AD-A044 461 NAVAL UNDERWATER SYSTEMS CENTER NEWPORT R I F/G 17/1

ZOOM FFT - AN APPROXIMATE VERNIER FREQUENCY ALGORITHM.(U)

NOV 72 C W NAWROCKI, J F FERRIE

UNCLASSIFIED NUSC-TM-SA2201-584-72 NL

1 OF 1

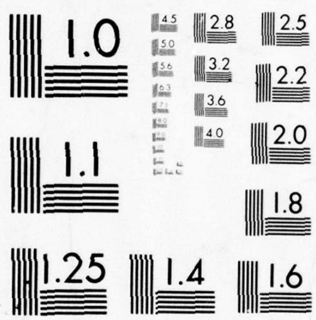
AD A044461

END

DATE FILMED

10-77

DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MOST Project-3 NW

TM NO. NUSC-TM-SA2201-584-72

NAVAL UNDERWATER SYSTEMS CENTER
Newport, Rhode Island 02840

Technical Memorandum

ZOOM FFT - AN APPROXIMATE VERNIER FREQUENCY ALGORITHM

DATE: 22 November 1972

36p.

Prepared by: C. W. Nawrocki
Surface Ship and Surveillance
Sonar Department

James F. Ferrie
Sonar Technology Department

C.W. Nawrocki
James F. Ferrie

Approved for public release; distribution unlimited

Encl (/) to NUSC/NL ltr ser SA2201-0268 of Apr 13

406068

TM NO.
SA2201-584-72

ABSTRACT

This report describes a method for obtaining fine frequency resolution. The technique employs the partitioning of a large data sequence by performing small size FFTs and other digital signal processing techniques to achieve the finer resolution. The method yields approximate results. The accuracy appears to be dependent on Vernier bandwidth, signal-to-noise ratio and signal spectrum. A FORTRAN program is available in the Appendix.

ADMINISTRATIVE INFORMATION

This memorandum was prepared in support of project work sponsored by the Naval Ship Systems Command, PMS-302-32, Program Manager is A. LaPointe.

The authors of this memorandum are located at the New London Laboratory, Naval Underwater Systems Center, New London, Conn. 06320

1.2 TABLE OF CONTENTS

1.3 LIST OF FIGURES

1.4 GLOSSARY

$\{x(n)\}$	Time data sequence
$X(k)$	Discrete Fourier coefficient at K^{th} frequency
Δf	Resolution bandwidth (in Hertz)
fs	Sampling frequency
N	Time sequence length
M	Number of partitions
L	Vernier bandwidth (in Hertz)
FFT	Fast Fourier Transform

2.0 INTRODUCTION

The concept of the ZOOM FFT was first suggested to the authors by Mr. Stephen Phanos, Sperry Gyroscope, Great Neck, New York. Much of the original analysis at the New London Laboratory was done by Dr. Nasir Ahmed, Kansas State University, a summer employee at NUSC, New London.

The Zoom technique increases the sample size of a vernier band, by creating a new narrowband time sequence from information taken from successive large size but course resolution FFTs. Thus finer frequency resolution is obtained that is an approximation of the true spectral estimate arrived at by a Discrete Fourier transform (DFT) computation of the original wide band time sequence.

3.0 MATHEMATICAL DESCRIPTION

Define an N point real sequence $\{X(n)\}$, where

$$\{X(n)\} = X(0),\ X(1),\ \ldots\ X(N-1) \qquad (3.1)$$

The DFT (1) yields,

$$\overline{X}(k) = \sum_{n=0}^{N-1} X(n)\,\mathrm{EXP}\left(\frac{-j2\pi nk}{N}\right) \qquad (3.2)$$

The resultant frequency resolution Δf is then given by:

$$\Delta f = \frac{fs}{N} \qquad (3.3)$$

where fs is the sampling frequency.

Consider now a new sequence $\{X(m)\}$, which is 4 times as long as the original sequence, and is defined by

$$\{X\}(m) = X(0),\ X(1),\ X(2)\ \ldots\ X(n-1),\ X(n)\ \ldots\ X(M-1)$$

where $M = 4N$. (3.4)

The choice of 4 partitions is, of course, arbitrary and used for illustrative purposes. In the general case, M is equal to RN, or

$$M = RN \tag{3.5}$$

where R is an interger.

By taking DFT's of the 4 partitions each of length N successively, the resultant four sequences are defined as;

$$\{X_1(k)\} = X_1(0), X_1(1), \ldots \ldots X_1(N-1)$$

$$\{X_2(k)\} = X_2(0), X_2(1), \ldots \ldots X_2(N-1)$$

$$\{X_3(k)\} = X_3(0), X_3(1), \ldots \ldots X_3(N-1)$$

$$\{X_4(k)\} = X_4(0), X_4(1), \ldots \ldots X_4(N-1) \tag{3.6}$$

The rationale for partitioning is twofold. First, the available FFT hardware may be constrained to do only an N point DFT. Second, in order to perform the computations in real time it may be desirable to begin processing each partition as it becomes available or in parallel with other operations where each sequence has a frequency resolution equal to Δ f.

Now define a limited frequency band consisting of L frequency points, and using the same band for each of the 4 previously defined sequences, write these sequences as;

$$\{X_r(\ell)\} = \begin{cases} X_r(k) & \text{where } k = P, P+1 \cdots P+L-1 \\ 0 & \text{elsewhere} \end{cases} \tag{3.7}$$

and r = 1,2,3,4

The inverse FFT of each successive L point sequence is computed. Even and odd symmetry is forced on the real and imaginary Fourier coefficients such that the resultant time sequence is real. The following relationship is used:

$$X_r(n) = \sum_{\ell=P}^{P+L-1} \mathbb{X}_r(\ell)\,EXP\left(\frac{j2\pi n\ell}{L}\right) \qquad (3.8)$$

and r = 1,2,3,4 $\qquad n = 0, 1, --- (L-1)$

Then we juxtapose the resultant time sequences. Defining this new sequence as

$$X(k) = \left[\{\mathbb{X}_1(n)\}, \{\mathbb{X}_2(n)\}, \{\mathbb{X}_3(n)\}, \{\mathbb{X}_4(n)\}\right]$$

and by taking the DFT of this new sequence according to $\qquad (3.9)$

$$\mathbb{X}(k) = \sum^{4L-1} X(n)\,EXP\left[-\frac{j2\pi kn}{4L}\right] \qquad (3.10)$$

$$k = 0, 1, \ldots (4L-1)$$

results in a frequency resolution given by

$$\Delta f = \frac{fs}{4N} \qquad (3.11)$$

or 4 times as fine as the original resolution. In effect, the Zoom FFT has performed a large size FFT, for a Vernier bandwidth with resolution Δf!

A more rigourous mathematical presentation can now be offered to illustrate the frequency approximation process.

Define an N point time sequence, and partition the sequence into M partitions, the resultant M FFT's can be represented by

$$\bar{X}_m(n) = \sum_{K=0}^{\frac{N}{M}-1} X(m+KM)\, EXP\left[\frac{-j2\pi nK}{N/M}\right]$$

TM NO.
SA2201-584-72

where m = 0, 1, ..., M - 1

n = 0, 1, ..., $\frac{N}{M}$ - 1 (3.12)

which yields M frequency sequences of size $\frac{N}{M}$ - 1. Discarding certain frequency components by selecting $X_m(R)$ where R is defined as

$n_1 < n < n$ where

$n_1 - n_2 = L$ and (3.13)

$R = 0, 1, \ldots (L-1)$

This relation yields M frequency sequences each of size L. Taking the inverse transform of each $X_m(R)$ by

$$X_m(b) = \sum_{R=0}^{L-1} \bar{X}_m(R)\, EXP\left[\frac{j2\pi bR}{L}\right] \qquad b = 0, 1 \cdots (L-1) \qquad (3.14)$$

which yields a M point time sequences each of size L. By forming a new time sequence of size ML by juxtaposing in time according to

$$X(d) = \sum_{m=0}^{M-1} X_m(b) \qquad (3.15)$$

where d = b + ML for (3.16)

b = 0, 1 - - (L-1);
m = 0, 1 - - - - (n-1)
and d = 0, 1 . . . (L-1) . . (ML)

and taking the FFT of the time sequence of size ML

$$\bar{X}(S) = \sum_{d=0}^{ML-1} X(d)\, EXP\left[\frac{-j2\pi dS}{ML}\right] \qquad (3.17)$$

S = 0, 1 . . . (ML-1)

Equation (3.17) yields a frequency sequence of size ML.

Given equations 3.7 through 3.8, the Zoom FFT process can be written as:

$$X(s) = \sum_{d=0}^{ML-1} EXP\left(\frac{-j2\pi ds}{ML}\right) \sum_{m=0}^{M-1} \sum_{R=0}^{L-1} EXP\left(\frac{j2\pi bR}{L}\right) \sum^{N/m-1} X(m+kM)\, EXP\left[\frac{-j2\pi hK}{N/M}\right] \quad (3.18)$$

By setting the R equal to $\frac{N}{M}-1$, the degenerate case, it can be seen that equation (3.18) yields the following equation (2) for a partitioned DFT case namely

$$X(h) = \sum_{m=0}^{M-1} EXP\left(\frac{-j2\pi mh}{N}\right) \sum_{k=0}^{\frac{N}{M}-1} X(m+KM)\, EXP\left[\frac{-j2\pi hK}{N/M}\right] \quad (3.19)$$

Thus the only time the Zoom FFT is equivalent to the DFT is for the degenerate case. Any lesser size of R, other than $\frac{N}{M}-1$, yields an approximate answer.

In the limit, as $R \longrightarrow (\frac{N}{M}-1)$, the approximation goes to the exact answer. The distortion in the algorithm is introduced by attempting perfect filtering in the frequency domain, thus by increasing the filter bandwidth, less error is introduced in the time domain.

Examples of the Zoom FFT

To present the effect of changing the inverse FFT window (L), two synthetic tones were digitally produced, one at 430 Hz and another at 401 Hz, the latter 10 dB lower than the former. A DFT and Zoom FFT was performed on this date.

Figure 3.1 presents the plot of the DFT case using an FFT algorithm (2), with 1 Hz resolution. This output was used as a reference for comparison purposes. The Zoom FFT was then performed on the same time data sequence of 1024 points.

First the sequence was partitioned into four smaller data sequence of 256 points each. Next an FFT was taken of each sequence; this FFT had a resolution of 4 Hz. This frequency domain data was then used as an input to the new Zoom algorithm to obtain 1 Hz resolution.

The vernier band of interest was selected to provide a vernier for the 401 Hz tone. Vernier bandwidths of 32 Hz, 64 Hz, 128 Hz, and 256 Hz were examined and shown in Figures 3.2 through 3.5. The degree of approximation can be compared with the DFT, quantitatively, in Appendix B.

Upon examining these values and figures, it is clear that as the Vernier window is increased, the error of the approximation to the DFT is less.

To prove this conclusively, the window was made equivalent to the partition DFT and the Zoom FFT was performed. The results are shown on Figure 3.6 and are exactly equivalent to the DFT reference case. The results substantiate the mathematical interpertation derived in the previous section.

An interesting aspect of the attempt a perfect filtering in the frequency domain, is the possibility of reducing the resultant side lobe structure by Cosine windowing (4). This window was applied to the real and imaginary frequency sequences before the inverse FFT was taken. The degenerate case (where the Zoom FFT is equivalent to the partitioned DFT) was examined. The results for one, two, and four partitions are shown in Figures 3.7 through 3.9. It is apparent that windowing in the frequency domain in this manner, increases the amplitude of the side lobe structure; the amplitude also appears to increase with the number of partitions used in operating the original time sequence.

Figures 3.10 through 3.12 shows DFT and the Zoom FFT, with vernier bandwidths of 64 Hz and 128 Hz using the two tone case plus additive random noise with a mean 10 dB below the 401 Hz tone. As expected the Zoom FFT performed best for the larger vernier bandwidth.

4.0 SUMMARY

The Zoom FFT algorithm can be used as a Vernier FFT algorithm with approximate results. This approximation approaches to a partition DFT case, as the Vernier bandwidth increases. Cosine windowing in the frequency domain appear to yield negative results.

4.1 CONCLUSION

The Zoom FFT appears to be applicable in cases where a Vernier FFT algorithm is needed and a real time constraint is such that other (2.5) Vernier algorithms cannot be implemented. By degrading the results, the Zoom FFT can be implemented using very small FFT's, thus saving appreciable time. If the time constraint is such that the desired frequency resolution can only be attained by reducing the Vernier bandwidth, the accuracy will degenerate rapidly in cases of low signal-to-noise ratio and closely spaced spectral lines.

5.0 ACKNOWLEDMENTS

The authors wish to acknowledge the assistance of G. C. Carter, A. H. Nuttall, and F. S. White at the New London Laboratory NUSC and L. Garda at Sperry Gyroscope, Great Neck, N. Y.

5.1 REFERENCES

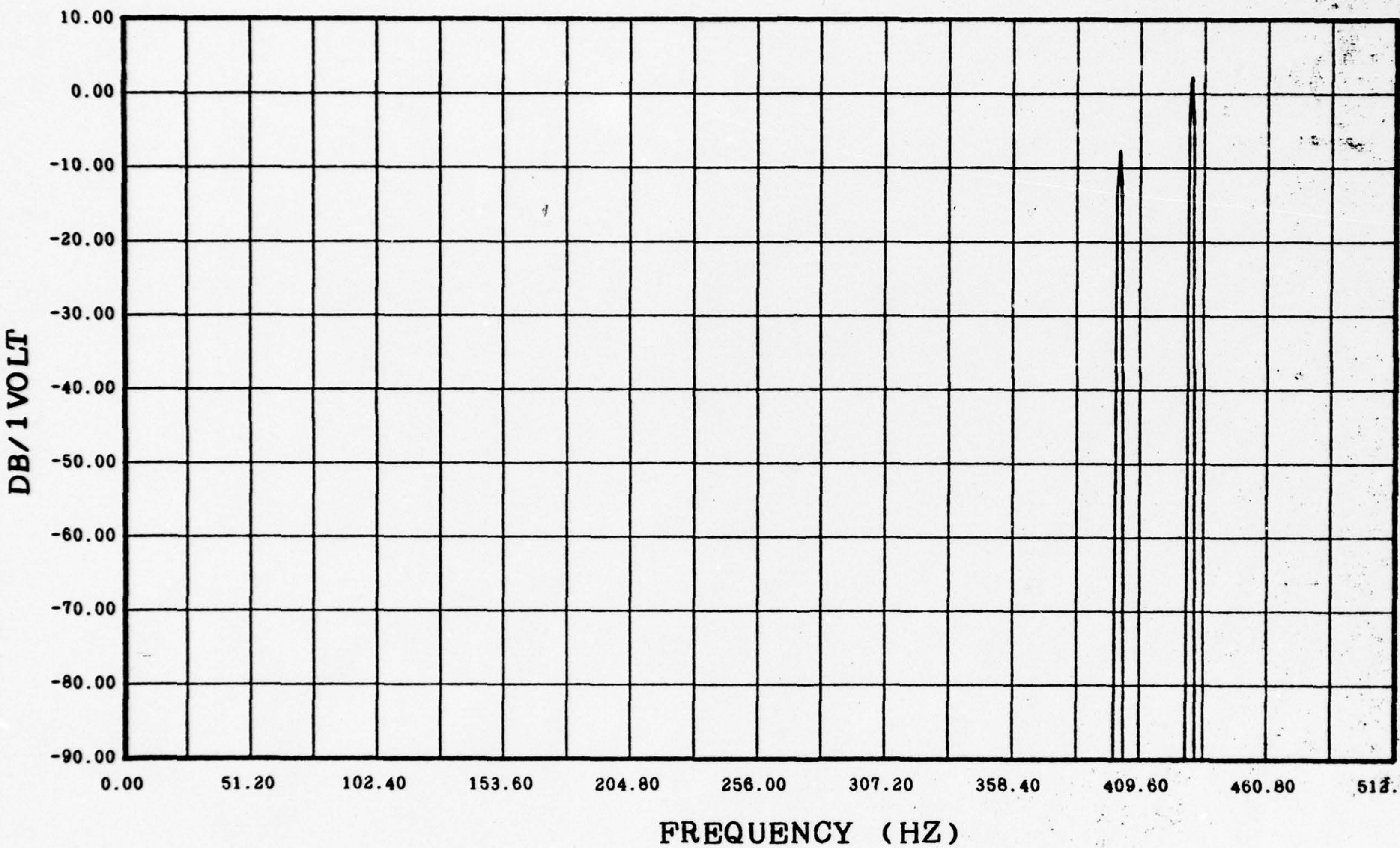
1. B. Gold and C. M. Rader, "Digital Processing of Signals", McGraw-Hill Co., New York 1969

2. A. H. Nuttall, "Two Methods of Performing a Large Size FFT By Means of Several Smaller FFTS", NUSC TM TC-185-72

3. C. R. Arnold, G. C. Carter, and J. F. Ferrie, "Unpublished FORTRAN Computer Program for Power Spectral Estimation"

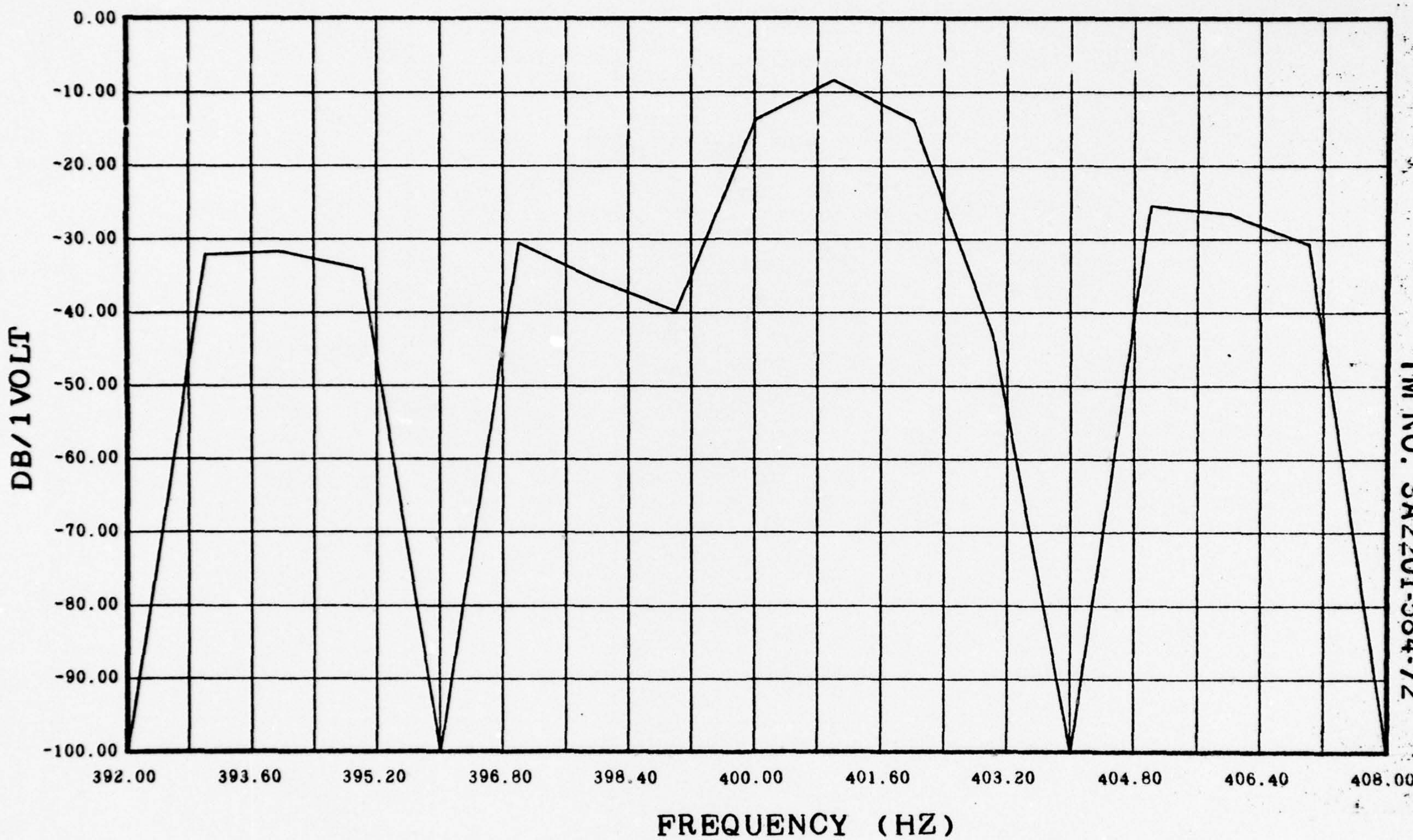
4. A. H. Nuttall, "Spectral Estimation by Means of Overlapped Fast Fourier Transform Processing of Windowed Data", NUSC Rpt 4169, 13 Oct 1971

5. J. F. Ferrie, C. W. Nawrocki, and G. C. Carter, "Implementation and Results of the Modified Chrip-Z Transform", NUSC Tech Memo TC-191-72, 18 Oct 1972



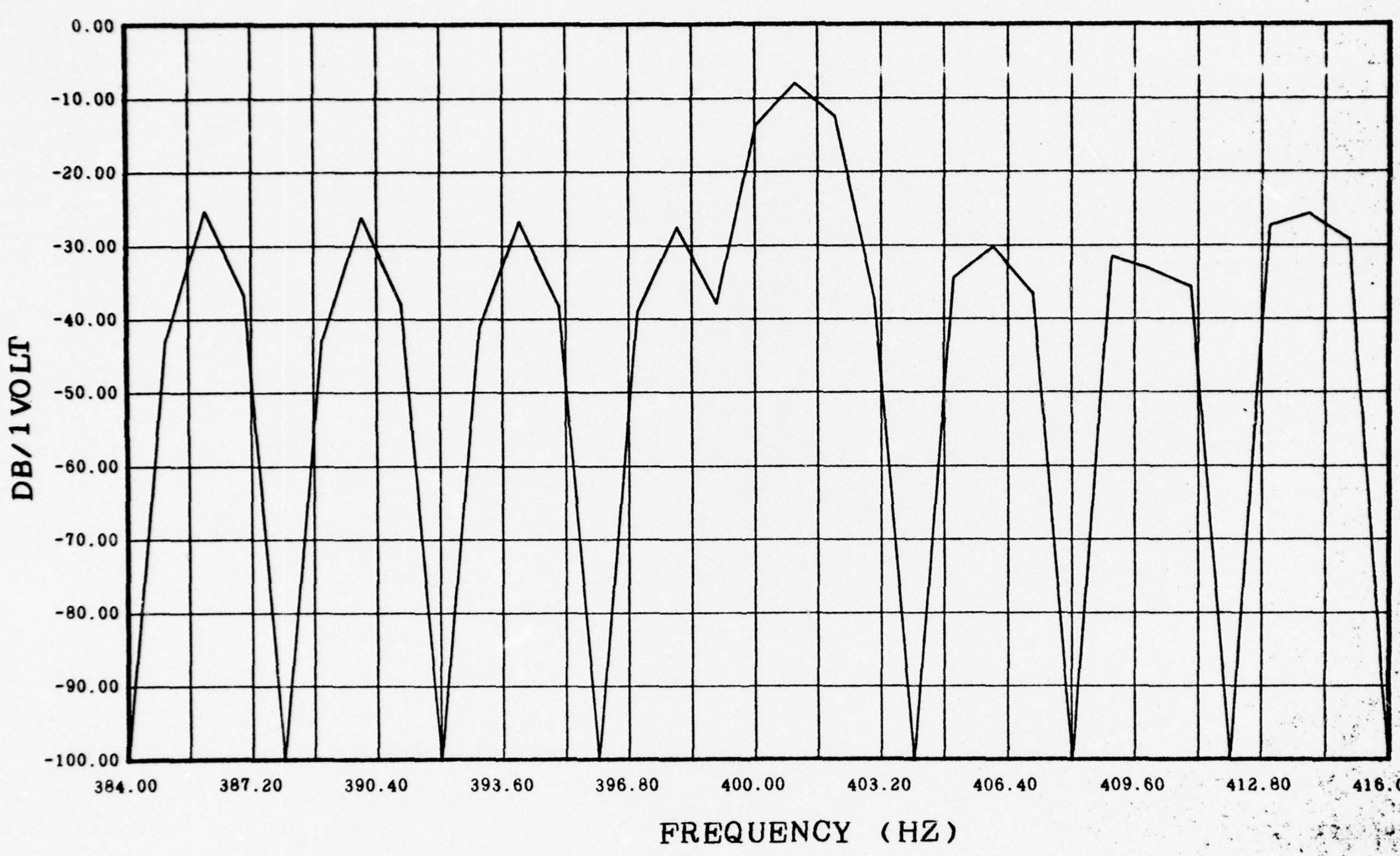
1024 POINT FFT OF TONES AT 401 AND 430 HZ

FIGURE 3.1



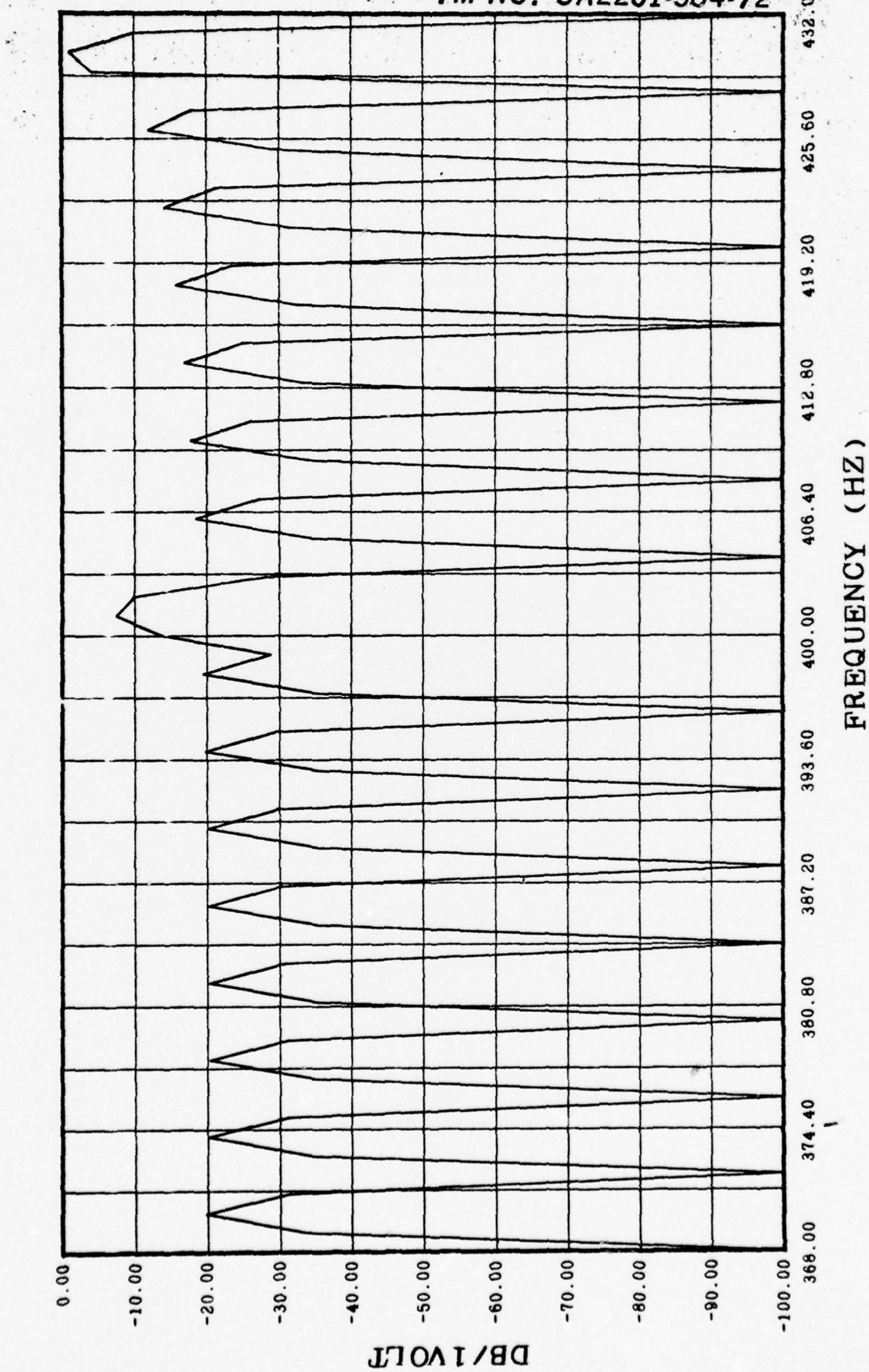
ZOOM FFT WITH VERNIER BANDWIDTH OF 16 HZ

FIGURE 3.2



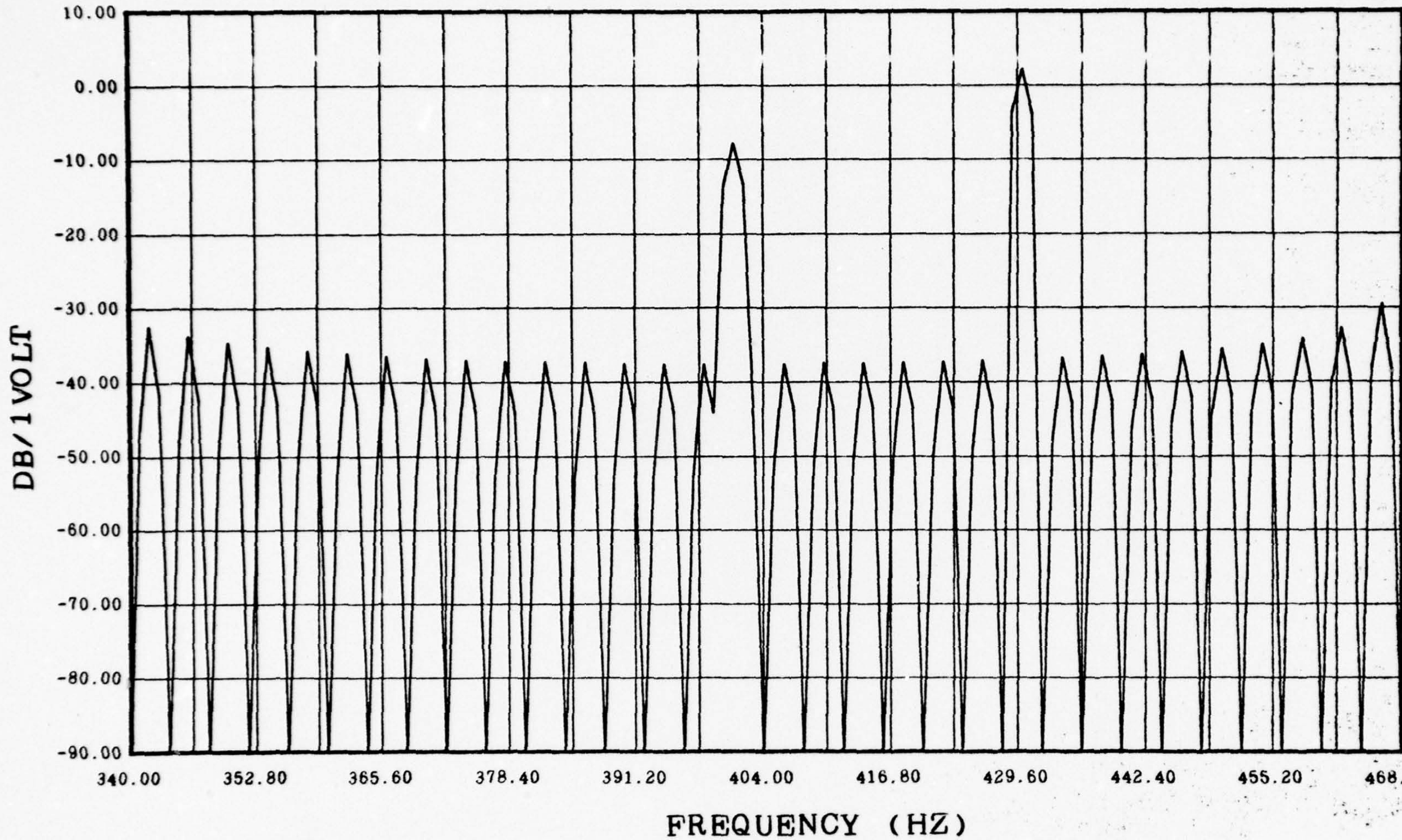
ZOOM FFT WITH VERNIER BANDWIDTH OF 64 HZ

FIGURE 3.3



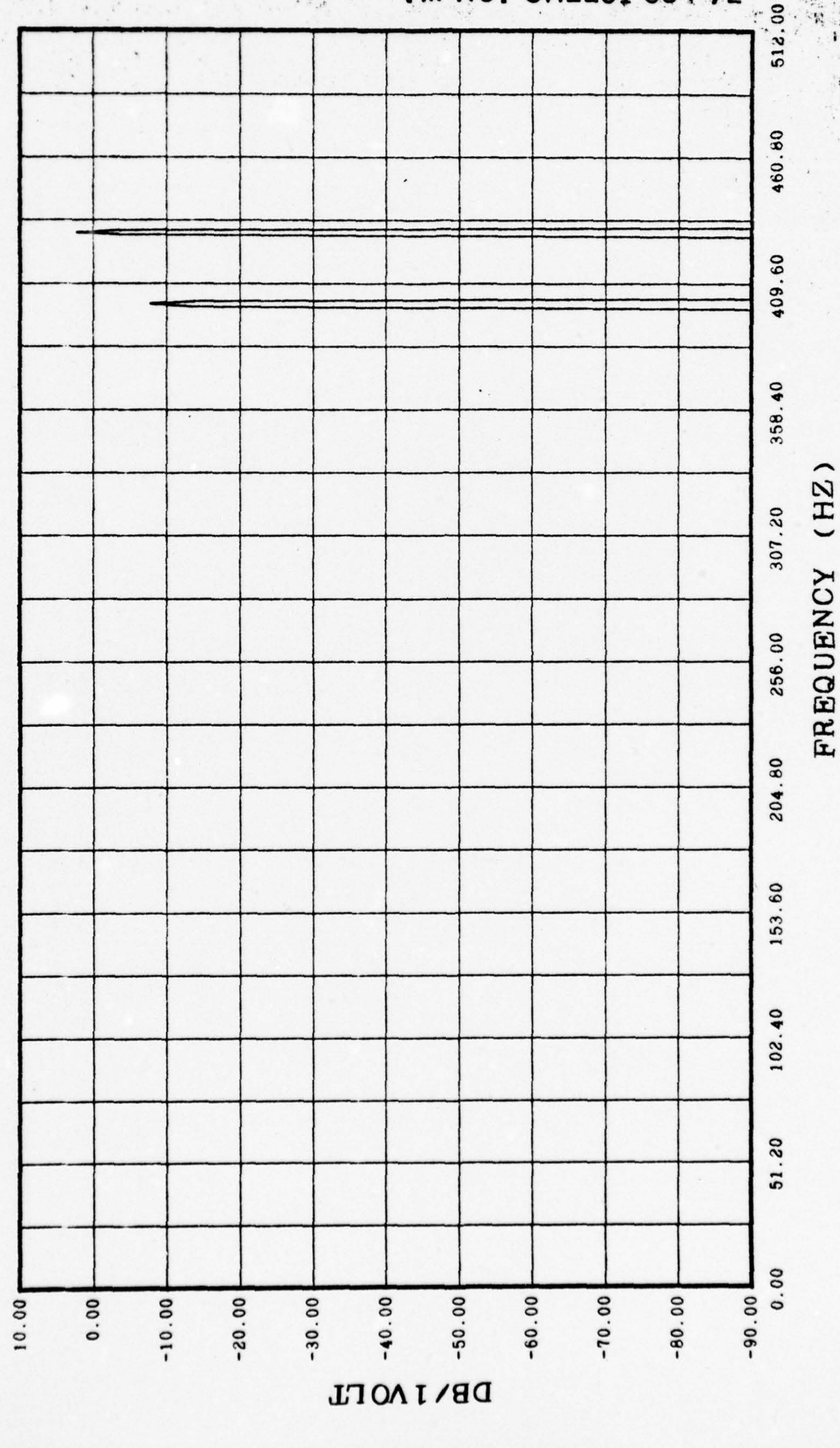
ZOOM FFT WITH VERNIER BANDWIDTH OF 128 HZ

FIGURE 3.4



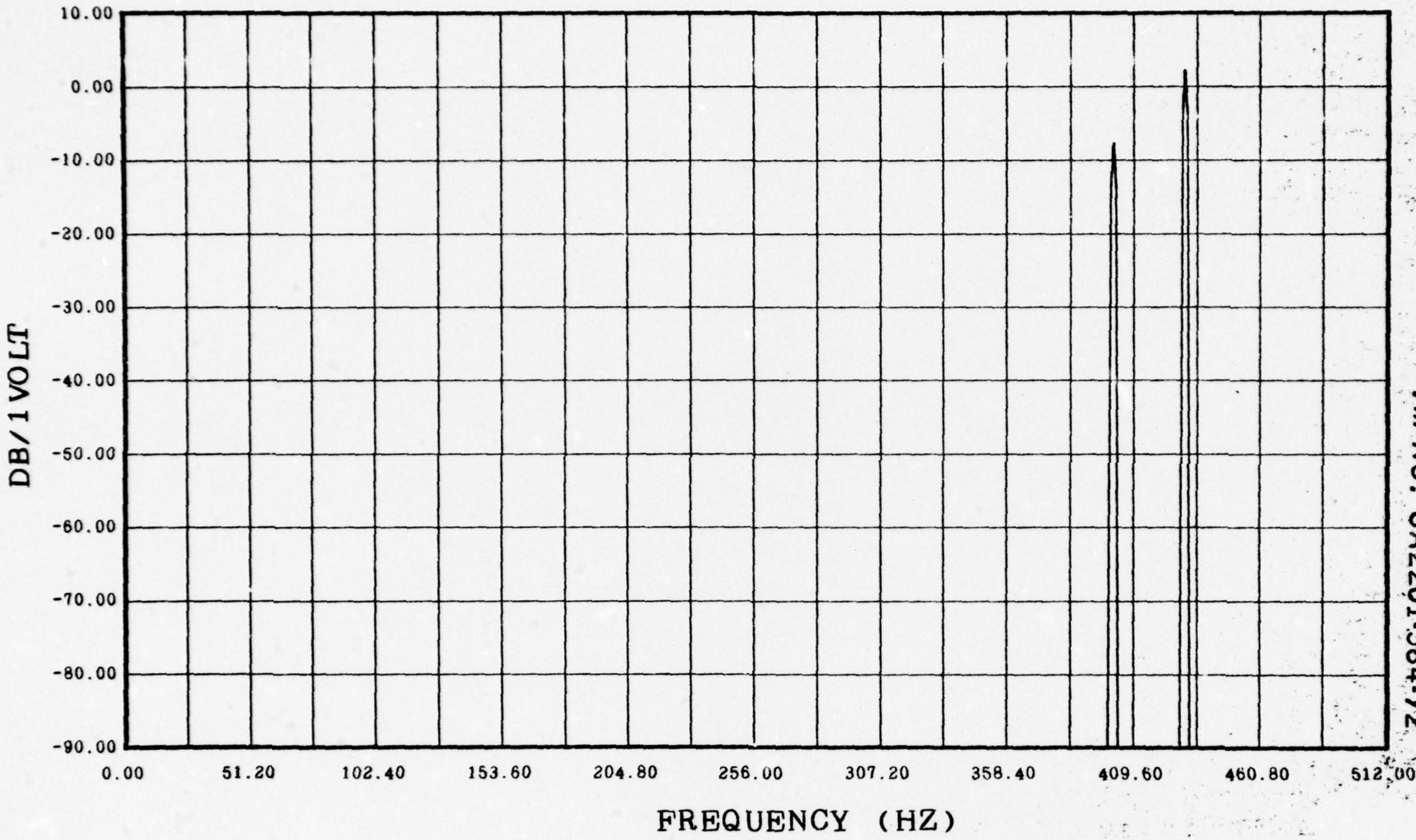
ZOOM FFT WITH VERNIER BANDWIDTH OF 256 Hz

FIGURE 3.5



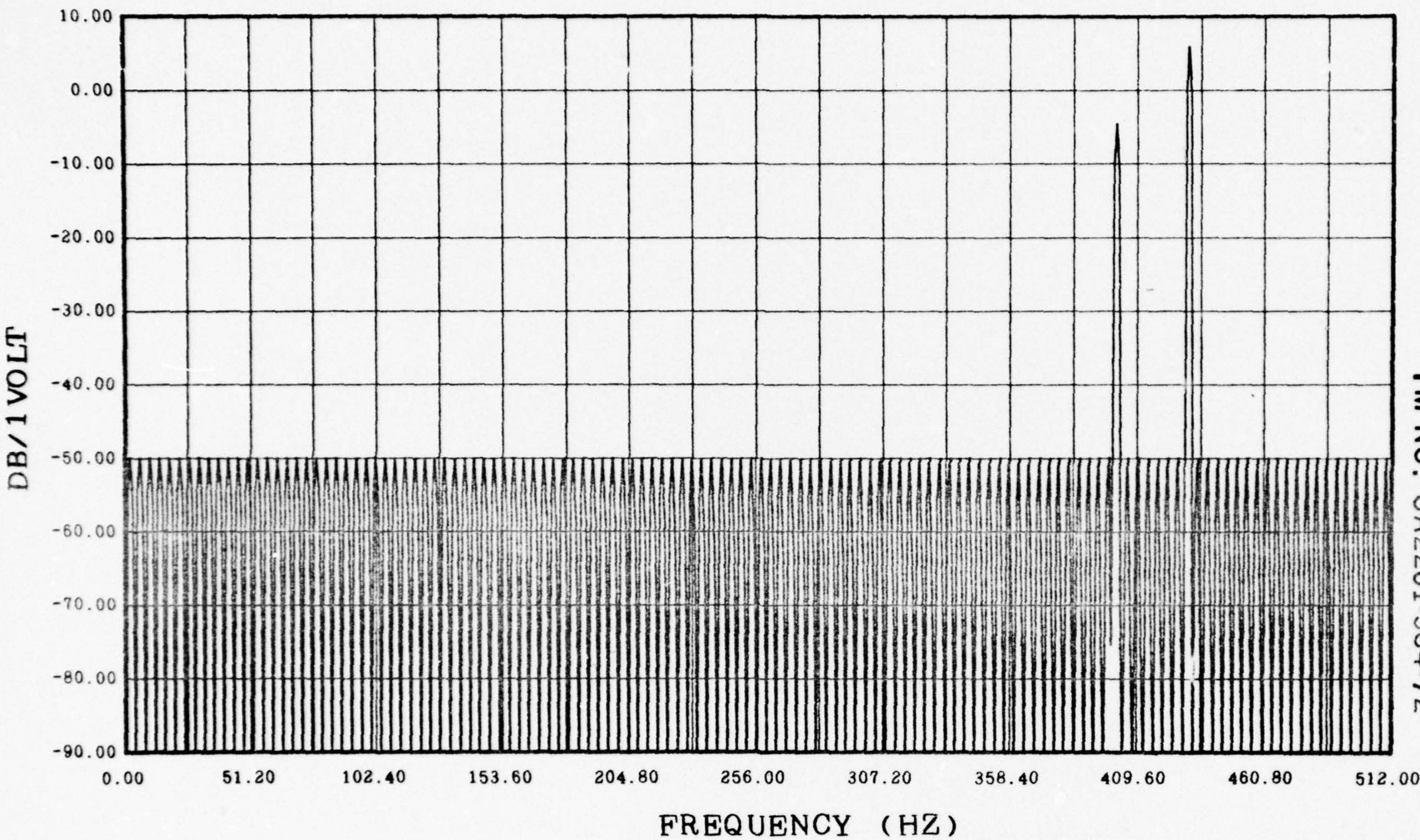
ZOOM FFT WITH VERNIER BANDWIDTH EQUAL TO BANDWIDTH OF PARTITIONED DFT

FIGURE 3.6



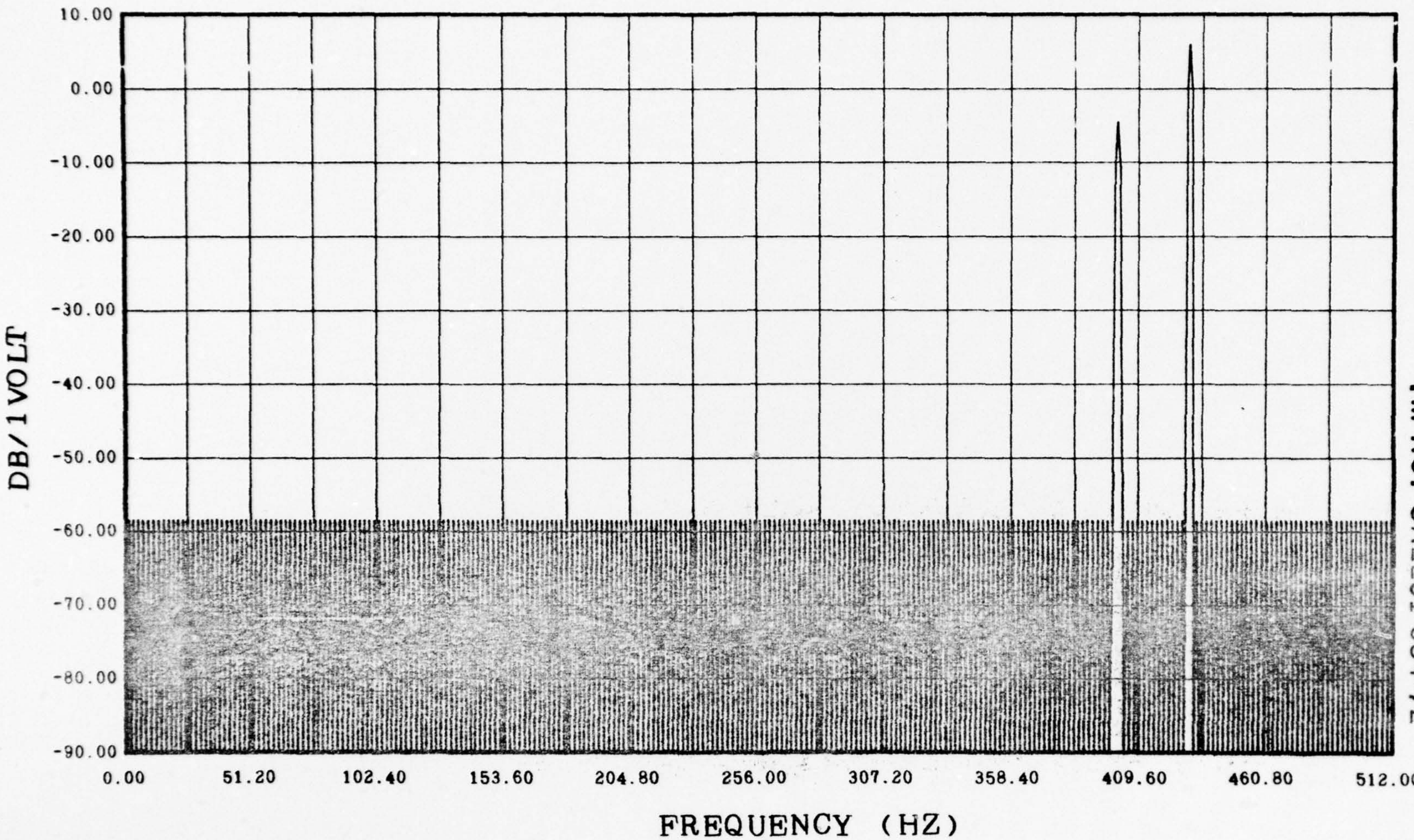
EFFECT OF COSINE WINDOW, IN THE FREQUENCY DOMAIN ON THE DEGENERATED ZOOM FFT WITH NO PARTITIONS

FIGURE 3.7



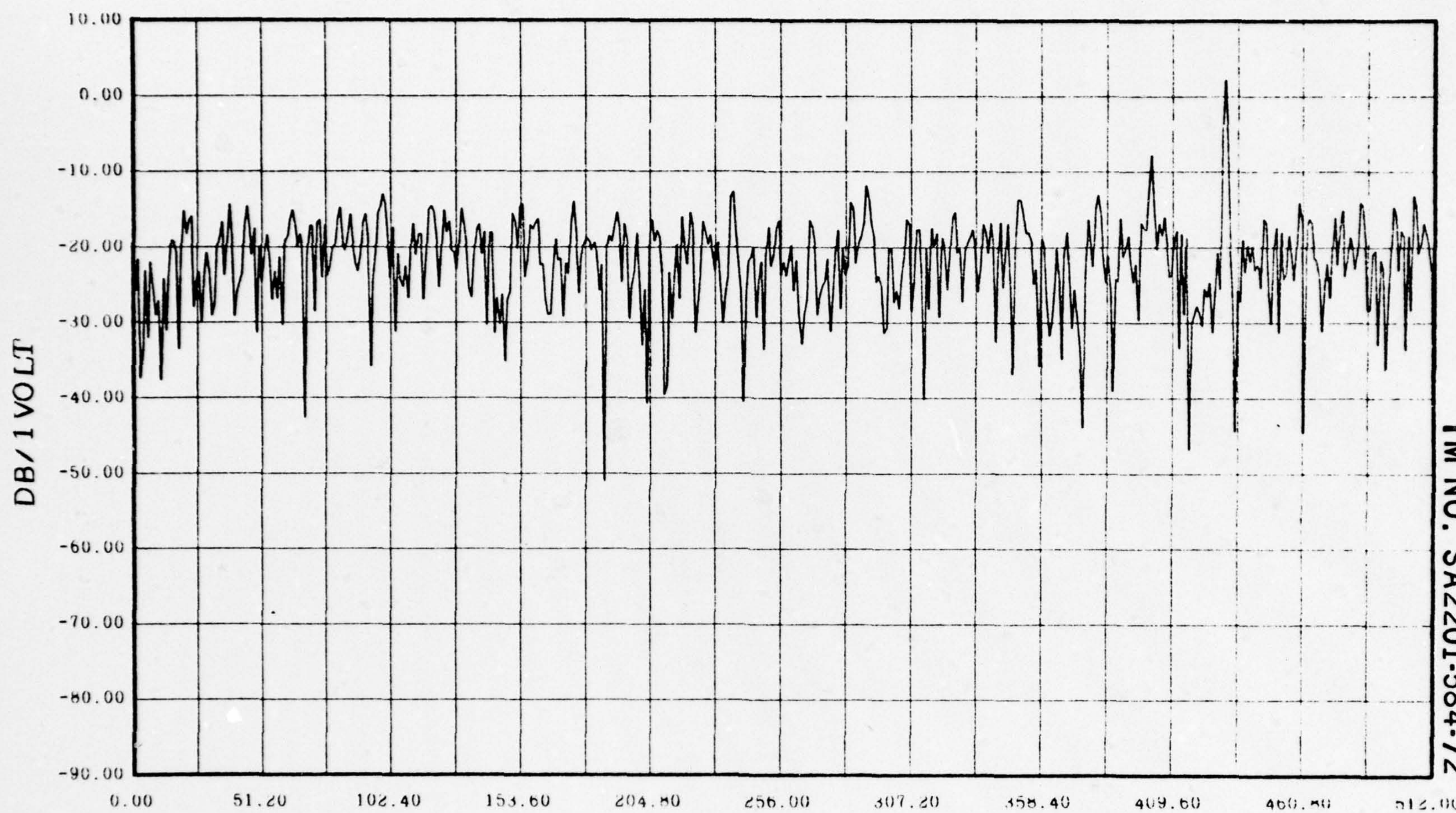
EFFECT OF COSINE WINDOWING, IN THE FREQUENCY DOMAIN, ON THE DEGENERATED ZOOM FFT WITH FOUR PARTITIONS

FIGURE 3.8



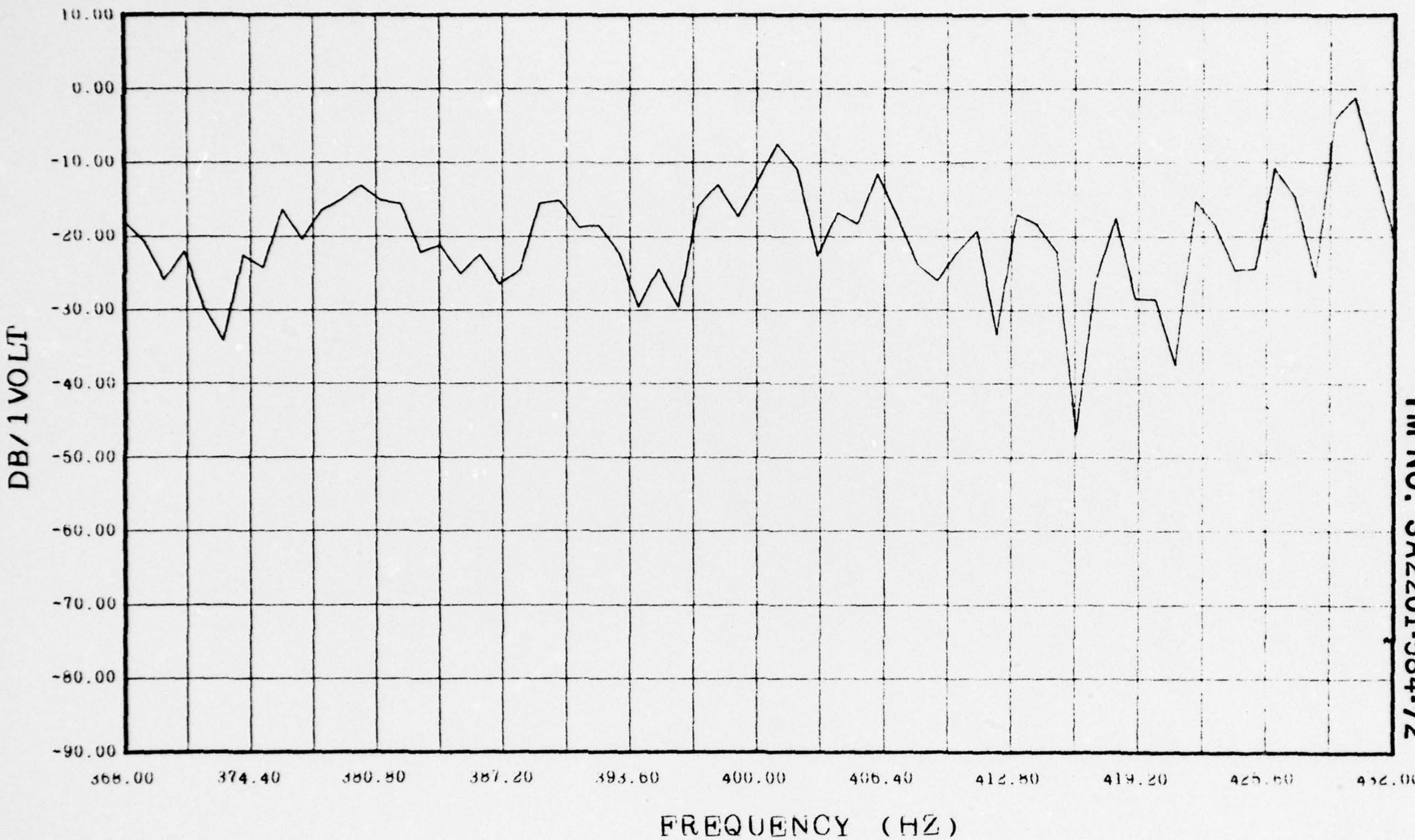
EFFECT OF COSINE WINDOWING, IN THE FREQUENCY DOMAIN, ON THE DEGENERATE ZOOM FFT WITH TWO PARTITIONS

FIGURE 3.9



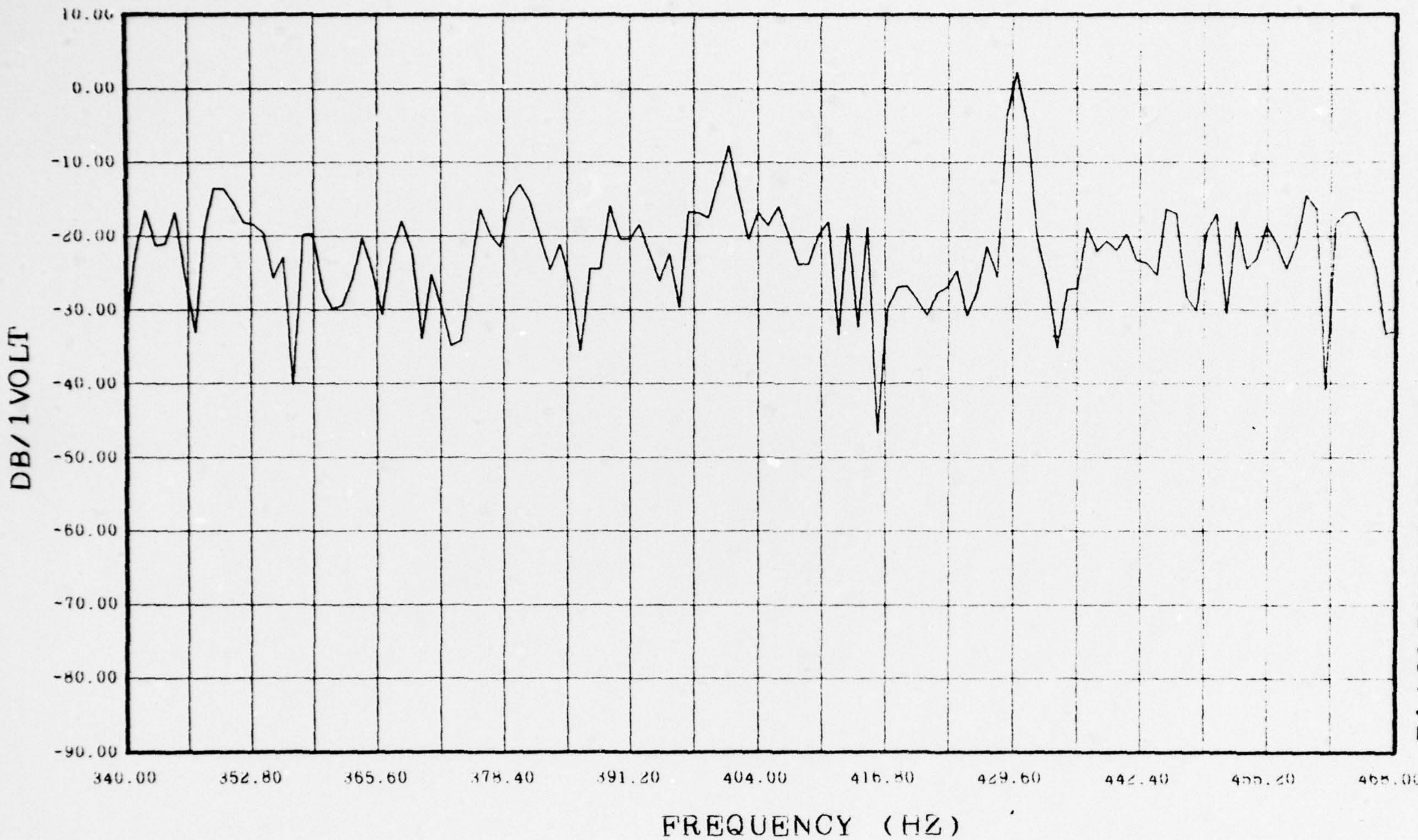
DFT OF TWO TONES PLUS NOISE

FIGURE 3.10



ZOOM FFT WITH VERNIER BANDWIDTH OF 64 HZ, FOR TWO TONES PLUS NOISE

FIGURE 3.11



ZOOM FFT WITH VERNIER BANDWIDTH OF 128 HZ, FOR TWO TONES PLUS NOISE

FIGURE 3.12

```
C     ZOOM FFT *** J.FERRIE , C.NAWROCKI *** JANUARY 3,1973.
C
C
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C     *** OPERATING INSTRUCTIONS *** ZOOM FFT

C     CARD 1 MUST BE REPEATED FOR EACH DATA SAMPLE
C
C     CARD  COLUMNS  FORMAT  ARGUMENT
C     ----  -------  ------  --------
C      1     1-10     I10       NNN - NUMBER OF INPUT VALUES TO BE PROCESSED
C           11-20     I10       ISR - INTEGER SAMPLING RATE
C           21-30     I10      NPAR - NUMBER OF PARTITIONS
C           31-40     I10     LFBIN - LOWER FREQUENCY BIN OF INTEREST
C           41-50     I10     UFBIN - HIGHER FREQUENCY BIN OF INTEREST
C           51-60     F10.5   FBASE - FREQUENCY OF VARIABLE TONE IN HERTZ
C           61-80     BLANK
C
C     LAST   1-80     BLANK
C
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C
C           SPECIFICATION AND TYPE STATEMENTS
C
      DIMENSION XX(1024),YY(1024),ZX(1024),ZY(1024)
      DIMENSION PHI(1024)
      DIMENSION TITLEX(3),TITLEY(3)
      COMMON AMODES(200)
      DIMENSION TIME(2)
      INTEGER UFBIN
      DEFINE F(II,KK)=II*KK+((1-SIGN(1,II*KK))/2)*34359738367
C
C
C           INITIALIZE PROGRAM UNITS AND IGS PLOTTER
C
      INCARD=3
      IPRNTR=4
      CALL MODESU(AMODES,0)
      ICASE=1
C
C
C           SET UP DB RANGE AND X,Y TITLES FOR PLOTS
C
      DBMAX=10.0
      DBMIN=-90.0
      TITLEX(1)='  FREQ'
      TITLEX(2)='UENCY '
      TITLEX(3)='(HZ)  '
      TITLEY(1)='DB//1 '
      TITLEY(2)='VOLTS'
      TITLEY(3)='0./HZ '
C
```

```
C
C          READ CONTROL PARAMETERS FROM DATA CARD 1
C
  100 READ(INCARD,102) NNN,ISR,NPAR,LFBIN,UFBIN,FBASE
  102 FORMAT(5I10,F10.5)
      IF(NNN.GT.1024) STOP NNN
      IF(NNN.EQ.0) GO TO 900
      WRITE(IPRNTR,104)NNN,ISR,NPAR,LFBIN,UFBIN,FBASE
  104 FORMAT(1H1,///10X,'NNN =',I6,5X,'ISR =',I8,5X,'NPAR =',I5,5X,'LFBI
     1N =',I5,5X,'UFBIN =',I5,5X,'FBASE =',F10.5,///)
C          CALCULATE CONSTANTS
C
      DF=FLOAT(ISR)/FLOAT(NNN)
      DT=1.0/FLOAT(ISR)
      DFF=DF*FLOAT(NPAR)
      CONST=DT/FLOAT(NNN)
      ISPAR=NNN/NPAR
      IDIFF=UFBIN-LFBIN
      IDIFF1=IDIFF+1
      IDIFF2=2*IDIFF
      NRES=IDIFF2*NPAR
      NDIS=NRES/2+1
      FLOW=FLOAT(LFBIN-1)*DFF
      FHIGH=FLOW+NDIS*DF-1.0
      FIDIFF=1.0/FLOAT(IDIFF2)
      WRITE(IPRNTR,110) ISPAR,IDIFF2,NRES,NDIS,FLOW,FHIGH,DF
  110 FORMAT(/,10X,'ISPAR =',I5,4X,'IDIFF2 =',I5,4X,'NRES =',I5,4X,'NDIS
     1 =',I5,4X,'FLOW =',F10.3,4X,'FHIGH =',F10.3,4X,'DF =',F12.8,///)
      IF(IDIFF2.GT.ISPAR) STOP DIF2
C
C
C          GENERATE INPUT DATA
C
      WRITE(IPRNTR,120)
  120 FORMAT(///10X,'INPUT DATA',///)
      PI=3.141592654
      GAIN=SQRT(10.0)
      FREQ2=430.0
      KK=5**15
      II=5281
      GAIN2=SQRT(10.0)
      DO 157 J=1,NNN
      XX(J)=SIN(2.0*PI*FBASE*DT*FLOAT(J))
      TEMP=GAIN*SIN(2.0*PI*FREQ2*FLOAT(J)*DT)
      XX(J)=XX(J)+TEMP
      II=F(II,KK)
      V=TINORM(FLOAT(II)/34359738367.,$126)
      GO TO 128
  126 CONTINUE
      WRITE(IPRNTR,127)
  127 FORMAT(1H ,'PROBLEM')
  128 XX(J)=XX(J)+V*GAIN2
      YY(J)=0.0
      ZX(J)=XX(J)
      ZY(J)=YY(J)
  157 CONTINUE
      NTOPRT=200
      IF(NNN.LT.NTOPRT) NTOPRT=NNN
      WRITE(IPRNTR,224) (XX(J),YY(J),J,J=1,NTOPRT)
  224 FORMAT(5(2F9.5,I4,4X))
```

```
C
C
C         COSINE WINDOW INPUT DATA
C
      CALL COSMTH(XX,NNN)
      IF(ICASE.GT.1) GO TO 230
      CALL COSMTH(ZX,NNN)
C
C
C         COMPUTE DFT OF SIGNAL TO CHECK ZOOM FFT ALGORITHM
C
      CALL FIRST
      CALL FFT(ZX,ZY,NNN,NNN,NNN,-1)
      CALL SECOND(TIME)
      WRITE(IPRNTR,225)
  225 FORMAT(////,' DFT CHECK CASE')
      WRITE(IPRNTR,242)
      ND2P1=NNN/2+1
      DO 226 J=1,ND2P1
      PHI(J)=CONST*(ZX(J)**2+ZY(J)**2)
      FREQ=FLOAT(J-1)*DF
      DB=10.0*ALOG10(MAX(PHI(J),1.0E-30))
      WRITE(IPRNTR,248) ZX(J),ZY(J),J,FREQ,PHI(J),DB
      PHI(J)=DB
  226 CONTINUE
      WRITE(IPRNTR,344) TIME
      CALL LINPLT(0.0,DBMIN,FREQ,DBMAX,PHI,1,ND2P1,TITLEX,TITLEY,0)
      CALL PAGEG(AMODES,0,1,1)
      ICASE=2
  230 CONTINUE
C
C
C         CALCULATE ZOOM FFT
C
      CALL FIRST
      DO 240 K=1,NPAR
      ISPAR1=ISPAR*(K-1)
      ISPAR2=IDIFF2*(K-1)
      INDEX1=ISPAR1+1
      INDEX2=ISPAR1+LFBIN-1
C         FORWARD FFT OF INDIVIDUAL PARTITIONS
      CALL FFT(XX(INDEX1),YY(INDEX1),ISPAR,ISPAR,ISPAR,-1)
C         PICK OUT DESIRED BINS AND TAKE INVERSE FFT
      DO 234 L=1,IDIFF1
      ZX(L)=XX(INDEX2+L)
      ZY(L)=YY(INDEX2+L)
  234 CONTINUE
C         GENERATE EVEN AND ODD SYMMETRY BEFORE INVERSE FFT
      ZY(IDIFF1)=0.0
      DO 238 L=2,IDIFF
      NEG=IDIFF2-L+2
      ZX(NEG)=ZX(L)
      ZY(NEG)=-ZY(L)
  238 CONTINUE
      CALL FFT(ZX,ZY,IDIFF2,IDIFF2,IDIFF2,+1)
      DO 239 I=1,IDIFF2
      XX(ISPAR2+I)=ZX(I)*FIDIFF
      YY(ISPAR2+I)=ZY(I)*FIDIFF
  239 CONTINUE
  240 CONTINUE
```

```
C             FORWARD FFT OF LFBIN TO UFBIN BINS
      CALL FFT(XX,YY,NRES,NRES,NRES,-1)
      CALL SECOND(TIME)
C


C
C             DISPLAY RESULTS
C
      WRITE(IPRNTR,242)
  242 FORMAT(///10X,'COMPLEX COEFFICIENTS',//12X,'REAL',16X,'IMAG',13X,'
     1M',7X,'FREQ',11X,'POWER',10X,'DB',/)
      DO 250 J=1,NDIS
      PHI(J)=CONST*(XX(J)**2+YY(J)**2)
      FREQ=FLOW+FLOAT(J-1)*DF
      DB=10.0*ALOG10(MAX(PHI(J),1.0E-30))
      WRITE(IPRNTR,248) XX(J),YY(J),J,FREQ,PHI(J),DB
  248 FORMAT(2F20.8,110,3F15.8)
      PHI(J)=DB
  250 CONTINUE
      WRITE(IPRNTR,344) TIME
  344 FORMAT(///,10X,'EXECUTION TIME ',2A6,' SECONDS')
      CALL LINPLT(FLOW,DBMIN,FHIGH,DBMAX,PHI,1,NDIS,TITLEX,TITLEY,0)
      CALL PAGEG(AMODES,0,1,1)
C
C
C             GO BACK FOR NEXT CASE
C
      GO TO 100
C
C
C             TERMINATE PROGRAM
C
  900 CALL EXITG(AMODES)
      STOP ZOOM
      END


C *** SUBROUTINE COSMTH *** NORMALIZED TIME DOMAIN COSINE WINDOW  A.H.NUTTALL
      SUBROUTINE COSMTH(XX,NNN)
      DIMENSION XX(1)
      T=6.283185307/FLOAT(NNN)
      DO 1 I=1,NNN
    1 XX(I)=XX(I)*(1.-COS(T*FLOAT(I-1)))*0.816496581
      RETURN
      END
```

APPENDIX B
DFT CASE

391.00000000	.00000000	-146.13865280
392.00000000	.00000000	-148.12798309
393.00000000	.00000000	-145.08983803
394.00000000	.00000000	-132.95911789
395.00000000	.00000000	-131.25646591
396.00000000	.00000000	-137.40895081
397.00000000	.00000000	-123.81223202
398.00000000	.00000000	-120.45800877
399.00000000	.00000000	-121.02896309
400.00000000	.04166749	-13.80202663
401.00000000	.16666679	-7.78150934
402.00000000	.04166590	-13.80219221
403.00000000	.00000000	-124.35157871
404.00000000	.00000000	-133.63653946
405.00000000	.00000000	-130.35192490
406.00000000	.00000000	-131.80841446
407.00000000	.00000000	-134.32049370
408.00000000	.00000000	-143.15020752
409.00000000	.00000000	-142.63742447
410.00000000	.00000000	-134.74664497
411.00000000	.00000000	-132.38457108
412.00000000	.00000000	-138.01848412
413.00000000	.00000000	-131.02366638
414.00000000	.00000000	-125.16299820
415.00000000	.00000000	-131.05418968
416.00000000	.00000000	-138.69378090
417.00000000	.00000000	-136.85265732
418.00000000	.00000000	-133.30390549
419.00000000	.00000000	-131.73151016
420.00000000	.00000000	-133.18720627
421.00000000	.00000000	-132.74911880
422.00000000	.00000000	-123.90300465
423.00000000	.00000000	-128.11253548
424.00000000	.00000000	-130.92435265
425.00000000	.00000000	-124.83635426
426.00000000	.00000000	-120.85968781
427.00000000	.00000000	-120.61209965
428.00000000	.00000000	-121.37391567
429.00000000	.41667493	-3.80202630
430.00000000	1.66666658	2.21848726
431.00000000	.41665835	-3.80219907
432.00000000	.00000000	-121.91281891
433.00000000	.00000000	-120.18064499
434.00000000	.00000000	-121.98831367
435.00000000	.00000000	-126.57015610
436.00000000	.00000000	-129.03431892
437.00000000	.00000000	-135.20573616
438.00000000	.00000000	-124.93441391
439.00000000	.00000000	-130.26271057
440.00000000	.00000000	-135.79203606

ZOOM FFT WITH VERNIER BANDWIDTH
OF 16 HZ; 1 HZ RESOLUTION

FREQ	POWER	DB
392.00000000	.00000000	-148.22646713
393.00000000	.00060436	-32.18701506
394.00000000	.00066398	-31.77844524
395.00000000	.00038748	-34.11746168
396.00000000	.00000000	-137.35529900
397.00000000	.00087759	-30.56709361
398.00000000	.00027502	-35.60634804
399.00000000	.00010563	-39.76223183
400.00000000	.04166749	-13.80202675
401.00000000	.14367197	-8.42627954
402.00000000	.04155525	-13.81374073
403.00000000	.00005242	-42.80472851
404.00000000	.00000000	-133.67460251
405.00000000	.00281936	-25.49850011
406.00000000	.00221985	-26.53676772
407.00000000	.00084729	-30.71970296
408.00000000	.00000000	-148.15300941

ZOOM FFT WITH VERNIER BANDWIDTH
OF 32 HZ; 1 HZ RESOLUTION

FREQ	POWER	DB
384.00000000	.00000000	-143.58618164
385.00000000	.00005125	-42.90313005
386.00000000	.00293665	-25.32147932
387.00000000	.00021421	-36.69154072
388.00000000	.00000000	-153.53866386
389.00000000	.00005102	-42.92224503
390.00000000	.00240252	-26.19332552
391.00000000	.00015874	-37.99326468
392.00000000	.00000000	-148.06138992
393.00000000	.00007590	-41.19755411
394.00000000	.00204946	-26.88360810
395.00000000	.00015091	-38.21289301
396.00000000	.00000000	-137.40611649
397.00000000	.00012189	-39.14017534
398.00000000	.00173565	-27.60538626
399.00000000	.00016308	-37.87609577
400.00000000	.04166749	-13.80202663
401.00000000	.15651207	-8.05452156
402.00000000	.05712260	-12.43192029
403.00000000	.00018715	-37.27802038
404.00000000	.00000000	-133.69916153
405.00000000	.00035623	-34.48267174
406.00000000	.00095036	-30.22111678
407.00000000	.00022076	-36.56080675
408.00000000	.00000000	-143.19004631
409.00000000	.00069765	-31.56365204
410.00000000	.00046813	-33.29635000
411.00000000	.00027215	-35.65199280
412.00000000	.00000000	-137.96994781
413.00000000	.00180507	-27.43505621
414.00000000	.00261760	-25.82096124
415.00000000	.00117454	-29.30130863
416.00000000	.00000000	-142.34171867

ZOOM FFT WITH VERNIER BANDWIDTH
OF 64 HZ; 1 HZ RESOLUTION

375.00000000	.00074553	-31.27533031
376.00000000	.00000000	-144.56781960
377.00000000	.00031923	-34.95895863
378.00000000	.00906725	-20.42524529
379.00000000	.00078177	-31.06918693
380.00000000	.00000000	-148.43681908
381.00000000	.00029906	-35.24242973
382.00000000	.00905400	-20.43159628
383.00000000	.00083644	-30.77565813
384.00000000	.00000000	-143.71667480
385.00000000	.00028903	-35.39052916
386.00000000	.00924582	-20.34054399
387.00000000	.00091046	-30.40737224
388.00000000	.00000000	-154.66052818
389.00000000	.00028675	-35.42501068
390.00000000	.00963630	-20.16089869
391.00000000	.00100819	-29.96458721
392.00000000	.00000000	-148.61285400
393.00000000	.00029138	-35.35540724
394.00000000	.01024772	-19.89372516
395.00000000	.00113724	-29.44146633
396.00000000	.00000000	-137.52787018
397.00000000	.00030311	-35.18393993
398.00000000	.01113142	-19.53449607
399.00000000	.00131020	-28.82661724
400.00000000	.04166749	-13.80202675
401.00000000	.17847740	-7.48416781
402.00000000	.09828197	-10.07526124
403.00000000	.00154734	-28.10414982
404.00000000	.00000000	-133.78762436
405.00000000	.00035306	-34.52148199
406.00000000	.01413954	-18.49564648
407.00000000	.00188445	-27.24815679
408.00000000	.00000000	-143.04985237
409.00000000	.00039742	-34.00747442
410.00000000	.01669311	-17.77462769
411.00000000	.00238694	-26.22159362
412.00000000	.00000000	-138.04380608
413.00000000	.00046325	-33.34181929
414.00000000	.02055604	-16.87060618
415.00000000	.00318809	-24.96469426
416.00000000	.00000000	-138.77390671
417.00000000	.00056520	-32.47796774
418.00000000	.02681831	-15.71568513
419.00000000	.00459815	-23.37416792

ZOOM FFT WITH VERNIER BANDWIDTH
OF 128 HZ; 1 HZ RESOLUTION

392.00000000	.00000000	-148.16699600
393.00000000	.00000633	-51.98672581
394.00000000	.00017162	-37.65429115
395.00000000	.00003836	-44.16107035
396.00000000	.00000000	-137.40612030
397.00000000	.00000655	-51.83911848
398.00000000	.00017081	-37.67474937
399.00000000	.00003879	-44.11311865
400.00000000	.04166749	-13.80202687
401.00000000	.16495706	-7.82629091
402.00000000	.04677598	-13.29977119
403.00000000	.00003936	-44.04945660
404.00000000	.00000000	-133.66323853
405.00000000	.00000724	-51.40287304
406.00000000	.00017198	-37.64523792
407.00000000	.00004013	-43.96532536
408.00000000	.00000000	-143.28040886
409.00000000	.00000774	-51.11419535
410.00000000	.00017394	-37.59594727
411.00000000	.00004108	-43.86320400
412.00000000	.00000000	-137.98767090
413.00000000	.00000837	-50.77159023
414.00000000	.00017682	-37.52461386
415.00000000	.00004224	-43.74276209
416.00000000	.00000000	-139.13274765
417.00000000	.00000917	-50.37527418
418.00000000	.00018067	-37.43108559
419.00000000	.00004362	-43.60283470
420.00000000	.00000000	-133.16780281
421.00000000	.00001019	-49.91934681
422.00000000	.00018555	-37.31549263
423.00000000	.00004525	-43.44398117
424.00000000	.00000000	-130.86851501
425.00000000	.00001148	-49.40131044
426.00000000	.00019158	-37.17651129
427.00000000	.00004717	-43.26312876
428.00000000	.00000000	-121.38550854
429.00000000	.42011832	-3.76628375
430.00000000	1.63444418	2.13370091
431.00000000	.40855811	-3.88746163
432.00000000	.00000000	-121.93789768
433.00000000	.00001527	-48.16065788
434.00000000	.00020800	-36.81927919
435.00000000	.00005199	-42.84105206
436.00000000	.00000000	-129.02977180
437.00000000	.00001809	-47.42601252
438.00000000	.00021912	-36.59309101
439.00000000	.00005502	-42.59447956
440.00000000	.00000000	-135.82561111

APPENDIX C

It is apparent from the report, that for a fixed FFT and a desired resolution, the largest Vernier bandwidth possible should be used to improve accuracy. If the algorithm is to be exercised in real time, a constraing to the bandwidth is made by the hardware multiply times.

We can model the FFT^{-1} and FFT respectively; by [C-1]

$$LM\ln L + LM\ln LM = Tm \qquad [C\text{-}1]$$

Where L is the vernier bandwidth
M is the number of juxaposed sequences
and Tm is the total number of multiplies

The above equation is unseperable by ordinary techniques, however we allow M to take on interger values (M = 1, 2, 4, 8, 16, 32) and increment L, so that the product of $LM < 1024$ (where 1024 is a typical hardware FFT size), constant multiply lines can be drawn, as shown in Figure C-1.

Thus, if the per multiply time of the processor was α, and a 8 times finer resolution were desired, by multiplying α times point picked from the constant multiply so that the real time constraint was satisfactory and the largest modulo -2 L was selected.

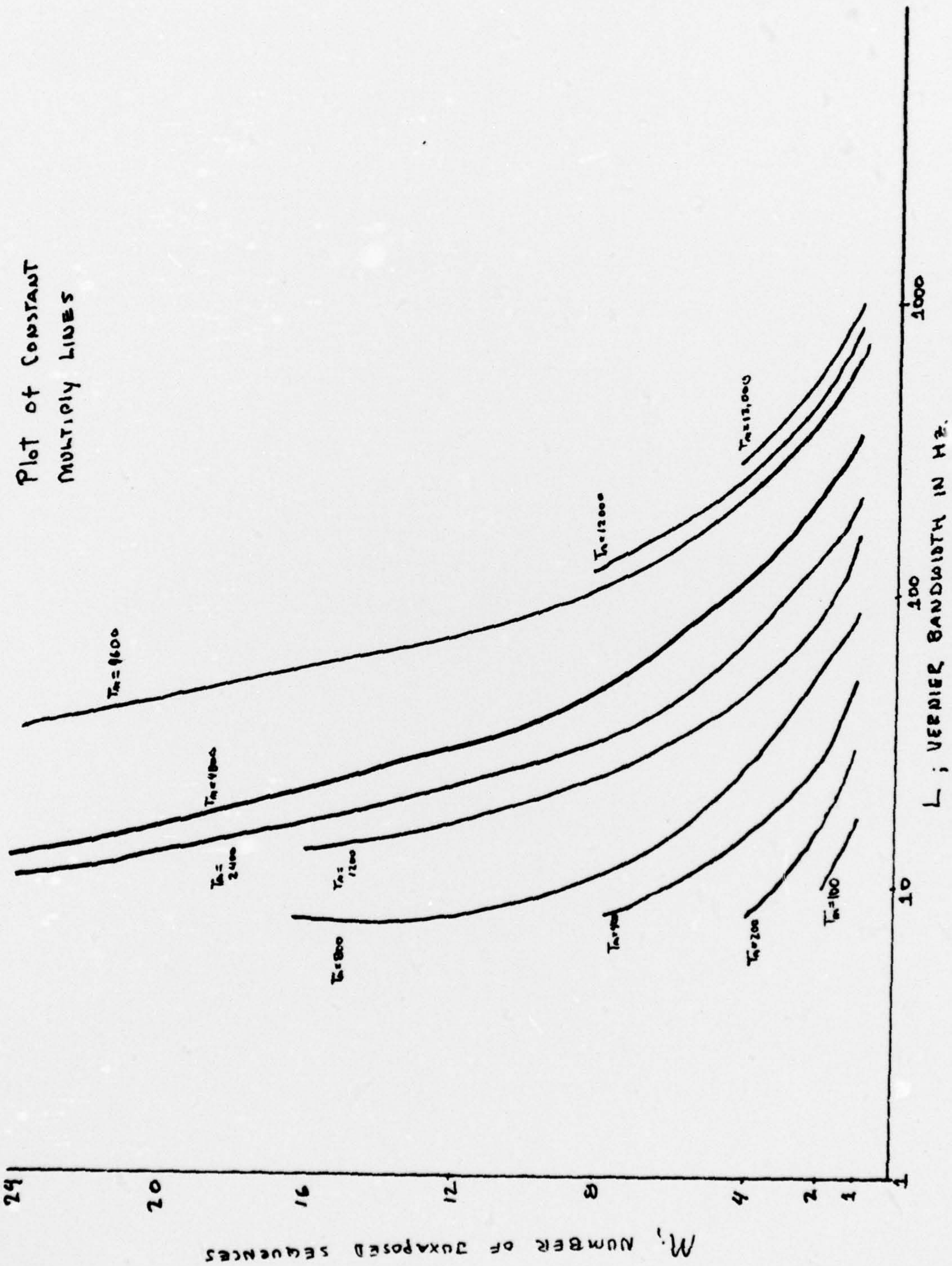

Plot of Constant
Multiply Lines
L, Vernier Bandwidth in Hz.
M, Number of Juxtaposed Sequences
1
10
100
1000
24
20
16
12
8
4
2
1

Figure C-1