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STRUCTURE OF THE ASTEROID BELT (STRUKTURA POYASA ASTEROIDOV),(U)
AUG 77 G A CHEBOTAREV, V A SHOR
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STRUCTURE OF THE ASTEROID BELT

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Abstract. This paper reviews the present-day data on the spacial distribution of orbits and the motion of characteristic objects in the asteroid belt. The distribution of asteroids as a function of their stellar magnitudes, radii, and masses is also considered. Particular attention is devoted to the influence of the selection effects on the statistics and to explanation of the characteristics of the distribution in terms of the gravitational theory and collisional fragmentation of bodies. This paper contains original data in the form of tables and graphs and an extensive bibliography. (19 figures, 20 tables, and 76 references) [60

INTRODUCTION

A large number of minor planets or asteroids in motion around the sun in orbits located between Mars and Jupiter form the asteroid belt of the planetary system. Although the asteroid belt is a unique formation of the solar system, nevertheless its origin is apparently not random, but is related to factors determining the development of the Solar system during the earlier stage of its formation. Numerous arguments can be given to prove that the formation of planets from the primordial gaseous dust cloud in the region near the solar space between Mars and Jupiter ceased at an intermediate stage and that the asteroid belt is a relic of that stage. The present-day spacial and physical structure of the asteroid belt was formed as a result of a prolonged evolutionary development caused by gravitational and inertial forces. A study of the present-day distribution of orbits and masses of asteroids makes it possible to refine the nature of interaction between bodies of the minor planet system and forms the foundation for the development of sufficiently accurate models for the behavior of the system at cosmogenically long time intervals. One of the principal aims in constructing such models is the reconstruction of the initial state of the system of asteroids during the planet-forming stage. There are reasons to believe that this may make it possible to achieve better understanding of the planet-forming process from a system of objects existing before formation of the planets. Knowledge of the distribution of asteroid bodies as a function of their masses and their orbital elements is necessary in considering problems such as the relationship between asteroids and other space objects (meteorites, meteor particles, craters on the surfaces of celestial bodies, etc.).

Numbers in right margin indicate pagination in original text

The interest concerning the structure of the asteroid belt has increased lately as a result of studies of the danger posed by meteorites to spacecraft traveling beyond the orbit of Mars. Statistical analysis used in this paper is based on elements of asteroid orbits published in "Ephemeris of Minor Planets for 1973" (Chebotarev, 1972) containing elements of 1795 numbered minor planets, from No. 1 to No. 1796 (No. 864 is blank), and elements of three unnumbered planets (Apollo, Adonis, and Hermes).

1. POSITION OF THE ASTEROID BELT IN THE SOLAR SYSTEM. BOUNDARIES OF THE BELT.

The mean distance between a planet and the sun (mean between its perihelian and aphelion distances) is usually characterized by the semimajor axis of the orbit a, or the mean diurnal motion of the planet n, which is related to the semimajor axis through Kepler's third law:

$$a^3 n^3 = k^3 (m_0 + m),$$

where m_0 and m are masses of the sun and the planet respectively, k is the Gaussian constant numerically equal to $3548''.188$ when the semimajor axis of the orbit is expressed in astronomical units, and where mean motion is expressed in arcsec/day and solar mass is set equal to unity.

Most of the asteroid orbits are located near the ecliptic plane, [61
between the orbit of Mars (a = 1.52 AU, n = $1886''.5$) and Jupiter (a = 5.20 AU, n = $299''.1$). The arithmetical mean of the semimajor axis of numbered asteroids is 2.78 AU, which is close to the value given by the Titius-Bode relation published in 1772 for the hypothetical planet between Mars and Jupiter.

According to the empirical Titius-Bode relation, the mean distance between the planets and the sun expressed in astronomical units is given by the formula

$$a_k = 0.4 + 0.3 \times 2^k$$

with $k = -\infty, 0, 1, 2, \ldots$ A comparison of the values of semimajor axes of planets with values calculated from this relation is given in Table 1. When k = 3, this relation gives a value of a_3 corresponding to the asteroid belt.

Table 1

Comparison of Mean Distances from the Major Planets to the Sun with those determined from the Titius-Bode Relation

Planet	k	a	a_k
Mercury	-	0.39	0.4
Venus	0	0.72	0.7
Earth	1	1.00	1.0
Mars	2	1.52	1.6
-	3	-	2.8
Jupiter	4	5.20	5.2
Saturn	5	9.55	10.0
Uranus	6	19.18	19.6
Neptune	7	30.03	38.8
Pluto	8	39.67	77.2

The Titius-Bode relation, which has played a definite role in the discovery of minor planets and the planet Neptune (even though for the latter it is not justified), is at the present time frequently considered by astronomers to be no more than a historical curio. On the contrary, it should be considered a manifestation (perhaps not most adequately expressed) of certain important properties of the planetary system. Typically, a number of cosmogenic hypothesis (for example Weizsäcker, Schmidt, etc.) lead to formation of planets at distances obeying the Titius-Bode or analogous relations.

Another specific feature of the location of the asteroid belt is [62
that it separates two groups of planets with very different physical characteristics: terrestial planets from the major planets. Apparently, such location of the asteroid belt is not random, but rather, a consistent result of the planet-forming process which occurred approximately 4.5 billion years ago.

The mean motion of approximately 98 percent of all numbered asteroids is in the range between 400" and 1200" ($2.06 < a < 4.29$ AU). The mean motion of only twenty numbered planets located near the inner boundary of the belt exceeds 1200". The Hungaria, Amor, and Apollo groups (Table 2) are usually included among these asteroids. Most of the planets in the Hungaria group have low eccentricities and do not even cross the orbit of Mars. The Apollo group asteroids are characterized by rapid motion and high eccentricities. At perihelia they cross the orbit of the Earth and (1566) Icarus, which is closest to the sun, has a perihelion distance less

Table 2

Asteroids with Mean Motion n > 1200"*

Asteroid	ω	Ω	i	φ	n	a, AU	q, AU
Hungaria group							
(1025) Riema	348°5	163°0	26°9	2°2	1274"1	1.98	1.90
(1139) Atami	205.6	213.2	13.1	14.8	1305.8	1.95	1.45
(434) Hungaria	123.4	175.0	22.5	4.2	1308.8	1.94	1.80
(1103) Sequoia	77.1	267.5	17.9	5.4	1319.4	1.93	1.75
(1750) 1950 NA_1	108.3	273.5	19.1	10.0	1327.0	1.93	1.59
(1019) Strackea	121.6	144.0	27.0	4.1	1342.7	1.91	1.77
(1235) Schorria	43.0	12.8	25.0	8.8	1343.9	1.91	1.62
(1453) Fennia	254.0	6.7	23.7	1.6	1358.1	1.90	1.84
(1556) 1942 *EC*	286.9	175.3	25.1	7.1	1379.2	1.88	1.65
(1509) Esclangona	267.3	283.2	22.3	1.9	1391.6	1.87	1.81
(1727) 1965 *BA*	312.5	132.6	22.9	5.9	1405.5	1.85	1.66
(1355) Magoeba	340.5	224.9	22.7	2.8	1406.5	1.85	1.76
(1600) 1947 *UC*	49.8	60.3	21.2	2.2	1411.2	1.85	1.78
(1747) 1947 *NH*	339.9	267.9	21.4	6.3	1588.1	1.71	1.52
Amor group							
(1221) Amor	25.9	171.1	11.9	25.9	1333.0	1.92	1.08
(1627) Ivar	167.0	132.9	8.4	23.4	1394.0	1.86	1.12
(433) Eros	178.1	304.0	10.8	12.9	2015.3	1.46	1.13
Apollo group							
Adonis ***	39.0	353.1	1.4	49.8	1387.4	1.87	0.44
Hermes ***	90.7	353.3	6.2	38.6	1690.6	1.64	0.62
Apollo ***	284.8	36.1	6.3	34.0	1988.0	1.47	0.65
(1685) Toro	126.6	274.0	9.4	25.8	2218.4	1.37	0.77
(1620) Geographos	276.3	336.9	13.3	19.6	2557.5	1.24	0.83
(1566) Icarus	30.9	87.8	23.0	55.8	3171.4	1.08	0.19

* The following standard notation is used in this and all other tables: Ω - longitude of the ascending node of the orbit; i-inclination of the orbit; ω - perihelion latitude; φ - angle defined through the relationship $e = \sin\varphi$ where e is the eccentricity; $q = a(1 - e)$ - perihelion distance.

**The Amor group also includes the asteroids (887) Alinda and 1580 Betulia ($q < 1.15$ AU) the mean motion of which is less than 1200" (Table 10).

***Unnumbered planet

than that of Mercury. The Apollo asteroid group includes (1221) Amor, (1627) Ivar, and (433) Eros which at the present have perihelion distances exceeding slightly 1 AU ($1.00 < q < 1.15$).

Asteroids with mean motion less than 400" are found near the inner boundary of the belt. They include 15 Trojans and (944) Hidalgo (Table 3). All Trojans have orbits close to that of Jupiter. Hidalgo is the only asteroid whose orbit not only crosses the orbit of Jupiter but even reaches

Table 3

Asteroids with Mean Motion n < 400"

Asteroid	ω	Ω	i	φ	n	a
(944) Hidalgo	57°.6	21°.0	42°.5	41°.0	252".7	5.82
Trojans						
(1583) Antilochus	186.3	221.1	28.3	3.1	292.7	5.28
(1749) Telamon	105.9	340.8	6.1	6.4	293.5	5.27
(659) Nestor	334.9	350.5	4.5	6.3	294.5	5.26
(1647) Menelaus	290.1	239.8	5.6	1.6	297.3	5.22
(1143) Odysseus	234.7	220.6	3.1	5.3	298.2	5.21
(588) Achilles	127.6	316.1	10.3	8.5	298.3	5.21
(1404) Ajax	58.7	332.3	18.1	6.5	298.5	5.21
(617) Patroclus	303.8	43.9	22.1	8.1	298.6	5.21
(884) Priamus	332.3	301.0	8.9	6.9	300.5	5.19
(1173) Anchises	33.2	283.9	6.9	8.1	302.2	5.17
(1208) Troilus	294.1	48.0	33.7	5.4	302.2	5.17
(1172) Aneas	47.2	246.8	16.7	5.9	302.2	5.17
(911) Agamemnon	80.5	337.3	21.9	3.8	303.3	5.15
(624) Hektor	182.1	342.1	18.3	1.4	306.2	5.12
(1437) Diomedes	130.5	315.1	20.6	2.6	309.7	5.08

the orbit of Saturn (However, as a result of a large inclination of Hidalgo's orbit to the orbit of Saturn, it never approaches that planet).

2. DISTRIBUTION OF ASTEROIDS WITH RESPECT TO MEAN DIURNAL MOTION

The distribution of asteroids as a function of the mean motion or, what is equivalent,as a function of the semimajor axes of the orbits has a clearly expressed nonuniformity (Table 4, Fig. 1). It can be seen from the table and the histogram that ranges of mean motion with a concentration of a considerable number of asteroids alternate with ranges of a small number or a total lack of planets. Such ranges characterized by the deepest local minima in the distribution are known as Kirkwood gaps, named after D Kirkwood who has discovered them and has justifiably attributed their origin to the influence of Jupiter.

The connection between gaps and the perturbing influence of planet's atraction, most importantly Jupiter, definitely follows from the fact that the gaps are localized near the values of mean motion which are commensurable [63 with the mean motion of the perturbing planet. The mean motion of the asteroid n is commensurable with the mean motion of the planet n_1, when

$$n_1/n = p/q \,, \qquad (1)$$

Table 4

Distribution of Asteroids Over the Mean Diurnal Motion (400" < n < 1200")

n	N *	n	N	n	N	n	N
400—410″	1	600—610″	5	800—810″	45	1000—1010″	13
410—420	—	610—620	43	810—820	49	1010—1020	11
420—430	—	620—630	84	820—830	33	1020—1030	13
430—440	—	630—640	125	830—840	45	1030—1040	8
440—450	8	640—650	83	840—850	34	1040—1050	16
450—460	13	650—660	42	850—860	38	1050—1060	40
460—470	2	660—670	30	860—870	33	1060—1070	49
470—480	—	670—680	85	870—880	36	1070—1080	30
480—490	1	680—690	45	880—890	11	1080—1090	18
490—500	—	690—700	11	890—900	1	1090—1100	27
500—510	—	700—710	18	900—910	6	1100—1110	13
510—520	1	710—720	36	910—920	18	1110—1120	—
520—530	1	720—730	44	920—930	20	1120—1130	—
530—540	2	730—740	35	930—940	29	1130—1140	—
540—550	5	740—750	4	940—950	20	1140—1150	1
550—560	6	750—760	30	950—960	26	1150—1160	—
560—570	18	760—770	64	960—970	25	1160—1170	—
570—580	11	770—780	55	970—980	37	1170—1180	1
580—590	1	780—790	51	980—990	16	1180—1190	—
590—600	1	790—800	25	990—1000	11	1190—1200	—

*N is the number of asteroids

where p and q are reciprocal whole prime numbers.

Apparently, when condition (1) is satisfied, certain configurations of the asteroid and the planet are repeated every q rotations of the asteroid and p rotations of the planet. Condition (1) is also referred to as the resonance condition, and the corresponding orbits, as resonance orbits. The quantity $|p - q|$ is known as the order of commensurability. The [64 relationship between the gap positions and commensurabilities of mean motion is revealed only in the case of commensurabilities of the lowest orders.

In Fig. 1, the values of the mean motion corresponding to the major commensurabilities with Jupiter are marked by arrows. It can be seen clearly that the most noticeable gaps occur near the values of n = 598″.3, 747″.8, and 897″.4, corresponding to 1:2, 2:5, and 1:3 commensurabilities with mean motion of Jupiter (n_1 = 299″.1). This figure also shows the gaps at commensurabilities 3:7 (698″.0), 3:8 (797″.7), and 2:7 (1046″.9). A deep gap in the range 647" - 674" is associated with three nearby commensurabilities 6:13 (648″.1), 5:11 (658″.1), and 4:9 (673″.0) (Fig. 2).

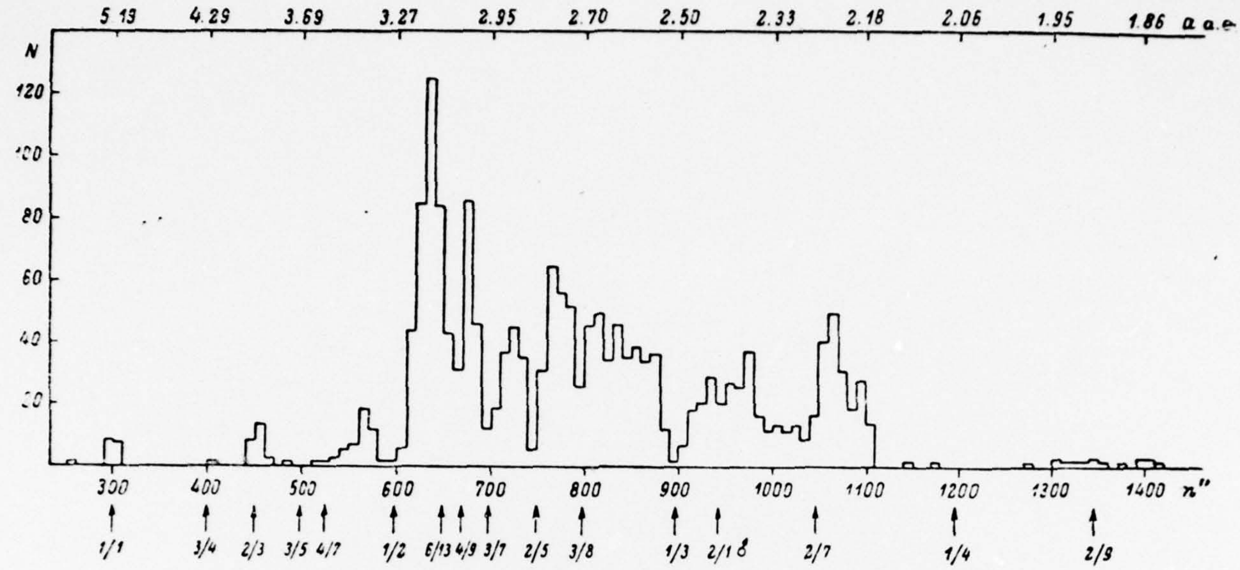

Fig. 1. Distribution of asteroids over ten-second ranges of mean motion.

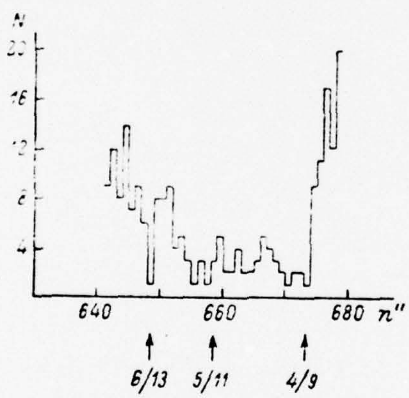

Fig. 2. Fine structure of the distribution of asteroids over the mean motion in the vicinity of gaps corresponding to 6:13, 5:11, and 4:9 commensurabilities.

The histogram in Fig. 1 also shows a sector of a relative density decrease near the value n = 943", apparently caused by commensurability with Mars (n_1 = 1886".5). It should be noted, that similarly to the density decreases in the distribution of asteroids over the mean motion, the width of the space intervals characterized by a total absence of asteroids depends on the order of commensurability: the lower the order of commensurability and the smaller the numbers p and q, the wider and deeper the gaps. The presence of wide gaps associated with pronounced commensurabilities 1:2, 2:5, 1:3 makes it possible to quite naturally subdivide the asteroid belt into zones (Table 5).

Table 5
Subdivision of the Asteroid Belt into Zones

Zone	Zone Width		N
	a, AU	n	
A	< 2.17	>1119"	22
I	2.17-2.50	1110-900	446
II	2.50-2.82	900-750	550
III	2.82-3.27	750-600	690
B	> 3.27	< 600	87

The above noted subdivision into zones reflects the principal [65
dharacteristics of the present-day dynamic structure of the asteroid belt and, therefore, is convenient for its description. In the range n < 500", commensurabilities with Jupiter of the lowest orders 1:1 (299".1), 2:3 (448".7), and 3:4 (398".8) correspond not to the sectors with decreasing, but rather to sectors with increasing mean motion of asteroids. The mean motion of the 15 Trojans is close to 300" (Table 3). A dense group of planets, the Hilda group, the mean motion of which falls in the range 443" < n < 464" is located near the 2:3 commensurability. Finally, the mean motion of the (279) Thule asteroid is close to 400" (Table 6).

A typical Trojan has an orbit close to that of Jupiter. During motion, the asteroid remains near the apex of an equilateral triangle with Jupiter and the Sun located at the other two apexes. One of the two Trojan groups is ahead of, and the other, behind Jupiter (Fig. 3). The motion of Trojans is in approximate agreement with the partial periodic solution of the restricted three body problem. Solution of this problem requires determination of the motion of a body with an infinitely small mass, under the influence of two other bodies of finite mass, in circular orbits around a common center of inertia. The above-noted partial periodic

Table 6

(279) Thule and Hilda Group Asteroids

Asteroid	ω	Ω	i	φ	n	a	Q*
(279) Thule	200°6	74°8	2°3	1°8	400″3	4.28	4.42
Hilda group							
(1529) 1938 *BC*	303.4	101.5	9.0	11.2	443.9	4.00	4.77
(1754) 1935 *FE*	120.9	164.0	12.0	9.4	444.6	3.99	4.64
(1345) Potomac	338.2	137.2	11.4	10.3	446.4	3.98	4.70
(1439) Vogtia	111.8	36.1	4.2	6.6	446.7	3.98	4.44
(1180) Rita	215.5	88.3	7.2	10.0	446.7	3.98	4.67
(153) Hilda	49.2	228.4	7.8	8.8	447.6	3.98	4.59
(1746) Brouwer	54.5	323.3	8.4	11.3	448.0	3.97	4.75
(499) Venusia	180.9	257.1	2.1	12.8	449.7	3.96	4.84
(1578) Kirkwood	6.2	73.5	0.8	12.9	450.5	3.96	4.84
(1212) Francette	351.8	149.3	7.6	10.6	451.2	3.95	4.68
(190) Ismene	281.1	176.6	6.2	9.8	451.5	3.95	4.62
(748) Simeisa	186.3	266.8	2.3	10.1	451.8	3.95	4.64
(1202) Marina	313.0	50.9	3.4	11.4	451.9	3.95	4.73
(958) Asplinda	97.6	344.1	5.7	11.0	453.0	3.94	4.70
(1162) Larissa	223.1	40.1	1.9	6.3	453.2	3.94	4.38
(1269) Rollandia	34.2	135.5	2.7	4.2	454.0	3.94	4.23
(1038) Tuckia	308.0	58.2	3.3	13.8	454.4	3.94	4.87
(361) Bononia	71.6	19.1	12.7	12.4	454.9	3.93	4.77
(1512) Oulu	248.1	10.7	6.6	9.3	454.9	3.93	4.57
(1268) Libya	139.8	352.4	4.4	6.1	455.3	3.93	4.35
(1748) Mauderli	203.1	125.7	3.4	13.5	457.3	3.92	4.84
(1256) Normannia	124.9	239.3	4.1	4.6	460.5	3.90	4.21
(334) Chicago	163.7	131.5	4.6	3.3	463.3	3.89	4.11

* $Q = a\,(1 + e)$ -- the aphelion distance

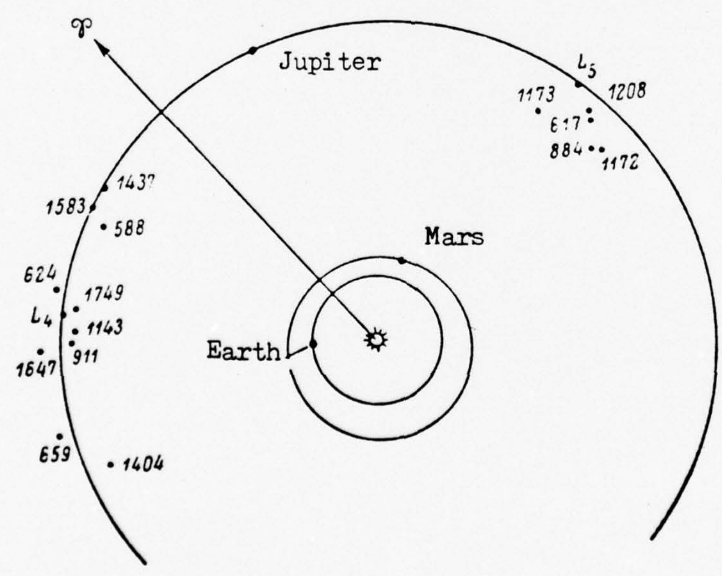

Fig. 3. Position of Trojans on 15 November 1950

solutions of this problem first obtained by Lagrange correspond to the position of an infinitely small mass at one of the libration points L_4 or L_5, located in the plane of motion of the finite masses and forming equilateral triangles with them. The Trojans are actually in real motion in the neighborhood of the libration point, with the asteroid tracing a certain trajectory relative to the equilibrium position. Very roughly, this trajectory can be represented as a highly elongated ellipse, with the semiminor axis pointing in the direction of the Sun. The difference between the heliocentric longitudes of the Trojan and the libration point undergoes temporal oscillations with a characteristic period of 150 years, upon which are superposed shorter-period oscillations. The amplitude of the oscillations of certain asteroids (Diomedes, Aneas) may reach, and even [66
exceed, 30°. However, as a result of the libration type motion, catalogued Trojans never approach Jupiter closer than 2.6 AU (Belyaev et. al, 1968).

The Hilda and Thule groups are also in motion in the vicinity of a stable periodic solution of the restricted three body problem. Theoretical considerations of the limited three body problem (Sun, Jupiter, asteroid) predict (Shwartzschild, 1898) the existence of a family of periodic solutions characterized by repetition of the mutual position of bodies, when the mean motions are commensurable. In the case when all three bodies are in motion in the same plane, symmetric periodic solutions have the property that the configuration when all three bodies lie along the same line coincides with the crossing of the body, having an infinitely small mass, of aphelion or perihelion of the orbit. Theory shows that at the 2:3 commensurability stable solutions are those for which conjunction (asteroid and Jupiter) occurs at the perihelion of the asteroid's orbit, and opposition, at aphelion (Fig. 4). In the case of the 3:4 commensurability, stable solutions are those for which conjunction and opposition occur at the asteroid crossing of the perihelion (Fig. 5).

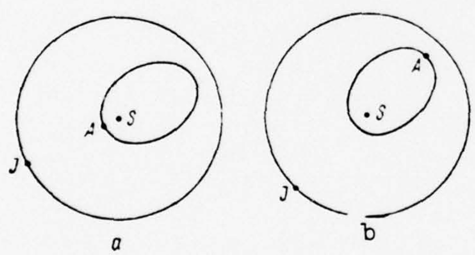

Fig. 4 Relative position of three bodies (Sun, Jupiter, asteroid) in conjunction (a) and in opposition (b) for the case of stable periodic motion at 2:3 commensurability

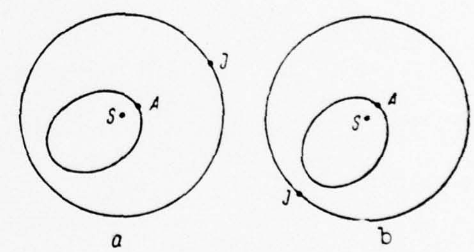

Fig. 5 Relative position of three bodies in conjunction (a) and in opposition (b) for the case of stable periodic motion at 3:4 commensurability

In investigating the problem of the closeness of asteroid orbit to the periodic solution it is convenient to consider the following quantity

$$\xi = p\lambda - q\lambda_1 - (p - q)\pi$$

where λ and λ_1 are the mean heliocentric longitudes of the asteroid and Jupiter, respectively and π is the perihelion longitude of the asteroid. The quantity ξ is known as the critical argument. In the case of a periodic solution, the critical argument is a periodic function with the period close to that of the q rotations of the asteroid. The mean value of the critical argument is either 0 or 180°, depending on the orbit point at which conjunction and opposition of the asteroid with Jupiter takes place. Near the stable periodic solution, the critical argument also undergoes long-period oscillations (libration) at frequencies proportional to $\mu^{\frac{1}{2}}$ and μ, where μ is the mass of Jupiter (corresponding periods of oscillation are several hundred and several thousand years). At small libration amplitudes, conjunction and opposition of bodies occur near the line of apsides of the asteroid and the stability of motion is not disturbed.

Schubart (1964) has shown that the critical argument of most planets in the Hilda group undergoes libration oscillations near zero with a period of approximately 250 years. This indicates closeness of the orbits of these planets to the stable periodic solution. For planets of this group, conjunction with Jupiter occurs near the perihelion (Fig. 4) and, since the orbit eccentricities are not low, close approaches of the planet to Jupiter does not occur. The minimal distance from Jupiter is observed in the case of (1269) Rollandia, however, it is still 1.4 AU. Two planets from the Hilda group--(334) Chicago and (1256) Normannia not undergoing libration have low eccentricities preventing their approach to Jupiter. Investigation of the motion of (279) Thule has shown that it is also controlled by the libration mechanism (Takenouchi, 1962).

The libration mechanism plays an important role in the accumulations of asteroids near the 1:1, 2:3, and 3:4 commensurabilities of mean motion of planets with Jupiter. The asteroids apparently can not remain in orbits approaching Jupiter's orbit for a long period of time when the libration mechanism by itself can not keep them at sufficiently long distances from this large planet. [67

Now let's consider the distribution of asteroids near commensurabilities of their mean motion with Jupiter at n > 500". Table 7 shows certain results of numerical integration of the equations of motion of ten asteroids from those being closest to the 2:5 commensurability (it should be noted that asteroids close to 2:5 commensurability are frequently called the Minerva type asteroids, named after the largest asteroids in this group). The table compares the extreme values of the mean motion,

Table 7

Boundaries of Changes of Asteroid Elements Close to Commensurability 2:5 (n_o = 747″8) During the Time Period 1660-2060

Asteroid	n min	n max	φ min	φ max	i min	i max	$n_m \pm n$	$n_m - n_o$
1597	737″1	739.8	4.7	5.3	11.8	11.9	738″4±1″4	−9″4
385	737.3	740.3	7.0	7.5	13.5	13.8	738.8±1.5	−9.0
430	636.1	741.6	14.5	15.4	14.5	14.7	739.4±2.3	−8.4
1570	738.9	740.4	3.2	3.6	1.5	1.7	739.6±0.8	−8.2
804	741.2	742.8	7.8	8.2	15.3	15.4	742.0±0.8	−5.8
975	742.5	744.2	1.7	2.2	2.5	2.7	743.4±0.9	−4.4
541	749.1	752.4	2.8	3.1	5.9	6.0	750.8±1.7	+3.0
673	750.4	751.9	0.5	1.0	2.7	2.9	751.2±0.8	+3.4
716	751.2	753.8	4.8	5.4	8.4	8.5	752.5±1.3	+4.7
403	752.2	754.1	5.5	6.0	9.0	9.2	753.2±1.0	+5.4

Note: $n_m = \frac{n_{min} + n_{max}}{2}$, $\Delta n = \left| n_m - n_{min,max} \right|$

Table 8

Evolution of (887) Allinda's Orbit

Epoch	n	$n - n_0$	φ	i	π	Ω	q	ξ	Δ_1
1660 II	910″1	+2″7	34°2	9°1	100°4	113°3	1.08	78°	+70°
1700 XII	916.6	+19.2	34.8	9.1	100.2	112.9	1.06	148	+88
1750 VI	910.5	+13.1	34.4	9.1	99.7	112.6	1.08	236	+33
1800 I	895.2	−2.2	33.2	9.0	99.9	112.1	1.13	269	−48
1850 VIII	882.5	−14.9	32.1	8.9	99.8	111.7	1.18	221	−90
1900 IV	879.7	−17.7	32.0	8.9	99.3	111.4	1.19	131	−73
1949 XII	889.1	−8.3	32.9	9.1	99.3	110.8	1.15	58	− 5
2000 I	906.1	+8.7	34.3	9.3	100.0	110.0	1.08	53	+91
2059 XI	916.1	+18.7	35.9	9.5	100.0	109.4	1.04	144	—

the angle of eccentricity, and the inclination of the orbit to the ecliptic during the time period between years 1660 and 2060. In this period, the mean motion of planets undergoes a number of short-period oscillations with amplitudes up to several seconds, however, the mean motion always remains on either one or the other side of $n_o = 747''.8$, corresponding to the exact commensurability.

The behavior of the majority of asteroids near the 1:3 commensurability, $n_o = 897''.4$ (Hestia type asteroids), does not differ qualitatively from that near the 2:5 commensurability. The only exception is (887) Allinda, the mean motion of which during the period investigated changes from one side of the exact commensurability $n_o = 897''.4$ to the other. Allinda has an orbit differing considerably from the orbits of the remaining Hestia type asteroids. Owing to its very high eccentricity, this planet at aphelion approaches closely the Jupiter orbit (Q = 3.9 AU). While at perihelion, it approaches the orbit of the Earth. The evolution of the orbit of Allinda during a period of 400 years is shown in Table 8. In addition to Kepler's orbital elements, the table also shows the perihelion distance q, critical argument $\xi = \lambda - 3\lambda_1 + 2\pi$, and its first difference Δ_1. While the critical argument of other asteroids near the 2:5 and 1:3 commensurabilities undergoes secular variation, Allinda is characterized [68
by libration of the argument near 161° with a period of approximately 360 years. The equilibrium value of the critical argument is sufficiently close to the theoretically predicted value of 180° for a stable periodic solution of the restricted three body problem near the 1:3 resonance. Among the Hecuba type asteroids (1:2 resonance), only (1362) Grikva is characterized by libration (Sinclair, 1969,; Schwezer, 1969). While the mechanism responsible for the accumulation of asteroids near the 1:1, 2:3 and 3:4 commensurabilities can be considered known, completely satisfactory theory for the formation of the gaps does not yet exist. Attempts to explain the existence of gaps were begun with the assumption that the motion of objects along orbits close to the resonance are unstable. Verification of this was seen in the fact that the series representing the motion of asteroids close to the resonance are characterized by the appearance of small devisors. However, expansions used in celestial mechanics do not converge and, therefore, can not be used to solve the problem of the stability of motion. Further investigations have shown that proximity to resonance not only does not disturb stability of motion but, just the opposite, increases it, provided perihelia of the perturbing and the perturbed body orbits are located in the proper manner (Hagihara, 1961).

According to Brown (1912), the existence of gaps can be explained as follows: the mean motion of asteroids while oscillating relative to its equilibrium position, on the average remains for considerably shorter time near the exact commensurability than away from it. However, Roy (1954) and Ovenden (1955) and then Schweizer (1969) have shown that such an explanation is invalid. Hirayama (1928 b) attempted to explain the appearance of gaps by the presence of a viscous medium in which motion of asteroids occurs. However, later on he refuted this assumption. Higihara (1961) expressed a completely new assumption that the gaps arise as a result of mutual perturbations of small planets during close approaches. However, this hypothesis has not been mathematically justified. Finally, Jefferys (1967) has shown that collsions between asteroids may play an important role in formation of gaps.

3. DISTRIBUTION OF ORBITS OF ASTEROIDS AS A FUNCTION OF ECCENTRICITIES AND INCLINATION TO THE PLANE OF THE ECLIPTIC.

Distribution of orbits as a function of eccentricities. Table 9 and Fig. 6 show the distribution of orbits of asteroids as a function of angle of eccentricity φ, where φ is defined through the relationship $\sin \varphi = e$. The arithmetical mean of φ is $8^{\circ}.7$ ($e = 0.15$) and the median is $8^{\circ}.2$. The maximum of the distribution function is in the range $6^{\circ} < \varphi < \sin 7^{\circ}$. The number of orbits decreases especially rapidly away from the maximum and toward lower eccentricities. Very noticeable is the almost complete absence of orbits with very low eccentricities ($\varphi < 1^{\circ}$). The number of orbits with "comet" eccentricities is also small. Only 11 asteroids (Table 10) have $\varphi > 25^{\circ}$ ($e > 0.42$).

The distribution of orbits of asteroids with angle φ reveals characteristic differences in different zones (Table 9, Fig. 7). The median of the distribution increases regularly with increasing distance of the zone from Jupiter. In zone A, nearest to Mars, the eccentricities of most orbits are also fairly low. These facts apparently indicate direct dependence of the stability of orbits on strong perturbations. The closer is the zone to Jupiter, the more unfavorable were the conditions in the zone during evolution for the disappearance of orbits with considerable high eccentricities. This conclusion is not contradicted by the presence of a group of orbits with eccentricities exceeding 0.15 ($\varphi > 8^{\circ}.6$) in zone B. Most of these orbits belong to the asteroids in the Hilda group. This group of orbits also includes the orbit of (1362) Grikva. As was shown in the previous paragraph, these planets maintain their stability only as a result of their proximity to resonance with Jupiter. In this specific case, high eccentricities only increase the stability of motion.

It should be noted that with the exception of Trojans and Hidalgo, not a single asteroid, the asteroidal nature of which can not be questionned, presently reaches the boundaries of Jupiter's orbit. A similar pattern is found at the inner boundary of the asteroid belt. The number of asteroids with low perihelion distances decreases rapidly upon approaching the boundary of the orbit of Mars ($q = 1.52$ AU, $Q = 1.67$ AU). While the number

Table 9

Distribution of Asteroid Orbits Over the Angle φ

φ	N'*	Zones				
		A	I	II	III	B
0–2°	56	2	4	12	32	6
2–4	202	3	25	48	111	15
4–6	291	4	57	68	146	16
6–8	312	3	77	78	135	19
8–10	294	2	96	84	100	12
10–12	270	0	93	91	77	9
12–14	174	1	47	69	51	6
14–16	95	1	27	45	22	0
16–18	52	0	13	26	12	1
18–20	20	1	2	13	2	2
20–22	12	0	4	6	2	0
22–24	3	1	0	2	0	0
24–26	4	2	0	2	0	0
26–28	2	0	0	2	0	0
28–30	2	0	1	1	0	0
30–32	0	0	0	0	0	0
32–34	3	0	0	3	0	0
≥34	2	1	0	0	0	1
0–56°	1794 *	21 *	446	550	690	87
Arithmetic mean	8°.7	11°.2	9°.4	9°.9	7°.3	7°.4
Median	8.2	6.6	9.3	9.6	6.7	6.4

*N' is the total numbered orbits
** (330) Adalberta asteroid is not included in the statistics

Table 10

Asteroids with Highest Eccentricities

Asteroid	ω	Ω	i	φ	n	a	q	Q
(1685) Toro	126°.6	274°.0	9°.4	25°.8	2218″.4	1.37	0.77	1.96
(1221) Amor	25.9	171.1	11.9	25.9	1333.0	1.92	1.08	2.76
(1009) Sirene	183.4	230.0	15.8	27.0	833.0	2.63	1.43	3.82
(1134) Kepler	330.4	6.4	15.0	27.8	807.6	2.68	1.43	3.93
(1474) Beira	82.4	324.9	26.8	29.3	785.4	2.73	1.39	4.07
(1580) Betulia	159.1	61.8	52.0	29.4	1090.6	2.20	1.12	3.27
(719) Albert	151.9	186.1	10.8	32.7	854.3	2.58	1.19	3.98
(1036) Ganymed	131.1	216.3	26.3	32.8	818.6	2.66	1.22	4.10
(887) Alinda	348.4	110.8	9.1	32.9	889.2	2.52	1.15	3.88
(944) Hidalgo	57.6	21.0	42.5	41.0	252.7	5.82	2.00	9.64
(1566) Icarus	30.9	87.8	23.0	55.8	3171.4	1.08	0.19	1.97

of asteroids with perihelion distances in range 1.7 to 1.8 AU is 74, their number in the range 1.6 to 1.7 AU is only 23, and in the range 1.5 to 1.6 AU it is only 9. Nevertheless, as a result of fairly high eccentricities, the [70
orbits of 41 asteroids lie partially within the orbit of Mars and three numbered asteroids are within the orbit of the Earth. The asteroids in such orbits are in a potentially unstable state. The latter is caused by the general nature of disturbances experienced by orbits of bodies in the Solar system; perihelia and nodes of orbits of asteroids undergo progressive changes, at rates which depend on the size of the semimajor axes. As a result of these changes, the orbits of asteroids periodically (with periods on the order of 10^3-10^5 years) cross the orbits of the major planets which are located at distances from the Sun intermediate between the perihelion and aphelion distances of the asteroid. Near the crossing points of orbits, an asteroid may come into contact with a planet or may experience strong perturbation as a result of which its orbit will undergo a radical change.

The mean lifetime of an asteroid in such an unstable orbit can be evaluated using probability methods (Opik, 1963, 1966; Weidenschilling, 1975). [72
According to Opik, the mean lifetime of a body in orbit crossing the orbit of Mars is several billion years. The lifetime of members of the Apollo group is on the average 10^8 years. For a body crossing the orbit of Jupiter, the lifetime is only 10^6 years. As a rule, the asteroid either impacts the surface of Jupiter or is ejected from the Solar system. It should be noted, that Opik's theory does not take into account the possibility of existence of mechanisms, such as libration, preventing approaches of bodies (Janiczek, et. al, 1972).

<u>2. Distribution of orbits as a function of inclination.</u> Distribution of orbits of asteroids as a function of inclination to the plane of the ecliptic is shown in Table 11 and in Fig. 8. The arithmetic mean of inclinations is 9.4°, the median is 8.4°, and the maximum of the distribution falls in the range $5° < i < 6°$. The orbits of asteroids "avoid" both very low and [73
very high inclinations to the plane of the ecliptic: only 28 asteroids have inclinations lower than 1° and only 10 asteroids have inclination of orbits higher than 30° (Table 12).

A comparison of inclination of orbits in different zones reveals interesting characteristics (Table 11, Fig. 9). While a sharp maximum in the distribution in zone 1 falls in the range 5 to 6° (median is 5°4), zone II is characterized by a more or less equal distribution of inclinations in the range 4 to 16° (median is 8°2), and finally, relatively high inclinations predominate in zone III (median is 9°8). Thus, one observes a gradual increase in inclinations upon transition from zone I to zone III, where the median in the distribution is almost two times larger than in zone I. This conclusion refers not only to the mean inclination of orbits, but also (and even to a larger degree) to the thickness of the asteroid belt in the direction perpendicular to the ecliptic. The latter is apparently connected with the fact that an increase in the mean inclination of orbits in transition from zone I to zone III is accompanied by an increase in the semimajor axes of orbits.

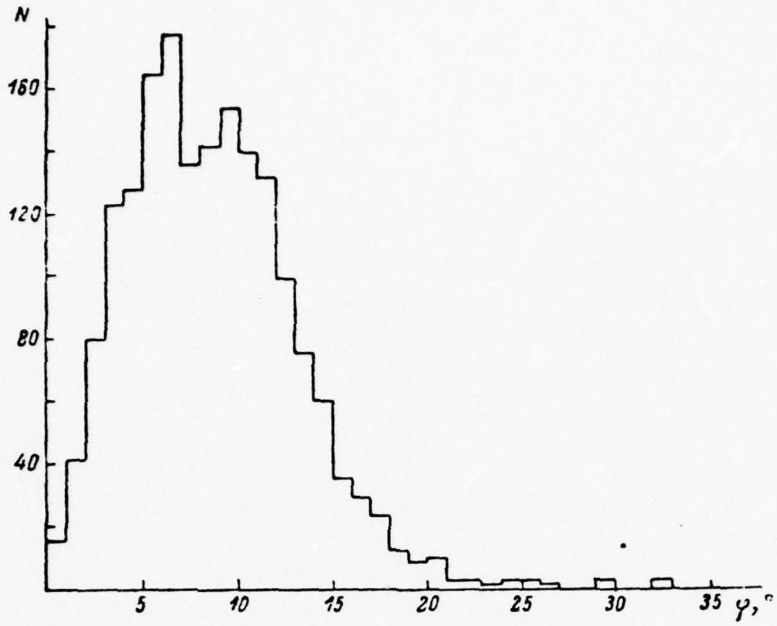

Fig. 6. Distribution of asteroid orbits as a function of angle φ

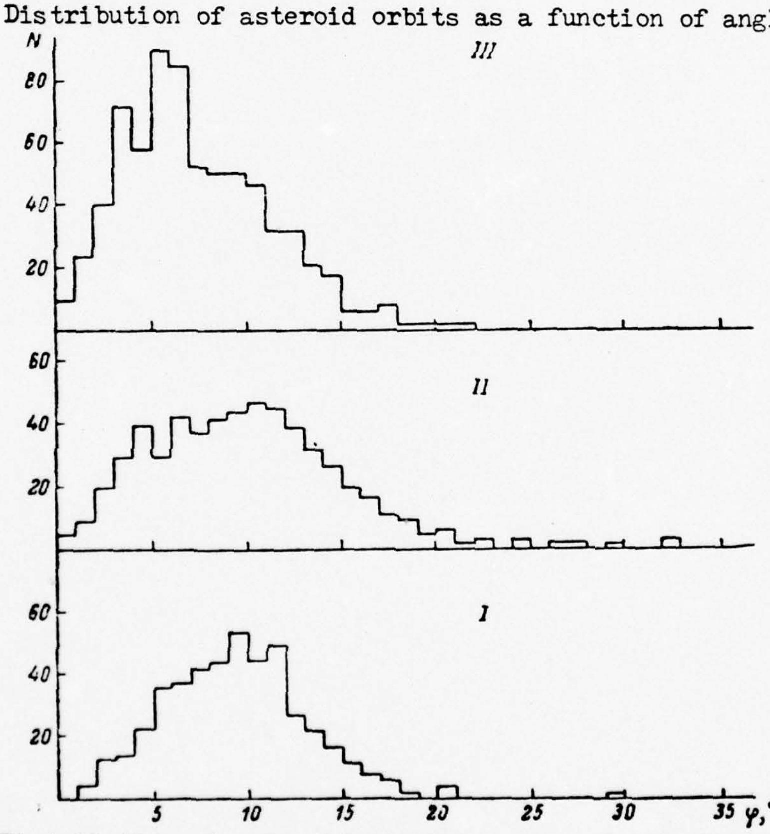

Fig. 7. Distribution of asteroid orbits in zones I-III as a function of angle φ

Table 11
Distribution of Asteroid Orbits as a Function of Inclination

i	N'	Zones				
		A	I	II	III	*B*
0–2°	110	1	27	10	68	4
2–4	244	0	90	46	96	12
4–6	283	0	152	75	42	14
6–8	218	0	69	82	55	12
8–10	224	2	35	82	95	10
10–12	206	2	23	68	107	6
12–14	147	2	8	68	66	3
14–16	130	0	7	61	59	3
16–18	58	1	0	16	39	2
18–20	53	2	1	17	27	6
20–22	39	2	8	4	18	7
22–24	28	6	11	3	6	2
24–26	29	2	13	6	7	1
26–28	11	2	1	3	4	1
28–30	5	0	0	3	1	1
30–32	0	0	0	0	0	0
32–34	5	0	0	4	0	1
34–36	2	0	0	2	0	0
⩾36	3	0	1	0	0	2
0–52°	1795	22	446	550	690	87
Arithmetic mean	9°4	18°6	7°1	10°3	9°7	11°0
median	8°4	21°2	5°4	8°2	9°8	8°4

Table 12
Asteroids with the Highest Inclination (i > 30°)

Asteroid	ω	Ω	i	φ	n	a
(582) Olympia	309°8	155°7	30°0	13°0	841"0	2.61
(594) Mireille	76.8	155.2	32.6	20.5	831.6	2.63
(945) Barcelona	160.7	318.1	32.9	9.3	827.4	2.64
(1208) Troilus	294.1	48.0	33.7	5.4	302.2	5.17
(1301) Ivonne	301.6	161.7	33.9	16.0	771.9	2.76
(1262) Celestia	63.0	140.8	33.9	11.7	802.7	2.69
(531) Zerlina	56.2	198.3	34.1	11.1	763.5	2.78
(2) Pallas	310.1	172.8	34.8	13.7	770.2	2.77
(1373) Cincinnati	99.1	298.1	38.9	18.8	563.2	3.41
(944) Hidalgo	57.6	21.0	42.5	41.0	252.7	5.82
(1580) Betulia	159.1	61.8	52.0	29.4	1090.6	2.20

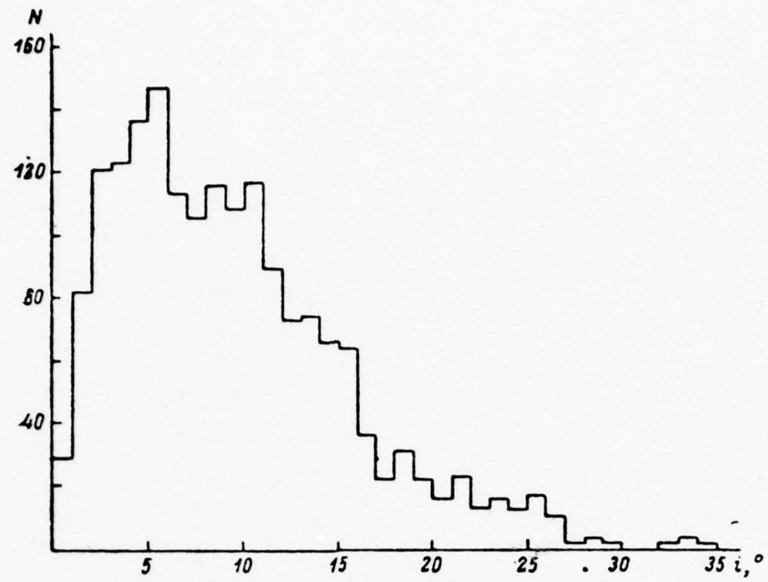

Fig. 8. Distribution of asteroid orbits as a function of inclination

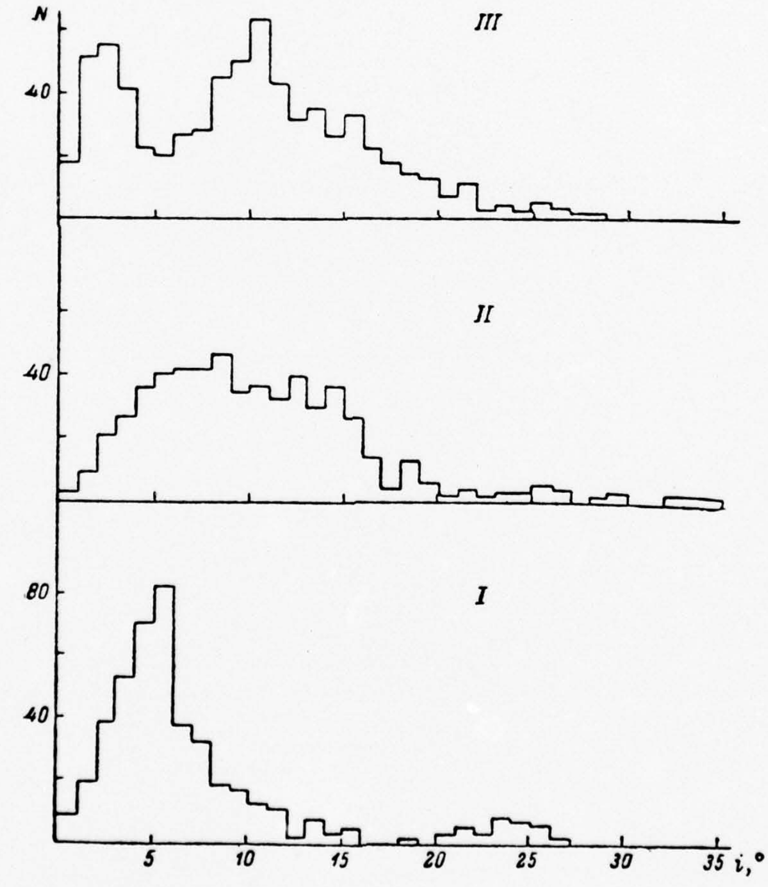

Fig. 9. Distribution of asteroid orbits in zones I-III as a function of inclination

A high inclination of orbits in zone A is an independent phenomenon. Most planets in this zone belong to the Hungaria group (Table 2). High inclinations of orbits for planets of this group apparently are connected with considerable secular perturbations in inclination experienced by asteroids with semimajor axes close to 1.9 AU (Charlier, 1927; Williams 1971). Considering all asteroids in the belt, rather than individual groups of asteroids, one can generally notice the presence of correlation between inclinations and eccentricities of orbits: on the average, high eccentricities correspond to high inclinations (Table 13). This correlation is even more noticeable in the case of high eccentricities and inclinations.

Table 13
Correlation Between Eccentricity and Inclination

φ	N	i	φ	N	i
0 - 5°	386	9°.0	20-25	17	17°.0
5 - 10	770	8.6	25-30	6	21.8
10 - 15	504	9.8	≥ 30	5	22.3
15 - 20	107	12.4			

4. DISTRIBUTION OF ORBITS OF ASTEROIDS WITH PERIHELION LONGITUDE AND LONGITUDE OF THE ASCENDING NODE.

The distribution of orbits of asteroids with perihelion longitude is given in Table 14 and is shown in Fig. 10. It displays a clearly expressed nonuniformity: the number of perihelia in the first and fourth quadrants is almost twice their number in the second and third quadrants. The highest density of perihelia are in the ranges 10 to 20° and 40 to 50°, the lowest density, in the ranges 170 to 180° and 200 to 210°. The smoothed distribution curve displays a maximum in the range 0 to 20° and a minimum in the range 180 to 200°. This distribution nonuniformity can be characterized as a concentration of perihelia of asteroid orbits near the perihelion of the orbit of Jupiter ($\pi_j = 13°$). The distributions of perihelia in the three principal zones (Fig. 11) do not differ considerably from each other.

The distribution of asteroid orbits as a function of the ascending node is also characterized by a nonuniformity which, however, is not as pronounced as in the case of perihelia (Table 15, Fig. 12). The most noticeable feature of the distribution is the presence of a minimum in the range 270-280°, corresponding to the descending node of the orbit of Jupiter ($\Omega_j = 100°$). With some minor variations, the distributions of nodes in the three principal zones (Fig. 13) are similar to the distribution throughout the whole belt. An explanation of the nonuniformity in the distribution

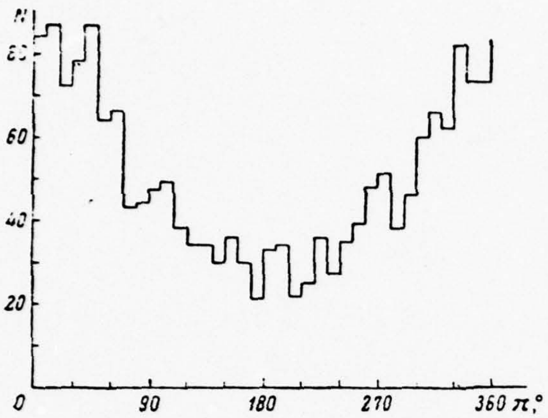

Fig. 10. Distribution of asteroid orbits as a function of perihelion longitude

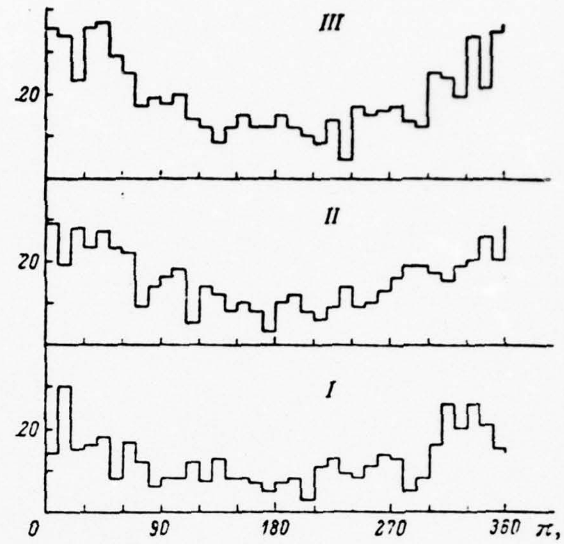

Fig. 11. Distribution of asteroid orbits in zones I-III as a function of perihelion longitude

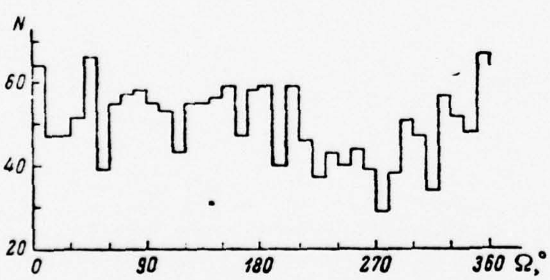

Fig. 12. Distribution of asteroid orbits as a function of nodal longitude

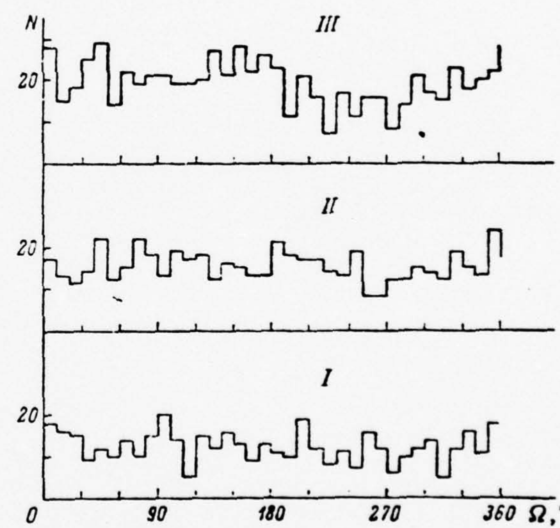

Fig. 13. Distribution of asteroid orbits in zones I-III as a function of nodal longitude

Table 14
Distribution of Asteroid Orbits as a Function of Perihelion Longitude

π	N°	Zones				
		A	I	II	III	B
0—30°	243	1	59	76	93	14
30—60	220	3	42	73	102	9
60—90	153	1	35	45	61	11
0—90°	625	5	136	194	256	34
90—120	134	3	28	39	52	12
120—150	98	1	28	34	32	3
150—180	87	1	20	21	39	6
90—180°	319	5	76	94	123	21
180—210	89	3	18	30	37	1
210—240	88	0	33	29	26	0
240—270	122	5	33	32	48	4
180—270°	299	8	84	91	111	5
270—300	135	2	26	54	42	11
300—330	188	0	62	51	68	7
330—360	228	1	62	66	90	9
270—360°	551	3	150	171	200	27

Table 15
Distribution of Asteroid Orbits as a Function of Nodal Longitude

Ω	N°	Zones				
		A	I	II	III	B
0—30°	158	2	49	41	61	5
30—60	156	0	31	48	68	9
60—90	170	2	39	55	62	12
0—90°	484	4	119	144	191	26
90—120	151	0	39	49	59	4
120—150	166	3	43	45	68	7
150—180	164	4	35	41	76	8
90—180°	481	7	117	135	203	19
180—210	158	0	40	56	55	7
210—240	126	2	31	44	40	9
240—270	123	2	35	35	43	8
180—270°	407	4	106	135	138	24
270—300	118	4	28	39	43	4
300—330	138	1	31	45	55	6
330—360	167	2	45	52	60	8
270—360°	423	7	104	136	158	18

of periehlia and nodes was given by Newcombe (1862) based on the theory of secular perturbations of planets (see section 6).

It follows from this theory that under the influence of attraction by the large planets, the mean orbital elements of asteroids undergo long-period perturbations with characteristic periods of several tens of thousands of years. The results of integration of equations determining the perturbed values of the eccentricity and perihelion longitude can be represented in the following form:

$$e \cos \pi = A \cos \pi_1 + p \cos \alpha,$$
$$e \sin \pi = A \sin \pi_1 + p \sin \alpha. \qquad (2)$$

The perturbed values of inclination and of the ascending node longitude are determined by analogous relations [74

$$\sin i \cos \Omega = B \cos \theta_1 + q \cos \beta$$
$$\sin i \sin \Omega = B \sin \theta_1 + q \sin \beta \qquad (3)$$

In expressions (2) and (3), A and B are the new constants of integration known as the proper eccentricity and proper inclination of the asteroid's orbit. The proper perihelion longitude (π_1) increases, while the proper longitude of the node (θ_1) decreases at a constant rate which is identical for both quantities. The period of revolution of π_1 and θ_1 and quantities p, q, α and β depend on the semimajor axis of the asteroid. The period of revolution and the quantities $p \cos \alpha$, $p \sin \alpha$, $q \cos \beta$, $q \sin \beta$ for several values of the semimajor axis are given in Table 16. The relationships (2) may be interpreted as projections of the vector equality $\overline{OP} = \overline{OO'} + \overline{O'P}$ on the coordinate axes, with $|\overline{OO'}| = p$, $|\overline{O'P}| = A$, $|\overline{OP}| = e$ (Fig. 14).

From the previous it follows that vector $\overline{O'P}$ makes one revolution around point O' during approximately 4000 to 40,000 years. (The time required depends on the value of the semimajor axis a) In this time interval, the eccentricity e undergoes an oscillation with the amplitude depending on [75
the distance of the point O' from the center of the coordinates. The perihelion of the orbit either increases constantly, if the proper eccentricity A exceeds p, or oscillates within more-or-less narrow limits, if A is smaller than p. Since p and α do not remain constant (in Table 16

Table 16

a	Period of Rotation	p cos α	p sin α	q cos β	q sin β
2.15	41 400	+0.0567	-0.0363	+0.0108	+0.0031
2.60	26 300	+0.0302	-0.0056	-0.0006	+0.0181
3.15	14 400	+0.0374	+0.0058	-0.0029	+0.0210
4.00	4 400	+0.0421	+0.0093	-0.0038	+0.0222

the quantities are given for the present epoch), the point O' is slowly displaced, tracing a certain curve with a proper period of abous 300,000 years, about the initial coordinate point. As can be seen from Table 16, position of points O' for different asteroids in the present epoch are concentrated near the horizontal axis (from which the longitudes are counted) at distances close to 0.04, to the right of the coordinate origin. The fact that the center of concentration of points O' is in the direction of Jupiter's perihelion is not a random phenomenon because the principal contribution to the components of vectors OO' is made by quantities $e_j \cos \pi_j$ and $e_j \sin \pi_j$ of Jupiter.

It can be expected that for different asteroids the position of points P has to have central symmetry relative to the center of concentration of points O'. The possible random deviations from the symmetry can only be temporary because they will disappear due to differences in the periods of rotation of points P. The position of points P has no symmetry with respect to the center O. As a result of this, concentration of perihelia should occur in the direction toward the center of concentration of points O'. Taking into account the present position of the center, it is to be expected that the distribution of perihelia will reach [76 a maximum in the vicinity of 0° and a minimum, in the vicinity of 180°.

The relationship (3) can be interpreted in a completely analogous manner (Fig. 15). As can be seen from Table 16, in most cases points O' corresponding to asteroids with different axes fall near the vertical axis, above the origin of the coordinate system and at a distance of about 0.02 from it. As a result of this, the center of symmetry of points P lies in the direction close to $\Omega = 90°$. From this it follows directely that the distribution of nodes of the asteroids has to have a maximum near $\Omega = 90°$ and a minimum near $\Omega = 270°$. Because the modulus of the vector $\overline{OO'}$ is smaller and the angular velocity of points O', determined by the rate of motion of the node of the orbit of Jupiter, at this point is approximately six times greater, it can be expected that the maximum and the minimum will not be as clearly defined as they are in the case of perihelia.

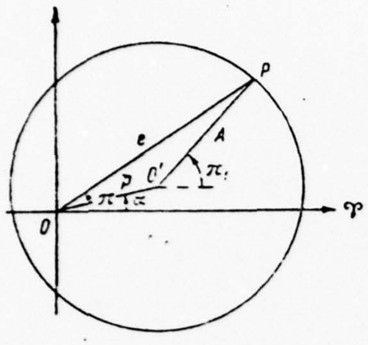

Fig. 14

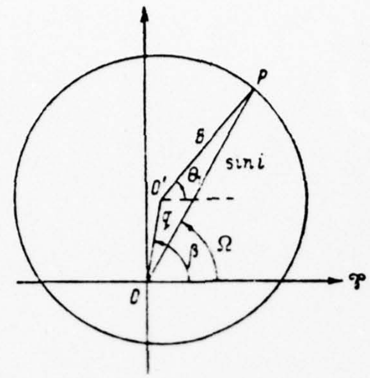

Fig. 15

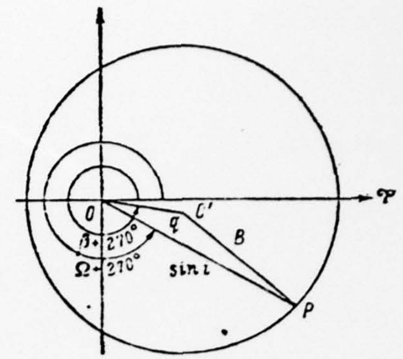

Fig. 16

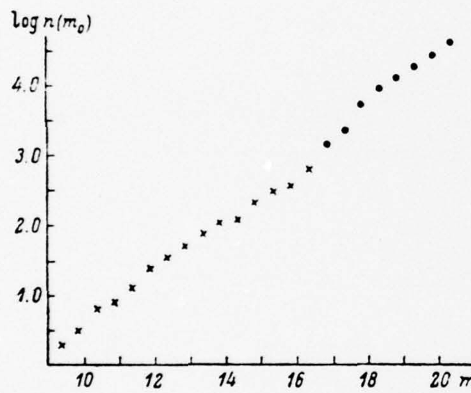

Fig. 17. Logarithm of the number of asteroids in the range $m_o - \frac{1}{4}$ -- $m_o + \frac{1}{4}$ as a function of mean opposition stellar magnitude m_o

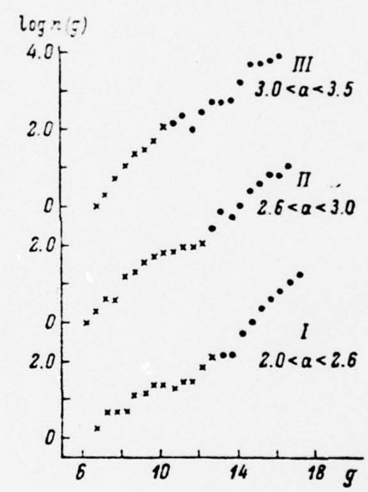

Fig. 18. Logarithm of number of asteroids in the range $g - \frac{1}{4}$ -- $g + \frac{1}{4}$ as a function of absolute stellar magnitude g in three zones

Therefore, the theory of secular perturbations qualitatively accounts for the characteristics of the distribution of perihelia and nodes of minor planets. However, the qualitative agreement between the observed and theoretical distributions, where the latter can be obtained using the theory described above and the experimentally found distributions of eccentricities and inclinations,is still being questionned. The difference between two distributions should be revealed by different percent distribution of perihelia or nodes in different quadrants, or smaller sectors of the circumference. The reason for such differences, provided they are more than just random fluctuations, is probably due to 1. deficiencies in the presently used theory of secular perturbations; 2. the influence of some perturbing forces, apparently, of nongravitational nature; or, finally, 3. in that the empirically derived distributions of elements do not reflect with sufficient accuracy the actual distribution (observational selection).

The theory of secular perturbations described above is based on simplified assumptions. The relationships (2) and (3) were obtained by integrating equations in which terms higher than second order in eccentricities and inclinations were neglected. Unquestionnable application of this theory to minor planets, the eccentricities and inclinations of which are generally not low, may be unjustified. On the other hand, undoubtedly, the distributions obtained by statistical processing of elements of numbered asteroids are burdened by the observational selection. Presently, almost all asteroids with mean opposition magnitudes $m_o < 15$, have been discovered and catalogued. Their number is only slightly less than 750, which is approximately 95 % of the total number of such asteroids [77
in the belt. In the range $15 < m_o < 16$, the asteroid catalogue is quite incomplete. Only a fraction of asteroids with $m_o \geq 16$ are known. This fraction was discovered under conditions favorable to their observation. In the case of faint asteroids close to the limiting stellar magnitude, which can still be observed with instruments, such favorable conditions occur when opposition coincides with the time when the asteroid crosses the perihelion. If the orbit of the asteroid is sufficiently elliptical, its apparent stellar magnitude may be $0^m.5$ to 2^m lower than the stellar mean opposition magnitude. This effect causes a larger percentage of asteroids with high eccentricities to be observed among numbered asteroids with $m_o \geq 16$ (Kiang, 1966a).

Another important factor responsible for the observational selection is the usual practice of searching for asteroids in regions of low ecliptic latitudes. Owing to this factor, the observed asteroids are those with low orbit inclinations which enhances their discovery. This effect Kiang (1966a) has shown that this factor is responsible for systematic deviation in the distribution of inclinations of orbits of asteroids with $m_o \geq 15$ toward smaller inclinations.

The season factor in the observational effect is associated with geographic position and climatic conditions at principal observatories conducting systematic searches for planets. A large number of these observatories are located in the northern hemisphere, at latitudes 40 to 55°. In these observatories, the most favorable observation periods are those near the autumnal and vernal equinoxes. A month by month plot of discovery of asteroids shows sinusoidal behavior with two deep minima in December-January and June-July and two maxima: a higher peak in August-September and a somewhat lower peak in February-March. This indicates that a very meticulous search for asteroids was conducted in the region of the ecliptical belt near 0° and 180° longitude. Combination of this selection effect with the previously described observational selection effect is responsible for favorable conditions for 1. discovery of asteroids with perihelion longitudes differing little from 0 or 180° and, 2. discovery of asteroids having ascending or descending nodes near the same longitudes. These circumstances definitely influence distribution functions of asteroids as a function of the perihelion longitude and the ascending node longitude in the range of variation of m_o, within the limits of which statistics of asteroids is insufficiently complete. In the distribution of perihelia of numbered asteroids, the selection effect leads to a considerable increase in the concentration of perihelia due to secular perturbations. The influence of selection on the distribution of nodes of numbered asteroids is to camouflage the concentration of ascending nodes near $\Omega = 90°$ due to secular perturbations and, just the opposite, to emphasize the decrease in nodes near $\Omega = 270°$ (see Fig. 12). The influence of selection on the distribution of elements was considered in detail by Kiang. In his opinion, even the distribution of perihelia with the selection effect removed, does not exactly correspond to the accepted theory of secular perturbations.

The existence of preferred orientation of orbits of individual asteroids can be considered to be due to the influence of ellipticity of the asteroid belt as a whole. Analogously, the nonuniform distribution of nodes as a function of longitude indicates that the mean plane of the belt does not coincide with the plane of the ecliptic. Such general characteristics of the belt as eccentricity, position of the perihelion, ascending node longitude, and inclination of the mean plane to the ecliptic are determined from the orbital elements of individual asteroids. We will describe how this can be done. Let's refer to Fig. 15 and rotate it 270° around point O, leaving the coordinate axis unchanged (Fig. 16). The points O, O', and P in this figure can be considered projections of certain points of heliocentric celestial spheres onto the plane of the ecliptic. Point O can be considered a projection of the pole of the ecliptic. Point P is the projection of the pole P_1 of the plane with the ascending node longitude Ω and inclination i. Finally, point O' is the projection of the pole O'_1 of the plane with the ascending node longitude β and inclination $\psi = \arcsin q$. Rotation of point P in the plane of the ecliptic around O corresponds to the displacement of P_1 around O'_1 along the celestial sphere.

In view of the proximity of O'_1 to the plane of the ecliptic, the motion of pole P_1 differs very little from a uniform rotation along the cirumference centered at O'_1. Having established this, in order to avoid unnecessary complications we will not differentiate between poles and their projections.

Therefore, the orbit of an asteroid maintains constant inclination to a certain plane, the pole of which O' is slowly displaced relative to the pole of the ecliptic. The position of pole O' is a function of time and the semimajor axis of the asteroid. In our epoch, the center of concentration O* of the poles O', corresponding to different values of semimajor axis, lies near the axis $O\gamma$, at a distance of approximately 0.2 to [78
the right of the origin of the coordinate system. The poles of the asteroid orbits are located more or less symmetric relative to O*, so that if all the poles are given a unit weight, the center of gravity will be close to O*.

The plane, the pole of which coincides with the center of gravity of poles P, can be naturally considered to be the mean plane of the asteroid belt. Therefore, position of the pole of the mean plane is determined by the coordinates

$$X = 1/n \sum_{k=1}^{n} \sin i_k \cos (\Omega_k + 270^\circ),$$

$$Y = 1/n \sum_{k=1}^{n} \sin i_k \sin (\Omega_k + 270^\circ).$$

Summation is over all asteroids for which selection is not important. Limiting our case to $n_o < 15$, we find $X = +0.0206$, $Y = +0.0030$. From this it follows that the mean plane is inclined to the ecliptic at an angle $i = 1\overset{\circ}{.}2$ and the longitude of its descending node $\Omega = 98\overset{\circ}{.}2$.

Analogous considerations lead to conclusions that eccentricity e and position of the perihelion π of the asteroid belt, considered as an entity, are determined by the position of the center of gravity of points P of individual asteroids (Fig. 14).

$$X = 1/n \sum_{k=1}^{n} e_k \cos \pi_k$$

$$Y = 1/n \sum_{k=1}^{n} e_k \sin \pi_k.$$

Summing over all numbered asteroids with $m_0 <$ sin 15^m, we find $\tilde{e} = 0.047$ and $\pi = 9\overset{\circ}{.}2$. As was to be expected, the eccentricity, inclination, perihelion longitude, and the ascending node longitude of the asteroid belt considered to be a single entity are close to the corresponding values for Jupiter. Similar results using different methods were obtained by a number of authors (Plummer, 1916; Oppenheim, 1924; Kresak, 1967; Duriez, 1971).

Let's consider in somewhat more detail certain manifestations of ellipticity of the asteroid belt. Let $r(\lambda)$ be the heliocentric distance of the asteroid with longitude λ. Following Kresak (1966) let's consider the quantity

$$\rho(\lambda) = \frac{r(\lambda)}{r(\lambda) + r(\lambda + 180^\circ)} .$$

Neglecting inclination of the asteroid orbit, the true anomaly $\nu = \lambda - \pi$ and

$$\rho(\lambda) = \frac{1 - e \cos (\lambda - \pi)}{2} .$$

The mean value of $r(\lambda)$ for the asteroids

$$\bar{\rho} = 1/n \sum_{k=1}^{n} {}_k$$

can be considered as a convenient measure of radial asymmetry of the belt in the direction of λ. It is easy to see that

$$\bar{\rho} = \frac{1 - e \cos (\lambda - \pi)}{2} .$$

It is obvious that $\bar{\rho}$ assumes the lowest value, equal to 0.48 when $\lambda = \tilde{\pi}$ and the largest value, when $\lambda = \pi + 180^\circ$. Therefore, on the average, in the direction of perihelion of the belt, the asteroid orbits approach the Sun considerably closer than in the opposite direction. From the law of areas, it follows that the regions of belt in the direction of Jupiter's [79
aphelion are populated more densely than the regions adjacent to the perihelion. The mean angular velocity of asteroids at the aphelion of the belt is 0.83 of the mean angular velocity at the perihelion.

As a result of the inclination of the mean plane of the belt to the ecliptic, the aphelion part of the belt is located above the plane of the ecliptic, while the perihelion part is above it. In view of the above discussion, the number of asteroids above the plane of the ecliptic should on the average exceed their number under the plane. Calculations made by Nairn (1966) for the first 1563 asteroids in the time period 1965 to 1995 have revealed just the opposite effect; on the average, more than half of the asteroids are located under the plane of the ecliptic. However, as was shown by Kresak (1967), the result obtained by Nairn is a consequence of observational selection. The geographic selection effect is responsible for the favorable conditions for the discovery of those asteroids, the perihelia of orbits of which are located north of the ecliptic. At the present time, this result in a noticeable predominance of asteroids with similar orbits among asteroids with $m_0 \geq 16$. This selection effect masks the actual asymmetry of the belt due to gravitational forces.

5. DISTRIBUTION OF ASTEROIDS AS A FUNCTION OF STELLAR MAGNITUDES, RADII, AND MASSES

The most complete data on the distribution of asteroids as a function of stellar magnitude (frequency-magnitude relation) were obtained from two systematic surveys initiated by Kuiper and undertaken during 1950 to 1952 and in 1961. The first survey conducted by members of the McDonald and Yerkes Observatories (Kuiper, et. al., 1958) included asteroids up to stellar magnitude 16.0 within a 40°-wide ecliptic belt. A considerable part of the ecliptic belt was photographed twice. More than 1100 numbered asteroids and a large number of unnumbered asteroids were found on plates.

The second survey initially intended to expand the earlier survey to lower stellar magnitudes, was conducted jointly by three organizations: Lunar and Planetary Laboratory in Tucson, Arizona; Leiden Observatory; and Cincinatti Observatory(van Houten et al.,1970). The 126-cm Schmidt camera at the Palomar Mountain Observatory was used in obtaining 130 plates of a sector of the sky with an area of 18° x 12°, near the vernal equinox. A relationship between the two sectors was established by multiplying the number of asteroids found, by the ratio of the areas covered in the two investigations, taking into account the decrease in the number of objects with increasing distance from the ecliptic.

As a result of the McDonald survey, stellar magnitudes of all asteroids were for the first time reduced to a single photometric scale- the International Photographic system. It was established that logarithms of a number of asteroids of a specific stellar magnitude (magnitude distribution) increases linearly with increasing stellar magnitude. If m_0

is the photographic stellar mean opposition magnitude* and $n(m_0)$ is the total number of asteroids in the belt between limits $m_0 - \frac{1}{2}$ and $m_0 + \frac{1}{2}$,

$$\log n\ (m_0) = -\ 2.38 + 0.35m_0. \tag{4}$$

In determining parameters of formula (4), a correction for the incompleteness of statistics was introduced into the actually observed number of asteroids.

From the number of traces of faint asteroids up to 19^m on four maps of the Palomer atlas of the sky and critical processing of data acquired by other investigators, Kiang (1962) obtained an analogous relationship:

$$\log n(m_0) = -\ 2.69 + 0.375m_0$$

with similar parameter values. The Palomer Leiden survey established that the logarithmic increase in the number of asteroids with increasing stellar magnitude can be extended to $20^m.5$. The results of the two surveys are shown in Fig. 17. The crosses and points in this figure give the logarithm of the number of asteroids plotted in steps of $\frac{1}{2}$ stellar magnitude. The following values were selected to be centers of intervals

$$m_0 = 9.35 + 0.5\ k\ (k = 0,\ 1,\ \ldots 22).$$

The relationship between the mean opposition magnitude m_0 and the number of asteroids between limits $m_0 - \frac{1}{4}$ and $m_0 + \frac{1}{4}$ is adequately described by the following formula: [80

$$\log n(m_0) = -\ 3.30 + 0.390\ m_0. \tag{5}$$

The greatest deviation from the straight line (5) takes place in the range $13 < m_0 < 17^m.5$, where the formulas give higher values for the number of asteroids (almost two times higher at $m_0 = 16$). The total number of asteroids in the belt up to $m_0 = 20^m.6$ is, according to (5), approximately 120,000.

The mean opposition magnitude of the asteroid depends on its size, reflecting properties of the surface (albedo), and the semimajor axis of its orbit. As a result of the difference in orbits of asteroids, the

*i.e. the stellar magnitude of the asteroid at a distance a from the Sun, a - 1 from the Earth and at a zero phase angle.

possible variations in the distributions of asteroids with respect to their radii and masses are smoothed out when changing to distributions in mean stellar magnitude. Therefore, it is of considerable interest to investigate the statistics of absolute stellar magnitudes g, i.e. stellar magnitudes reduced to unit distances from the Sun and the Earth. [81

The distribution of numbered asteroids with respect to the absolute stellar magnitude (magnitude distribution) is shown in Table 17. As a

Table 17
Distribution of Asteroids as a Function of Stellar Magnitude

B(1,0)*	Photometric radius, km**	N	Zones				
			A	I	II	III	B
4.0—5.0	300—190	2	0	1	1	0	0
5.0—6.0	190—120	1	0	0	1	0	0
6.0—7.0	120—76	6	0	22	2	2	0
7.0—8.0	76—48	19	0	5	4	9	1
8.0—9.0	48—30	82	0	13	32	20	8
9.0—10.0	30—19	183	0	16	71	71	25
10.0—11.0	19—12	329	0	31	99	170	29
11.0—12.0	12—7.6	406	0	32	109	244	21
12.0—13.0	7.6—4.8	370	2	79	144	143	2
13.0—14.0	4.8—3.0	261	5	164	71	21	0
14.0—15.0	3.0—1.9	113	10	91	10	1	1
⩾15.0	⩽1.9	23	5	12	6	0	0

*B(1,0) is the absolute stellar magnitude of an asteroid in the B photometric system (blue stellar magnitudes in the UBV system) published annually in Ephemerides of Minor Planets sinces 1971. With a sufficient accuracy, the relationship between the stellar magnitudes of asteroids in the B system and the International Photographic System is given by B = m + 0.10, B(1,0) = q + 0.1 (Gehrels, 1970)
**Asteroid radii, corresponding to the limiting values of the absolute stellar magntidues, were calculated from formula (9)

result of the differences in distances of the zones from the Sun, the incomplete statistics begins to be revealed in different zones at different values of absolute magnitudes. Nevertheless, the difference in the rate of increase in the number of asteroids is clearly noticeable. The distribution of asteroids as a function of absolute magnitudes in different zones was a subject of a special investigation both in the McDonald and Palomar-Leiden surveys. In both surveys, statistics were referred to the following zones: $2.0 < a < 2.6$, $2.6 < a < 3.0$, $3.0 < a < 3.6$, which do not differ considerably from the three central zones accepted in this paper. The McDonald survey has shown that considerable differences exist in the three zones in the distributions as a function of the absolute magnitude in the three zones (Fig.18). As can be seen from this figure, the frequency-magnitude curves for all three zones consists of linear sectors separated by a point of inflection in the range $10 < g < 12$. The rates of increase in the number of asteroids with increasing g in the three zones are quite different. In the range of small absolute magnitudes

$$\log n(g) = c + bg, \qquad (6)$$

where n(g) is the number of asteroids between limits $g - \frac{1}{4}$ to $g + \frac{1}{4}$. For the three zones, b = 0.33, 0.56, and 0.75. The data from the McDonald survey show that for the zone most distant from the Sun in the range of large absolute magnitudes, b is very large (b = 0.73). However, the Palomer-Leiden survey shows that the rapid increase in the number of asteroids in the outer zone is the result of incomplete statistics. In reality, in the range of large absolute magnitudes the rate of increase in the logarithm of the number of asteroids is approximately the same for both the central and the outer zones, but is somewhat larger for the inner zone. The value of b averaged over the three zones is close to 0.4 ($6 < g < 17$).

According to the authors of the McDonald survey, the point of inflection in the magnitude distribution curve noticeable for all three zones may separate asteroids produced by accretion from asteroids which are products of collisional fragmentation. This concept was further developed by Anders (1955) who has reconstructed the initial mass distribution of asteroids in the zone $2.16 < a < 2.6$ AU and also by Hartmann and Hartmann (1968) and Kiang (1966b).

Let's establish the connection between the absolute stellar magnitude of an asteroid and its radius. The following formula follows from the definition of the geometrical albedo (Vokuler, 1974)

$$\log p = 0.4 (m_o - j) - 2 \log \sigma, \qquad (7)$$

where m_o is the stellar magnitude of the Sun, g is the absolute stellar magnitude of the planet, and σ is the angular radius of the planet at a unit distance. Assuming that in the UBV system, m_o has the value V(1,0) = 26.77, b - V = + 0.63, and expressing the radius r in kilometers, we find

$$\log r = 2.95 - 0.5 \log p - 0.2\, B(1, 0). \qquad (8)$$

The relationship (8) can be used to calculate the radii of asteroids, provided the albedo is known, or for the solution of the inverse problem, provided the angular diameters of asteroids are known.

Three principal methods (micrometric, radiometric, and polarimetric) are presently used for the determination of the albedo and diameters of the asteroid. The filar micrometer measurements were used to determine diameters of only the first four asteroids (Barnard, 1902; Dolfus, 1974). The corresponding values of albedo lie in the range between 0.11 (Pallas) and 0.40 (Vesta) (Gehrels, 1974). The mean value is equal to 0.21, with not more than three times the error on either side. Using this value of albedo one finds

$$\log r = 3.28 \pm 0.24 - 0.2\, B(1.0) \qquad (9)$$

Assuming that the mean density of asteroids is 3.5 gm/cm^3, which is close to the density of stellar meteorites, the mass of asteroids (in grams) can be expressed in terms of the absolute magnitude through the following relationship

$$\log m = 26.1 - 0.6\, B\,(1,\ 0) \qquad (10)$$

The error in the log value calculated from formula (10) does not exceed unity.

The radiometric method based on measurements of the heat flux from the asteroid and the polarimetric method based on the dependence of albedo of a body on polarization of light reflected by it began to be used only recently (Veverka and Noland, 1973; Jones and Morrison, 1974; Zellner, Gehrels and Grady, 1974). Both methods, but especially the second one, can be applied to a large number of objects with a high degree of accuracy. It is interesting to note that the diameters of the minor planets determined using these methods are systematically (approximately 1.2 to 1.5 times) higher than the values found from micrometric measurements (Table 18). [82

Table 18
Diameters of First Four Asteroids

Asteroid	Diameter, km M*	P**	R***	P****
(1) Ceres	770	1050	1040	580
(2) Pallas	490	570	570	350
(3) Juno	195	222	254	200
(4) Vesta	390	490	528	530

*Micrometric values, from Bernard
**Polarimetric values, from Zellner, Gehrels, and Gradie (1974)
***Radiometric values, from Jones and Morrison (1974)
****Calculated from formula (9)

Formulas (9) and (10) can be utilized to find the distribution functions of asteroids as a function of radii and masses. Letting $n(g)$ denote the number of asteroids per unit range of measurement of the absolute magnitude and defining $n(r)$ and $n(m)$ in an analogous manner

$$n(g)dg = n(r)dr = n(m)\,dm. \tag{11}$$

Taking into account that (9) can be represented in the form

$$g = \text{const} - 5 \log r,$$

then from (6) and (11) it follows that

$$n(r) = c_1 r^{-\alpha}, \tag{12}$$

$$n(m) = c_2 m^{-\beta}, \tag{13}$$

where the exponential factors α and β are connected with the coefficient B in formula (6) by means of the relationship

$$\alpha = 5b + 1, \qquad \beta = 5/3\, b + 1.$$

Substituting the value $b = 0.4$ we find that $\alpha = 3$ and $\beta = 5/3$.

The results of processing of the McDonald and Palomar-Leiden surveys by Dochnanyi (1969, 1972) have shown that the mass distribution of asteroids both in the range of small absolute magnitudes $(g < 11)$ and in the range of large absolute magnitudes $(g > 13)$ is of the form given by (13) with the exponential factor β in both cases close to 1.8. However, the constant c_2 was found to be considerably larger for large asteroids. A gradual transition from one distribution to another occurs in the range $11 < g < 13$, apparently reflecting the influence of the point of inflection in the distribution curve for the absolute asteroid magnitudes. Thus, the distribution functions for visible asteroids with respect to radii and masses are power functions with exponential factors being close to between - 3 and -3.5 and - 5/3 and - 11/16, respectively. Similar distributions are also found for systems of other bodies: meteors, meteorites, and craters on surfaces of celestial bodies. This seems to indicate the existence of an internal force between these objects. In order to identify this force, it is of considerable interest to determine whether the distribution of asteroids obeys a power law outside the boundaries of objects known to us. The answer to this question can be found

only when the causes responsible for this or that distribution are known. Presently, most of the specialists are of the opinion that the observed distribution of bodies in the belt is the result of collisional fragmentation of asteroids and asteroids with bodies having dimensions of meteorites. A number of authors have constructed physical and mathematical models of this process, which more or less completely take into account the various aspects of the process (Piotrowsky, 1953; Jones, 1968, Safronov, 1969; Dohnanyi, 1969, 1970; Hellyer, 1970, 1971). The distribution of bodies with respect to masses (or radii) is a particular solution of an integro-differential equation describing behavior of the system. The distributions turn out to be somewhat different for large masses near their upper boundary and for small masses, even though in both cases the solution is a power function. For example, solutions for different ranges of variation of masses may have different values of the exponential factor β in formula (13), which for large masses is 5/3, while for small masses it is close to 1.8 [83
(Hellyer, 1970). In the Dohnanyi model (Dohnanyi, 1970), β has the same values for all masses ($\beta = 11/6$). However, for the largest asteroids the distribution function contains an additional factor which approaches unity with decreasing mass. It is shown, that the asteroid population could have reached such a state during a sufficiently long period of time, independently of its initial distribution.

Thus, the models constructed are in good qualitative and quantitative agreement with the observational data and, consequently, reflect with sufficient accuracy the state reached in the asteroid belt during its evolution. The models considered make it possible to reach one more important conclusion concerning the mass distribution in the belt. It is obvious that the total mass of all bodies the masses of which are between limits m and m + dm, is equal to $mn(m)dm$. Substituting $n(m) = cm^{-\beta}$ and integrating gives the total mass between m_1 and m_2 ($m_2 > m_1$):

$$M = \frac{c}{2 - \beta} \left(m_2^{2 - \beta} - m_1^{2 - \beta}\right).$$

Since in the asteroid belt β is less than two, the first term predominates and, subsequently, most of the mass is concentrated in the largest bodies. This characteristic of the distribution of the total mass between individual members of the asteroid belt, which follows from the power law nature of the distribution at $\beta < 2$, is even more distinctly expressed because three of the largest asteroids (Ceres, Pallas, and Vesta) have excess mass in comparison with the smoothed out distribution (Kuiper et al., 1958). Schubart (1971b, 1974) estimates the mass of Sorora to be $(5.9 \pm 0.3) \times 10^{-10}$ $= 1.2 \times 10^{24}$ gm. This was obtained from analysis of the perturbing influence of Sorora on the motion of Pallas. Hertz (1968) determined the mass of Vesta from analysis of the perturbations of (197) Arete to be $(1.2 \pm 0.8) \times 10^{-10}$ M = 0.24×10^{24} gm.

Assuming that the diameters of Sorora and Vesta are equal to the values determined by Bernard (1902), their densities are equal to 4.9gm/cm^3 and 7.7 gm/cm^3, respectively. These values, particularly the second one, are considerably higher than the density of stony meteorites, frequently considered to be the probable density of asteroids. Without a doubt, the given values of density can have considerable relative errors primarily due to determination of diameters. Assuming that the diameters of Sorora and Vesta are equal to those determined by polarimetric or radiometric methods (Table 18), the densities turn out to be 2.3 to 2.5 times smaller.

The masses of the great majority of asteroids the diameters of which have now been measured can be calculated from formula (10) or an analogous formula, using the accepted values of albedo and density. This technique was frequently used to calculate the total mass of the known asteroids. Bauschinger and Neugebauer (1901) found that, assuming the mean density to be equal to the density of the Earth, the total mass of 450 asteroids, M = 3.4×10^{-9} M_o = 6.8×10^{24} gm. As was to be expected, addition of new asteroids changes little in this evaluation. Schubart (1971a), using his own value for the mass of Sorora and a calculation of the contribution of asteroids of various stellar magnitudes obtains the value M = 2.4×10^{24}gm. Using the theoretically calculated value $\beta = 1.837$ in (13) and determining the coefficient c_2 by comparison with the results of the McDonald Survey, Dohnanyi (1969) found the value of 0.2×10^{24} gm for the mass of all asteroids (excluding the three largest members). The total mass of all asteroids was calculated by Dahnanyi to be 0.5×10^{24}gm.

In the past, numerous attempts were made to evaluate the total mass of the belt from the perturbations caused by it in the motion of planets. These evaluations invariably led to high values of mass, on the order of 0.1 M⊕ = 6.0×10^{26} g. Apparently, this can be attributed to the fact that the influence of the belt included effects caused by other factors not taken into account with sufficient accuracy.

6. ASTEROID FAMILIES AND STREAMS

The differential equations determining the changes of the planetary orbits under the influence of mutual perturbations of planets are too complex to be integrated in a finite form. The usual technique used for determining perturbations of the orbital elements is the method of successive approximations. The expressions for perturbations obtained using this technique, in the form of segments of time power series are sufficiently accurate for calculation of positions of planets during time periods

spanning tens to hundreds of years. However, these expressions provide no data on the properties of motion during this span of very large periods of time. In order to study the changes of planetary orbits during periods exceeding tens of thousands of years it is desirable to obtain solutions [84
of equations determining perturbations, without expanding the solution into a time series. Lagrange has shown that this can be achieved by simplifying the equations by neglecting all periodic terms and also all non-periodic terms higher than second order in eccentricities and inclinations in the right side of the equations. Integration of these reduced equations makes it possible to find, with a somewhat limited degree of accuracy, the so called secular perturbations, which are most important during very long time intervals. The result of integration of Lagrange's equation determining the perturbed values of the mean elements (i.e. elements free from short-period perturbations) can be represented in the following form (e.g. see Brauer and Clemens, 1964)

$$e \cos \pi = A \cos (gt + C) + \Sigma G_j \cos (g_j t + \beta_j),$$

$$e \sin \pi = A \sin (gt + C) + \Sigma G_j \sin (g_j t + \beta_j),$$

$$\sin i \cos \Omega = B \cos (-gt + D) + \Sigma H_j \cos (h_j t + \gamma_j),$$

$$\sin i \sin \Omega = B \sin (-gt + D) + \Sigma H_j \sin (h_j t + \gamma_j),$$

where e, i, π and Ω are the eccentricity, inclination, perihelion longitude, and nodal longitude of the asteroid's orbit; A, B, C, and D are new arbitrarily assigned constants; g_j, h_j, β_j, and γ_j, are known constants in the theory of motion of the major planets, which is assumed to be developed; and g, G_i, and H_i are known functions of the semimajor axis of the asteroid. The arbitrary constants A, B, C, and D can be found from equalities (14), provided the values of the mean elements e, i, π, and Ω are known at a certain time t and the sums in the right side of (14), have been calculated. In practice, in calculating constants A, B, C, and D, instead of the mean elements one uses the usual osculating elements, which actually does not introduce substantial errors.

For asteroids with semimajor axes in the range $1.9 \leq A \leq 4.3$ AU, the secular variation of the argument g is a function of the semimajor axis (Brouwer, 1951) and varies between 26" and 534" per year. The annual variation of g_j, h_j in the trigonometric functions, under the summation sign in the right sides of (14), is on the average considerably smaller. Introducing the notations $p \cos \alpha$, $p \sin \alpha$, $q \cos \beta$, $q \sin \beta$ for these sums and assuming $gt + C = \pi_1$, $-gt + D = \theta_1$, the first two equations in (14) can be written in the form (2), while the two latter ones in the form (3). Thus, the geometrical interpretation of the variation of the mean elements of asteroid orbits over large time intervals described in section 4 is based on Lagrange's theory for the secular perturbations of asteroids in the form given by (14).

Relationships (14) express the fact that Lagrangian elements of asteroid orbits, i.e. $e \cos \pi$, $e \sin \pi$, $\sin i \cos \Omega$, $\sin i \sin \Omega$ undergo temporal changes due to superposition of induced oscillations caused by disturbing forces (terms under the summation sign) over natural oscillations (terms containing arbitrary constants). The arbitrary constants A, B, C, and D, are elements which , better than Kepler elements, reflect the general properties of asteroid motion during prolonged periods of time. This is the basis for referring to them and the semimajor axis of the orbit as proper elements of the asteroid. In a number of papers, Hirayama (1923, 1928a, 1933) calculated the proper eccentricity A and proper inclination B of a large number of asteroids. By comparing proper elements, Hirayama identified a certain number of asteroid groups or, in his terminology, families the member of which have similar values of semimajor axes, proper eccentricities, and proper inclinations. These families were named Themis, Eros, Coronis, Maria, Phocoea and Flora, after the first planet in each group.

Considerable contribution to the investigation of asteroid families was made by Brouwer (1951). Using an improved theory of secular perturbations of large planets, he calculated the proper elements of 1537 asteroids. Brouwer found a considerable number of new members in the Themis, Eros, Coronis, Maria, and Flora families. According to Brouwer, the Flora family can be separated into 4 individual families. The asteroid families after Brouwer are listed in Table 19. The mean values of proper elements for each family are given together with the mean square deviation for

Table 19
Asteroid Families After Brouwer

No.	Family	Mean Motion Range	No. of Planets	a	A	B
1	Themis	622".6-658".0	53	3.1367 ±0.0267	0.1550 ± 0.0148	0.0239 ±0.0042
2	Eros	673.8-682.9	58	3.0149 ± 63	0.0747 ± 99	0.1757 ± 48
3	Coronis	716.8-743.5	33	2.8753 ± 178	0.0491 ± 63	0.0371 ± 16
4	Maria	804.8-882.3	17	2.5460 ± 107	0.0991 ± 188	0.2596 ± 41
5	Phocoea	923.6-1024.8	21	2.3653 ± 532	0.2413 ± 366	0.4015 ± 253
6	Flora I	1047.8-1106.5	23	2.2175 ± 244	0.1253 ± 163	0.0511 ± 78
7	Flora II	1049.8-1108.5	62	2.2182 ± 215	0.1368 ± 188	0.0801 ± 78
8	Flora III	1052.0-1089.3	9	2.2167 ± 180	0.1448 ± 87	0.0976 ± 24
9	Flora IV	1006.8-1107.2	31	2.2528 ± 369	0.1582 ± 171	0.1129 ± 88

each value. It can be seen from the table, that in a number of cases the degree of concentration of proper elements is very large (Eros, Coronis). Fig. 19 shows the high concentration of the Eros family in the phase plane A, B.

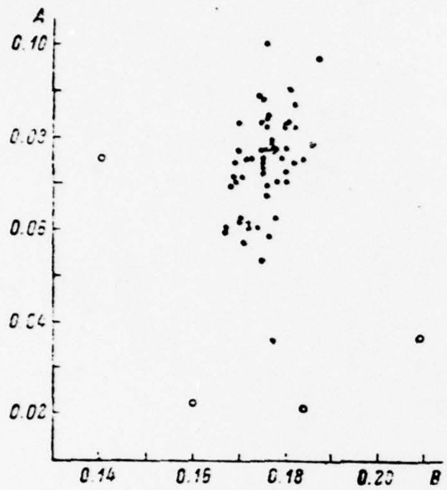

Fig. 19. Eros family. Points in the phase plane A, B.denote members of the family (Table 19); circles denote asteroids in the range 673".8 -- 682".9, which are not members of the family

Brouwer noted that within the framework of the theory of secular perturbations, the sum of quantities $\pi_1 = gt + C$ and $\theta_1 = -gt + D$ (to which he refers to as proper perihelion and nodal longitudes, respectively) always remain equal to C + D. Assuming that at the time of formation the constants C and D for all members of a certain group of asteroids had similar values, then, inside this group the sum $\pi_1 + \theta_1$ will always remain within narrow limits. Based on these considerations, Brouwer has separated 19 asteroid groups, characterized by concentration of values of a, A, B, and $\pi_1 + \theta_1$. These groups of asteroids were numbered from 11 to 29. All of these groups had only a few members: their number varied between 4 and 11. In addition to this, considerable number of members in these groups had only a weak concentration of $\pi_1 + \theta_1$ within a specific part of the circumference. [85

The Palomar-Leiden survey has extended considerably the search for asteroid families. Authors of the survey assigned 389 planets, or 40 percent of the total number of new planets with reliably determined orbits, into the families. A large number of new members was found to belong to the Themis, Eros, Coronis, Maria and Flora families. Five new families (No. 30 to 34) with a number of members between 26 and 77 were identified. New members were identified for 7 of the 19 Brouwer groups. However, the new members of these groups did not reveal concentration of $\pi_1 + \theta_1$ in a specific part of the circumference. Subsequent investigations have shown that most of the Brouwer groups have only a few members.

Alfven (1969) identified jet streams i.e. groups of planets in the asteroid belts, with similar values of each of the five proper elements a, A, B, π_1, and θ_1. Alfven discovered Flora A (Table 20) in the Flora I

Table 20
Flora A Stream After Alfven

Asteroid	a	A	B	π_1	θ_1
				circumference sector	
(960) Birgitt	2.2482	0.1188	0.0640	0.930	0.689
(1422) Stromgrenia	2.2472	0.1309	0.0556	0.064	0.582
([illegible]) Demoulina	2.2421	0.1246	0.0498	0.054	0.514
(315) Constantia	2.2416	0.1241	0.0456	[illegible]	0.487
([illegible]) Skuld	[illegible]	0.1487	0.0480	[illegible]	[illegible]
([illegible]) 1939 SE	2.2044	0.1525	[illegible]	0.042	[illegible]
([illegible]) [illegible]	2.1985	[illegible]	0.0565	0.925	0.691
(1150) Achaia	2.1910	0.1512	0.0515	0.969	0.582
(1494) Savo	2.1900	[illegible]	0.0523	0.115	0.555
(810) Atossa	2.1891	0.1315	0.0504	0.989	0.453
(703) Noemi	2.1749	0.1165	0.0529	0.145	0.593
(244) Sita	2.1746	0.1026	0.0597	0.103	0.583

family. He also separated Flora B and Flora C from Flora II, III, IV families. The members of streams are in similar orbits, thus resembling meteor streams. In all of the studies discussed so far, identification of families and streams from the large number of asteroids was intuitive. Attempts to establish certain objective criteria for determining the orbits, or their concentration in a certain region of the phase space and, to classify asteroid groups on this basis were made by Arnold (1969), Lindblad and Southwort (1971), and Danielsson (1971). Highlights of the method used by Arnold to separate families and streams can be described as follows. Depending on the number of orbit elements considered, each asteroid is assigned a point in the three-or five - dimensional rectangular box having a definite volume. The number of asteroids included in the box is compared with a certain stipulated value calculated on the assumption that the known asteroids are distributed equally throughout the volume, which can be assumed to be equal to the volume of the belt. If the ratio of these quantities is larger than a certain previously specified quantity, the population of asteroids included in the box is considered to be a family (stream) or part of a family.

The use of this technique verified the existence of the first 9 families, which also gained new members as a result of addition of newly numbered asteroids. In Arnold's work, the small Brouwer families (No. 11 to 29) underwent considerable revision. The composition of such families as, [86 for example, No. 19 to 24, was changed so radically that, new designations were introduced for these families. In addition to this, Arnold has separated previously unidentified families with a number of members between 7 and 26. Seven asteroid streams with between 7 and 32 members were also identified. The largest of these includes Flora A and part of the Flora B discovered by Alfven.

Limdblad and Southworth define the "distance" between two orbits in the five-dimensional space by means of the following formula:

$$d^2 = (e_1 - e_2)^2 + (q_1 + q_2)^2 + 2 \sin \left(\frac{i_1 - i_2}{2} \right)^2 + \sin i_1 \sin i_2$$
$$\times \left(2 \sin \frac{\Omega_1 - \Omega_2}{2} \right)^2 + \left(\frac{e_1 + e_2}{2} \times 2 \sin \frac{\pi_1 + \pi_2}{2} \right)^2$$

where e, i, π, Ω, $q = a(1 - e)$ are the oscullating, or proper elements of orbits. A program which searches for streams, sequentially calculates distances between all possible pairs of orbits and, if for any pair it is less than a preassigned rejection level, this pair is considered to be a stream. This process is gradually expanded to include ever increasing numbers of orbits into the stream. Only the first three terms in formula (15) are used to search for the families.

The algorithm described made it easy to discover the existence of the first nine families among the numbered asteroids. In most cases, their composition turned out to be close to that discovered by Brouwer and Arnold. Limdblad and Southwort also discovered 16 of 19 Brouwer's small families. However, in most cases the number of members in these families is between two and four and is statistically insignificant. Several of Arnold's families were also identified. However, in a number of cases, a considerable shuffling of members or splitting of families into independent groups had taken place. All five families (No. 30 to 34) among the Palomar-Leiden asteroids discovered earlier by the authors of that survey were sorted out.

The results of analysis of a population of 2652 asteroids consisting of numbered asteroids and Palomar-Leiden asteroids with first class orbits, performed by Limdblad and Southwort are of particular interest. Out of the total population it was possible to identify families No. 1 to 9, 16, 18 to 22, 27, and 30 to 34. At the assumed values of d, families No. 11 to 15, 17, 23, and 26 were not identified. Ten new families with a number of members between 5 and 7 were also discovered. It is interesting to note that the average "distance" d between members of these families is shorter than for most of the members in the Brouwer family.

In analyzing the osculating elements of 2929 asteroids (numbered asteroids plus Palomar-Leiden asteroids with first and second quality class orbits), Limdblad and Southworth identified 36 streams with a number of members exceeding 3 and 13 streams with 7 or more members. Flora A and Flora C with a somewhat different composition can be identified among the discovered streams. The Flora B stream can also be identified however it is subdivided into two small statistically not too significant [87
groups (5 and 4 members). A comparison of the observed streams with those discovered by Arnold shows that only three are common to both investigations.

In evaluating the distance between orbits, Danielsson introduces the concept of the average distance D. If s(λ) is the shortest distance between points of two orbits with heliocentric longitudes λ, then D is defined as a result of averaging s over λ:

$$D = \left(1/2\pi \int_0^{2\pi} s^2 (\lambda) (d\lambda)\right)^{\frac{1}{2}}.$$

Assuming that the asteroid stream was formed by objects the mean distance between orbits of which does not exceed 0.15 AU, Danielsson separated three streams (Flora A, B, and C) containing 10, 9, and 10 members respectively. However, their composition differs considerably from the composition of the streams determined by other authors in their earlier papers.

The above considerations show agreement between the results of different investigations concerning the existence of the largest asteroid families with a large number of members and a high concentration of proper elements. The existence of these families is statistically significant. Each such grouping is apparently connected through a common origin of its members or similarity of their history. On the other hand, in different investigations many of the less significant groupings are less clearly defined. This indicates the absence of clear boundaries separating these groupings from the background and low concentration . The small number of members and low concentration appear to indicate that a certain number of the groups established are random distribution fluctuations. The probability of this concept has hardly been investigated mathematically and the results obtained have a limited value (Danielsson, 1971). The asteroid streams are of particular interest. Different investigations revealed that only one or two of them have more or less stable composition. Nevertheless, the probability of random formation of even these streams is low (Danielsson, 1969, 1971). As was noted at the beginning of this section, the annual variation of the proper perihelion longitude and proper nodal longitude are functions of the semimajor axis. Differences in the semimajor axes of orbits of members of the same stream are such that, the longitudes of their perihelia at nodes during the time of approximately 100,000 to 1,000,000 years should be distributed more or less equally over the whole circumference, even if they were close to each other at a specific time (see section 4). The existence of streams at the present time requires a special explanation.

It is most natural to assume that the stream has appeared quite recently ($t \ll 1,000,000$ years), as a result of fragmentation of a large body into smaller objects with approximately identical orbits. However, short duration of the existence of a stream produced in such a manner contradicts their possible abundance. The concentration of orbits in the streams can be explained as a result of constant influence of certain forces (for example, inelastic particle collisions) which result in shorter distances between orbits inside the stream (Alfven, Arrhenius, 1970a, 1970b; Trulsen, 1971). Finally, one can not exclude the possibility that the observed concentration of orbits, particularly concentration of perihelia and nodes, in certain parts of the circumference is a consequence of the observation selection (Kresak, 1971). The origin of asteroid streams is apparently closely associated with the origin of families the existence and abundance of which unlike that of streams is a definitely established fact. Two opposite trends have appeared for considerable time in attempts to solve the problem of the origin of families. Some of the researchers assume that the formation of families occurred as a result of approaches of different-type orbits (Brown, 1932; Rabe, 1956). In particular, Brown notes that Hirayama families are apparently groups of relatively stable orbits formed by forces of attraction from a large number of different configurations. Representatives of the opposite point of view consider formation of families as a result of fragmentation of "parent" bodies (Hirayama, 1923, 1928a; Brauwer, 1951).

The most likely cause of fragmentation is collision of asteroids (Kuiper, 1950). The values of mean relative velocity between objects on the order of 5 km/s correspond to mean eccentricities and inclinations in the asteroid belt of 0.15 and $9^\circ.4$, respectively (Piotrowsky, 1953; [88
Dohnanyi, 1969). At these relative velocities, head-on collisons of asteroids of comparable dimensions should result in a total destruction of both bodies. However, in the case when one of the bodies is considerably smaller than the other ($r/r_2 \leq 0.05$), complete destruction of the larger body does not occur and, a large number of fragments with masses exceeding two to three time the mass of the smaller body are formed. In all cases, most of the fragments acquire relative velocities which are considerably lower than the orbital velocities of colliding bodies. The heliocentric orbits of fragments will form a single stream until the inflence of planetary perturbations will redistribute the perihelia and nodes more or less equally in all directions.

The concept that a large number of relatively small bodies rather than one large planet was formed during formation of the Solar system planets in the region between Mars and Jupiter finds wider acceptance in the in the present day cosmogeny (Schmidt, 1950, 1954; Kuiper, 1953). Under these conditions, the probability of the first collision resulting in a large number of fragments is not so low. Further collisions at a constantly increasing rate resulted in formation of the asteroid belt in its present-day form.

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