATE SPHERICAL BESSEL FUNCTIONS JN(MX) AND NN(MX)            MIE0027
      CALL BESS(AJR,Z,N,CQ,STOPR)                                               MIE0028
C ***         GENERATE SPHERICAL BESSEL FUNCTIONS JN(X) AND NN(X)              MIE0029
      DP(1)=-DCOS(X)/X                                                          MIE0030
      DP(2)=DP(1)/X-DSIN(X)/X                                                   MIE0031
      DO 15 M=2,N                                                               MIE0032
      AN2=2*M-1                                                                 MIE0033
      DP(M+1)=AN2/X*DP(M)-DP(M-1)                                               MIE0034
      IF (DABS(DP(M+1)).GE.STOPR) GO TO 20                                      MIE0035
   15 CONTINUE                                                                  MIE0036
   20 NP1=M+1                                                                   MIE0037
      N=M                                                                       MIE0038
      P(NP1)=0.D0                                                               MIE0039
      P(NP1-1)=-1.DO/(X*X*DP(NP1))                                              MIE0040
      NM2=NP1-2                                                                 MIE0041
      DO 25 M=1,NM2                                                             MIE0042
      I=NP1-M                                                                   MIE0043
      AN2=2*I-1                                                                 MIE0044
   25 P(I-1)=AN2/X*P(I)-P(I+1)                                                  MIE0045
      RQ=DSIN(X)/(X*P(1))                                                       MIE0046
      DO 30 M=1,NP1                                                             MIE0047
   30 P(M)=RQ*P(M)                                                              MIE0048
      NM1=N-1                                                                   MIE0049
      NM2=N-2                                                                   MIE0050
      CQ=1.DO/CQ                                                                MIE0051
      RQ=1.DO/RQ                                                                MIE0052
C ***         PRINT OUT TITLE, BASIC INPUT DATA, AND ERROR ANALYSIS DATA       MIE0053
      PRINT 35,FREQ,ALAMDA,EO,SIGM1,EPS,D,TIME,AM,NM1,X,RQ,CQ                   MIE0054
   35 FORMAT  ('1DEPOSITION OF POWER INSIDE A HOMOGENEOUS SPHERE IMMERSEDMIE0055
     1D IN AN ELECTROMAGNETIC FIELD'/'-FREQUENCY =',F9.2,' MHZ      WAVEMIE0056
     1LENGTH =',F15.5,' CM       FIELD STRENGTH =',F7.2,' V/M'/'0CONDUCTMIE0057
     1IVITY =',E10.6,' MHO/M      RELATIVE DIELECTRIC CONSTANT =',F7.2,'MIE0058
     1      DIAMETER =',F7.2,' CM      TIME =',F7.2,' SEC'/'0REFRACTIVE MIE0059
     1INDEX =',1P2E13.5/'0THE HIGHEST ORDER NEUMANN FUNCTION COMPUTED ISMIE0060
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     1 OF ORDER =',I3,'     SIZE PARAMETER =',E13.5/'0RATIO OF THE FIRSTMIE0061
     1 CALCULATION OF THE ZERO ORDER SHERICAL BESSEL FUNCTION OF ARGUMENMIE0062
     1T X'/' TO THE VALUE OF SIN(X)/X =',E15.5/'0RATIO OF THE FIRST CALCMIE0063
     1ULATION OF THE ZERO ORDER SHERICAL BESSEL FUNCTION OF ARGUMENT Z'/MIE0064
     1' TO THE VALUE OF SIN(Z)/Z =',2F15.5//)                          MIE0065
C ***       GENERATE COEFFICIENTS CN AND DN                           MIE0066
      DO 40 NN=1,NM2                                                   MIE0067
      NP1=NN+1                                                         MIE0068
      RPHI=X*P(NP1)                                                    MIE0069
      RPSI=DCMPLX(RPHI,-X*DP(NP1))                                     MIE0070
      CPHI=Z*AJR(NP1)                                                  MIE0071
      AN=NN                                                            MIE0072
      FACT=1.D0/(2.D0*AN+1.D0)                                         MIE0073
      RPHIP=P(NP1)+X*(FACT*(AN*P(NP1-1)-(AN+1.D0)*P(NP1+1)))           MIE0074
      RKIP=-DP(NP1)-X*(FACT*(AN*DP(NP1-1)-(AN+1.D0)*DP(NP1+1)))        MIE0075
      RPSIP=DCMPLX(RPHIP,RKIP)                                         MIE0076
      CPHIP=AJR(NP1)+Z*(FACT*(AN*AJR(NP1-1)-(AN+1.D0)*AJR(NP1+1)))     MIE0077
      Q=(RPHIP*RPSI-RPHI*RPSIP)                                        MIE0078
      CN(NN)=Q/(CPHIP*RPSI-AM*CPHI*RPSIP)                              MIE0079
   40 DN(NN)=Q/(AM*CPHIP*RPSI-CPHI*RPSIP)                              MIE0080
      IF (NOC.EQ.0) GO TO 65                                           MIE0081
      DO 60 J=1,NOC                                                    MIE0082
      READ 10, R,THETAD,PHID                                           MIE0083
      D1=2.D0*R                                                        MIE0084
      PHI=PHID/RAD                                                     MIE0085
      IF (THETAD.EQ.OLDTH) GO TO 45                                    MIE0086
      THETA=THETAD/RAD                                                 MIE0087
      SINTH=DSIN(THETA)                                                MIE0088
      CALL PL(THETA,N,P,DP)                                            MIE0089
   45 IF (D1.EQ.OLDD1) GO TO 50                                        MIE0090
      RKR=ALAMDA/(PIE*D1)                                              MIE0091
      Z=AM*PIE*D1/ALAMDA                                               MIE0092
      CALL BESS(AJR,Z,NC,Q,STOPR)                                      MIE0093
      NC=NC-2                                                          MIE0094
      IF (NC.GT.NM2)NC=NM2                                             MIE0095
C           ABSORBED POWER DENSITY AT GIVEN INTERNAL POINT OF SPHERE   MIE0096
   50 CALL EVEC(PD)                                                    MIE0097
      PD=.05D0*SIGMA*PD                                                MIE0098
C ***       PRINT OUT INTERIOR POINT PARTICULARS                       MIE0099
      PRINT 55,R,THETAD,PHID,PD                                        MIE0100
   55 FORMAT (' INTERNAL POINT:  RADIUS =',F8.3,' CM    THETA =',F7.2,' MIE0101
     1DEG    PHI =',F7.2,' DEG     ABSORBED POWER DENSITY =',F12.8,' WATMIE0102
     1TS/M**3')                                                        MIE0103
   60 CONTINUE                                                         MIE0104
   65 IF (IOPT.EQ.0) GO TO 5                                           MIE0105
C           TOTAL ABSORBED POWER AND AVERAGE ABSORBED POWER DENSITY    MIE0106
      TOTPOW=.05D-6*SIGMA*GAUSS3(M3)                                   MIE0107
      PAVG=TOTPOW*1.D6/((4.D0/3.D0)*PIE*(D/2.D0)**3)                   MIE0108
C ***       PRINT OUT ABSORBED TOTAL POWER AND AVERAGE ABSORBED        MIE0109
C           POWER DENSITY                                              MIE0110
      PRINT 75,TOTPOW,PAVG                                             MIE0111
   75 FORMAT ('0',9X,'TOTAL ABSORBED POWER=',1PE13.5,' WATTS'/'0',9X,'AVMIE0112
     1ERAGE ABSORBED POWER DENSITY=',E13.5,' WATTS/M**3')             MIE0113
      GO TO 5                                                          MIE0114
   80 STOP                                                             MIE0115
      END                                                              MIE0116
      SUBROUTINE EVEC(PD)                                              MIE0117
C           COMPUTES RADIAL,COLATITUDE,AND AZIMUTHAL COMPONENTS OF     MIE0118
C           ELECTRIC FIELD VECTOR E AND THE SCALAR PRODUCT E.E*        MIE0119
      IMPLICIT REAL*8 (A-H,O-Z)                                        MIE0120
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      COMMON CN(100),DN(100),AJR(100),AM,AMK,CEX,Z,DP(100),P(101),ALAMDAMIE0121
     1,D1,EO,PHI,PI2,RKT,STOPR,SINTH,THETA,NC,NM2                      MIE0122
      COMPLEX*16 CN,DN,AJR,AM,AMK,CEX,Z,UE,UT,UP,URE,URT,URP,URRE,URRT,UMIE0123
     1RRP,DERJN1,AMZ,SUM,ER,ETHETA,EPHI,TE,TT,TP,T1,T2,T3,ZSQ          MIE0124
      DATA EPS/1.D-10/                                                 MIE0125
      ZSQ=Z*Z                                                          MIE0126
      UE=DCMPLX(0.D0,0.D0)                                             MIE0127
      URE=DCMPLX(0.D0,0.D0)                                            MIE0128
      URRE=DCMPLX(0.D0,0.D0)                                           MIE0129
      UT=DCMPLX(0.D0,0.D0)                                             MIE0130
      URT=DCMPLX(0.D0,0.D0)                                            MIE0131
      URRT=DCMPLX(0.D0,0.D0)                                           MIE0132
      UP=DCMPLX(0.D0,0.D0)                                             MIE0133
      URP=DCMPLX(0.D0,0.D0)                                            MIE0134
      URRP=DCMPLX(0.D0,0.D0)                                           MIE0135
      IE=0                                                             MIE0136
      IT=0                                                             MIE0137
      IP=0                                                             MIE0138
      NCK=1                                                            MIE0139
      DO 70 NN=1,NC                                                    MIE0140
      NNP1=NN+1                                                        MIE0141
      FAC1 = 2*NN+1                                                    MIE0142
      FAC2=NN*NNP1                                                     MIE0143
      FAC3=FAC1/FAC2                                                   MIE0144
      DERJN1=(NN*AJR(NN)-NNP1*AJR(NN+2))/FAC1                          MIE0145
      IF (IE.EQ.1) GO TO 5                                             MIE0146
      TE=CN(NN)*FAC3*P(NN)                                             MIE0147
      T1=TE*AJR(NN+1)                                                  MIE0148
      CALL TERM(NCK,T1)                                                MIE0149
      UE=UE+T1                                                         MIE0150
      T2=TE*DERJN1                                                     MIE0151
      CALL TERM(NCK,T2)                                                MIE0152
      URE=URE+T2                                                       MIE0153
      T3=TE*(-2.D0*DERJN1/Z-(1.D0-FAC2/ZSQ)*AJR(NN+1))                 MIE0154
      CALL TERM(NCK,T3)                                                MIE0155
      URRE=URRE+T3                                                     MIE0156
      IF (CDABS(T1).GT.CDABS(UE)*EPS) GO TO 5                          MIE0157
      IF (CDABS(T2).GT.CDABS(URE)*EPS) GO TO 5                         MIE0158
      IF (CDABS(T3).LE.CDABS(URRE)*EPS) IE=1                           MIE0159
    5 IF (IT.EQ.1) GO TO 45                                            MIE0160
      IF ((THETA.GE.1.0D-6).AND.(THETA.LT.3.141592D0)) GO TO 20        MIE0161
      TT=DN(NN)*FAC1/2.D0                                              MIE0162
      IF (THETA.GE.3.141592D0)TT=(-1.D0)**NNP1*TT                      MIE0163
      GO TO 25                                                         MIE0164
   20 TT=DN(NN)*FAC3*P(NN)/SINTH                                       MIE0165
   25 T1=TT*AJR(NN+1)                                                  MIE0166
      CALL TERM(NCK,T1)                                                MIE0167
      UT=UT+T1                                                         MIE0168
      TT=CN(NN)*FAC3*DP(NN)                                            MIE0169
      T2=TT*AJR(NN+1)                                                  MIE0170
      CALL TERM(NCK,T2)                                                MIE0171
      URT=URT+T2                                                       MIE0172
      T3=TT*DERJN1                                                     MIE0173
      CALL TERM(NCK,T3)                                                MIE0174
      URRT=URRT+T3                                                     MIE0175
      IF (CDABS(T1).GT.CDABS(UT)*EPS) GO TO 45                         MIE0176
      IF (CDABS(T2).GT.CDABS(URT)*EPS) GO TO 45                        MIE0177
      IF (CDABS(T3).LE.CDABS(URRT)*EPS) IT=1                           MIE0178
   45 IF (IP.EQ.1) GO TO 68                                            MIE0179
      TP=DN(NN)*FAC3*DP(NN)                                            MIE0180
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      T1=TP*AJB(NN+1)                                                   MIE0181
      CALL TERM(NCK,T1)                                                 MIE0182
      UP=UP+T1                                                          MIE0183
      IF ((THETA.GE.1.0D-6).AND.(THETA.LT.3.141592D0)) GO TO 60         MIE0184
      TP=CN(NN)*FAC1/2.D0                                               MIE0185
      IF (THETA.GE.3.141592D0)TP=(-1.D0)**NNP1*TP                       MIE0186
      GO TO 65                                                          MIE0187
   60 TP=CN(NN)*FAC3*P(NN)/SINTH                                        MIE0188
   65 T2=TP*AJB(NN+1)                                                   MIE0189
      CALL TERM(NCK,T2)                                                 MIE0190
      URP=URP+T2                                                        MIE0191
      T3=TP*DERJN1                                                      MIE0192
      CALL TERM(NCK,T3)                                                 MIE0193
      URRP=URRP+T3                                                      MIE0194
      IF (CDABS(T1).GT.CDABS(UP)*EPS) GO TO 68                          MIE0195
      IF (CDABS(T2).GT.CDABS(URP)*EPS) GO TO 68                         MIE0196
      IF (CDABS(T3).LE.CDABS(URRP)*EPS)IP=1                             MIE0197
   68 IF (IP+IP.EQ.3) GO TO 71                                          MIE0198
      NCK=NCK+1                                                         MIE0199
   70 IF (NCK.GT.4)NCK=1                                                MIE0200
   71 PCO=DCOS(PHI)                                                     MIE0201
      AMZ=AM*Z                                                          MIE0202
      ER=CEX*PCO*(2.D0*AM*URR+AMZ*(URRP+UP))                            MIE0203
      ER=EO*DCMPLX(-DIMAG(ER),DREAL(ER))                                MIE0204
      SUM=CEX*PCO*(RKR*URT+AM*URRT)                                     MIE0205
      ETHETA=EO*(AM*CEX*PCO*UT+DCMPLX(-DIMAG(SUM),DREAL(SUM)))          MIE0206
      PSI=DSIN(PHI)                                                     MIE0207
      SUM=-CEX*PSI*(RKR*URP+AM*URRP)                                    MIE0208
      EPHI=EO*(-AM*CEX*PSI*UP+DCMPLX(-DIMAG(SUM),DREAL(SUM)))           MIE0209
      PD=DREAL(ER*DCONJG(ER))+DREAL(ETHETA*DCONJG(ETHETA))+DREAL(PPHI*DCMIE0210
     1ONJG(EPHI))                                                       MIE0211
      RETURN                                                            MIE0212
      END                                                               MIE0213
      SUBROUTINE TERM(NCK,T)                                            MIE0214
C           COMPUTES (-J)**N*(N-TH TERM IN SERIES)                      MIE0215
      IMPLICIT REAL*8 (A-H,O-Z)                                         MIE0216
      COMPLEX*16 T                                                      MIE0217
      GO TO (5,10,15,20),NCK                                            MIE0218
    5 T=DCMPLX(DIMAG(T),-DREAL(T))                                      MIE0219
      GO TO 20                                                          MIE0220
   10 T=-T                                                              MIE0221
      GO TO 20                                                          MIE0222
   15 T=DCMPLX(-DIMAG(T),DREAL(T))                                      MIE0223
   20 RETURN                                                            MIE0224
      END                                                               MIE0225
      SUBROUTINE BESS(AJ,Z,N,Q,STOPB)                                   MIE0226
C           GENERATES SPHERICAL BESSEL FUNCTIONS JN(MX) AND NN(MX)      MIE0227
      IMPLICIT REAL*8 (A-H,O-Z)                                         MIE0228
      DIMENSION AJ(100)                                                 MIE0229
      COMPLEX*16 Z,AJ,Y,Y0,Y1,Q                                         MIE0230
      Y0   =-CDCOS(Z)/Z                                                MIE0231
      Q=CDSIN(Z)                                                        MIE0232
      Y1   =(Y0-Q)/Z                                                    MIE0233
      DO 5 M=3,100                                                      MIE0234
      Y=(2*M-3)/Z*Y1-Y0                                                 MIE0235
      IF (CDABS(Y      ).GE.STOPB) GO TO 10                             MIE0236
      Y0=Y1                                                             MIE0237
    5 Y1=Y                                                              MIE0238
   10 IF (M.GT.3) GO TO 30                                              MIE0239
C ***       PRINT OUT ERROR MESSAGE                                     MIE0240
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      PRINT 25,Z                                                        MIE0241
   25 FORMAT ('0**** PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z =',1P2DMIE0242
     1 15.7)                                                            MIE0243
      STOP                                                              MIE0244
   30 AJ(M)=DCMPLX(0.D0,0.D0)                                           MIE0245
      AJ(M-1)=-1.D0/(Z*Z*Y)                                             MIE0246
      NM2=M-2                                                           MIE0247
      DO 35 I=1,NM2                                                     MIE0248
      L=M-I                                                             MIE0249
   35 AJ(L-1)=(2*L-1)/Z*AJ(L)-AJ(L+1)                                   MIE0250
      Q=Q/(Z*AJ(1))                                                     MIE0251
      NM1=M-1                                                           MIE0252
      DO 40 M=1,NM1                                                     MIE0253
      AJ(M)=Q*AJ(M)                                                     MIE0254
      IF (CDABS(AJ(M)).LT.1.D-25) RETURN                                MIE0255
   40 CONTINUE                                                          MIE0256
      RETURN                                                            MIE0257
      END                                                               MIE0258
      SUBROUTINE PL(THETA,M,P,DP)                                       MIE0259
C           GENERATES ASSOCIATED LEGENDRE FUNCTIONS AND THEIR           MIE0260
C           FIRST DERIVATIVES                                           MIE0261
      IMPLICIT REAL*8 (A-H,O-Z)                                         MIE0262
      DIMENSION P(101),DP(100)                                          MIE0263
      SNJ=DSIN(THETA)                                                   MIE0264
      CNJ=DCOS(THETA)                                                   MIE0265
      P(1)=SNJ                                                          MIE0266
      P(2)=3.D0*SNJ*CNJ                                                 MIE0267
      DP(1)=CNJ                                                         MIE0268
      DO 10 N=2,M                                                       MIE0269
      A=N                                                               MIE0270
      MP1=M+1                                                           MIE0271
      P(M+1)=(2.D0*A+1.D0)/A*CNJ*P(M)-(A+1.D0)/A*P(M-1)                 MIE0272
      IF ((THETA.GE.1.0D-6).AND.(THETA.LT.3.141592D0)) GO TO 5          MIE0273
      DP(M)=M*MP1/2                                                     MIE0274
      IF (THETA.GE.3.141592D0) DP(M)=(-1.D0)**M*DP(M)                   MIE0275
      GO TO 10                                                          MIE0276
    5 DP(M)=(A*CNJ*P(M)-(A+1.D0)*P(M-1))/SNJ                            MIE0277
   10 CONTINUE                                                          MIE0278
      RETURN                                                            MIE0279
      END                                                               MIE0280
      FUNCTION GAUSS3(M3)                                               MIE0281
C      PRODUCT RULE: M-POINT GAUSS-LEGENDRE QUADRATURE FORMULAS         MIE0282
      IMPLICIT REAL*8 (A-H,O-Z)                                         MIE0283
      COMMON /GAUSS/D,M1,M2,N                                           MIE0284
      COMMON CN(100),DN(100),AJR(100),AM,AMK,CEX,Z,DP(100),P(101),ALAMDAMIE0285
     1,D1,EO,PHI,PIE,RKR,STORE,SINTH,THETA,NC,NM2                       MIE0286
      COMPLEX*16 CN,DN,AJR,AM,AMK,CEX,Z,Q                               MIE0287
      DIMENSION NPOINT(9),KEY(10),Y(33),WT(33),ARG2(2),ARG3(2)          MIE0288
      DATA NPOINT/2,3,4,5,6,8,10,12,14/                                 MIE0289
      DATA KEY/1,2,4,6,9,12,16,21,27,34/                                MIE0290
      DATA Y/0.5773502691896 3D0,0.0000000000000 0D0,0.7745966924148 8D0,0.MIE0291
     1339981043584 86D0,0.8611363115940 5D0,0.0000000000000 0D0,0.538469310MIE0292
     1105680D0,0.9061798459386 6D0,0.2386191860832 0D0,0.6612093864662 6D0,0MIE0293
     1.9324695142031 5D0,0.1834346424956 5D0,0.5255324099163 3D0,0.7966664 7MIE0294
     1741363D0,0.9602898564975 4D0,0.1488743389816 3D0,0.4333953941292 5D0,MIE0295
     1.6794095682990 2D0,0.8650633666889 8D0,0.9739065285171 7D0,0.1252334MIE0296
     1085114 7D0,0.3678314989981 8D0,0.5873179542866 2D0,0.7699026741943 0D0MIE0297
     1,0.9041172563704 7D0,0.9815606342467 2D0,0.1080549487073 4D0,0.3 9112MIE0298
     136892789D0,0.5152486363581 5D0,0.6872929048116 9D0,0.8272013150697 6DMIE0299
     10,0.9284348836635 7D0,0.9862838086968 1D0,/                       MIE0300
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      DATA WT/1.000000000000000D0,0.888888888888889D0,0.555555555555555D0,0MIE0301
     1.652145154862555D0,0.347854845137145D0,0.568888888888889D0,0.47862867MIE0302
     104993700,0.236926885056190D0,0.467913934572690D0,0.360761573048140D0,MIE0303
     10.171324492379170D0,0.362683783378360D0,0.313706645877890D0,0.222381GMIE0304
     13445337D0,0.101228536290380D0,0.295524224714750D0,0.269266719310000D0MIE0305
     1,0.219086362515980D0,0.149451349150580D0,0.066671344308690D0,0.249147MIE0306
     104581340D0,0.233492536538350D0,0.203167426723070D0,0.160078328543350MIE0307
     10,0.106939325995320D0,0.047175336386510D0,0.215263853463160D0,0.20519MIE0308
     18463721300D0,0.185538397477940D0,0.157203167158190D0,0.121518570687900MIE0309
     1D0,0.080158087159760D0,0.035119460331750D0/                          MIE0310
      DO 5 I=1,9                                                           MIE0311
      IF (M3.EQ.NPOINT(I)) GO TO 20                                        MIE0312
    5 CONTINUE                                                             MIE0313
   10 PRINT 15,M1,M2,M3                                                    MIE0314
   15 FORMAT ('-ERROR IN INTEGRATION CONTROLS.  M1 =',I6,'  M2 =',I6,'   MIE0315
     1M3 =',I6)                                                           MIE0316
      GAUSS3=0.D0                                                          MIE0317
      RETURN                                                               MIE0318
   20 JP3   =KEY(I)                                                        MIE0319
      JL3   =KEY(I+1)-1                                                    MIE0320
      DO 25 I=1,9                                                          MIE0321
      IF (M2.EQ.NPOINT(I)) GO TO 30                                        MIE0322
   25 CONTINUE                                                             MIE0323
      GO TO 10                                                             MIE0324
   30 JP2   =KEY(I)                                                        MIE0325
      JL2   =KEY(I+1)-1                                                    MIE0326
      DO 35 I=1,9                                                          MIE0327
      IF (M1.EQ.NPOINT(I)) GO TO 40                                        MIE0328
   35 CONTINUE                                                             MIE0329
      GO TO 10                                                             MIE0330
   40 JP1   =KEY(I)                                                        MIE0331
      JL1   =KEY(I+1)-1                                                    MIE0332
C           INTEGRATE OVER THETA                                           MIE0333
      PD2=PI2/2.D0                                                         MIE0334
      R=D/4.D0                                                             MIE0335
      SUM3=0.D0                                                            MIE0336
      DO 85 J3=JP3,JL3                                                     MIE0337
      IF (Y(J3).NE.0.D0) GO TO 45                                          MIE0338
      ARG3(1)=PD2                                                          MIE0339
      I3=1                                                                 MIE0340
      GO TO 50                                                             MIE0341
   45 ARG3(1)=PD2+PD2*Y(J3)                                                MIE0342
      ARG3(2)=PD2-PD2*Y(J3)                                                MIE0343
      I3=2                                                                 MIE0344
   50 DO 85 L=1,I3                                                         MIE0345
      THETA=ARG3(L)                                                        MIE0346
      SINTH=DSIN(THETA)                                                    MIE0347
      CALL PL(THETA,N,P,DP)                                                MIE0348
C           INTEGRATE OVER RADIUS                                          MIE0349
      SUM2=0.D0                                                            MIE0350
      DO 80 J2=JP2,JL2                                                     MIE0351
      IF (Y(J2).NE.0.D0) GO TO 55                                          MIE0352
      ARG2(1)=R                                                            MIE0353
      I2=1                                                                 MIE0354
      GO TO 60                                                             MIE0355
   55 ARG2(1)=R+Y(J2)*R                                                    MIE0356
      ARG2(2)=R-Y(J2)*R                                                    MIE0357
      I2=2                                                                 MIE0358
   60 DO 80 I=1,I2                                                         MIE0359
      D1=2.D0*ARG2(I)                                                      MIE0360
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      RKR=ALAMDA/(PIE*D1)                                              MIE0361
      Z=AM*PIE*D1/ALAMDA                                               MIE0362
      CALL BESS(AJR,Z,NC,Q,STOPR)                                      MIE0363
      NC=NC-2                                                          MIE0364
      IF (NC.GT.MM2)NC=MM2                                             MIE0365
C           INTEGRATE OVER PHI                                         MIE0366
      SUM1=0.D0                                                        MIE0367
      DO 75 J=JP1,JL1                                                  MIE0368
      IF (Y(J).NE.0.D0) GO TO 65                                       MIE0369
      PHI=PIE                                                          MIE0370
      XMUL1=1.D0                                                       MIE0371
      GO TO 70                                                         MIE0372
   65 PHI=PIE*Y(J)*PIE                                                 MIE0373
C           FUNC(180+PHI)=FUNC(180-PHI)                                MIE0374
      XMUL1=2.D0                                                       MIE0375
   70 CALL EVEC(PD)                                                    MIE0376
   75 SUM1=SUM1+XMUL1*WT(J)*PD                                         MIE0377
   80 SUM2=SUM2+WT(J2)*SUM1*ARG2(I)**2                                 MIE0378
   85 SUM3=SUM3+WT(J3)*SUM2*SINTH                                      MIE0379
      GAUSS3=SUM3*PD2*B*PIE                                            MIE0380
      RETURN                                                           MIE0381
      END                                                              MIE0382
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