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THE DETERMINATION OF DEFLECTION AND STRESS DISTRIBUTION FOR A TRANSPARENT LAMINATED BEAM

Douglas Aircraft Company
McDonnell Douglas Corporation
3855 Lakewood Boulevard
Long Beach, California 90846

MARCH 1977

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Air Force Flight Dynamics Laboratory
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Air Force Systems Command
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This report documents the theory and procedures for calculating the elastic deflection and stress distribution of a flat, laminated beam with fixed ends or simply-supported ends comprised of transparency materials subjected to a normal point load at the midpoint of the beam. The development of a computer program using the basic theory is presented along with illustrative examples and a "user's manual." The program results have been favorably compared with actual test data and with a finite element method.

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FOREWORD

This report is one of a series of reports that describes work performed by Douglas Aircraft Company, McDonnell Douglas Corporation, 3855 Lakewood Blvd., Long Beach, California 90846, under the Windshield Technology Demonstrator Program. This work was sponsored by the U.S. Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, under Contract F33615-75-C-3105, Project 2202/0201.

Captain D. C. Chapin (AFFDL/FEW) was the Air Force Project Manager who monitored the program and provided reference material on a timely basis.

Mr. J. H. Lawrence, Jr. was Program Director for the Douglas Aircraft Company.

Mr. P. H. Denke and Mr. J. B. Hoffman were the principal investigators and co-authors for this report.

Mr. P. H. Denke of Douglas Aircraft Company developed and solved the differential equations presented in this report.

Mr. J. F. Burke and Mr. R. C. Morris of Douglas Aircraft Company and Mr. L. Chahinian of the McDonnell Douglas Automation Company provided valuable assistance in the preparation of the Computer Program User's Manual in this report.

Mr. J. C. Thomsen of Douglas Aircraft Company developed the finite element model presented in this report.

Mr. T. Price of McDonnell Douglas Automation Company wrote the computer program described in this report.

Dr. P. Poirier (ASD/ADDS) and Dr. P. J. Nikolai (AFFDL/FB), Wright-Patterson Air Force Base, Dayton, Ohio, provided the EISPACK subroutines utilized in the Source Coding, Appendix B, in this report.

This report was submitted to the Air Force in November 1976 and covers work performed during the period of January 1976 and March 1977.

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LIST OF ABBREVIATIONS AND SYMBOLS

(Used in Analysis, Section II)

A	A square coefficient matrix in the governing matric differential equation
A_i	Cross-sectional area of i^{th} layer
$a_{1,2}$	k_A^{-1}
$a_{2,1}$	$\Delta_q \, k_\eta \, \Delta_q^T$
$a_{2,3}$	$-\Delta_q \, k_\eta \, \bar{t}$
$a_{5,4}$	$K_I^{-1} \bar{t}^T k_\eta \bar{t}$
$a_{5,2}$	$K_I^{-1} a_{2,3}^T a_{1,2}$
B	Coefficient matrix in the boundary condition equation
b	Beam width
C	A column matrix of arbitrary constants
$\bar{C}$	Column matrix of unit elements
C_{A_k}	Coefficient of k^{th} algebraic mode
C_{g_k}	$\bar{C}_{g_k} e^{\lambda g k \ell}$

C_k	k^{th} arbitrary constant
C_ℓ, C_g, C_A	Column matrices of the arbitrary constants C_{ℓ_k}, C_{g_k}, and C_{A_k}
C_{ℓ_k}, $\overline{C}_{g_k}$	Arbitrary constants in the expressions for $Y_{E\ell}$ and Y_{Eg}
e	Base of natural logarithms
E_i	Young's modulus of i^{th} layer
EI_{eff}	Effective beam bending stiffness
F	Load at beam centerline
$F_{g_k}(x)$	$e^{\lambda_{\ell k}(\ell - x)}$
$F_\ell(x)$, $F_g(x)$	Column matrices of the $F_{\ell_k}(x)$'s and the $F_{g_k}(x)$'s
$F_{\ell_k}(x)$	$e^{\lambda_{\ell k} x}$
$F_{\ell D}(x)$, $F_{gD}(x)$	$F_\ell(x)$, $F_g(x)$ diagonalized
G	An eigenvector of A
G_{g_k}	k^{th} eigenvector of A corresponding to λ_{g_k}

G_i	Shear modulus of the i^{th} layer
G_k	k^{th} eigenvector of A
G_{ℓ_k}	k^{th} eigenvector of A corresponding to λ_{ℓ_k}
G_{u_k}, G_{H_k}; G_{v_k}, G_{θ_k}; G_{β_k}, G_{ϕ_k}	Partitions of G_k corresponding to the partitions u, H, v, θ, β, and ϕ of Y
H	Column matrix of structural ply axial loads
H_i	Axial load in i^{th} layer
H_ℓ, H_g	Rectangular matrices consisting of the modal columns G_{ℓ_k} and G_{g_k}
I	Unit (identity) matrix
I_i	Cross-sectional moment of inertia of i^{th} layer
k_A	Diagonal matrix of structural ply axial stiffnesses
K_I	Sum of cross-sectional moments of inertia of structural plies
k_I	Column matrix of structural ply bending stiffnesses
k_ℓ	Column matrix of interlayer shear stiffnesses
k_η	Diagonal matrix of interlayer shear stiffnesses

L	Length of beam
ℓ	L/2
M	Column matrix of structural ply bending moments
M_i	Bending moment in i^{th} layer
M_{xq}	A matrix of structural ply thicknesses
0	Null matrix
P	F/2
Q	A Boolean integrating matrix
q	Column matrix of interlayer shear flows
q_i	Shear flow on lower surface of i^{th} layer
S	$a_{2,1}$
$S(x)$	Total shear on the beam cross section
$S_{1,1}, S_{1,2}$, $S_{2,1}, S_{2,2}$	Partitions of S
T	(Superscript) Indicates transposed matrix
T	${}^{k}A^{u}(0)$
$\bar{t}$	$t_\ell + \bar{t}_s$

t_i	Thickness of i^{th} layer
t_ℓ	Column matrix of interlayer thicknesses
t_s	Column matrix of structural ply thicknesses
$\bar{t}_s$	Column matrix of average structural ply thicknesses
T_1, T_2	Partitions of T
u	Column matrix of longitudinal displacements at layer centerlines
u_i	Longitudinal displacement at centerline of i^{th} layer
$u_{(0)}$	A column matrix of longitudinal displacements of structural ply centerlines in the rigid body translation parallel to x algebraic mode
$u_{(1)}$	A column matrix of longitudinal displacements of structural ply centerlines in the rigid body rotation about y algebraic mode
$u_{(2)}$	A column matrix of additional longitudinal displacements of structural ply centerlines in the shear parallel to z algebraic mode
$\bar{u}_{(2)}$	A partition of $u_{(2)}$
V	Column matrix of structural ply transverse shears
V_i	Shear in i^{th} layer

v_i, v	Vertical displacement at centerline of i^{th} layer
W	$k_A Q\bar{t}$
w_i	Normal load per unit length on lower surface of i^{th} layer
W_1, W_2	Partitions of W
x, y, z	Coordinates with origin at beam centerline
$Y_A(x)$	Rectangular matrix of the algebraic modes
$Y_{A_k}(x)$	k^{th} algebraic mode
$Y_{Au}(x), Y_{AH}(x)$, $Y_{Av}(x), Y_{A\theta}(x)$, $Y_{A\beta}(x), Y_{A\phi}(x)$	Partitions of $Y_A(x)$ corresponding to the partitions u, H, v, θ, β, and ϕ of Y
Y_E	A column matrix representing the exponential beam response
Y_{E_ℓ}	A column matrix representing exponential beam responses corresponding to eigenvalues λ_{ℓ_k}
Y_{E_g}	A column matrix representing exponential beam responses corresponding to eigenvalues λ_{g_k}
$Y(x)$	A column matrix of beam responses

α	A scalar coefficient of $u_{(0)}$ in the equation for $u_{(1)}$
β_i, β	Curvature of i^{th} layer = $d\theta_i/dx$
γ_i	Shear strain of i^{th} layer
Δ_η	A longitudinal displacement differencing matrix
Δq	A shear flow differencing matrix
Δ_{ts}	A Boolean coefficient matrix
$\bar{\Delta}_\eta$	Column matrix of longitudinal displacement differences at layer interfaces
ε_{x_i}	Axial strain of i^{th} layer
ζ_i	Vertical displacement of lower surface of i^{th} layer
η	Column matrix of longitudinal displacements at layer interfaces
η_i	Longitudinal displacement of lower surface of i^{th} layer
θ_i, θ	Slope of i^{th} layer = dv_i/dx
λ	An eigenvalue of A
λ_{g_k}	k^{th} eigenvalue of A greater than zero

λ_k	k^{th} eigenvalue of A
λ_{ℓ_k}	k^{th} eigenvalue of A less than zero
Σ	Summation sign
Σ_u	A Boolean coefficient matrix
Φ	$d\beta/dx$
$\bar{\Phi}_i$	Rotation of a vertical section of the i^{th} layer
ψ	Column matrix of constants in the boundary condition equation

SECTION I

INTRODUCTION

Bird impact hazards to high speed, low flying, aircraft have become one of the major flight safety problems of the jet age. The Douglas Aircraft Company's Windshield Technology Demonstrator Program is part of an effort by the Air Force Flight Dynamics Laboratory Improved Windshield Protection Program to develop technologies that will allow the design of aircraft transparent enclosures to increase protection against birdstrikes.

One of the basic objectives of the Windshield Technology Demonstrator Program is to develop windshield design technology for application to high performance military aircraft. Pursuant to this effort, a need arose for a simplified analytical method of rapidly assessing the structural effectiveness of candidate laminate configurations.

Typical laminated configurations consist of alternating structural plies and interlayers. Structural plies are composed of high modulus materials such as polycarbonate, glass or acrylic. Interlayers are composed of materials that may be less stiff by several orders of magnitude.

A realistic method of static analysis for a laminated beam carrying a concentrated load at the center, representing a bird impact, and having various types of end supports, would be a useful tool. Such a method should avoid the frequently made Bernoulli-Euler assumption that plane sections remain plain during deformation, because shear deformations in the soft interlayers are large and must be considered. The significance of these deformations is shown experimentally in Reference 1. Other rough assumptions that would unnecessarily diminish the effectiveness of the analysis should also be avoided. The method should accommodate laminates composed of nine layers or more, with different material properties for each layer.

Apparently no method meeting these requirements exists in the literature, although related work is described. Reference 2 contains an analysis of a three layered beam based on the assumption that the angle of rotation between a cross section of the center layer and its undeformed position is a constant factor times the slope along the length of the beam. This assumption is considered unnecessary, and its effects on the reliability of the results are hard to assess. The solution is based on the energy method, which introduces further approximations. Reference 3 presents methods of analysis of multiple layered beams based on the Bernoulli-Euler hypothesis; consequently, the approach is not applicable to beams with soft interlayers. Reference 4 offers an analysis of three layered beams involving simplifying assumptions which are also not considered acceptable for the present application.

Therefore, a new approach to the problem was developed, and is presented in subsequent sections of this report. The approach is based on the assumption that each structural ply can be treated as a beam to which the Bernoulli-Euler hypothesis is applicable, but no such assumption is applied to the cross section as a whole. Structural plies can bend and stretch, but shear deformations of these plies are considered negligible. Interlayers are assumed to carry shear, but not axial loads or bending moments. Deformations through the thickness of the beam are considered negligible, but stresses normal to the layers are assumed to exist. The equations of equilibrium and compatibility are written. The resulting set of differential equations is then expressed as a single matric differential equation, which is solved exactly.

The analytical results have been translated into an efficient Fortran program. This program applies to nine layer laminates, which is an adequate number in most cases. The program can be easily extended to cover more layers, if necessary. Fewer layers can be accommodated by introducing negligible stiffness properties (E and G) for some of the layers. The method applies to fixed ended beams, but the program can be modified to be applicable to other boundary conditions. Extension of the method to other loading conditions is possible.

The most significant simplification involved in the analysis is the assumption that deformations through the thickness are negligible. This assumption greatly simplifies the analysis without slighting the primary feature of windshield laminate behavior, which is the relative freedom of structural plies to slide past each other because of the low stiffness of interlayer materials. The only negative effect of the assumption is that stresses normal to the layers are not correctly predicted. This is believed to be a localized effect confined to the center and ends of the beam where loads are applied. The elimination of this assumption is a possible subject for additional research. An analysis which accounts for transverse deformations might provide data that would be useful in defining adhesive strength needed to prevent delamination in regions of high transverse loads.

Results of the analysis have been correlated with finite element results and test data as described in a subsequent section. The comparisons are good and verify the validity of the basic assumptions.

Although the present method is intended to apply to fixed ended beams, it can also be applied to a beam with pinned ends by considering a fixed ended beam twice as long as the beam under consideration. The data output by the computer program between the quarter points of the fixed ended beam is applicable to the pin ended case.

SECTION II

ANALYSIS

DERIVATION OF EQUATIONS

This analysis applies to the fixed ended laminated beam of Figure 1. As the figure shows, L is the length of the beam, and F is the concentrated load acting at the center.

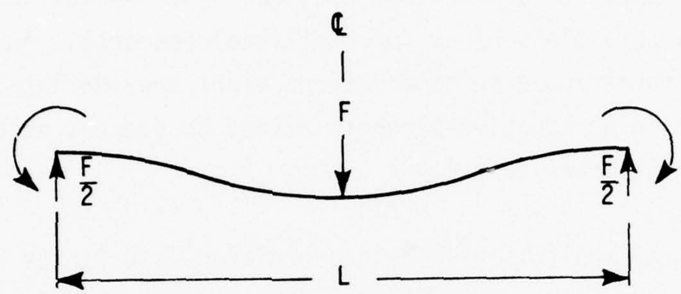


Figure 1. Fixed ended laminated beam.

The cross section is shown in Figure 2. The cross hatched layers are called "structural plies", while the other layers are "interlayers". The width of the beam is b, and the thickness of the ith layer is t_i. Different material properties can be assigned to each layer. The flexibilities of the interlayers are assumed to be large compared to the structural plies. The analysis can be applied to a beam having fewer than nine layers by assigning negligible values of Young's modulus, E, and the shear modulus, G, to some of the layers.

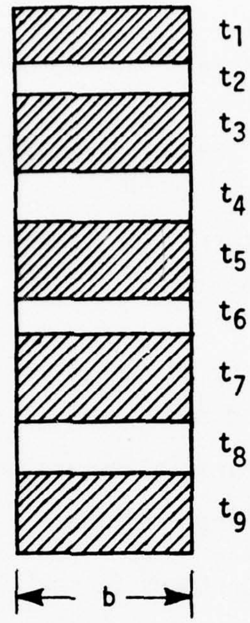


Figure 2. A typical transparency nine ply cross section.

Because of symmetry, consider one-half the beam as shown in Figure 3. In the figure $\ell = L/2$, and $P = F/2$. The figure also shows the xyz reference frame, and the definition of the vertical displacement v.

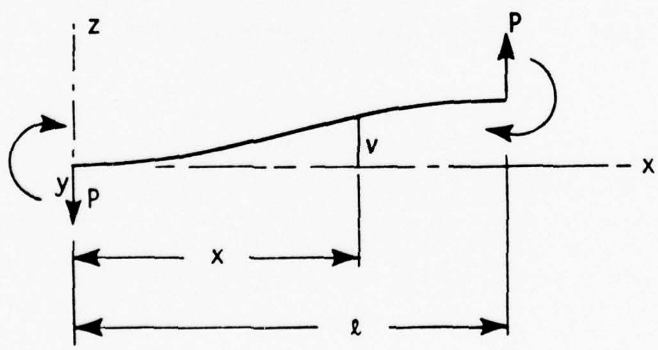


Figure 3. Half-beam.

Assumptions

1. Structural plies carry axial load, bending moment, shear, and normal load.

2. Interlayers carry shear and normal stresses only.

3. Plane sections of structural plies remain plane and normal to their elastic axes during deformation.

4. Normal strains through the thickness for structural plies and interlayers are negligible and can be considered equal to zero.

5. Shear strains for structural plies are negligible.

6. All stress-strain relations are linear.

7. Structural ply bending conforms to small displacement theory.

Equilibrium, Structural Ply Element, i Odd

Figure 4 shows the equilibrium of an element of the ith structural ply. H_i, V_i and M_i are the axial load, shear and bending moment acting on the element cross section. The transverse load per inch, w_i, and the shear flow, q_i, act upon the lower face. The corresponding forces on the upper face are w_{i-1} and q_{i-1}, since the i-1st element is next above. Note that i is odd for structural plies.

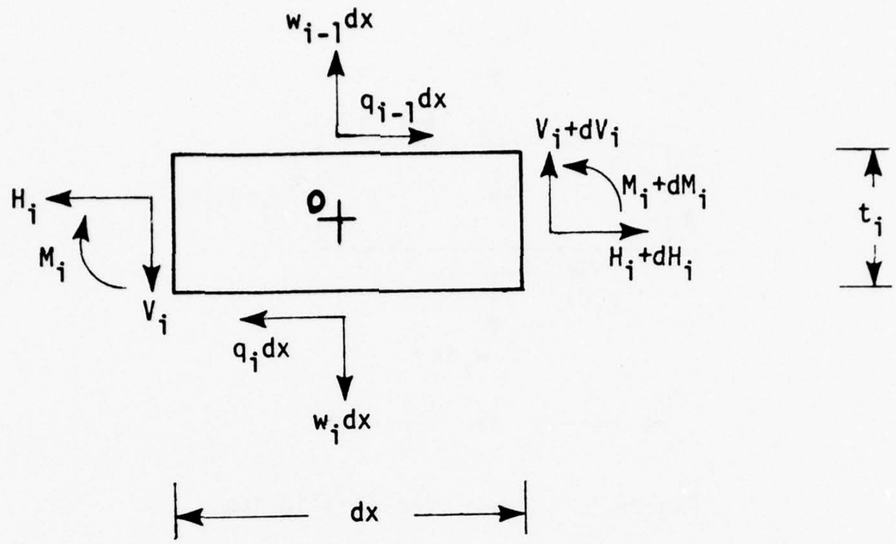


Figure 4. Structural ply equilibrium.

Summing forces in two directions and moments about 0 gives:

$$\begin{cases} \dfrac{dH_i}{dx} + q_{i-1} - q_i = 0 & \text{(i odd)} \qquad (1) \\ \dfrac{dV_i}{dx} + w_{i-1} - w_i = 0 & \text{(i odd)} \qquad (2) \\ \dfrac{dM_i}{dx} + V_i - q_{i-1}\dfrac{t_i}{2} - q_i\dfrac{t_i}{2} = 0 & \text{(i odd)} \qquad (3) \end{cases}$$

where $1/2\ dV_i$ has been deleted from Equation 3 to reduce complexity of problem. This is allowable due to insignificance of the term.

Equilibrium, Interlayer Element, i Even

Figure 5 shows the equilibrium of an element of the ith interlayer. The notation is consistent with the notation established for structural plies, although no axial loads or bending moments are acting, because of the assumption that these forces are negligible for interlayers. Note that i is even for these plies.

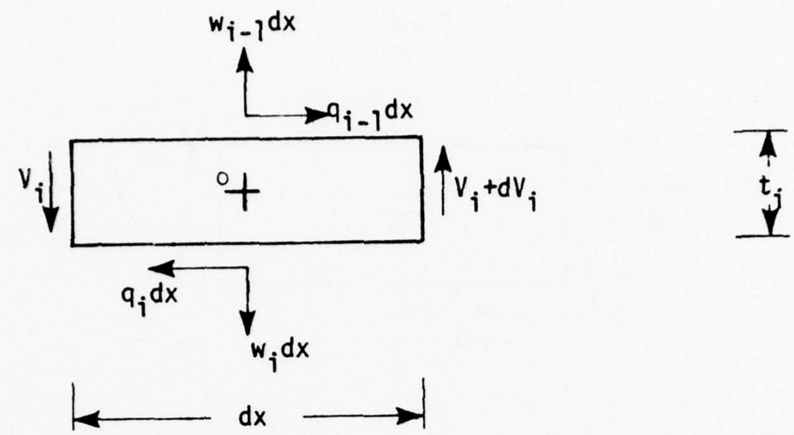


Figure 5. Interlayer equilibrium.

Summing forces in two directions and moments about 0 gives:

$$q_{i-1} - q_i = 0 \qquad \text{(i even)} \qquad (4)$$

$$\frac{dV_i}{dx} + w_{i-1} - w_i = 0 \qquad \text{(i even)} \qquad (5)$$

$$V_i - q_{i-1}\frac{t_i}{2} - q_i\frac{t_i}{2} = 0 \qquad \text{(i even)} \qquad (6)$$

From Equation 4, $q_{i-1} = q_i$, and, from Equation 6, $V_i = q_i t_i$.

Therefore, from Equation 5,

$$t_i\frac{dq_i}{dx} + w_{i-1} - w_i = 0 \qquad \text{(i even)} . \qquad (6a)$$

Also note that

$$q_0 = q_9 = 0 \qquad (6b)$$

$$w_0 = w_9 = 0 \qquad (6c)$$

since transverse forces and shear flows are assumed to be zero on the upper and lower surfaces of the beam.

Compatibility, Structural Plies, i Odd

Figure 6 shows the relationships between the displacements of the centerline of a structural ply and the displacements of the upper and lower surfaces of the same layer. In the figure, u_i , v_i and θ_i are the longitudinal and transverse displacements and the slope of the centerline of the ith ply. The longitudinal and transverse displacements of the lower surface of the ith ply are denoted η_i and ζ_i. Consequently the corresponding displacements of the upper surface are η_{i-1} and ζ_{i-1} , since the upper surface of the ith ply is the lower surface of the i-1st ply.

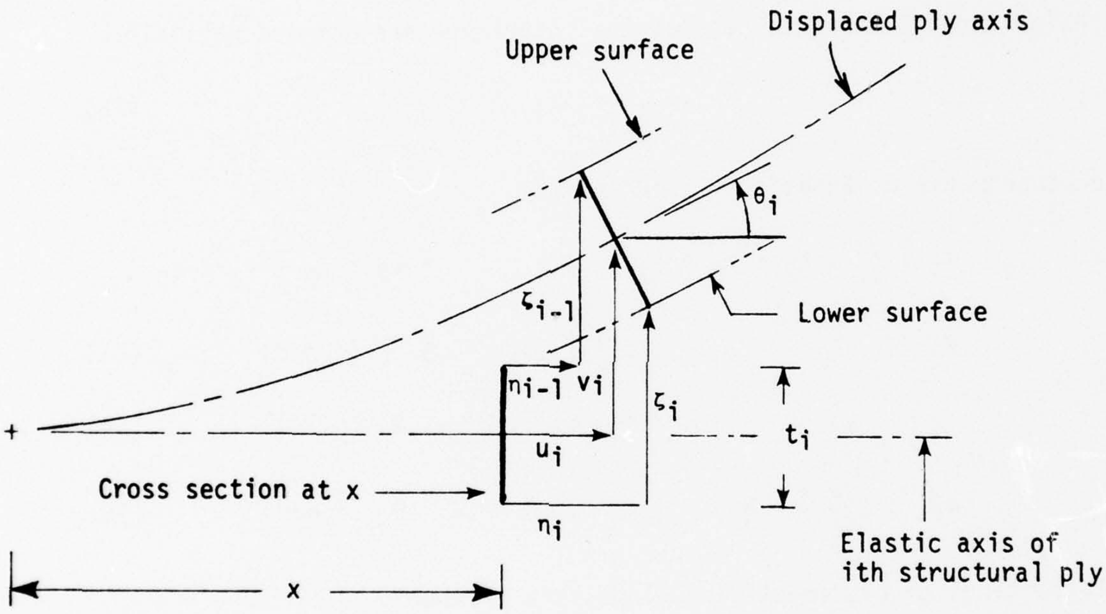


Figure 6. Compatibility.

Then from Figure 6, $$\eta_{i-1} = u_i - \theta_i \frac{t_i}{2} \quad (7)$$

and $$\eta_i = u_i + \theta_i \frac{t_i}{2} \text{ for small displacements.} \quad (8)$$

These equations are consistent with the assumption that cross sections of the structural ply remain normal to the elastic axis during deformation.

Also from Figure 6,

$$\zeta_{i-1} = v_i \ , \qquad (9)$$

$$\zeta_i = v_i \ , \qquad (10)$$

and $$\theta_i = \frac{dv_i}{dx} \ . \qquad (11)$$

Equations 9 and 10 hold because strains through the thickness of structural plies are considered negligible. Equation 11 is consistent with the assumption that structural ply bending conforms to the theory of small displacements.

Compatibility, Interlayers, i Even.

Strains through the thickness of the interlayer are assumed negligible.

$$\therefore \qquad \zeta_{i-1} = \zeta_i \ . \qquad (12)$$

In consequence of Equations 9, 10 and 12:

$$\zeta_0 = v_1 = \zeta_1 = \zeta_2 = v_3 = \zeta_3 = \zeta_4 = \ \cdots \ = \zeta_7 = \zeta_8 = v_9 = \zeta_9 \ ,$$

$$\therefore \qquad v_i = v \qquad i = 1, 3, \ldots 9 \quad (i \text{ odd}) \qquad (13)$$

$$\zeta_i = v \qquad i = 0, 1, 2 \ldots 9 \quad (\text{all } i) \qquad (14)$$

$$\theta_i = \frac{dv}{dx} = \theta \qquad i = 1, 3 \ \ldots \ 9 \quad (i \text{ odd}) \ . \qquad (15)$$

All vertical displacements and slopes are the same.

Strains, Structural Plies, i Odd.

Longitudinal strain, ε, is equal to the derivative of longitudinal displacement:

$$\varepsilon_{x_i} = \frac{du_i}{dx} \qquad \text{axial strain} \ .$$

In small displacement theory, curvature β is equal to the rate of change of slope, therefore,

$$\beta_i = \frac{d\theta_i}{dx} = \frac{d\theta}{dx} = \beta \ . \qquad (15a)$$

The notation, $\frac{d\beta}{dx} = \phi$, (15b)
to denote the rate of change of beam curvature, is useful subsequently.

Strains, Interlayers, i Even

Figure 7 shows the relationships between the shear strain of the ith interlayer, γ_i , the slope of the beam, θ , and the longitudinal displacement of the ith interlayer upper and lower surfaces. In the figure, $\overline{\phi}_i$ denotes the rotation of a cross section of the ith interlayer relative to its undisplaced position.

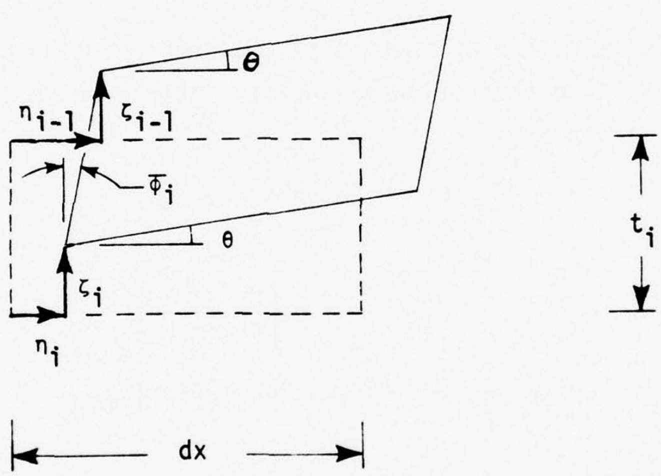


Figure 7. Interlayer shear strain.

From Figure 7, $\gamma_i = \theta + \overline{\phi}_i$ (15c)

$$\therefore \quad \gamma_i = \frac{dv}{dx} + \frac{\eta_{i-1} - \eta_i}{t_i} \ . \qquad (15d)$$

Force-Deformation Relations

The longitudinal force on the ith interlayer is given in terms of the longitudinal strain by

$$H_i = E_i A_i \epsilon_{x_i} .$$

$$\therefore \quad H_i = E_i A_i \frac{du_i}{dx} \quad \text{(i odd)} \tag{16}$$

where E_i and A_i are the values of Young's modulus and the cross-sectional area of the ith structural ply. The bending moment in the ith structural ply is given in terms of the curvature by

$$M_i = E_i I_i \beta_i .$$

$$\therefore \quad M_i = E_i I_i \frac{d^2v}{dx^2} \quad \text{(i odd)} \tag{17}$$

where I_i is the moment of inertia of the cross section of the ith structural ply. The shear flow in the ith interlayer is given in terms of the shear strain by

$$q_i = G_i b \gamma_i .$$

$$\therefore \quad q_i = G_i b \left(\frac{dv}{dx} + \frac{\eta_{i-1} - \eta_i}{t_i} \right) \quad \text{(i even)} \tag{18}$$

where G_i is the shear modulus of the ith interlayer.

Matric Formulation

Equation 1 can be written for each structural ply thus:

$$\frac{dH_1}{dx} + q_0 - q_1 = 0$$

$$\frac{dH_3}{dx} + q_2 - q_3 = 0$$

$$\frac{dH_5}{dx} + q_4 - q_5 = 0$$

$$\frac{dH_7}{dx} + q_6 - q_7 = 0$$

$$\frac{dH_9}{dx} + q_8 - q_9 = 0$$

But, from Equations 4 and 6b:

$$q_0=0 \qquad q_1=q_2 \qquad q_3=q_4 \qquad q_5=q_6 \qquad q_7=q_8 \qquad q_9=0$$

$$\therefore \quad \frac{dH_1}{dx} - q_2 = 0 \, ,$$

$$\frac{dH_3}{dx} + q_2 - q_4 = 0 \, ,$$

$$\frac{dH_5}{dx} + q_4 - q_6 = 0 \, ,$$

$$\frac{dH_7}{dx} + q_6 - q_8 = 0 \, ,$$

$$\frac{dH_9}{dx} + q_8 = 0 \, .$$

Writing these equations in matric form gives:

$$\underbrace{\begin{Bmatrix} \frac{dH_1}{dx} \\ \frac{dH_3}{dx} \\ \frac{dH_5}{dx} \\ \frac{dH_7}{dx} \\ \frac{dH_9}{dx} \end{Bmatrix}}_{\frac{dH}{dx}} + \underbrace{\begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}}_{\Delta_q} \underbrace{\begin{Bmatrix} q_2 \\ q_4 \\ q_6 \\ q_8 \end{Bmatrix}}_{q} = 0 \qquad (18a)$$

Equation 18a has been derived in detail to show the method of converting the scalar differential equations into matric form. Subsequent matric equations are derived in a similar manner, although the details are omitted.

The matrices in Equation 18a are denoted according to the symbols written under the equation. Thus, H is a column matrix of structural ply longitudinal forces, dH/dx is its derivative, q is a column matrix of interlayer shear flows, and Δ_q is a Boolean differencing matrix.

Therefore,

$$\frac{dH}{dx} + \Delta_q\, q = 0 \,. \tag{19}$$

From Equations 3, 4 and 6b,

$$\underset{\dfrac{dM}{dx}}{\begin{Bmatrix} \frac{dM_1}{dx} \\ \frac{dM_3}{dx} \\ \frac{dM_5}{dx} \\ \frac{dM_7}{dx} \\ \frac{dM_9}{dx} \end{Bmatrix}} + \underset{V}{\begin{Bmatrix} V_1 \\ V_3 \\ V_5 \\ V_7 \\ V_9 \end{Bmatrix}} - \underset{M_{xq}}{\begin{bmatrix} \frac{t_1}{2} & & & \\ \frac{t_3}{2} & \frac{t_3}{2} & & \\ & \frac{t_5}{2} & \frac{t_5}{2} & \\ & & \frac{t_7}{2} & \frac{t_7}{2} \\ & & & \frac{t_9}{2} \end{bmatrix}} \underset{q}{\begin{Bmatrix} q_2 \\ q_4 \\ q_6 \\ q_8 \end{Bmatrix}} = 0 \,.$$

The matrices are denoted according to the symbols written under the equation. The symbol M denotes a column matrix of structural ply bending moments, dM/dx is the derivative of M, V is a column matrix of structural ply shear forces, and M_{xq} is a rectangular matrix of structural ply thicknesses.

$$\therefore \quad \frac{dM}{dx} + V - M_{xq}\, q = 0 \,. \tag{22}$$

From Equations 2, 6a and 6c:

$$
\left\{
\begin{aligned}
&\frac{dV_1}{dx} && && - w_1 && = 0 \\
t_2 &\frac{dq_2}{dx} && + w_1 && - w_2 && = 0 \\
&\frac{dV_3}{dx} && + w_2 && - w_3 && = 0 \\
t_4 &\frac{dq_4}{dx} && + w_3 && - w_4 && = 0 \\
&\frac{dV_5}{dx} && + w_4 && - w_5 && = 0 \\
t_6 &\frac{dq_6}{dx} && + w_5 && - w_6 && = 0 \\
&\frac{dV_7}{dx} && + w_6 && - w_7 && = 0 \\
t_8 &\frac{dq_8}{dx} && + w_7 && - w_8 && = 0 \\
&\frac{dV_9}{dx} && + w_8 && && = 0
\end{aligned}
\right.
$$

Adding these equations gives

$$\frac{dV_1}{dx} + \frac{dV_3}{dx} + \frac{dV_5}{dx} + \frac{dV_7}{dx} + \frac{dV_9}{dx} + t_2 \frac{dq_2}{dx} + t_4 \frac{dq_4}{dx} + t_6 \frac{dq_6}{dx} + t_8 \frac{dq_8}{dx} = 0 \; . \quad (23)$$

The matric form of this equation is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} dV_1/dx \\ dV_3/dx \\ dV_5/dx \\ dV_7/dx \\ dV_9/dx \end{Bmatrix} + \begin{bmatrix} t_2 & t_4 & t_6 & t_8 \end{bmatrix} \begin{Bmatrix} dq_2/dx \\ dq_4/dx \\ dq_6/dx \\ dq_8/dx \end{Bmatrix} = 0$$

$$\bar{C}^T \qquad \frac{dV}{dx} \qquad t_\ell^T \qquad \frac{dq}{dx}$$

The matrices are denoted according to the symbols written under the equation. The matrices dV/dx and dq/dx are the derivatives of the column matrices V and q previously defined, $\bar{C}$ is a Boolean column matrix, and t_ℓ is a column matrix of interlayer thicknesses. The superscript T indicates a transposed matrix.

$$\therefore \qquad \bar{C}^T \frac{dV}{dx} + t_\ell^T \frac{dq}{dx} = 0 \,. \qquad (23a)$$

From Equation 16:

$$\begin{Bmatrix} H_1 \\ H_3 \\ H_5 \\ H_7 \\ H_9 \end{Bmatrix} = \begin{bmatrix} E_1A_1 & & & & \\ & E_3A_3 & & & \\ & & E_5A_5 & & \\ & & & E_7A_7 & \\ & & & & E_9A_9 \end{bmatrix} \begin{Bmatrix} du_1/dx \\ du_3/dx \\ du_5/dx \\ du_7/dx \\ du_9/dx \end{Bmatrix}$$

$$H \qquad k_A \qquad \frac{du}{dx}$$

The matrices are denoted according to the symbols written under the equation. The symbol u denotes a column matrix of longitudinal displacements of structural ply centerlines, du/dx is the derivative of u, and k_A is a diagonal matrix of structural ply axial stiffnesses.

$$\therefore \qquad H = k_A \frac{du}{dx} \qquad (24)$$

From Equation 17:

$$\begin{Bmatrix} M_1 \\ M_3 \\ M_5 \\ M_7 \\ M_9 \end{Bmatrix} = \begin{Bmatrix} E_1 I_1 \\ E_3 I_3 \\ E_5 I_5 \\ E_7 I_7 \\ E_9 I_9 \end{Bmatrix} \frac{d^2 v}{dx^2}$$

$$M \qquad\qquad k_I$$

The matrices are denoted according to the symbols written under the equation. The symbol k_I denotes a column matrix of structural ply bending stiffnesses.

$$\therefore \qquad M = k_I \frac{d^2 v}{dx^2} \qquad (25)$$

From Equation 18:

$$\begin{Bmatrix} q_2 \\ q_4 \\ q_6 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} G_2 b \\ G_4 b \\ G_6 b \\ G_8 b \end{Bmatrix} \frac{dv}{dx} + \begin{bmatrix} \frac{G_2 b}{t_2} & & & \\ & \frac{G_4 b}{t_4} & & \\ & & \frac{G_6 b}{t_6} & \\ & & & \frac{G_8 b}{t_8} \end{bmatrix} \begin{Bmatrix} \eta_1 - \eta_2 \\ \eta_3 - \eta_4 \\ \eta_5 - \eta_6 \\ \eta_7 - \eta_8 \end{Bmatrix}$$

$$q \qquad\qquad k_\ell \qquad\qquad k_\eta \qquad\qquad \overline{\Delta\eta}$$

The matrices are denoted according to the symbols written under the equation. The symbol $\overline{\Delta\eta}$ denotes a column matrix of longitudinal displacement differences at layer interfaces, k_ℓ is a column matrix of interlayer shear stiffnesses, and k_η is a diagonal matrix of interlayer shear stiffnesses divided by interlayer thicknesses.

$$\therefore \qquad q = k_\ell \frac{dv}{dx} + k_\eta \overline{\Delta\eta}$$

But $\qquad \overline{\Delta\eta} = \Delta_\eta \eta$

where $\qquad \Delta_\eta = \begin{bmatrix} 1 & -1 & & & & & & \\ & & 1 & -1 & & & & \\ & & & & 1 & -1 & & \\ & & & & & & 1 & -1 \end{bmatrix}$

and

$$\eta = \left\{ \eta_1 \mid \eta_2 \mid \eta_3 \mid \eta_4 \mid \eta_5 \mid \eta_6 \mid \eta_7 \mid \eta_8 \right\} \qquad \text{(column)} .$$

$\Delta\eta$ is a rectangular differencing matrix, and η is a column matrix of longitudinal displacements of layer interfaces.

$$\therefore \quad q = k_\ell \frac{dv}{dx} + k_\eta \Delta_\eta \eta . \qquad (26)$$

From Equations 7, 8 and 15,

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_3 \\ u_3 \\ u_5 \\ u_5 \\ u_7 \\ u_7 \\ u_9 \end{Bmatrix} + \begin{Bmatrix} t_1/2 \\ -t_3/2 \\ t_3/2 \\ -t_5/2 \\ t_5/2 \\ -t_7/2 \\ t_7/2 \\ -t_9/2 \end{Bmatrix} \theta$$

$$\therefore \quad \underset{\eta}{\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_8 \end{Bmatrix}} = \underset{\Sigma_u}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & 1 & & \\ & & & 1 & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}} \underset{u}{\begin{Bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ u_9 \end{Bmatrix}} + \frac{1}{2} \underset{\Delta_{ts}}{\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & 1 & & & \\ & & -1 & & \\ & & 1 & & \\ & & & -1 & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}} \underset{t_s}{\begin{Bmatrix} t_1 \\ t_3 \\ t_5 \\ t_7 \\ t_9 \end{Bmatrix}} \theta$$

The matrices are denoted according to the symbols written under the equation. The symbol η denotes a column matrix of longitudinal displacements, t_s is a column matrix of structural ply thicknesses, and Σ_u and Δ_{ts} are Boolean coefficient matrices.

$$\therefore \quad \eta = \Sigma_u u + \frac{1}{2} \Delta_{ts} t_s \frac{dv}{dx} \qquad (27)$$

From Equation 24,

$$\frac{du}{dx} - a_{1,2} H = 0 \tag{28}$$

where $$a_{1,2} = k_A^{-1} . \tag{29}$$

Eliminate η from Equations 26 and 27:

$$q = (k_\ell + k_\eta \cdot \frac{1}{2}\Delta_\eta \Delta_{ts} t_s) \frac{dv}{dx} + k_\eta \Delta_\eta \Sigma_u u$$

But $k_\ell = k_\eta t_\ell$, $\frac{1}{2}\Delta_\eta \Delta_{ts} t_s = \bar{t}_s$, and $\Delta_\eta \Sigma_u = -\Delta_q^T$,

where
$$\bar{t}_s = \begin{Bmatrix} \frac{1}{2}(t_1 + t_3) \\ \frac{1}{2}(t_3 + t_5) \\ \frac{1}{2}(t_5 + t_7) \\ \frac{1}{2}(t_7 + t_9) \end{Bmatrix}$$

and $\bar{\mathbf{t}}_s$ is a column matrix of average structural ply thicknesses.

$$\therefore \quad q = k_\eta \bar{t} \frac{dv}{dx} - k_\eta \Delta_q^T u \tag{30}$$

where
$$\bar{t} = t_\ell + \bar{t}_s = \begin{Bmatrix} \frac{1}{2}t_1 + t_2 + \frac{1}{2}t_3 \\ \frac{1}{2}t_3 + t_4 + \frac{1}{2}t_5 \\ \frac{1}{2}t_5 + t_6 + \frac{1}{2}t_7 \\ \frac{1}{2}t_7 + t_8 + \frac{1}{2}t_9 \end{Bmatrix} .$$

Eliminate q and $\frac{dv}{dx}$ from Equations 15, 19 and 30

$$\therefore \quad \frac{dH}{dx} - a_{2,1} u - a_{2,3} \theta = 0 \tag{31}$$

where $$a_{2,1} = \Delta_q \, k_\eta \, \Delta_q^T \tag{32}$$

and $$a_{2,3} = -\Delta_q \, k_\eta \bar{t} \quad . \tag{33}$$

Eliminate V from Equations 22 and 23a.

$$\therefore \quad \bar{t}_s^T \frac{dq}{dx} - \bar{C}^T \frac{d^2M}{dx^2} + t_\ell^T \frac{dq}{dx} = 0 , \qquad (34)$$

since $\bar{C}^T M_{xq} = \bar{t}_s^T$. Eliminate M from Equations 25 and 34.

$$\text{Then,} \quad \bar{t}^T \frac{dq}{dx} - K_I \frac{d^4v}{dx^4} = 0 \qquad (35)$$

where $K_I = (EI)_1 + (EI)_3 + (EI)_5 + (EI)_7 + (EI)_9$. (36)

Therefore, from Equations 15, 15a, 15b, and 35

$$\bar{t}^T \frac{dq}{dx} - K_I \frac{d\phi}{dx} = 0 . \qquad (37)$$

Eliminate q from Equations 30 and 37. Then,

$$\frac{d\phi}{dx} - K_I^{-1} a_{2,3}^T \frac{du}{dx} - a_{5,4} \frac{d^2v}{dx^2} = 0 \qquad (38)$$

where $a_{5,4} = K_I^{-1} \bar{t}^T k_n \bar{t}$. (38a)

Therefore, from Equations 15, 15a, 28 and 38,

$$\frac{d\phi}{dx} - a_{5,2}H - a_{5,4}\,\beta = 0 , \qquad (39)$$

where $a_{5,2} = K_I^{-1} a_{2,3}^T a_{1,2}$. (40)

Equations 15, 15a, 15b, 28, 31 and 39 are a set of 14 linear, ordinary, first order, homogeneous differential equations with constant coefficients, in the 14 dependent variables u_i, H_i, v, θ , β and ϕ . The number of variables is 14 since the matrices u and H each contain five variables. The theory of differential equations shows that, for a set of equations of this kind, n linearly independent solutions exist, where n is the number of equations, equal to 14 in this case. See Reference 5, article 12-8.

Matric Differential Equation

From Equations 15, 15a, 15b, 28, 31 and 39,

$$\underbrace{\begin{Bmatrix} du/dx \\ dH/dx \\ dv/dx \\ d\theta/dx \\ d\beta/dx \\ d\phi/dx \end{Bmatrix}}_{\frac{dY}{dx}} - \underbrace{\begin{bmatrix} & a_{1,2} & & & & \\ a_{2,1} & & a_{2,3} & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & a_{5,2} & & a_{5,4} & & \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} u \\ H \\ v \\ \theta \\ \beta \\ \phi \end{Bmatrix}}_{Y} = 0 \qquad (41)$$

The matrices are denoted according to the symbols written under the equation. The symbol Y denotes a column matrix of beam responses, dY/dx is its derivative with respect to x, and A is a square matrix of constant coefficients.

$$\therefore \qquad \frac{dY}{dx} - AY = 0 \qquad (42)$$

This linear matric differential equation can be solved by taking

$$Y = Ge^{\lambda x} \qquad (43)$$

where G is an unknown column matrix, λ is an unknown scalar and e is the base of natural logarithms. Eliminating Y from Equations 42 and 43 gives

$$(A-\lambda I)G = 0 \ . \qquad (44)$$

Evidently, λ and G are an eigenvalue and an eigenvector of the characteristic Equation 44. Therefore the solution of Equation 42 can be written

$$Y = \sum_{k=1}^{14} C_k \, G_k \, e^{\lambda_k x} \qquad (45)$$

where G_k, and λ_k are the kth eigenvector and eigenvalue of Equation 44, and the C_k's are arbitrary constants needed to satisfy the boundary conditions.

Nature of the Roots of the Characteristic Equation

The eigenvalues (roots) of Equation 44 are expected to be all real, from the nature of the physical problem. The roots can be easily shown to occur in pairs equal in magnitude and opposite in sign, such that if λ_α and λ_β are a pair of roots, then $\lambda_\beta + \lambda_\alpha = 0$. (See Appendix A.) It is shown subsequently that at least six repeated roots, equal to zero, exist. No more than six such roots are expected, because of the nature of the problem.

Algebraic Modes

The solutions of Equation 42 of the form given by Equation 43 are here called exponential solutions or exponential modes. The following paragraphs show that other solutions exist which are algebraic functions of x . Consequently, these solutions are called algebraic solutions or algebraic modes. Six linearly independent algebraic modes are shown to exist. These solutions are also linearly independent of the exponential solutions, since they are algebraic. Therefore, the characteristic equation (Equation 44) can yield at most only eight linearly independent exponential modes. Consequently, six of the roots of Equation 44 must be zero.

In view of the existence of the algebraic modes, the complete solution of Equation 42 can be written in the form

$$Y = \sum_{k=1}^{8} C_k G_k e^{\lambda_k x} + \sum_{k=1}^{6} C_{A_K} Y_{A_K} \tag{46}$$

where the C_{A_K}'s are arbitrary constants, and the Y_{A_K}'s (algebraic modes) are column matrices whose elements are algebraic functions of x. The only requirements for the Y_{A_K}'s is that they must satisfy Equation 42, and the G_k's and Y_{A_K}'s must be linearly independent. The following Y_{A_K}'s satisfy these requirements:

Rigid Body Translation Parallel to x —

$$Y_{A_1} = \begin{Bmatrix} u_{(0)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Big\} \text{5 rows} \qquad \text{where } u_{(0)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{47}$$

Uniform Extension Parallel to x —

$$Y_{A_2} = \left\{ u_{(0)}x \mid k_A u_{(0)} \mid 0 \mid 0 \mid 0 \mid 0 \right\} \text{ (Column)} \tag{48}$$

Rigid Body Translation Parallel to z —

$$Y_{A_3} = \{\underbrace{0}_{\substack{5\\ \text{rows}}} \mid \underbrace{0}_{\substack{5\\ \text{rows}}} \mid 1 \mid 0 \mid 0 \mid 0\} \text{ (Column)} \tag{49}$$

Rigid Body Rotation about y —

$$Y_{A_4} = \{u_{(1)} \mid \underbrace{0}_{\text{5 rows}} \mid x \mid 1 \mid 0 \mid 0\} \text{ (Column)} \tag{50}$$

$$\text{where} \quad u_{(1)} = Q\bar{t} + \alpha u_{(0)} \quad Q = \begin{bmatrix} -1 & -1 & -1 & -1 \\ & -1 & -1 & -1 \\ & & -1 & -1 \\ & & & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \alpha = \text{a scalar} \tag{51}$$

Bending about y —

$$Y_{A_5} = \left\{ u_{(1)}x \mid k_A u_{(1)} \mid x^2/2 \mid x \mid 1 \mid 0 \right\} \text{ (Column)} \tag{52}$$

Shear Parallel to z —

$$\gamma_{A_6} = \left\{ \tfrac{1}{2}u_{(1)}x^2 + u_{(2)} \,\middle|\, k_A u_{(1)} x \,\middle|\, \tfrac{x^3}{6} \,\middle|\, \tfrac{x^2}{2} \,\middle|\, x \,\middle|\, 1 \right\} \text{(Column)} \tag{53}$$

where $u_{(2)}$ = a column matrix with 5 rows.

α and $u_{(2)}$ must satisfy the equation

$$Su_{(2)} - T\alpha = W \tag{54}$$

where $S = \Delta_q k_n \Delta_q^T$, $T = k_A u_{(0)}$, $W = k_A Q\bar{t}$. (55)

The matrix S is singular. Equation 54 is a matrix equation equivalent to five scalar simultaneous equations in the five unknown elements of $u_{(2)}$ and the unknown α. However one element of $u_{(2)}$ may be chosen arbitrarily since S has one degree of singularity. In this respect $u_{(2)}$ has the character of an eigenvector. Partition S, T, W and $u_{(2)}$:

$$S = \underbrace{\begin{bmatrix} S_{1,1} \\ S_{2,1} \end{bmatrix}}_{\text{1 col.}}\!\!\underbrace{\begin{bmatrix} S_{1,2} \\ S_{2,2} \end{bmatrix}}_{\text{4 col.}} \begin{matrix} \}\text{ 1 row} \\ \}\text{ 4 rows} \end{matrix} \quad T = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \begin{matrix} \}\text{ 1 row} \\ \}\text{ 4 rows} \end{matrix} \quad W = \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} \begin{matrix} \}\text{ 1 row} \\ \}\text{ 4 rows} \end{matrix} \tag{56}$$

$$u_{(2)} = \begin{Bmatrix} 0 \\ \bar{u}_{(2)} \end{Bmatrix} \begin{matrix} \}\text{ 1 row} \\ \}\text{ 4 rows} \end{matrix} \tag{57}$$

where the element in the first row of $u_{(2)}$ has been chosen equal to zero.

From Equations 54, 56 and 57,

$$\begin{cases} S_{1,2}\,\bar{u}_{(2)} - T_1\alpha = W_1 \\ S_{2,2}\,\bar{u}_{(2)} - T_2\alpha = W_2 \end{cases} \tag{58}$$

$$\therefore \qquad \alpha = \frac{1}{T_1}\left[S_{1,2}\,\bar{u}_{(2)} - W_1\right] \tag{59}$$

and $$\bar{u}_{(2)} = (S_{2,2} - \frac{1}{T_1} T_2 S_{1,2})^{-1} (W_2 - \frac{1}{T_1} T_2 W_1) \tag{60}$$

Nature of the Algebraic Solutions

The first five algebraic solutions (Y_{A_1}, Y_{A_2}, ... Y_{A_5}) correspond to solutions provided by the engineering theory of tension and bending of beams. These solutions have been appropriately named to reflect this fact. Thus Y_{A_1} can be recognized as corresponding to a rigid body translation of the beam parallel to x, since the longitudinal motions of the centerlines of the structural plies are all equal to 1, and all other responses are zero. The mode Y_{A_4} can be recognized as a rigid body rotation about the centerline of the bottom structural ply, represented by $Q\bar{t}$, plus a rigid body translation parallel to x, represented by $\alpha u_{(0)}$. The mode Y_{A_6} corresponds to the responses calculated by engineering theory when the beam is subjected to a constant shear load. However the responses given by elementary theory involve an approximation, whereas the responses given by Y_{A_6} exactly satisfy the differential equations.

Response

Appendix A shows that the eigenvalues λ_k occur in pairs, such that for a pair λ_α and λ_β,

$$\lambda_\beta = -\lambda_\alpha \quad . \tag{61}$$

Appendix A also shows that the corresponding eigenvectors have the property that if

$$G_\alpha = \begin{Bmatrix} G_{u_\alpha} \\ G_{H_\alpha} \\ G_{V_\alpha} \\ G_{\theta_\alpha} \\ G_{\beta_\alpha} \\ G_{\phi_\alpha} \end{Bmatrix}, \quad \text{then} \quad G_\beta = \begin{Bmatrix} G_{u_\alpha} \\ -G_{H_\alpha} \\ -G_{V_\alpha} \\ G_{\theta_\alpha} \\ -G_{\beta_\alpha} \\ G_{\phi_\alpha} \end{Bmatrix} \tag{62}$$

where G_{u_α} , G_{H_α} etc. are partitions of G_α corresponding to the partitions u, H etc. of the matrix Y (Equation 41). Because half of the eigenvalues λ are positive, the corresponding functions $e^{\lambda x}$ can become very large as x increases. To avoid this difficulty, proceed as follows: See Equation 46. Let

$$Y_E = \sum_{k=1}^{8} C_k G_k e^{\lambda_k x} \qquad \text{(exponential response)}$$

$$= Y_{E\ell} + Y_{Eg} \tag{63}$$

where

$$Y_{E\ell} = \sum_{k=1}^{4} C_{\ell_k} G_{\ell_k} e^{\lambda_{\ell k} x}, \qquad Y_{Eg} = \sum_{k=1}^{4} \overline{C}_{g_k} G_{g_k} e^{\lambda_{g_k} x}, \tag{64}$$

and where λ_{ℓ_k} and λ_{g_k} are a pair of eigenvalues having the properties $\lambda_{\ell_k} = -\lambda_{g_k}$, $\lambda_{g_k} > 0$. G_{ℓ_k} and G_{g_k} are corresponding eigenvectors. Now

$$Y_{Eg} = \sum_{k=1}^{4} \overline{C}_{g_k} G_{g_k} e^{\lambda_{g_k}(x-\ell+\ell)} \tag{65}$$

$$= \sum_{k=1}^{4} \overline{C}_{g_k} G_{g_k} e^{-\lambda_{g_k}(\ell-x)} e^{\lambda_{g_k}\ell} \tag{66}$$

$$= \sum_{k=1}^{4} C_{g_k} G_{g_k} e^{\lambda_{\ell k}(\ell-x)} \tag{67}$$

$$\text{where} \quad C_{g_k} = \overline{C}_{g_k} e^{\lambda_{g_k}\ell}. \tag{68}$$

From Equation 46:

$$Y = \sum_{k=1}^{4} G_{\ell_k} e^{\lambda_{\ell_k} x} C_{\ell_k} + \sum_{k=1}^{4} G_{g_k} e^{\lambda_{\ell_k}(\ell - x)} C_{g_k} + \sum_{k=1}^{6} C_{A_k} Y_{A_k} . \quad (69)$$

Thus the troublesome eigenvalues λ_{g_k}, which are greater than zero, have been lumped into the arbitrary constants C_{g_k}, and thus eliminated. The functions $e^{\lambda_{\ell_k} x}$ and $e^{\lambda_{\ell_k}(\ell - x)}$ are less than or equal to 1 for $0 < x < \ell$, since $\lambda_{\ell_k} < 0$. Let

$$F_{\ell_k}(x) = e^{\lambda_{\ell_k} x} \quad \text{and} \quad F_{g_k}(x) = e^{\lambda_{\ell_k}(\ell - x)} \quad (70)$$

$$\therefore \quad Y = \sum_{k=1}^{4} G_{\ell_k} F_{\ell_k}(x) C_{\ell_k} + \sum_{k=1}^{4} G_{g_k} F_{g_k}(x) C_{g_k} + \sum_{k=1}^{6} C_{A_k} Y_{A_k} \quad (71)$$

$$\therefore \quad Y(x) = H_\ell F_{\ell D}(x) C_\ell + H_g F_{gD}(x) C_g + Y_A(x) C_A \quad (72)$$

where the symbols in this equation are defined by Equations 73,74,75 and 7

$$H_\ell = \left[G_{\ell_1} \middle| G_{\ell_2} \middle| G_{\ell_3} \middle| G_{\ell_4} \right] \qquad H_g = \left[G_{g_1} \middle| G_{g_2} \middle| G_{g_3} \middle| G_{g_4} \right] \quad (73)$$

$$F_\ell(x) = \begin{Bmatrix} F_{\ell_1}(x) \\ F_{\ell_2}(x) \\ F_{\ell_3}(x) \\ F_{\ell_4}(x) \end{Bmatrix} \qquad F_g(x) = \begin{Bmatrix} F_{g_1}(x) \\ F_{g_2}(x) \\ F_{g_3}(x) \\ F_{g_4}(x) \end{Bmatrix} \quad (74)$$

$F_{\ell D}(x)$, $F_{gD}(x)$ = $F_\ell(x)$, $F_g(x)$ diagonalized

$$C_\ell = \begin{Bmatrix} C_{\ell_1} \\ C_{\ell_2} \\ C_{\ell_3} \\ C_{\ell_4} \end{Bmatrix} \qquad C_g = \begin{Bmatrix} C_{g_1} \\ C_{g_2} \\ C_{g_3} \\ C_{g_4} \end{Bmatrix} \qquad C_A = \begin{Bmatrix} C_{A_1} \\ C_{A_2} \\ \vdots \\ C_{A_6} \end{Bmatrix} \quad (75)$$

$$Y_A(x) = \left[Y_{A_1}(x) \;\middle|\; Y_{A_2}(x) \;\middle|\; \cdots \;\middle|\; Y_{A_6}(x) \right] . \qquad (76)$$

Partition $Y(x)$, H_ℓ, H_g, $Y_A(x)$:

$$\underbrace{\begin{Bmatrix} u(x) \\ H(x) \\ v(x) \\ \theta(x) \\ \beta(x) \\ \phi(x) \end{Bmatrix}}_{Y(x)} = \underbrace{\begin{bmatrix} H_{u\ell} \\ H_{H\ell} \\ H_{v\ell} \\ H_{\theta\ell} \\ H_{\beta\ell} \\ H_{\phi\ell} \end{bmatrix}}_{H_\ell} F_{\ell D}(x)\, C_\ell + \underbrace{\begin{bmatrix} H_u \\ -H_{H\ell} \\ -H_{v\ell} \\ H_{\theta\ell} \\ -H_{\beta\ell} \\ H_{\phi\ell} \end{bmatrix}}_{H_g} F_{gD}(x)\, C_g + \underbrace{\begin{bmatrix} Y_{Au}(x) \\ Y_{AH}(x) \\ Y_{Av}(x) \\ Y_{A\theta}(x) \\ Y_{A\beta}(x) \\ Y_{A\phi}(x) \end{bmatrix}}_{Y_A(x)} C_A \qquad (77)$$

The matrices are denoted according to the symbols written under the equation. The partitions of $Y_A(x)$ are defined, from Equation 76, by the following equations:

$$Y_{Au}(x) = \left[u_{(0)} \;\middle|\; u_{(0)}x \;\middle|\; 0 \;\middle|\; u_{(1)} \;\middle|\; u_{(1)}x \;\middle|\; \tfrac{1}{2}u_{(1)}x^2 + u_{(2)} \right] \qquad (78)$$

$$Y_{AH}(x) = \left[0 \;\middle|\; k_A u_{(0)} \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; k_A u_{(1)} \;\middle|\; k_A u_{(1)}x \right] \qquad (79)$$

$$Y_{Av}(x) = \left[0 \;\middle|\; 0 \;\middle|\; 1 \;\middle|\; x \;\middle|\; x^2/2 \;\middle|\; x^3/6 \right] \qquad (80)$$

$$Y_{A\theta}(x) = \left[0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 1 \;\middle|\; x \;\middle|\; x^2/2 \right] \qquad (81)$$

$$Y_{A\beta}(x) = \left[0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 1 \;\middle|\; x \right] \qquad (82)$$

$$Y_{A\phi}(x) = \left[0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; 1 \right] \qquad (83)$$

Interlayer Shear Flows, Structural Ply Shears and Moments

From Equations 15 and 30

$$q = k_n \bar{t}\theta - k_n \Delta_q^T u \quad . \tag{84}$$

From Equations 15b, 22 and 25,

$$V = V_q q - k_I \phi \tag{85}$$

where $V_q = M_{xq}$.

From Equations 15a and 25

$$M = k_I \beta \quad . \tag{86}$$

Let $S(x)$ = total shear on the cross section at station x.

$$\therefore \quad S = \underset{u_{(0)}^T}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}} \underset{V}{\begin{Bmatrix} V_1 \\ V_3 \\ V_5 \\ V_7 \\ V_9 \end{Bmatrix}} + \underset{t_\ell^T}{\begin{bmatrix} t_2 & t_4 & t_6 & t_8 \end{bmatrix}} \underset{q}{\begin{Bmatrix} q_2 \\ q_4 \\ q_6 \\ q_8 \end{Bmatrix}} \tag{87}$$

The matrices are denoted according to the symbols written under the equation.

Therefore $$S = u_{(0)}^T V + t_\ell^T q \quad . \tag{88}$$

Eliminate V from Equations 85 and 88,

$$\therefore \quad S = \bar{t}^T q - K_I \phi \quad . \tag{89}$$

Eliminate q from Equations 84 and 89,

$$\therefore \quad S = a_{2,3}^T u + K_I a_{5,4} \theta - K_I \phi \quad . \tag{90}$$

Boundary Conditions

Figure 8 shows the boundary conditions for a fixed ended beam.

At $x = 0$ —

$$\begin{cases} u(0) = 0 & \text{(5 conditions)} \\ v(0) = 0 & \text{(1 condition)} \\ \theta(0) = 0 & \text{(1 condition)} \end{cases}$$

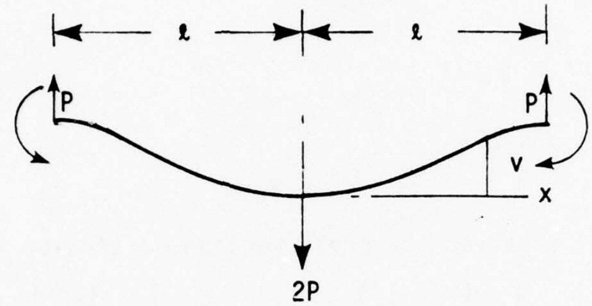


Figure 8. Boundary conditions.

For all x —

The total shear on the cross section is constant

$$\therefore \quad S(x) = P \; .$$

$S(x)$ is most easily evaluated at $X = 0$, therefore take

$$S(0) = P \quad \text{(1 condition)} \; .$$

From Equation 90,

$$-K_I \phi(0) = P \quad \text{and} \quad \phi(0) = -K_I^{-1} P \; , \tag{92}$$

since $u(0) = \theta(0) = 0$.

Condition 1. Fixed Ends at $x = \pm\ell$. For $x = \ell$ —

$$\begin{cases} u(\ell) = 0 & \text{(5 conditions)} \\ \theta(\ell) = 0 & \text{(1 condition)} \end{cases} \tag{93}$$

From Equation 77

$$\left.\begin{aligned}
H_{u\ell}F_{\ell D}(0)C_{\ell} + H_{u\ell}F_{gD}(0)C_g + Y_{Au}(0)C_A &= 0 \\
H_{v\ell}F_{\ell D}(0)C_{\ell} - H_{v\ell}F_{gD}(0)C_g + Y_{Av}(0)C_A &= 0 \\
H_{\theta\ell}F_{\ell D}(0)C_{\ell} + H_{\theta\ell}F_{gD}(0)C_g + Y_{A\theta}(0)C_A &= 0 \\
H_{\phi\ell}F_{\ell D}(0)C_{\ell} + H_{\phi\ell}F_{gD}(0)C_g + Y_{A\phi}(0)C_A &= -K_I^{-1}P \\
H_{u\ell}F_{\ell D}(\ell)C_{\ell} + H_{u\ell}F_{gD}(\ell)C_g + Y_{Au}(\ell)C_A &= 0 \\
H_{\theta\ell}F_{\ell D}(\ell)C_{\ell} + H_{\theta\ell}F_{gD}(\ell)C_g + Y_{A\theta}(\ell)C_A &= 0
\end{aligned}\right\} \qquad (94)$$

$$\therefore \underbrace{\begin{bmatrix}
H_{u\ell}F_{\ell D}(0) & H_{u\ell}F_{gD}(0) & Y_{Au}(0) \\
H_{v\ell}F_{\ell D}(0) & -H_{v\ell}F_{gD}(0) & Y_{Av}(0) \\
H_{\theta\ell}F_{\ell D}(0) & H_{\theta\ell}F_{gD}(0) & Y_{A\theta}(0) \\
H_{\phi\ell}F_{\ell D}(0) & H_{\phi\ell}F_{gD}(0) & Y_{A\phi}(0) \\
H_{u\ell}F_{\ell D}(\ell) & H_{u\ell}F_{gD}(\ell) & Y_{Au}(\ell) \\
H_{\theta\ell}F_{\ell D}(\ell) & H_{\theta\ell}F_{gD}(\ell) & Y_{A\theta}(\ell)
\end{bmatrix}}_{B}
\underbrace{\begin{Bmatrix} C_{\ell} \\ C_g \\ C_A \end{Bmatrix}}_{C}
= \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -K_I^{-1}P \\ 0 \\ 0 \end{Bmatrix}}_{\psi} . \qquad (95)$$

The matrices are denoted according to the symbols written under the equation. The symbol C denotes a column matrix of arbitrary constants, ψ is a column matrix of boundary conditions, and B is a coefficient matrix.

$$\therefore \qquad BC = \psi . \qquad (96)$$

From Equations 70 and 74

$$F_{\ell D}(0) = F_{gD}(\ell) = I \qquad (97)$$

$$\text{and} \qquad F_{\ell D}(\ell) = F_{gD}(0) \qquad (98)$$

where I is a unit (identity) matrix. Therefore; B =

$$
\begin{bmatrix}
H_{u\ell} & H_{u\ell}F_{\ell D}(\ell) & u_{(0)} & 0 & 0 & u_{(1)} & 0 & u_{(2)} \\
H_{v\ell} & -H_{v\ell}F_{\ell D}(\ell) & 0 & 0 & 1 & 0 & 0 & 0 \\
H_{\theta\ell} & H_{\theta\ell}F_{\ell D}(\ell) & 0 & 0 & 0 & 1 & 0 & 0 \\
H_{\phi\ell} & H_{\phi\ell}F_{\ell D}(\ell) & 0 & 0 & 0 & 0 & 0 & 1 \\
H_{u\ell}F_{\ell D}(\ell) & H_{u\ell} & u_{(0)} & u_{(0)}\ell & 0 & u_{(1)} & u_{(1)}\ell & \frac{1}{2}u_{(1)}\ell^2+u_{(2)} \\
H_{\theta\ell}F_{\ell D}(\ell) & H_{\theta\ell} & 0 & 0 & 0 & 1 & \ell & \frac{\ell^2}{2}
\end{bmatrix}
\quad \ldots (99)
$$

Effective Beam Stiffness

For a fixed-ended monolithic beam carrying a concentrated load at the center,

$$v(\ell) = \frac{P\ell^3}{12EI} \quad . \tag{100}$$

Therefore define the effective stiffness of the laminated beam as

$$EI_{eff} = \frac{P\ell^3}{12v(\ell)} \quad . \tag{101}$$

EQUATION SUMMARY

This section summarizes the equations for the laminated beam. Figure 8a shows the basic beam dimensions.

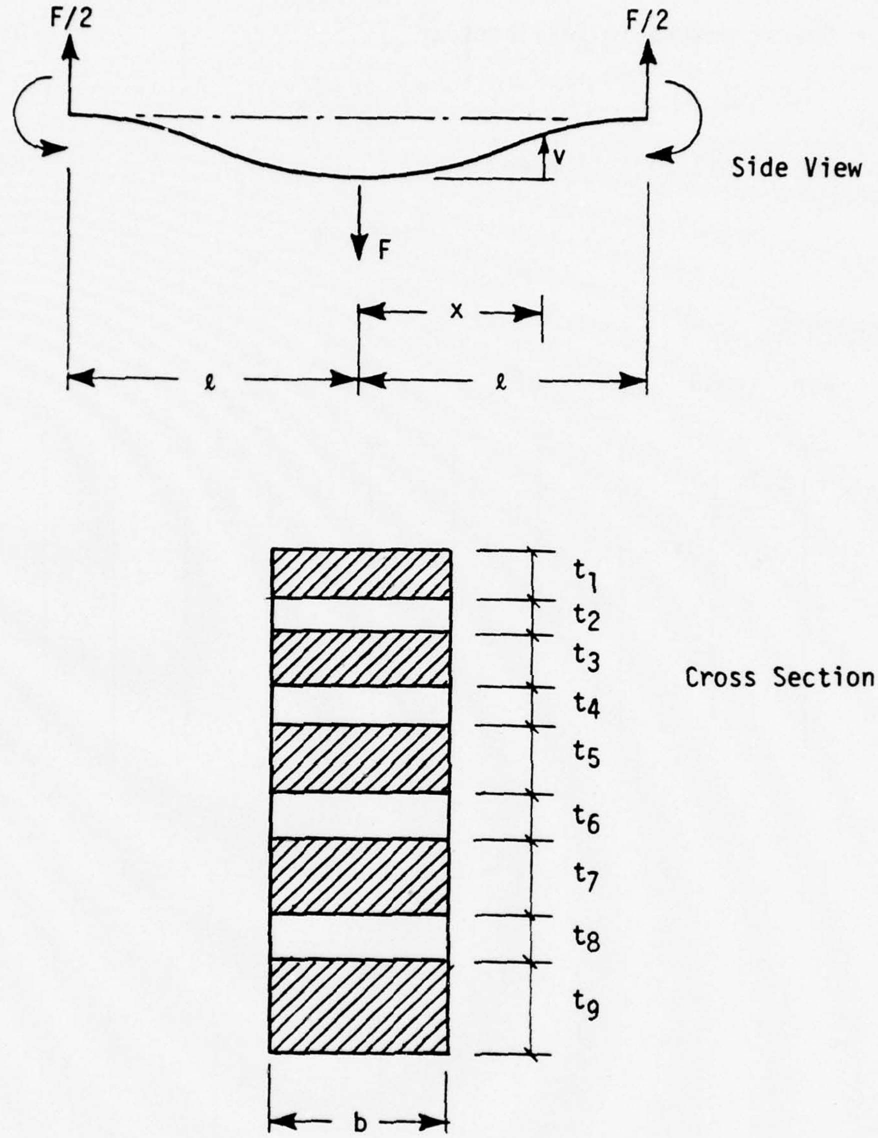


Figure 8a. Beam dimensions.

Odd numbered layers are structural plies.

Even numbered layers are interlayers.

Input

$F, \ \ell, \ b, \ x = x_j, \ j = 0,1,2 \ldots n_x$ where $x_o = 0, \ x_{n_x} = \ell$.

For the structural plies(i odd):

$$t_i \ , \ E_i$$

For the interlayers (i even):

$$t_i \ , \ G_i$$

Equations

For i odd: $A_i = bt_i \qquad I_i = \dfrac{bt^3{}_i}{12}$ (102)

$$k_A = \begin{bmatrix} E_1A_1 & & & & \\ & E_3A_3 & & & \\ & & E_5A_5 & & \\ & & & E_7A_7 & \\ & & & & E_9A_9 \end{bmatrix} \qquad k_n = \begin{bmatrix} \frac{G_2b}{t_2} & & & \\ & \frac{G_4b}{t_4} & & \\ & & \frac{G_6b}{t_6} & \\ & & & \frac{G_8b}{t_8} \end{bmatrix} \quad \ldots(103)$$

$$K_I = E_1I_1 + E_3I_3 + E_5I_5 + E_7I_7 + E_9I_9 \quad (104)$$

$$\bar{t} = \begin{Bmatrix} \frac{1}{2}t_1+t_2+\frac{1}{2}t_3 \\ \frac{1}{2}t_3+t_4+\frac{1}{2}t_5 \\ \frac{1}{2}t_5+t_6+\frac{1}{2}t_7 \\ \frac{1}{2}t_7+t_8+\frac{1}{2}t_9 \end{Bmatrix} \qquad \Delta_q = \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} \quad (105)$$

$$
\left.
\begin{aligned}
a_{1,2} &= k_A^{-1} \\
a_{2,1} &= \Delta_q k_n \Delta_q^T \\
a_{2,3} &= -\Delta_q k_n \bar{t} \\
a_{5,2} &= K_I^{-1} a_{2,3}^T a_{1,2} \\
a_{5,4} &= K_I^{-1} \bar{t}^T k_n \bar{t}
\end{aligned}
\right\} \qquad (106)
$$

$$
\underset{14\times14}{A} = \left[\begin{array}{c|c|c|c|c|c}
0 & a_{1,2} & 0 & 0 & 0 & 0 \\
a_{2,1} & 0 & 0 & a_{2,3} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & a_{5,2} & 0 & 0 & a_{5,4} & 0
\end{array}\right] \qquad (107)
$$

Find the eigenvalues and eigenvectors of

$$(A - \lambda I)G = 0 \ . \qquad (108)$$

Six of the eigenvalues should be zero. The remaining eight eigenvalues should be real and should occur in pairs such that for any pair, λ_ℓ and λ_g ,

$$\lambda_\ell < 0 \ , \quad \lambda_g > 0 \quad \text{and} \quad \lambda_\ell + \lambda_g = 0 \ . \qquad (109)$$

If the eigenvalues do not occur in this manner, print an error message. Otherwise, discard the eigenvalues which are equal to or greater than zero, and the corresponding eigenvectors. Denote the remaining eigenvalues and eigenvectors λ_{ℓ_i} and G_{ℓ_i} , $i = 1,2,3,4$.

Form

$$H_\ell = \left[G_{\ell_1} \middle| G_{\ell_2} \middle| G_{\ell_3} \middle| G_{\ell_4} \right] \quad F_\ell(\ell) = \begin{pmatrix} e^{\lambda_{\ell_1}\ell} \\ e^{\lambda_{\ell_2}\ell} \\ e^{\lambda_{\ell_3}\ell} \\ e^{\lambda_{\ell_4}\ell} \end{pmatrix} \tag{110}$$

$F_{\ell D}(\ell) = F_\ell(\ell)$ diagonalized. Partition H_ℓ :

$$H_\ell = \begin{bmatrix} H_{u\ell} \\ H_{H\ell} \\ H_{v\ell} \\ H_{\theta\ell} \\ H_{\beta\ell} \\ H_{\phi\ell} \end{bmatrix} \quad \text{Form } u_{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad Q = \begin{bmatrix} -1 & -1 & -1 & -1 \\ & -1 & -1 & -1 \\ & & -1 & -1 \\ & & & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{111}$$

$$S = \Delta_q k_n \Delta_q^T \; ; \quad T = k_A u_{(0)} \; ; \quad W = k_A Q \bar{t} \; . \tag{112}$$

Partition S, T and W :

$$S = \left[\begin{array}{c|c} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{array}\right] \begin{matrix} \}\text{ 1 row,} \\ \}\text{ 4 rows} \end{matrix} \quad T = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \begin{matrix} \}\text{ 1 row,} \\ \}\text{ 4 rows} \end{matrix} \quad W = \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} \begin{matrix} \}\text{ 1 row} \\ \}\text{ 4 rows} \end{matrix}$$

1 col. 4 cols. ....(113)

Discard $S_{1,1}$ and $S_{2,1}$.

$$\bar{u}_{(2)} = \left(S_{2,2} - \frac{1}{T_1} T_2 S_{1,2}\right)^{-1} \left(W_2 - \frac{1}{T_1} T_2 W_1\right) \tag{114}$$

$$\alpha = \frac{1}{T_1}\left[S_{1,2}\,\bar{u}_{(2)} - W_1\right] \tag{115}$$

$$u_{(1)} = Q\bar{t} + \alpha u_{(0)} \tag{116}$$

$$u_{(2)} = \left\{\begin{matrix} 0 \\ \bar{u}_{(2)} \end{matrix}\right\} \quad . \tag{117}$$

Form

$$B = \left[\begin{array}{c|c|c|c|c|c|c|c} H_{u\ell} & H_{u\ell}F_{\ell D}(\ell) & u_{(0)} & 0 & 0 & u_{(1)} & 0 & u_{(2)} \\ \hline H_{v\ell} & -H_{v\ell}F_{\ell D}(\ell) & 0 & 0 & 1 & 0 & 0 & 0 \\ H_{\theta\ell} & H_{\theta\ell}F_{\ell D}(\ell) & 0 & 0 & 0 & 1 & 0 & 0 \\ H_{\phi\ell} & H_{\phi\ell}F_{\ell D}(\ell) & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline H_{u\ell}F_{\ell D}(\ell) & H_{u\ell} & u_{(0)} & u_{(0)}\ell & 0 & u_{(1)} & u_{(1)}\ell & \frac{1}{2}u_{(1)}\ell^2 + u_{(2)} \\ \hline H_{\theta\ell}F_{\ell D}(\ell) & H_{\theta\ell} & 0 & 0 & 0 & 1 & \ell & \frac{\ell^2}{2} \end{array}\right] ,$$

...(118)

$$P = F/2, \text{ and} \tag{119}$$

$$\psi = \left\{ 0^* \;\middle|\; 0 \;\middle|\; 0 \;\middle|\; -K_I^{-1}P \;\middle|\; 0^* \;\middle|\; 0 \right\} \text{ (column)} . \tag{120}$$

Solve $BC = \psi$ for C . (121)

Form the following for $x = x_j$. The values of x_j are input.

Then $F_{\ell_k}(x_j) = e^{\lambda_{\ell_k} x_j}$ and $F_{g_k}(x) = e^{\lambda_{\ell_k}(\ell - x_j)}$. (122)

* These null matrices each have five rows.

Then

$$F_\ell(x_j) = \begin{Bmatrix} F_{\ell_1}(x_j) \\ F_{\ell_2}(x_j) \\ F_{\ell_3}(x_j) \\ F_{\ell_4}(x_j) \end{Bmatrix} \qquad F_g(x_j) = \begin{Bmatrix} F_{g_1}(x_j) \\ F_{g_2}(x_j) \\ F_{g_3}(x_j) \\ F_{g_4}(x_j) \end{Bmatrix} \tag{123}$$

$F_{\ell D}(x_j)$ and $F_{gD}(x_j)$ = $F_\ell(x_j)$ and $F_g(x_j)$ diagonalized. (124)

$$\left.\begin{aligned}
Y_{Au}(x_j) &= \left[u_{(0)} \,\middle|\, u_{(0)}x_j \,\middle|\, 0 \,\middle|\, u_{(1)} \,\middle|\, u_{(1)}x_j \,\middle|\, \tfrac{1}{2}u_{(1)}x_j^2+u_{(2)}\right] \\
Y_{AH}(x_j) &= \left[0 \,\middle|\, k_A u_{(0)} \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, k_A u_{(1)} \,\middle|\, k_A u_{(1)}x_j\right] \\
Y_{Av}(x_j) &= \left[0 \,\middle|\, 0 \,\middle|\, 1 \,\middle|\, x_j \,\middle|\, x_j^2/2 \,\middle|\, x_j^3/6\right] \\
Y_{A\theta}(x_j) &= \left[0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 1 \,\middle|\, x_j \,\middle|\, x_j^2/2\right] \\
Y_{A\beta}(x_j) &= \left[0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 1 \,\middle|\, x_j\right] \\
Y_{A\phi}(x_j) &= \left[0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 0 \,\middle|\, 1\right]
\end{aligned}\right\} \tag{125}$$

Partition C :

$$C = \begin{Bmatrix} C_\ell \\ C_g \\ C_A \end{Bmatrix} \begin{matrix} \}\text{ 4 rows} \\ \}\text{ 4 rows} \\ \}\text{ 6 rows} \end{matrix} \tag{126}$$

$$\begin{Bmatrix} u(x_j) \\ H(x_j) \\ v(x_j) \\ \theta(x_j) \\ \beta(x_j) \\ \phi(x_j) \end{Bmatrix} = \begin{bmatrix} H_{u\ell} \\ H_{H\ell} \\ H_{v\ell} \\ H_{\theta\ell} \\ H_{\beta\ell} \\ H_{\phi\ell} \end{bmatrix} F_{\ell D}(x_j)C_\ell + \begin{bmatrix} H_{u\ell} \\ -H_{H\ell} \\ -H_{v\ell} \\ H_{\theta\ell} \\ -H_{\beta\ell} \\ H_{\phi\ell} \end{bmatrix} F_{gD}(x_j)C_g + \begin{bmatrix} Y_{Au}(x_j) \\ Y_{AH}(x_j) \\ Y_{Av}(x_j) \\ Y_{A\theta}(x_j) \\ Y_{A\beta}(x_j) \\ Y_{A\phi}(x_j) \end{bmatrix} C_A \tag{127}$$

$$q(x_j) = k_\eta \bar{t}\theta(x_j) - k_\eta \Delta_q^T u(x_j) \tag{128}$$

$$V_q = \frac{1}{2}\begin{bmatrix} t_1 & & & \\ t_3 & t_3 & & \\ & t_5 & t_5 & \\ & & t_7 & t_7 \\ & & & t_9 \end{bmatrix} \qquad k_I = \begin{Bmatrix} E_1 I_1 \\ E_3 I_3 \\ E_5 I_5 \\ E_7 I_7 \\ E_9 I_9 \end{Bmatrix} \tag{129}$$

$$V(x_j) = V_q q(x_j) - k_I \phi(x_j) \tag{130}$$

$$M(x_j) = k_I \beta(x_j) \tag{131}$$

$$(EI)_{eff.} = \frac{P\ell^3}{12v(x_{n_x})} \tag{132}$$

<u>Output (printed)</u>

$u(x_j)$, $H(x_j)$, $v(x_j)$, $\theta(x_j)$, $\beta(x_j)$, $\phi(x_j)$

$q(s_j)$, $V(x_j)$, $M(x_j)$ for $j = 0, 1, 2 \ldots n_x$

$(EI)_{eff}$

ILLUSTRATIVE EXAMPLES

Four sample cases have been selected to illustrate the usefulness of this program. Cross sections and material properties for the four cases are shown in Figures 9, 10, 11 and 12. The end conditions have been defined as fixed. The load is one pound and has been applied at the beam center line. The outer glass shield is fixed at the ends, although for a typical windshield the glass floats on an interlayer and is free-floating at the ends. The program is set up to handle a total of nine plies. For beams with less than nine plies, such as Cases B, C and D, small values were assigned to the thickness, modulus of elasticity, and shear modulus to negate the influence of those fictitious plies.

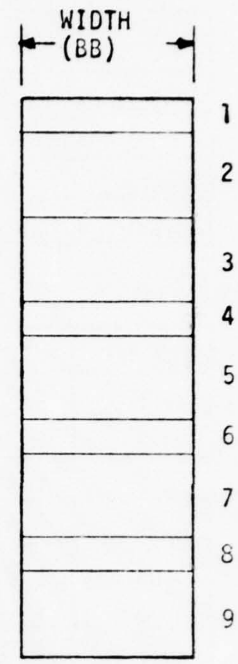
Input data and output data for five examples are presented in Appendix D. Case A was analyzed for a beam ten inches long and for a beam 30 inches long. Case D and Case C were analyzed for a beam ten inches long. Case B was analyzed for a beam 20 inches long. Table 1 is a summary and explanation of input data for Case A. Table 2 is a summary and explanation of output data for Case A. Table 3 lists a summary of deflections for the five examples as well as additional deflections for the four cases at various lengths measured at the beam center line. The deflections do not vary with the cube of the length and this deviation is shown in Figure 13, which compares a laminated beam to a multiply beam without interlayer for Cases A, B, C and D.

At the end of output data for each example in Appendix D, there is a printout for EI(EFF) which is defined as "effective stiffness". This value is calculated from the deflection at the beam center line using the formula for a beam with fixed ends and a center point load which is:

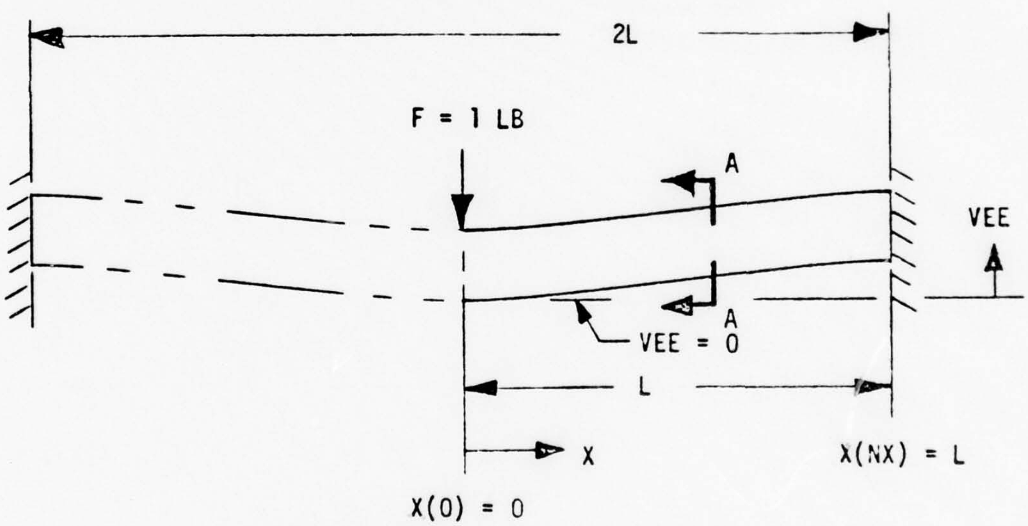
$$EI(EFF) = \frac{FL^3}{192V}$$

CASE A				
END COND	FIXED			
L	5	10	15	25
WIDTH	1			
TOT THICK	1.65			

PLY NO.	T	E	G	TYPE
1	0.100	10×10^6	4×10^6	GLASS
2	0.250	183	73	CIP SILICONE
3	0.250	340,000	109,000	POLYCARBONATE
4	0.100	183	73	CIP SILICONE
5	0.250	340,000	109,000	POLYCARBONATE
6	0.100	183	73	CIP SILICONE
7	0.250	340,000	109,000	POLYCARBONATE
8	0.100	183	73	CIP SILICONE
9	0.250	340,000	109,000	POLYCARBONATE



SECTION A-A

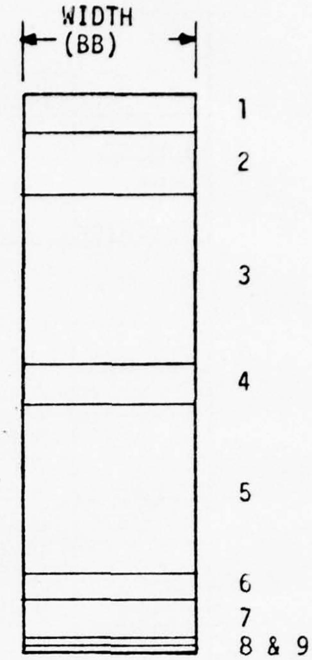


DEFLECTED SHAPE OF BEAM - NO SCALE

See Tables 1 and 2 for symbols

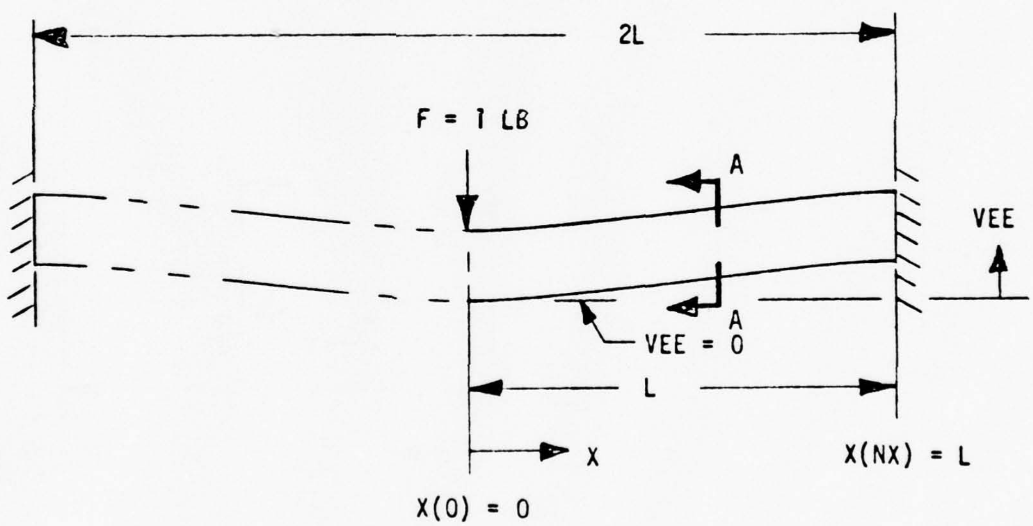
Figure 9. Cross sectional properties for Case A.

CASE B				
END COND	FIXED			
L	5	10	15	25
WIDTH	1			
TOT THICK	1.65			



SECTION A-A

PLY NO.	T	E	G	TYPE
1	0.110	10×10^6	4×10^6	GLASS
2	0.188	1,100	440	PPG 112
3	0.500	10×10^6	4×10^6	GLASS
4	0.120	1,100	440	PPG 112
5	0.500	10×10^6	4×10^6	GLASS
6	0.080	1,100	440	PPG 112
7	0.110	10×10^6	4×10^6	GLASS
8	0.021	1	1	FICTITIOUS
9	0.021	1	1	FICTITIOUS



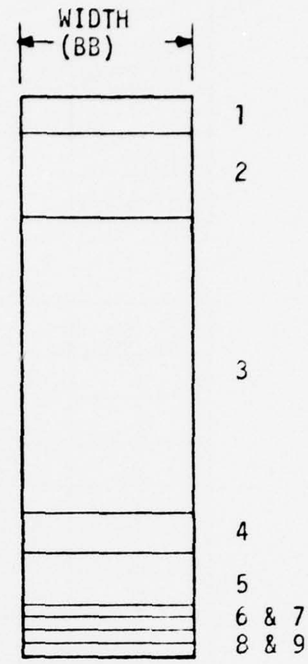
DEFLECTED SHAPE OF BEAM - NO SCALE

See Tables 1 and 2 for symbols

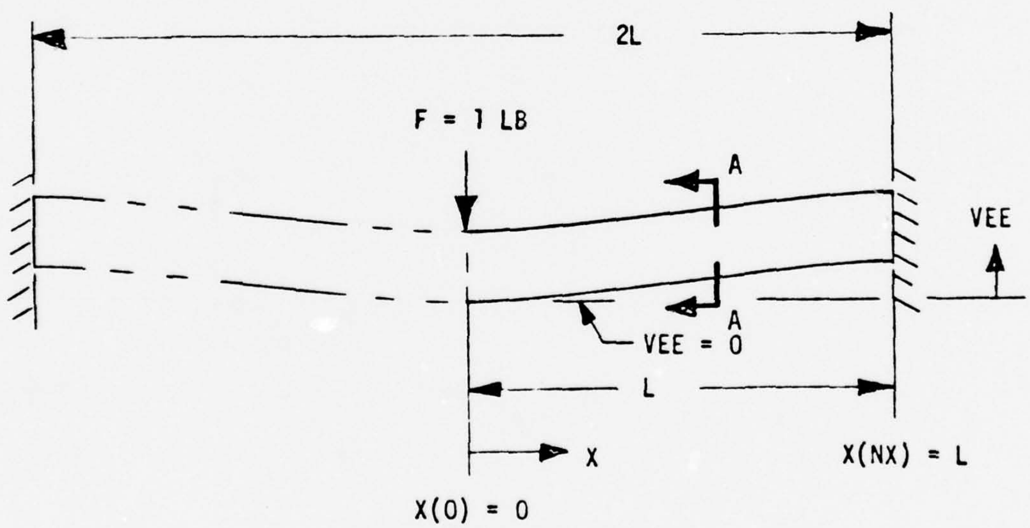
Figure 10. Cross sectional properties for Case B.

CASE C				
END COND	FIXED			
L	5	10	15	25
WIDTH	1			
TOT THICK	1.65			

PLY NO.	T	E	G	TYPE
1	0.105	10×10^6	4×10^6	GLASS
2	0.250	183	73	CIP SILICONE
3	0.870	340,000	109,000	POLYCARBONATE
4	0.120	183	73	CIP SILICONE
5	0.150	340,000	109,000	POLYCARBONATE
6	0.03875	1	1	FICTITIOUS
7	0.03875	1	1	FICTITIOUS
8	0.03875	1	1	FICTITIOUS
9	0.03875	1	1	FICTITIOUS



SECTION A-A



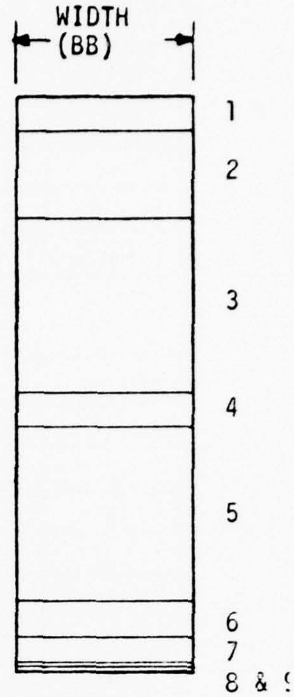
DEFLECTED SHAPE OF BEAM - NO SCALE

See Tables 1 and 2 for symbols

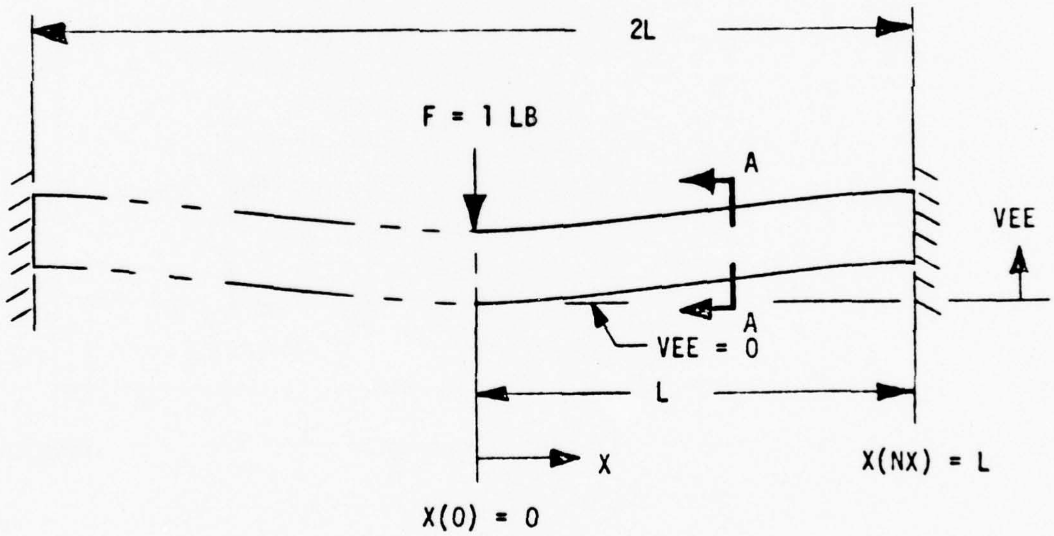
Figure 11. Cross sectional properties for Case C.

CASE D				
END COND	FIXED			
L	5	10	15	25
WIDTH	1			
TOT THICK	1.65			

PLY NO.	T	E	G	TYPE
1	0.100	10×10^6	4×10^6	GLASS
2	0.250	1,100	440	PPG 112
3	0.500	340,000	109,000	POLYCARBONATE
4	0.100	1,100	440	PPG 112
5	0.500	340,000	109,000	POLYCARBONATE
6	0.100	1,100	440	PPG 112
7	0.080	490,000	181,000	ACRYLIC
8	0.010	1	1	FICTITIOUS
9	0.010	1	1	FICTITIOUS



SECTION A-A



DEFLECTED SHAPE OF BEAM - NO SCALE

See Tables 1 and 2 for symbols

Figure 12. Cross sectional properties for Case D.

TABLE 1. SUMMARY OF INPUT DATA FOR CASE A

COMPUTER PROGRAM SYMBOLS	DESCRIPTION	INPUT DATA
NX	Number of Increments	10.0
F	Load (Pounds)	1.0
L	Half Length (Inches)	5.0
BB	Width (Inches)	1.0
T	Ply Thickness (Inches) (5 Structural Plies and 4 Interlayer Plies)	.10 .25, .25, .10, .25, .10, .25, .10, .25
E	Modulus of Elasticity (PSI)	10^7, 183, 340,000, 183, 340,000, 183, 340,000, 183, 340,000
G	Shear Modulus (PSI)	4×10^6, 73, 109,000, 73, 109,000, 73, 109,000, 73, 109,000
X	Distance from Center of Beam (Inches)	0, .50, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0

TABLE 2. SUMMARY OF OUTPUT DATA FOR CASE A

COMPUTER PROGRAM SYMBOLS	DESCRIPTION	OUTPUT DATA
X(0)	Increment Number	0
U	Horizontal Displacement of Center of Structural Plies (Inches)	0
H	Axial Load in Structural Plies (Pounds)	-.0959, -0979, -00192, -.00389, -.1918
VEE	Vertical Deflection (Inches) (Zero Deflection is Point of Maximum Deflection at Center)	0
QXJ	Shear Flow in Interlayer (Pounds Per Inch)	0
VXJ	Shear Force in Ply (Pounds)	.16, .085, .085, .085, .085
MXJ	Moment on Structural Ply (Inch-Pounds)	.322, .171, .171, .171, .171
THETA	Slope (Radians)	-5.551115E-17
BETA	Rate of Change of Slope	.000386
PHI	Rate of Change of BETA	-.000192

TABLE 3. DEFLECTION - (INCHES)

	OUTPUT FROM PROGRAM			
LENGTH OF BEAM - INCHES	10	20	30	50
<u>CASE A</u> 4 Ply Polycarbonate	.00153	.00733	.0152	.0339
<u>CASE B</u> 2-Ply Glass	.0000239	.000175	.000520	.00178
<u>CASE C</u> 1-Ply Polycarbonate	.000252	.00178	.00511	.0172
<u>CASE D</u> 2-Ply Polycarbonate	.000364	.00142	.00292	.00731

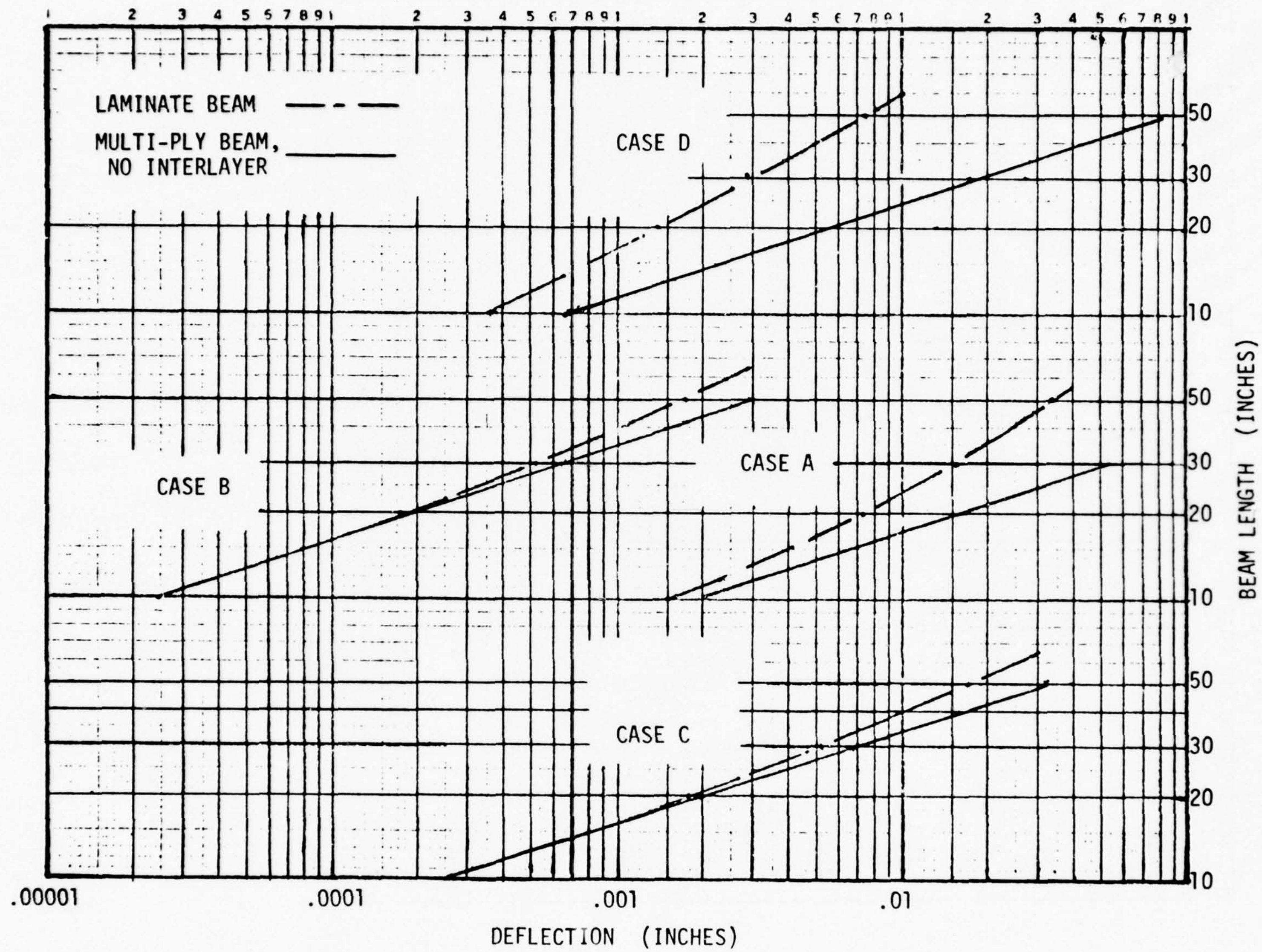


Figure 13. Deflection versus beam length.

where F is load, L is length and V is deflection. Effective stiffness varies with length. Table 4 is a comparison of effective stiffness, at various lengths for the four cases, to the stiffness if all the plies acted as individual beams. The reason why effective stiffness increases with length is that the interlayers are more effective in transferring shear between structural plies, as length increases. Thus, the behavior of the longer beams approaches monolithic.

Figure 14 is a plot of strain (inches per inch) versus thickness (inches) for Case A (4 ply polycarbonate) at the center of the beam with a one pound load applied at the center (beam length is ten inches). The values were calculated from the output data using the moment in the structural plies (MXJ) and the axial load in the structural plies (H).

Figure 15 shows a finite element model for a beam ten inches long with a cross section identical to Case A. The beam was analyzed using the computer aided Structural Design computer program* to provide a comparison. The deflections agreed within one percent. This comparison is presented as verification of the validity of the basic assumptions for the enclosed derivations.

An actual test (Reference 1) was conducted to determine the deflection of the midpoint of a beam with a cross section similar to Case C, Figure 11. The beam length was 34.7 inches. The face-ply (ply No. 1) was "free floating". The deflection of the test beam was .00882 inch. The data output for the program is given in Appendix D and shows a deflection of 0.00913 inch which agrees within 3.5 percent. This comparison is presented as additional verification of the validity of the basic assumptions for the enclosed derivations.

* Ref 9

TABLE 4. EFFECTIVE STIFFNESS (POUNDS-INCHES SQUARE)

	OUTPUT FROM PROGRAM				SINGLE ACTING BEAMS (NO INTERLAYER)
LENGTH OF BEAM - (INCHES)	10	20	30	50	
CASE A 4-Ply Polycarbonate	3,397	5,683	9,238	19,212	2,600
CASE B 2-Ply Glass	217,500	237,900	270,600	366,000	209,100
CASE C 1-Ply Polycarbonate	20,700	23,450	27,500	37,900	19,720
CASE D 2-Ply Polycarbonate	14,326	29,370	48,175	89,080	7,900

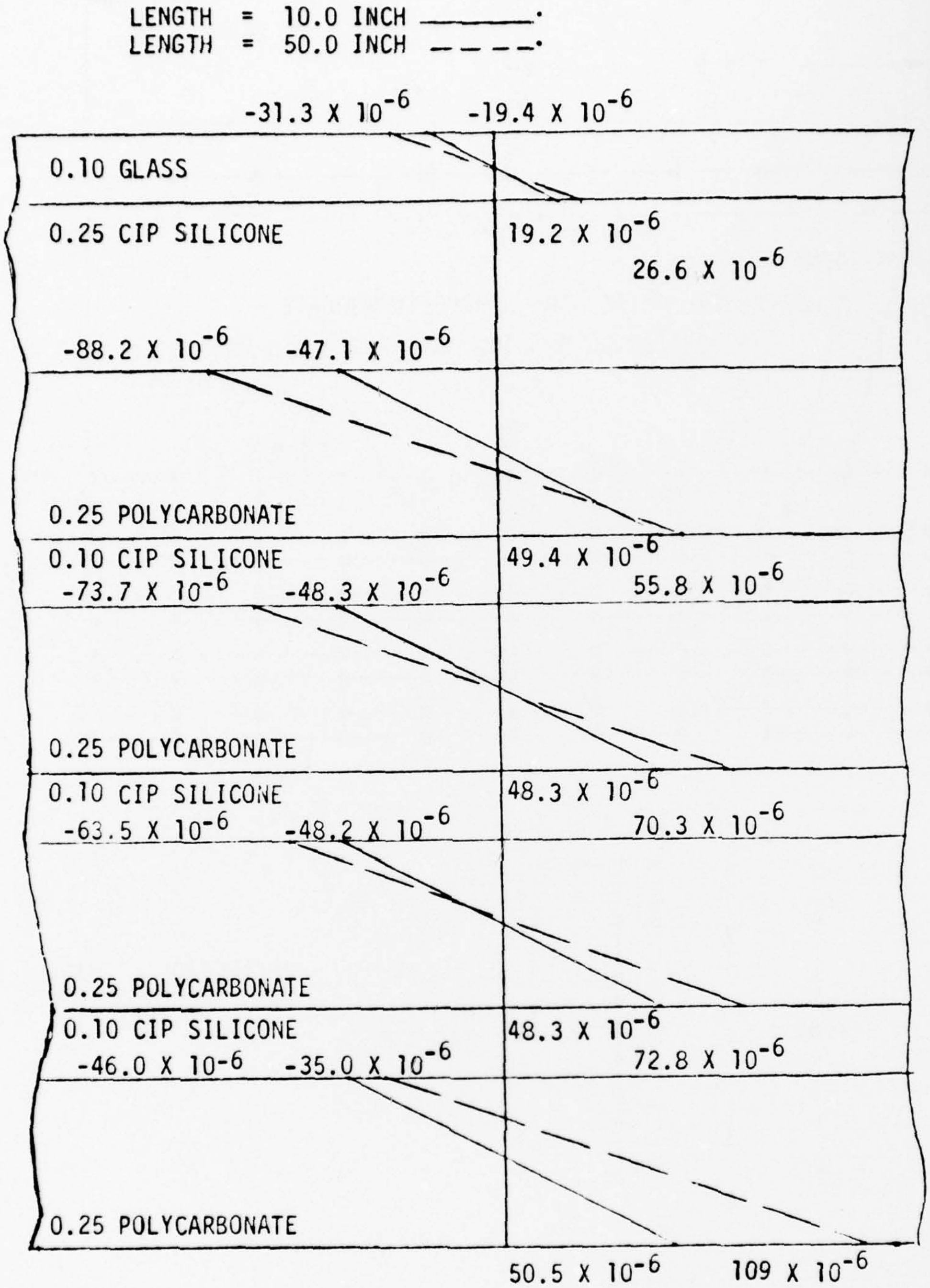


Figure 14. Strain (in/in) versus thickness for Case A at center of beam.

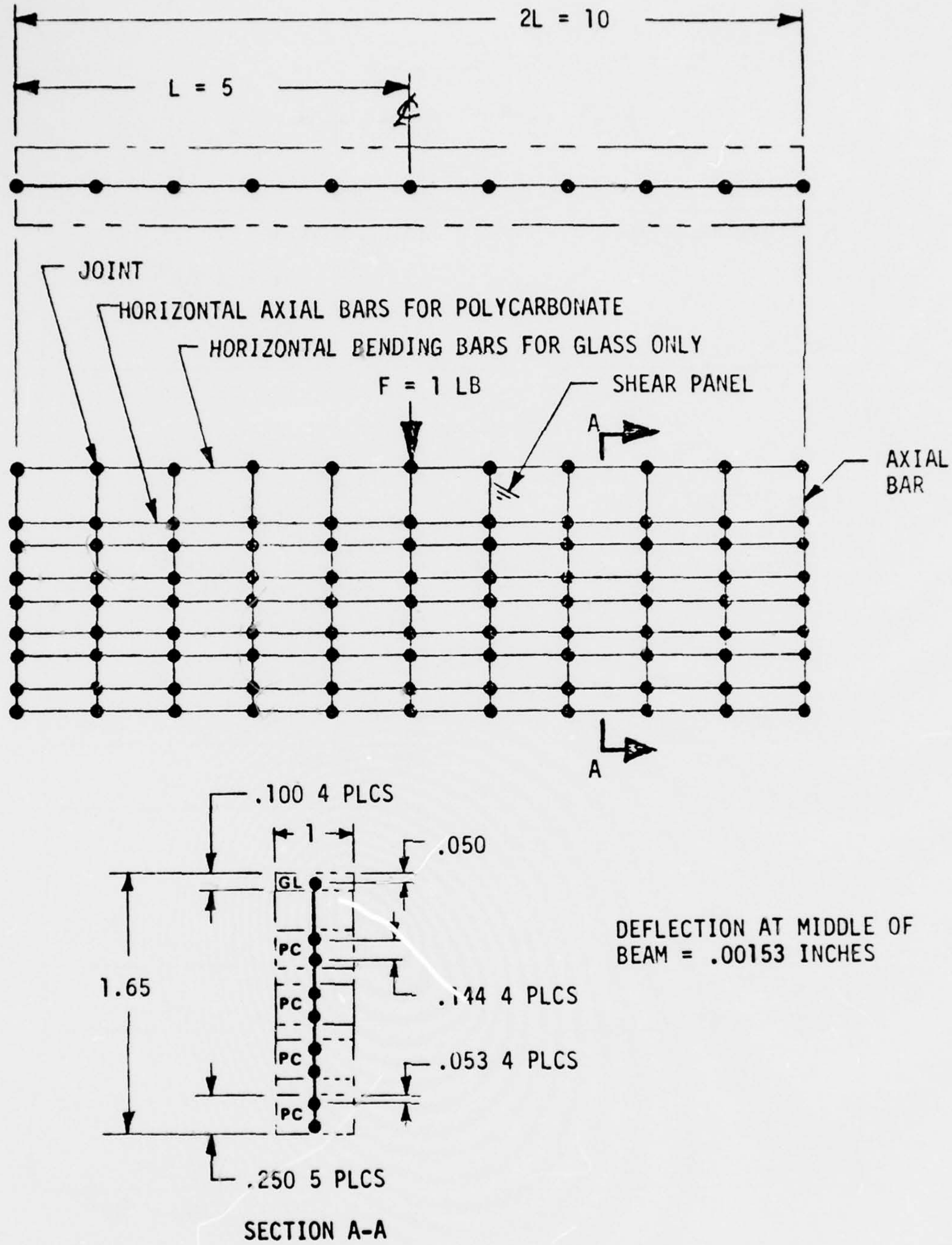


Figure 15. Finite element model of a fixed ended beam - Case A.

SECTION III
COMPUTER PROGRAM USER'S MANUAL

A FORTRAN computer program has been coded which determines the internal loads, deflections, and relative stiffness of a laminated, fixed ended beam with one concentrated load at the center. The program is based on the equations and procedures embodied in the preceding sections of this report. A copy of the source coding is contained in Appendix B. Appendix C lists the intermediate matrices which were run for Case A. The output for four illustrative problems and one test beam is contained in Appendix D. The program was written in CDC FORTRAN Extended Version 4-0 for utilization of a CDC6600 computer.

Figure 16 shows a complete card deck ready to load for a particular computer installation. Certain obvious changes to the information shown on the cards, such as time charge numbers, programmer's name, and program identification, would have to be made to suit the requirements of another installation. A review of the source coding shown in Appendix B may reveal other minor changes that might be needed at another installation.

The complete card deck consists of essentially five categories: a job card, two control cards, a source deck, input data cards, and separating cards, called 7, 8, 9 Cards and 6, 7, 8, 9 Cards. Figure 17 shows the relationship of these components to the complete deck.

The job card, control cards, and separating cards may be obtained by entering the appropriate data on the data coding sheet, Figure 18, and submitting the sheet to procure the key-punched cards. The line entries are described as follows:

Line 1 - This line describes the job card which identifies the user and lists other pertinent identifying data.

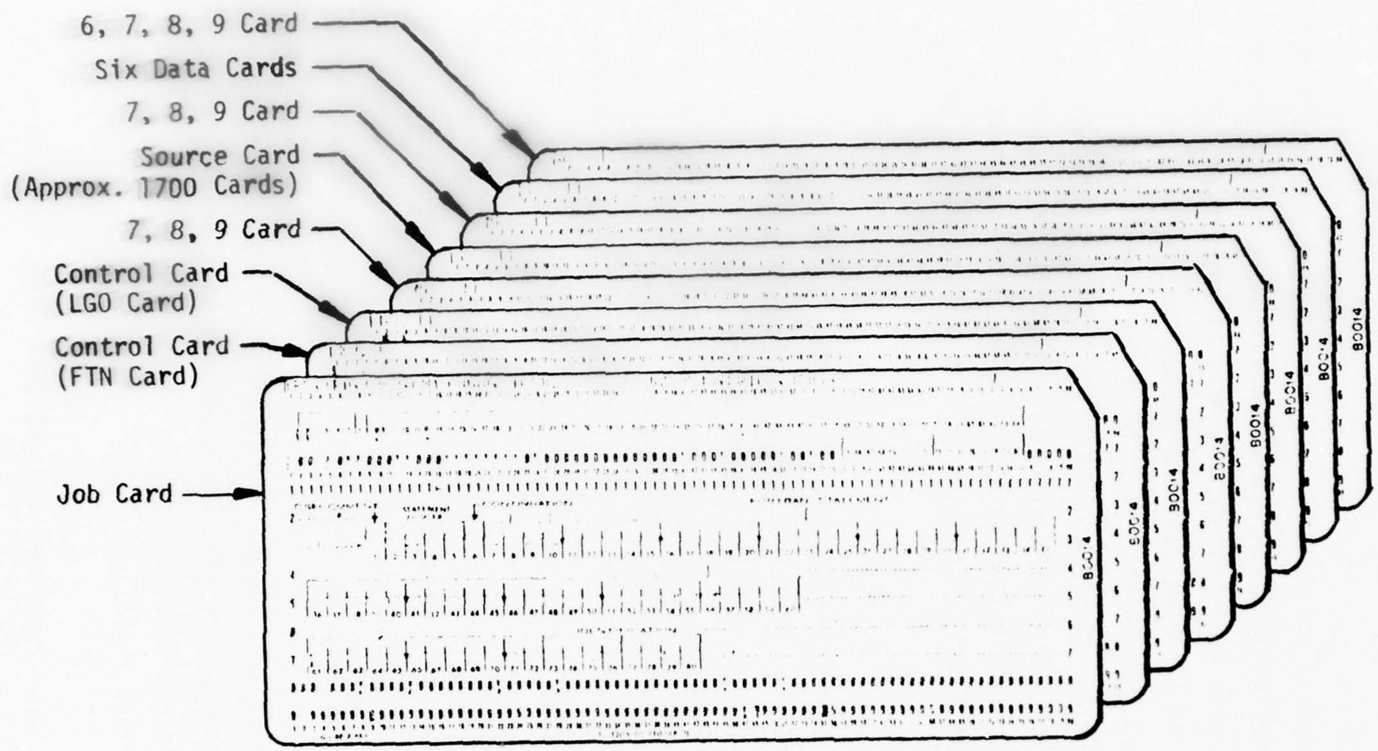


Figure 16. Complete card deck.

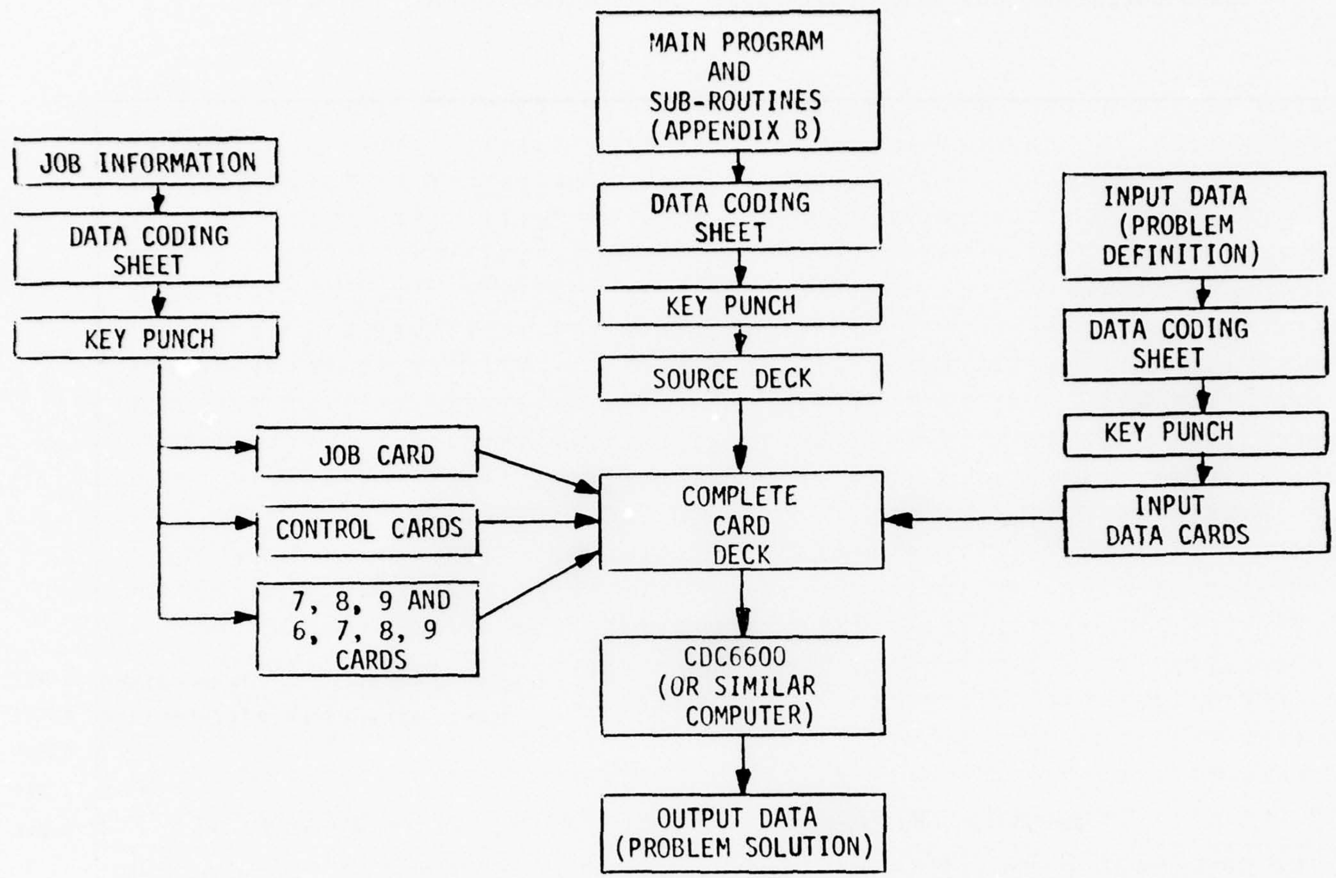


Figure 17. Card deck flow diagram.

```
LINE 1  TRP,T20,JØ50,CM100000,STCSB.          D750545,PRICE,32096
LINE 2  FTN
LINE 3  LGO
LINE 4  THIS CARD HAS 7, 8, 9 PUNCH IN CARD COLUMN 1.
LINE 5  THIS CARD HAS 6, 7, 8, 9 PUNCH IN CARD COLUMN 1.
```

Figure 18. Sample coding sheet for control cards and separating cards.

Line 2 - This line refers to the FTN control card and provides access to library subroutines and compilation of the FORTRAN source deck.

Line 3 - This line refers to an LGO control card, which is a load and go card. One LGO card must be inserted for every set of data cards.

Line 4 - This line refers to a card that separates the source deck from the data cards and control cards. A minimum of two cards is required and an additional 7, 8, 9 Card must be added for each additional set of data cards.

Line 5 - This line refers to the last card in the deck, a 6, 7, 8, 9 Card.

The source deck is a set of approximately 1,700 cards and may be compiled from the main program and subroutines listed in Appendix B. This information may be entered on data coding sheets and submitted to key-punch to procure a source deck.

The input data card deck is a set of six cards and may be obtained by entering the appropriate information on the data coding sheet, Figure 19. The data sheet may be submitted to procure the key-punched cards shown in Figures 20 and 21 for illustrative Case A, Table 1. The program statements are shown for reference. A set of input data cards must be included for each problem to be solved. The line entries shown in Figure 19 for Case A are described as follows:

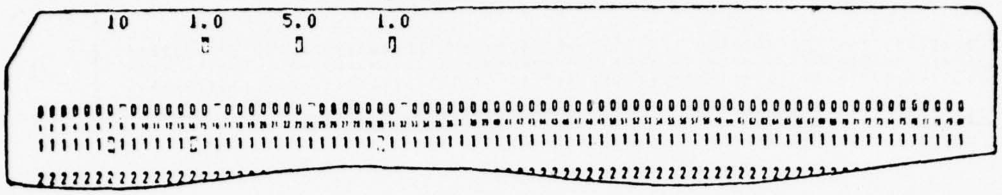
Line 1 - Columns 1 through 8 show the number of spanwise increments to the center line of the beam.

	1–8	9–16	17–24	25–32	33–40	41–48	49–56	57–64	65–72	73–80
LINE 1	1.0	1.0	5.0	1.0						
LINE 2	0.10	0.25	0.25	0.10	0.25	0.10	0.25	0.10	0.25	
LINE 3	10000000	183.0	340000.0	183.0	340000.0	183.0	340000.0	183.0	340000.0	
LINE 4	4000000.	73.0	109000.0	73.0	109000.0	73.0	109000.0	73.0	109000.0	
LINE 5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	
LINE 6	5.0									
LINE 7	MX	F	L	BB						
LINE 8	T(1)	T(2)	T(3)	T(4)	T(5)	T(6)	T(7)	T(8)	T(9)	
LINE 9	E(1)	E(2)	E(3)	E(4)	E(5)	E(6)	E(7)	E(8)	E(9)	
LINE 10	G(1)	G(2)	G(3)	G(4)	G(5)	G(6)	G(7)	G(8)	G(9)	
LINE 11	X(2)	X(3)	X(4)	X(5)	X(6)	X(7)	X(8)	X(9)	X(10)	
LINE 12	X(11)									

Figure 19. Sample input data coding sheet (Case A).

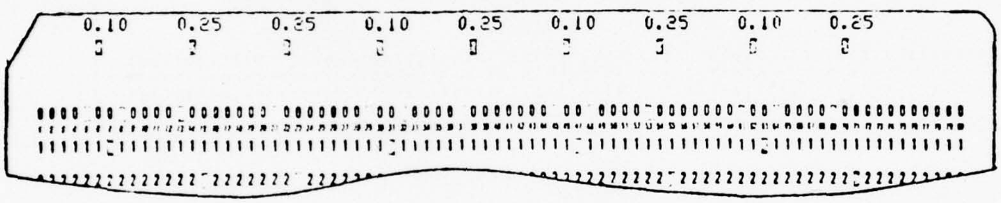
DATA CARD 1. INPUT NUMBER OF SPANWISE INCREMENTS TO CENTER LINE OF BEAM, FORCE (POUNDS), LENGTH TO CENTER LINE OF BEAM (INCHES), WIDTH (INCHES).

```
READ (5,10)NX,F,L,BB  }
10 FORMAT (I8,3F8.0)  }  PROGRAM STATEMENTS (REFERENCE ONLY)
```



DATA CARD 2. INPUT THICKNESS (INCHES) OF EACH LAYER.

```
READ (5,20) (T(J),J = 1,9)  }
20 FORMAT (9F8.0)           }  PROGRAM STATEMENTS (REFERENCE ONLY)
```



DATA CARD 3. INPUT YOUNG'S MODULUS FOR EACH LAYER.

```
READ (5,20) (E(J),J = 1,9)  }
20 FORMAT (9F8.0)           }  PROGRAM STATEMENTS (REFERENCE ONLY)
```

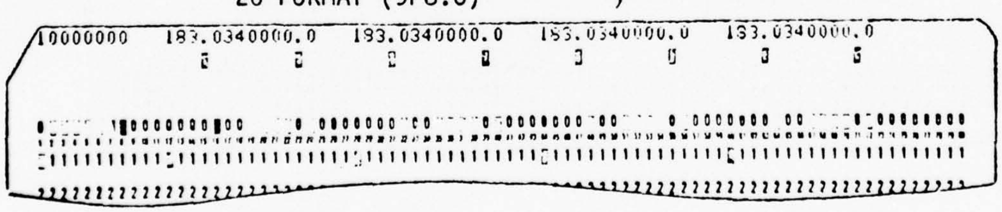
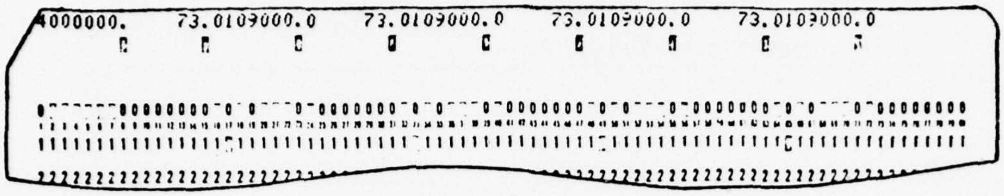


Figure 20. Sample data cards.

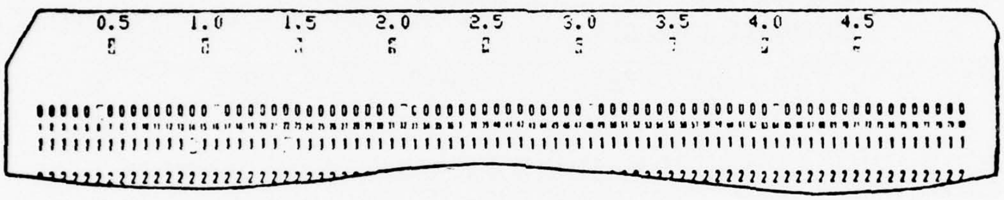
DATA CARD 4. INPUT SHEAR MODULUS FOR EACH LAYER.

READ (5,20) (G(J),J = 1,9) } PROGRAM STATEMENTS
20 FORMAT (9F8.0) } (REFERENCE ONLY)



DATA CARD 5. INPUT DISTANCE FROM CENTER LINE OF BEAM TO X INCREMENT. (NUMBER OF VALUES MUST EQUAL NX ON CARD 1 AND LAST VALUE MUST BE EQUAL TO L ON CARD 1.) X(1) = 0

READ (5,20) (X(J),J = 2, NX1) } PROGRAM STATEMENTS
20 FORMAT (9F8.0) } (REFERENCE ONLY)



DATA CARD 6. (CONTINUATION OF CARD 5.)

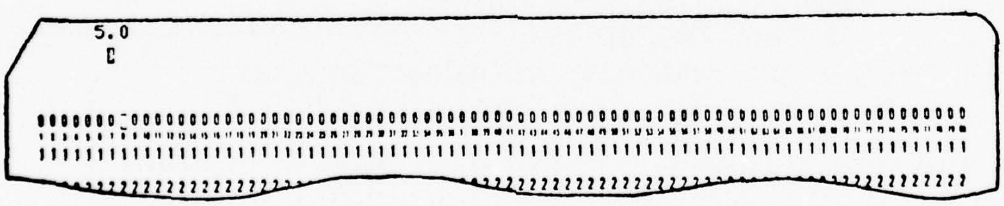


Figure 21. Sample data cards.

Line 1 - Columns 9 through 16 are the load in pounds.

Line 1 - Columns 17 through 24 are the length to the center line of the beam in inches.

Line 1 - Columns 25 through 32 are the width of the beam in inches.

Line 2 - Columns 1 through 8 are the thickness of the first ply in inches and the thickness of each of the remaining eight plies is shown in Columns 9 through 72.

Line 3 - Columns 1 through 8 are the modulus of elasticity (PSI) for Ply Number 1 and the modulus of elasticity for Ply Numbers 2 through 8 are shown in Columns 9 through 72.

Line 4 - Columns 1 through 8 are the shear modulus (PSI) for Ply Number 1 and the shear modulus for Ply Numbers 2 through 8 are shown in Columns 9 through 72.

Lines 5 and 6 - Columns 1 through 8 of Line 5 are the distance from the center of the beam to "X" increment in inches. The number of entries on Lines 5 and 6 correspond to the number of increments on Line 1, Columns 1 through 8. There are ten entries on Lines 5 and 6.

Lines 7 through 12 - These entries are the appropriate computer symbols corresponding to the entries on Lines 1 through 6 above and are for reference only.

Data cards must meet the following requirements:

- Each card field is eight (8) card columns wide.

- There cannot be any zeros or blank fields imbedded in the data (i.e., If a layer is to be dropped from the problem, its thickness and other parameters must be input to the problem as relatively small numbers to cause its effective elimination.).

- Numbers must be right justified in their fields.

- A trailing decimal point may be dropped.

SECTION IV
CONCLUSIONS

The analysis and computer program presented in this report were developed to provide a tool for evaluating laminated combinations of transparent materials during initial design of windshields and windows. Historically, formulas and theories from engineering handbooks have been used for this purpose, but these methods are highly approximate.

The laminated beam analyzed in the present approach is considered to represent a strip cut from a typical aircraft windshield transparency. The primary characteristic of such a laminated beam is the relative ease of structural plies to slide past each other because of the softness of interlayer materials. Equilibrium and compatibility equations are written for the beam, based upon assumptions that preserve the important features of laminated beam behavior. The resulting set of differential equations is solved exactly.

The computer code presented can expedite the application of the theory to practical problems. Utilizing this program, the deflections for a series of typical laminated beams were examined, and were compared to the results of a proven finite element computer program. The two sets of computed values agreed within one percent. Data obtained from another beam, tested as noted in Reference 1, were compared to calculated results provided by the present method. The outer ply of the test beam was "free floating" at the ends, rather than fixed, as assumed in the analysis. To simulate this condition, a negligible value of Young's modulus was assigned to the outer ply. The calculated deflection was 3.5 percent greater than the measured value. This agreement is considered reasonably close, in view of the approximate method used to account for the "free floating" outer ply.

The applications of the computer program described in Section II under the heading "Illustrative Examples" were run on the CDC6600 computer, employing 21,000 decimal words of core. Each case required 0.879 seconds CPU time and 7.588 seconds I/O time. The I/O time includes compilation of the entire program.

SECTION V
RECOMMENDATIONS

The new computer program is recommended as a tool for use in the early stages of aircraft windshield design as a means of screening laminate configurations. As an example, a strip of a certain width can be cut out of a transparency being studied. This strip can be flattened to form a laminated beam having pinned or fixed ends as appropriate. The width of the strip can be selected so that the resulting beam, supported only at the ends, roughly represents the transparency supported on all edges. A load can be applied at the center to represent a bird impact. The number of plies, structural ply and interlayer materials, and ply thicknesses can be varied, so that established stress and/or displacement constraints are met. The appropriate constraints are that ultimate stresses and maximum allowable displacements are not exceeded. Weights per square foot of transparency surface can then be calculated and compared. The laminates showing the most promise on this basis can be selected for further study.

Another use for the present code is to provide an effective bending stiffness for a laminate. This effective stiffness can be selected as the EI value of a monolithic beam necessary to give the same maximum deflection as a beam composed of the given laminate under the same load. The computer program outputs such a value. This value can then be employed as a means of developing a finite element model of the laminated transparency and supporting structure, in which the laminate is replaced by a single layer, as a means of reducing engineering, computing, and elapsed time to perform either a static or a dynamic analysis. Since effective stiffness depends upon beam length, the user should estimate the half length of the wave produced in the transparency when impact occurs. Such an estimate can be made on the basis of past experience, including previous studies employing the Bird Impact Math Model (Reference 6) or test results. This half wave length can then be taken as the length of the simply supported beam for which an effective EI value can be calculated. An approach of this kind is probably most useful for calculating transparency displacements and loads on edge attachments and supporting structure.

The feasibility of extending the method to cover other loadings and boundary conditions should be studied. For example, an option applicable to an end condition in which the layers are prevented from sliding relative to each other, as by a bolt, although the end tangent can rotate, might be useful. A beam uniformly loaded over part of its length is a candidate case. The feasibility of applying the approach to dynamic problems also should be studied, and the possibility of eliminating the assumption that transverse strains are negligible should be considered. Eliminating this assumption would mean that the method would provide reasonably reliable estimates of transverse stresses, and perhaps a means of forecasting delamination. Any of these improvements would increase the realism of the analytical results. Consequently, the usefulness of the approach as a design tool would be enhanced.

If any of these improvements were accomplished, they could be verified by comparison with results obtained by finite element analysis.

APPENDIX A

PROPERTIES OF THE EIGENVALUES AND EIGENVECTORS

The matric differential equation (Equation 42) can be written in the following form by rearranging the rows of Y :

$$\begin{Bmatrix} dY_a/dx \\ dY_b/dx \end{Bmatrix} = \left[\begin{array}{c|c} 0 & a_b \\ b_a & 0 \end{array}\right] \begin{Bmatrix} Y_a \\ Y_b \end{Bmatrix} \qquad (A.1)$$

where

$$b_a = \left[\begin{array}{c|c|c} a_{2,1} & a_{2,3} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \qquad Y_a = \begin{Bmatrix} u \\ \theta \\ \phi \end{Bmatrix} \qquad (A.2)$$

$$a_b = \left[\begin{array}{c|c|c} a_{1,2} & 0 & 0 \\ 0 & 1 & 0 \\ a_{5,2} & a_{5,4} & 0 \end{array}\right] \qquad Y_b = \begin{Bmatrix} H \\ \beta \\ v \end{Bmatrix} \qquad (A.3)$$

The corresponding characteristic equation is

$$\left[\begin{array}{c|c} -\lambda I & a_b \\ b_a & -\lambda I \end{array}\right] \begin{Bmatrix} G_a \\ G_b \end{Bmatrix} = 0 \qquad (A.4)$$

where $$G_a = \begin{Bmatrix} G_u \\ G_\theta \\ G_\phi \end{Bmatrix} \qquad G_b = \begin{Bmatrix} G_H \\ G_\beta \\ G_v \end{Bmatrix} \qquad (A.5)$$

$$\therefore \quad \begin{cases} -\lambda G_a + a_b G_b = 0 & (A.6) \\ b_a G_a - \lambda G_b = 0 & (A.7) \end{cases}$$

Changing the signs of λ and G_b leaves Equations A.5 and A.6 unchanged.

Therefore if λ and $\{G_a \mid G_b\}$ are an eigenvalue and an eigenvector of Equation A.4, then $-\lambda$ and $\{G_a \mid -G_b\}$ are also an eigenvalue and an eigenvector.

APPENDIX B

SOURCE CODING (MAIN PROGRAM AND SUB-ROUTINES)

```
      PROGRAM MAIN(INPUT=512,TAPE5=INPUT,OUTPUT=512,TAPE6=OUTPUT)
      EQUIVALENCE (AL,LA),(TEMP(1,1),WORK(1000))
      REAL L,KA,KETA,KI,LAMBL,MXJ
      COMPLEX EIGVAL(14),EIGVEC(14,14)
      DIMENSION WORK(3000),T(9),E(9),G(9),X(100),KA(5,5)
     2,        KETA(4,4),XKI(5),TBAR(4),DELTAQ(5,4),A12(5,5),A21(5,5)
     3,        A23(5),A52(5),A54(1),A(14,14),FLDL(4,4),Q(5,4),U1(5)
     4,        U2(5),B(14,14),HL(14,4),LAMBL(4),PSI(14),FLD(4,4)
     5,        FGD(4,4),ANS(14),QXJ(4)       ,VQ(5,4),VXJ(5),MXJ(5)
     6,        TEMP(14,6)
     7,        S22(4,4),S12(4),XT(5),W(5)
C     NX = NUMBER OF X'S.
C     F  = FORCE.
C     L  = LENGTH.
C     BB = WIDTH.
      WRITE(6,935)
      READ(5,10) NX,F,L,BB
   10 FORMAT(I8,3F8.0)
C     READ T,E,G.
      READ(5,20) (T(J),J=1,9)
      READ(5,20) (E(J),J=1,9)
      READ(5,20) (G(J),J=1,9)
      X(1) = 0.0
      NX1 = NX+1
C     READ X.
      READ(5,20) (X(J),J=2,NX1)
   20 FORMAT(9F8.0)
      WRITE(6,1) NX,F,L,BB
      WRITE(6,2) (T(J),J=1,9)
      WRITE(6,3) (E(J),J=1,9)
      WRITE(6,4) (G(J),J=1,9)
      WRITE(6,5) (X(J),J=1,NX1)
    1 FORMAT(11H NX,F,L,BB ,I5,1P3E16.6)
    2 FORMAT(3H T ,1P5E16.6/3H   ,1P4E16.6)
    3 FORMAT(3H E ,1P5E16.6/3H   ,1P4E16.6)
    4 FORMAT(3H G ,1P5E16.6/3H   ,1P4E16.6)
```

```
      5 FORMAT(3H X ,1P5E16.6/3H    ,1P6E16.6)
C       FORM VQ
        DO 90 I=1,5
        DO 90 J=1,4
     90 VQ(I,J) = 0.0
        DO 95 I=1,4
        VQ(I,I) = 0.5*T(2*I-1)
     95 VQ(I+1,I) = 0.5*T(2*I+1)
C       FORM KA
        DO 100 I=1,5
        DO 100 J=1,5
    100 KA(I,J) = 0.0
        DO 150 I=1,5
    150 KA(I,I) = E(2*I-1)*BB*T(2*I-1)
C       FORM KETA
        DO 160 I=1,4
        DO 160 J=1,4
    160 KETA(I,J) = 0.0
        DO 170 I=1,4
    170 KETA(I,I) = G(2*I)*BB/T(2*I)
C       FORM KI,XKI
        KI = 0.0
        DO 180 I=1,5
        XKI(I) = E(2*I-1)*BB*T(2*I-1)**3/12.0
    180 KI = KI+XKI(I)
C       FORM TBAR
        DO 190 I=1,4
    190 TBAR(I) = 0.5*T(2*I-1)+T(2*I)+0.5*T(2*I+1)
C       FORM DELTAQ
        DO 200 I=1,5
        DO 200 J=1,4
    200 DELTAQ(I,J) = 0.0
        DO 210 I=1,4
        DELTAQ(I,I) = -1.0
    210 DELTAQ(I+1,I) = 1.0
C       FORM A12
```

```
      DO 220 I=1,5
      DO 220 J=1,5
      A12(I,J) = 0.0
  220 IF(I .EQ. J) A12(I,J) = 1.0/KA(I,J)
C     FORM A21
      CALL  MULT(DELTAQ,5,4,KETA,4,4,WORK)
      CALL TMULT(WORK,5,4,DELTAQ,5,4,A21)
C     FORM A23
      DO 230 I=1,20
  230 WORK(I) = -WORK(I)
      CALL MULT(WORK,5,4,TBAR,4,1,A23)
C     FORM A52
      CALL LTMULT(A23,5,1,A12,5,5,A52)
      DO 240 I=1,5
  240 A52(I) = A52(I)/KI
C     FORM A54
      CALL LTMULT(TBAR,4,1,KETA,4,4,WORK)
      CALL   MULT(WORK,1,4,TBAR,4,1,A54)
      A54(1) = A54(1)/KI
C     FORM A
      DO 250 I=1,14
      DO 250 J=1,14
  250 A(I,J) = 0.0
      DO 260 I=1,5
      DO 260 J=1,5
      A(I+5,J) = A21(I,J)
  260 A(I,J+5) = A12(I,J)
      DO 270 J=1,5
  270 A(14,J+5) = A52(J)
      DO 280 I=1,5
  280 A(I+5,12) = A23(I)
      A(14,13) = A54(1)
      DO 290 I=11,13
  290 A(I,I+1) = 1.0
      EPSILN = 0.00001
      DO 299 J=1,14
```

```
      WRITE(6,298) (A(I,J),I=1,14)
  298 FORMAT(3H A ,1P7E16.6/1H ,1P7E16.6)
  299 CONTINUE
      IND = 1
C     CALL EIGEN ROUTINE.
      CALL RGEIG(14,14,A,IND,EIGVAL,EIGVEC,WORK)
      DO 310 I=1,14
  310 WORK(I) = REAL(EIGVAL(I))
      WRITE(6,311) (WORK(I),I=1,14)
  311 FORMAT(13H EIGENVALUES ,1P7E16.6/1H ,1P7E16.6)
C     TEST EIGENVALUES,
      DO 320 I=1,14
      COM = AIMAG(EIGVAL(I))
      IF(ABS(COM) .LE. EPSILN) GO TO 320
      WRITE(6,315) COM,I
  315 FORMAT(19H COMPLEX EIGENVALUE,1PE13.6,I10)
      GO TO 9999
  320 CONTINUE
  340 KK = 1
      KOUNT = 0
      NNEG  = 0
      NPOS  = 0
      DO 371 I=1,14
      IF(WORK(I) .GT. EPSILN) NPOS=NPOS+1
      IF(WORK(I) .LT. -EPSILN) NNEG=NNEG+1
      IF(ABS(WORK(I)) .GT. EPSILN) GO TO 350
      KOUNT = KOUNT+1
      GO TO 371
  350 ITEST = 0
      DO 360 J=1,14
      IF(ABS(WORK(I)+WORK(J)) .GT. EPSILN) GO TO 360
      ITEST = 1
      GO TO 361
  360 CONTINUE
  361 IF(ITEST .EQ. 1) GO TO 363
      WRITE(6,362)
```

```
  362 FORMAT(22H UNMATCHED EIGENVALUES)
      GO TO 9999
  363 IF(WORK(I) .GT. 0) GO TO 371
      IF(KK .LE. 4) GO TO 365
      WRITE(6,364)
  364 FORMAT(9H  KK GT 4)
      GO TO 9999
C     STORE DESIRED EIGENVALUES.
  365 LAMBL(KK) = WORK(I)
      DO 369 K=1,14
      COM = AIMAG(EIGVEC(K,I))
      IF(COM .LE. EPSILN) GO TO 369
      WRITE(6,368) COM,I,K
  368 FORMAT(16H COMPLEX VECTOR ,1PE13.6,2I5)
      GO TO 9999
  369 CONTINUE
C     STORE DESIRED EIGENVECTORS,
      DO 370 K=1,14
  370 HL(K,KK) = REAL(EIGVEC(K,I))
      KK = KK+1
  371 CONTINUE
      IF((KOUNT .EQ. 6) .AND. (NPOS .EQ. 4) .AND. (NNEG .EQ. 4))
     1GO TO 400
      WRITE(6,375) KOUNT,NPOS,NNEG
  375 FORMAT(17H KOUNT,NPOS,NNEG ,3I5)
      GO TO 9999
  400 CONTINUE
C     FORM FLDL
      DO 408 I=1,4
      DO 408 J=1,4
  408 FLDL(I,J) = 0.0
      DO 409 I=1,4
  409 FLDL(I,I) = EXP(LAMBL(I)*L)
C     FORM Q
      DO 410 I=1,5
      DO 410 J=1,4
```

```
  410 Q(I,J) = 0.0
      DO 420 I=1,4
      DO 420 J=I,4
  420 Q(I,J) = -1.0
C     FORM U1 AND U2
      DO 425 I = 1,5
      U2(I) = 0.0
      XT(I) = 0.0
      DO 425 J = 1,5
  425 XT(I) = XT(I) + KA(I,J)
      CALL MULT ( KA,5,5,Q,5,4,WORK(1) )
      CALL MULT ( WORK(1),5,4,TBAR,4,1,W )
      DO 426 I = 1,4
  426 S12(I) = A21(1,I+1)
      CALL MULT ( XT(2),4,1,S12,1,4,S22 )
      XT1I = 1.0 / XT(1)
      DO 427 I = 1,4
      U2(I+1) = W(I+1) - XT1I * XT(I+1) * W(1)
      DO 427 J = 1,4
  427 S22(I,J) = A21(I+1,J+1) - XT1I * S22(I,J)
      CALL SLEQ ( S22,U2(2),4,IER )
      CALL MULT ( S12,1,4,U2(2),4,1,ALPHA )
      ALPHA = XT1I * ( ALPHA - W(1) )
      CALL MULT ( Q,5,4,TBAR,4,1,U1 )
      DO 430 I = 1,5
  430 U1(I) = U1(I) + ALPHA
C     FORM B
      DO 435 I=1,14
      DO 435 J=1,14
  435 B(I,J) = 0.0
      DO 440 I=1,5
      DO 440 J=1,4
      B(I,J) = HL(I,J)
  440 B(I+8,J+4) = HL(I,J)
      DO 450 J=1,4
  450 B(6,J) = HL(11,J)
```

```
      DO 460 J=1,4
      B(7,J) = HL(12,J)
  460 B(14,J+4) = HL(12,J)
      DO 470 J=1,4
  470 B(8,J) = HL(14,J)
      DO 480 I=1,5
      B(I,9) = 1.0
      B(I+8,9) = 1.0
  480 B(I+8,10) = L
      B(6,11) = 1.0
      B(7,12) = 1.0
      B(8,14) = 1.0
      B(14,12) = 1.0
      B(14,13) = L
      B(14,14) = L*L/2.0
      DO 490 I=1,5
      B(I,12) = U1(I)
      B(I+8,12) = U1(I)
      B(I+8,13) = U1(I)*L
      B(I,14) = U2(I)
  490 B(I+8,14) = 0.5*U1(I)*L*L+U2(I)
      DO 500 J=1,4
      DO 500 I=1,5
  500 WORK(I+5*(J-1)) = HL(I,J)
      CALL MULT(WORK,5,4,FLDL,4,4,WORK(21))
      DO 510 J=1,4
      DO 510 I=1,5
      IJ = I+20+5*(J-1)
      B(I+8,J) = WORK(IJ)
  510 B(I,J+4) = WORK(IJ)
      DO 520 J=1,4
  520 WORK(J) = HL(12,J)
      CALL MULT(WORK,1,4,FLDL,4,4,WORK(5))
      DO 530 J=1,4
      B(14,J) = WORK(J+4)
  530 B(7,J+4) = WORK(J+4)
```

```
      DO 540 J=1,4
  540 WORK(J) = -HL(11,J)
      CALL MULT(WORK,1,4,FLDL,4,4,WORK(5))
      DO 550 J=1,4
  550 B(6,J+4) = WORK(J+4)
      DO 560 J=1,4
  560 WORK(J) = HL(14,J)
      CALL MULT(WORK,1,4,FLDL,4,4,WORK(5))
      DO 570 J=1,4
  570 B(8,J+4) = WORK(J+4)
      P = F/2.0
C     FORM PSI
      DO 580 J=1,14
  580 PSI(J) = 0.0
      PSI(8) = -P/KI
C     CALL SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS ROUTINE.
      CALL SLEQ(B,PSI,14,IER)
C     LOOP OVER ALL X.
      DO 1000 JELT=1,NX1
C     FORM FLD,FGD
      DO 700 I=1,4
      DO 700 J=1,4
      FLD(I,J) = 0.0
      FGD(I,J) = 0.0
      IF(I .NE. J) GO TO 700
      FLD(I,J) = EXP(LAMBL(I)*X(JELT))
      FGD(I,J) = EXP(LAMBL(I)*(L-X(JELT)))
  700 CONTINUE
C     FORM ANS
      DO 710 J=1,4
  710 WORK(J) = PSI(J)
      CALL MULT(FLD,4,4,WORK,4,1,WORK(5))
      CALL MULT(HL,14,4,WORK(5),4,1,ANS)
      DO 720 J=1,4
  720 WORK(J) = PSI(J+4)
      CALL MULT(FGD,4,4,WORK,4,1,WORK(5))
```

```
      DO 730 I=6,11
      DO 730 J=1,4
  730 HL(I,J) = -HL(I,J)
      DO 740 J=1,4
  740 HL(13,J) = -HL(13,J)
      CALL MULT(HL,14,4,WORK(5),4,1,WORK(10))
      DO 750 I=1,14
  750 ANS(I) = ANS(I)+WORK(I+9)
      DO 760 I=6,11
      DO 760 J=1,4
  760 HL(I,J) = -HL(I,J)
      DO 770 J=1,4
  770 HL(13,J) = -HL(13,J)
      DO 780 I=1,14
      DO 780 J=1,6
  780 TEMP(I,J) = 0.0
      DO 790 I=1,5
      TEMP(I,1) = 1.0
      TEMP(I,2) = X(JELT)
      TEMP(I,4) = U1(I)
      TEMP(I,5) = U1(I)*X(JELT)
  790 TEMP(I,6) = 0.5*TEMP(I,5)*X(JELT)+U2(I)
      DO 800 I=1,5
  800 TEMP(I+5,2) = KA(I,I)
      CALL MULT(KA,5,5,U1,5,1,A23)
      DO 810 I=1,5
      AB = A23(I)
      TEMP(I+5,5) = AB
  810 TEMP(I+5,6) = AB*X(JELT)
      DO 820 I=11,14
  820 TEMP(I,I-8) = 1.0
      DO 830 I=11,13
  830 TEMP(I,I-7) = X(JELT)
      TEMP(11,5) = X(JELT)*X(JELT)*0.5
      TEMP(12,6) = TEMP(11,5)
      TEMP(11,6) = TEMP(11,5)*X(JELT)/3.0
```

```
      DO 840 I=1,6
  840 WORK(I) = PSI(I+8)
      CALL MULT(TEMP,14,6,WORK,6,1,WORK(10))
      DO 850 I=1,14
  850 ANS(I) = ANS(I)+WORK(I+9)
C     FORM QXJ
      DO 860 I=1,4
  860 WORK(I) = TBAR(I)*ANS(12)
      DO 870 I=1,5
  870 A23(I) = ANS(I)
      CALL LTMULT(DELTAQ,5,4,A23,5,1,WORK(10))
      DO 880 I=1,4
  880 WORK(I) = WORK(I)-WORK(I+9)
      CALL MULT(KETA,4,4,WORK,4,1,QXJ)
C     FORM VXJ
      CALL MULT(VQ,5,4,QXJ,4,1,A23)
      DO 910 I=1,5
  910 A52(I) = XKI(I)*ANS(14)
      DO 920 I=1,5
  920 VXJ(I) = A23(I)-A52(I)
C     FORM MXJ
      DO 930 I=1,5
  930 MXJ(I) = XKI(I)*ANS(13)
      J1 = JELT-1
C     OUTPUT DESIRED MATRICES.
  935 FORMAT(1H1)
      WRITE(6,940) J1
  940 FORMAT(1H /3H X(,I5,1H))
      WRITE(6,950) (ANS(I),I=1,5)
  950 FORMAT(1H /3H U ,1P5E16.6)
      WRITE(6,960) (ANS(I),I=6,10)
  960 FORMAT(1H /3H H ,1P5E16.6)
      WRITE(6,970) (ANS(I),I=11,14)
  970 FORMAT(1H /5H VEE ,1PE16.6,7H THETA ,1PE16.6,6H BETA ,1PE16.6,5H P
     1HI ,1PE16.6)
      WRITE(6,980) (QXJ(I),I=1,4)
```

AD-A044 595 DOUGLAS AIRCRAFT CO LONG BEACH CALIF F/G 1/3
THE DETERMINATION OF DEFLECTION AND STRESS DISTRIBUTION FOR A T--ETC(U)
NOV 76 P H DENKE, J B HOFFMAN F33615-75-C-3105

UNCLASSIFIED MDC-J7173 AFFDL-TR-76-114 NL

2 OF 2

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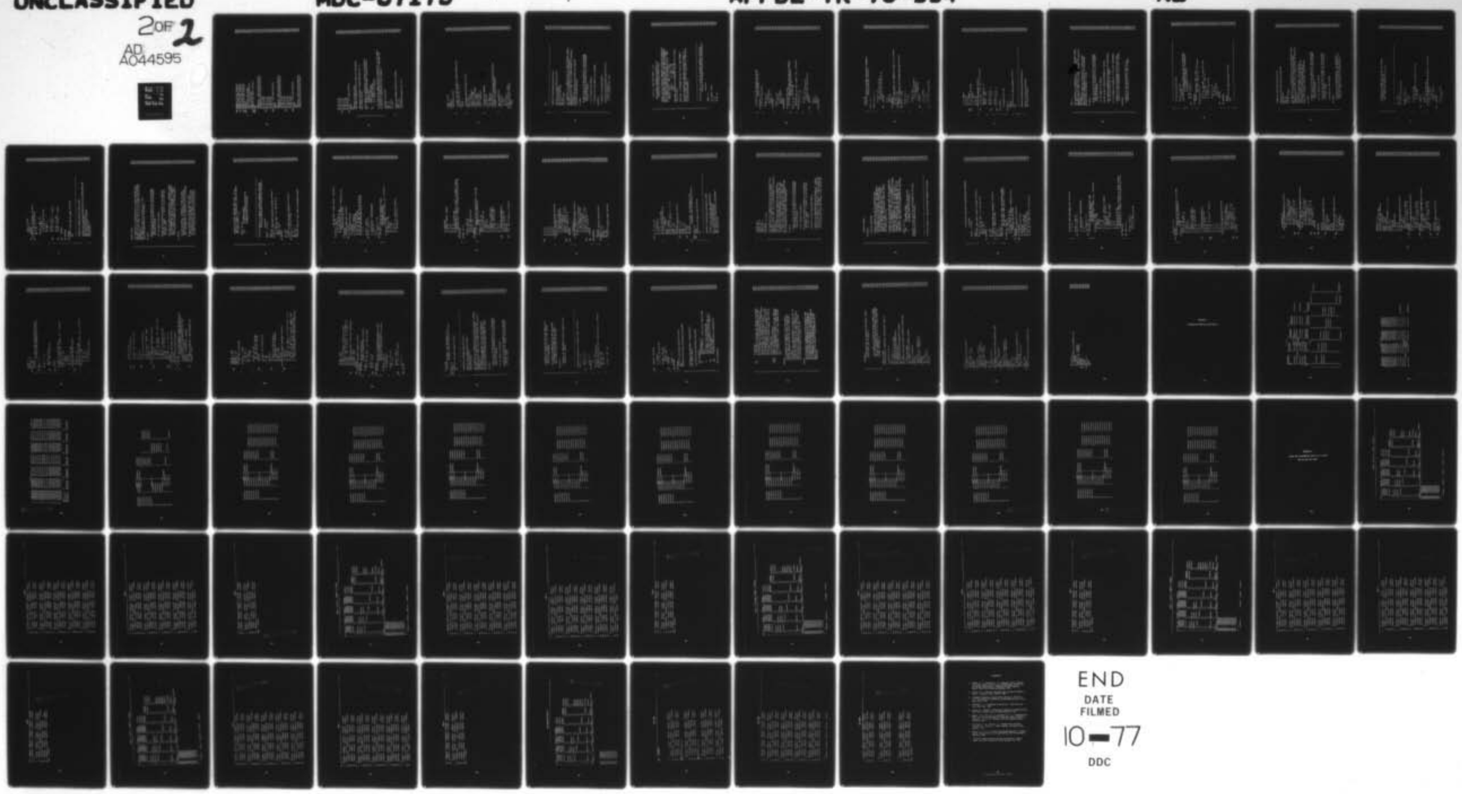

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```
  980 FORMAT(1H /5H QXJ  ,1P4E16.6)
      WRITE(6,990) (VXJ(I),I=1,5)
  990 FORMAT(1H /5H VXJ  ,1P5E16.6)
      WRITE(6,995) (MXJ(I),I=1,5)
  995 FORMAT(1H /5H MXJ  ,1P5E16.6)
 1000 CONTINUE
C     FORM EI(EFF) AND OUTPUT.
      EIEFF = P*L**3/(12.0*ANS(11))
      WRITE(6,1100) EIEFF
 1100 FORMAT(1H /9H EI(EFF) ,1PE16.6)
 9999 STOP
      END
      SUBROUTINE   MULT(A,N1,M1,B,N2,M2,C)
      DIMENSION A(N1,M1),B(N2,M2),C(N1,M2)
      DO 100 I=1,N1
      DO 100 J=1,M2
  100 C(I,J) = 0.0
      DO 200 I=1,N1
      DO 200 J=1,M2
      DO 200 K=1,M1
  200 C(I,J) = C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
      SUBROUTINE  TMULT(A,N1,M1,B,N2,M2,C)
      DIMENSION A(N1,M1),B(N2,M2),C(N1,N2)
      DO 100 I=1,N1
      DO 100 J=1,N2
  100 C(I,J) = 0.0
      DO 200 I=1,N1
      DO 200 J=1,N2
      DO 200 K=1,M1
  200 C(I,J) = C(I,J)+A(I,K)*B(J,K)
      RETURN
      END
      SUBROUTINE LTMULT(A,N1,M1,B,N2,M2,C)
      DIMENSION A(N1,M1),B(N2,M2),C(M1,M2)
```

```
      DO 100 I=1,M1
      DO 100 J=1,M2
  100 C(I,J) = 0.0
      DO 200 I=1,M1
      DO 200 J=1,M2
      DO 200 K=1,N1
  200 C(I,J) = C(I,J)+A(K,I)*B(K,J)
      RETURN
      END
      SUBROUTINE SLEQ(A,B,N,IER)
C     SLEQ FROM THE COMPUTER CENTER LIBRARY OF 6600 ROUTINES
C
C     SIMULTANEOUS SOLUTION OF LINEAR EQUATIONS, AX = B.
C     CALL SLEQ(A,B,N,IER)
C
C     A(N,N) MATRIX OF COEFFICIENTS, DESTROYED IN COMPUTATION.
C     B(N)   VECTOR OF ORIGINAL CONSTANTS, SOLUTION VECTOR X
C            RETURNED HERE.
C     N      NUMBER OF EQUATIONS.
C     IER    IF 0, NORMAL SOLUTION
C            IF 1, MATRIX A IS SINGULAR
C
C     IF NO PIVOT CAN BE FOUND EXCEEDING A TOLERANCE OF 0.0, THE MATRIX
C     IS CONSIDERED SINGULAR AND IER IS SET TO 1. THIS TOLERANCE CAN
C     BE MODIFIED BY REPLACING THE FIRST STATEMENT.
C
      DIMENSION A(N,N),B(N)
      TOL = 0.
      IER = 0
      DO 8 J=1,N
      J1 = J+1
C
C     SEARCH FOR MAXIMUM COEFFICIENT IN COLUMN J.
      PIVOT = 0.
      DO 2 I=J,N
      IF(ABS(PIVOT)-ABS(A(I,J))) 1,2,2
```

```
    1 PIVOT = A(I,J)
      IMAX = I
    2 CONTINUE
      WRITE(6,100) PIVOT
C
C     IF PIVOT LESS THAN TOLERANCE (SINGULAR MATRIX) EXIT.
      IF(ABS(PIVOT)-TOL) 3,3,4
    3 IER = 1
      RETURN
C
C     INTERCHANGE ROWS AND DIVIDE BY PIVOT.
    4 DO 5 K=J,N
      TEMP = A(J,K)
      A(J,K) = A(IMAX,K)
      A(IMAX,K) = TEMP
    5 A(J,K) = A(J,K)/PIVOT
      TEMP = B(IMAX)
      B(IMAX) = B(J)
      B(J) = TEMP/PIVOT
C
C     ELIMINATE NEXT VARIABLE
      IF(J-N) 6,9,6
    6 J1 = J+1
      DO 8 IROW=J1,N
      DO 7 JCOL=J1,N
    7 A(IROW,JCOL) = A(IROW,JCOL)-A(IROW,J)*A(J,JCOL)
    8 B(IROW) = B(IROW)-B(J)*A(IROW,J)
C
C     BACK SUBSTITUTION
    9 DO 10 K=2,N
      I = N+1-K
      I1 = I+1
      DO 10 J=I1,N
   10 B(I) = B(I)-A(I,J)*B(J)
  100 FORMAT(7H PIVOT ,1PE16.6)
      RETURN
```

```
      END
C
C     ------------------------------------------------------------------
C
      SUBROUTINE BALANC(NM,N,A,LOW,IGH,SCALE)
C
      INTEGER I,J,K,L,M,N,JJ,NM,IGH,LOW,IEXC
      REAL A(NM,N),SCALE(N)
      REAL C,F,G,R,S,B2,RADIX
C     REAL ABS
      LOGICAL NOCONV
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE BALANCE,
C     NUM. MATH. 13, 293-304(1969) BY PARLETT AND REINSCH.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 315-326(1971).
C     (REFERENCE 7)
C     THIS SUBROUTINE BALANCES A REAL MATRIX AND ISOLATES
C     EIGENVALUES WHENEVER POSSIBLE.
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT.
C
C        N IS THE ORDER OF THE MATRIX.
C
C        A CONTAINS THE INPUT MATRIX TO BE BALANCED.
C
C     ON OUTPUT-
C
C        A CONTAINS THE BALANCED MATRIX.
C
C        LOW AND IGH ARE TWO INTEGERS SUCH THAT A(I,J)
C          IS EQUAL TO ZERO IF
C           (1) I IS GREATER THAN J AND
```

```
C            (2) J=1,...,LOW-1 OR I=IGH+1,...,N,
C
C         SCALE CONTAINS INFORMATION DETERMINING THE
C            PERMUTATIONS AND SCALING FACTORS USED,
C
C      SUPPOSE THAT THE PRINCIPAL SUBMATRIX IN ROWS LOW THROUGH IGH
C      HAS BEEN BALANCED, THAT P(J) DENOTES THE INDEX INTERCHANGED
C      WITH J DURING THE PERMUTATION STEP, AND THAT THE ELEMENTS
C      OF THE DIAGONAL MATRIX USED ARE DENOTED BY D(I,J).  THEN
C         SCALE(J) = P(J),    FOR J = 1,...,LOW-1
C                  = D(J,J),      J = LOW,...,IGH
C                  = P(J)         J = IGH+1,...,N.
C      THE ORDER IN WHICH THE INTERCHANGES ARE MADE IS N TO IGH+1,
C      THEN 1 TO LOW-1.
C
C      NOTE THAT 1 IS RETURNED FOR IGH IF IGH IS ZERO FORMALLY.
C
C      THE ALGOL PROCEDURE EXC CONTAINED IN BALANCE APPEARS IN
C      BALANC  IN LINE.  (NOTE THAT THE ALGOL ROLES OF IDENTIFIERS
C      K,L HAVE BEEN REVERSED.)
C
C
C
C
C      -----------------------------------------------------------------
C
C      ********** RADIX IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C                 THE BASE OF THE MACHINE FLOATING POINT REPRESENTATION.
C
C                 **********
       RADIX = 2.
C
       B2 = RADIX * RADIX
       K = 1
       L = N
       GO TO 100
```

```
C         *********** IN-LINE PROCEDURE FOR ROW AND
C                     COLUMN EXCHANGE **********
   20 SCALE(M) = J
      IF (J .EQ. M) GO TO 50
C
      DO 30 I = 1, L
         F = A(I,J)
         A(I,J) = A(I,M)
         A(I,M) = F
   30 CONTINUE
C
      DO 40 I = K, N
         F = A(J,I)
         A(J,I) = A(M,I)
         A(M,I) = F
   40 CONTINUE
C
   50 GO TO (80,130), IEXC
C     *********** SEARCH FOR ROWS ISOLATING AN EIGENVALUE
C                 AND PUSH THEM DOWN **********
   80 IF (L .EQ. 1) GO TO 280
      L = L - 1
C     *********** FOR J=L STEP -1 UNTIL 1 DO -- **********
  100 DO 120 JJ = 1, L
         J = L + 1 - JJ
C
         DO 110 I = 1, L
            IF (I .EQ. J) GO TO 110
            IF (A(J,I) .NE. 0.0) GO TO 120
  110    CONTINUE
C
         M = L
         IEXC = 1
         GO TO 20
  120 CONTINUE
C
```

```
      GO TO 140
C     *********** SEARCH FOR COLUMNS ISOLATING AN EIGENVALUE
C                 AND PUSH THEM LEFT **********
  130 K = K + 1
C
  140 DO 170 J = K, L
C
         DO 150 I = K, L
            IF (I .EQ. J) GO TO 150
            IF (A(I,J) .NE. 0.0) GO TO 170
  150    CONTINUE
C
         M = K
         IEXC = 2
         GO TO 20
  170 CONTINUE
C     ********** NOW BALANCE THE SUBMATRIX IN ROWS K TO L *********
      DO 180 I = K, L
  180 SCALE(I) = 1.0
C     ********** ITERATIVE LOOP FOR NORM REDUCTION **********
  190 NOCONV = .FALSE.
C
      DO 270 I = K, L
         C = 0.0
         R = 0.0
C
         DO 200 J = K, L
            IF (J .EQ. I) GO TO 200
            C = C + ABS(A(J,I))
            R = R + ABS(A(I,J))
  200    CONTINUE
C
         G = R / RADIX
         F = 1.0
         S = C + R
  210    IF (C .GE. G) GO TO 220
```

```
      F = F * RADIX
      C = C * B2
      GO TO 210
220   G = R * RADIX
230   IF (C .LT. G) GO TO 240
      F = F / RADIX
      C = C / B2
      GO TO 230
C     ********** NOW BALANCE **********
240   IF ((C + R) / F .GE. 0.95 * S) GO TO 270
      G = 1.0 / F
      SCALE(I) = SCALE(I) * F
      NOCONV = .TRUE.
C
      DO 250 J = K, N
250   A(I,J) = A(I,J) * G
C
      DO 260 J = 1, L
260   A(J,I) = A(J,I) * F
C
  270 CONTINUE
C
      IF (NOCONV) GO TO 190
C
  280 LOW = K
      IGH = L
      RETURN
C     ********** LAST CARD OF BALANC **********
      END
C
C     ------------------------------------------------------------------
C
      SUBROUTINE BALBAK(NM,N,LOW,IGH,SCALE,M,Z)
C
      INTEGER I,J,K,M,N,II,NM,IGH,LOW
      REAL SCALE(N),Z(NM,M)
```

```
      REAL S
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE BALBAK,
C     NUM. MATH. 13, 293-304(1969) BY PARLETT AND REINSCH.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 315-326(1971).
C     (REFERENCE 7)
C     THIS SUBROUTINE FORMS THE EIGENVECTORS OF A REAL GENERAL
C     MATRIX BY BACK TRANSFORMING THOSE OF THE CORRESPONDING
C     BALANCED MATRIX DETERMINED BY  BALANC.
C
C     ON INPUT -
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT.
C
C        N IS THE ORDER OF THE MATRIX.
C
C        LOW AND IGH ARE INTEGERS DETERMINED BY  BALANC.
C
C        SCALE CONTAINS INFORMATION DETERMINING THE PERMUTATIONS
C          AND SCALING FACTORS USED BY  BALANC.
C
C        M IS THE NUMBER OF COLUMNS OF Z TO BE BACK TRANSFORMED.
C
C        Z CONTAINS THE REAL AND IMAGINARY PARTS OF THE EIGEN-
C          VECTORS TO BE BACK TRANSFORMED IN ITS FIRST M COLUMNS.
C
C     ON OUTPUT -
C
C        Z CONTAINS THE REAL AND IMAGINARY PARTS OF THE
C          TRANSFORMED EIGENVECTORS IN ITS FIRST M COLUMNS.
C
C
C
C
```

```
C       -------------------------------------------------------------------
C
        IF (IGH .EQ. LOW) GO TO 120
C
        DO 110 I = LOW, IGH
           S = SCALE(I)
C       ********** LEFT HAND EIGENVECTORS ARE BACK TRANSFORMED
C                  IF THE FOREGOING STATEMENT IS REPLACED BY
C                  S=1.0/SCALE(I). **********
           DO 100 J = 1, M
    100    Z(I,J) = Z(I,J) * S
C
    110 CONTINUE
C       *********- FOR I=LOW-1 STEP -1 UNTIL 1.
C                 IGH+1 STEP 1 UNTIL N DO -- **********
    120 DO 140 II = 1, N
           I = II
           IF (I .GE. LOW .AND. I .LE. IGH) GO TO 140
           IF (I .LT. LOW) I = LOW - II
           K = SCALE(I)
           IF (K .EQ. I) GO TO 140
C
           DO 130 J = 1, M
              S = Z(I,J)
              Z(I,J) = Z(K,J)
              Z(K,J) = S
    130    CONTINUE
C
    140 CONTINUE
C
        RETURN
C       ********** LAST CARD OF BALBAK **********
        END
C
C       -------------------------------------------------------------------
C
```

```
      SUBROUTINE ELMHES(NM,N,LOW,IGH,A,INT)
C
      INTEGER I,J,M,N,LA,NM,IGH,KP1,LOW,MM1,MP1
      REAL A(NM,N)
      REAL X,Y
C     REAL ABS
      INTEGER INT(IGH)
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE ELMHES,
C     NUM. MATH. 12, 349-368(1968) BY MARTIN AND WILKINSON.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 339-358(1971).
C     (REFERENCE 7)
C     GIVEN A REAL GENERAL MATRIX, THIS SUBROUTINE
C     REDUCES A SUBMATRIX SITUATED IN ROWS AND COLUMNS
C     LOW THROUGH IGH TO UPPER HESSENBERG FORM BY
C     STABILIZED ELEMENTARY SIMILARITY TRANSFORMATIONS.
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT.
C
C        N IS THE ORDER OF THE MATRIX.
C
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING
C          SUBROUTINE BALANC.   IF  BALANC  HAS NOT BEEN USED,
C          SET LOW=1, IGH=N.
C
C        A CONTAINS THE INPUT MATRIX.
C
C     ON OUTPUT-
C
C        A CONTAINS THE HESSENBERG MATRIX.  THE MULTIPLIERS
C          WHICH WERE USED IN THE REDUCTION ARE STORED IN THE
C          REMAINING TRIANGLE UNDER THE HESSENBERG MATRIX.
```

```
C
C        INT CONTAINS INFORMATION ON THE ROWS AND COLUMNS
C          INTERCHANGED IN THE REDUCTION.
C          ONLY ELEMENTS LOW THROUGH IGH ARE USED.
C
C
C
C
C     ------------------------------------------------------------------
C
      LA = IGH - 1
      KP1 = LOW + 1
      IF (LA .LT. KP1) GO TO 200
C
      DO 180 M = KP1, LA
         MM1 = M - 1
         X = 0.0
         I = M
C
         DO 100 J = M, IGH
            IF (ABS(A(J,MM1)) .LE. ABS(X)) GO TO 100
            X = A(J,MM1)
            I = J
  100    CONTINUE
C
         INT(M) = I
         IF (I .EQ. M) GO TO 130
C    ********** INTERCHANGE ROWS AND COLUMNS OF A **********
         DO 110 J = MM1, N
            Y = A(I,J)
            A(I,J) = A(M,J)
            A(M,J) = Y
  110    CONTINUE
C
         DO 120 J = 1, IGH
            Y = A(J,I)
```

```
            A(J,I) = A(J,M)
            A(J,M) = Y
  120    CONTINUE
C     ********** END INTERCHANGE **********
  130    IF (X .EQ. 0.0) GO TO 180
         MP1 = M + 1
C
         DO 160 I = MP1, IGH
            Y = A(I,MM1)
            IF (Y .EQ. 0.0) GO TO 160
            Y = Y / X
            A(I,MM1) = Y
C
            DO 140 J = M, N
  140       A(I,J) = A(I,J) - Y * A(M,J)
C
            DO 150 J = 1, IGH
  150       A(J,M) = A(J,M) + Y * A(J,I)
C
  160    CONTINUE
C
  180 CONTINUE
C
  200 RETURN
C     ********** LAST CARD OF ELMHES **********
      END
C
C     ------------------------------------------------------------------
C
      SUBROUTINE HQR(NM,N,LOW,IGH,H,WR,WI,IERR)
C
      INTEGER I,J,K,L,M,N,EN,LL,MM,NA,NM,IGH,ITS,LOW,MP2,ENM2,IERR
      REAL H(NM,N),WR(N),WI(N)
      REAL P,Q,R,S,T,W,X,Y,ZZ,MACHEP
C     REAL SQRT,ABS,SIGN
C     INTEGER MIN0
```

```
      LOGICAL NOTLAS
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE HQR.
C     NUM. MATH. 14, 219-231(1970) BY MARTIN, PETERS, AND WILKINSON.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 359-371(1971).
C     (REFERENCE 7)
C     THIS SUBROUTINE FINDS THE EIGENVALUES OF A REAL
C     UPPER HESSENBERG MATRIX BY THE QR METHOD.
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT.
C
C        N IS THE ORDER OF THE MATRIX.
C
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING
C          SUBROUTINE  BALANC.  IF  BALANC  HAS NOT BEEN USED,
C          SET LOW=1, IGH=N.
C
C        H CONTAINS THE UPPER HESSENBERG MATRIX.  INFORMATION ABOUT
C          THE TRANSFORMATIONS USED IN THE REDUCTION TO HESSENBERG
C          FORM BY  ELMHES  OR  ORTHES, IF PERFORMED, IS STORED
C          IN THE REMAINING TRIANGLE UNDER THE HESSENBERG MATRIX.
C
C     ON OUTPUT-
C
C        H HAS BEEN DESTROYED.  THEREFORE, IT MUST BE SAVED
C          BEFORE CALLING  HQR  IF SUBSEQUENT CALCULATION AND
C          BACK TRANSFORMATION OF EIGENVECTORS IS TO BE PERFORMED.
C
C        WR AND WI CONTAIN THE REAL AND IMAGINARY PARTS,
C          RESPECTIVELY, OF THE EIGENVALUES.  THE EIGENVALUES
C          ARE UNORDERED EXCEPT THAT COMPLEX CONJUGATE PAIRS
C          OF VALUES APPEAR CONSECUTIVELY WITH THE EIGENVALUE
```

```
C          HAVING THE POSITIVE IMAGINARY PART FIRST.  IF AN
C          ERROR EXIT IS MADE, THE EIGENVALUES SHOULD BE CORRECT
C          FOR INDICES IERR+1,...,N.
C
C        IERR IS SET TO
C          ZERO       FOR NORMAL RETURN.
C          J          IF THE J-TH EIGENVALUE HAS NOT BEEN
C                     DETERMINED AFTER 30 ITERATIONS.
C
C
C
C
C     ------------------------------------------------------------------
C
C     ********** MACHEP IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C                THE RELATIVE PRECISION OF FLOATING POINT ARITHMETIC.
C
C                **********
      MACHEP = 2.**(-47)
C
      IERR = 0
C     ********** STORE ROOTS ISOLATED BY BALANC **********
      DO 50 I = 1, N
         IF (I .GE. LOW .AND. I .LE. IGH) GO TO 50
         WR(I) = H(I,I)
         WI(I) = 0.0
   50 CONTINUE
C
      EN = IGH
      T = 0.0
C     ********** SEARCH FOR NEXT EIGENVALUES **********
   60 IF (EN .LT. LOW) GO TO 1001
      ITS = 0
      NA = EN - 1
      ENM2 = NA - 1
C     ********** LOOK FOR SINGLE SMALL SUB-DIAGONAL ELEMENT
```

```
C                 FOR L=EN STEP -1 UNTIL LOW DO -- **********
   70 DO 80 LL = LOW, EN
         L = EN + LOW - LL
         IF (L .EQ. LOW) GO TO 100
         IF (ABS(H(L,L-1)) .LE. MACHEP * (ABS(H(L-1,L-1))
     X      + ABS(H(L,L)))) GO TO 100
   80 CONTINUE
C     ********** FORM SHIFT **********
  100 X = H(EN,EN)
      IF (L .EQ. EN) GO TO 270
      Y = H(NA,NA)
      W = H(EN,NA) * H(NA,EN)
      IF (L .EQ. NA) GO TO 280
      IF (ITS .EQ. 30) GO TO 1000
      IF (ITS .NE. 10 .AND. ITS .NE. 20) GO TO 130
C     ********** FORM EXCEPTIONAL SHIFT **********
      T = T + X
C
      DO 120 I = LOW, EN
  120 H(I,I) = H(I,I) - X
C
      S = ABS(H(EN,NA)) + ABS(H(NA,ENM2))
      X = 0.75 * S
      Y = X
      W = -0.4375 * S * S
  130 ITS = ITS + 1
C     ********** LOOK FOR TWO CONSECUTIVE SMALL
C                SUB-DIAGONAL ELEMENTS.
C                FOR M=EN-2 STEP -1 UNTIL L DO -- **********
      DO 140 MM = L, ENM2
         M = ENM2 + L - MM
         ZZ = H(M,M)
         R = X - ZZ
         S = Y - ZZ
         P = (R * S - W) / H(M+1,M) + H(M,M+1)
         Q = H(M+1,M+1) - ZZ - R - S
```

```
      R = H(M+2,M+1)
      S = ABS(P) + ABS(Q) + ABS(R)
      P = P / S
      Q = Q / S
      R = R / S
      IF (M .EQ. L) GO TO 150
      IF (ABS(H(M,M-1)) * (ABS(Q) + ABS(R)) .LE. MACHEP * ABS(P)
     X   * (ABS(H(M-1,M-1)) + ABS(ZZ) + ABS(H(M+1,M+1)))) GO TO 150
  140 CONTINUE
C
  150 MP2 = M + 2
C
      DO 160 I = MP2, EN
         H(I,I-2) = 0.0
         IF (I .EQ. MP2) GO TO 160
         H(I,I-3) = 0.0
  160 CONTINUE
C     ********** DOUBLE QR STEP INVOLVING ROWS L TO EN AND
C                COLUMNS M TO EN **********
      DO 260 K = M, NA
         NOTLAS = K .NE. NA
         IF (K .EQ. M) GO TO 170
         P = H(K,K-1)
         Q = H(K+1,K-1)
         R = 0.0
         IF (NOTLAS) R = H(K+2,K-1)
         X = ABS(P) + ABS(Q) + ABS(R)
         IF (X .EQ. 0.0) GO TO 260
         P = P / X
         Q = Q / X
         R = R / X
  170    S = SIGN(SQRT(P*P+Q*Q+R*R),P)
         IF (K .EQ. M) GO TO 180
         H(K,K-1) = -S * X
         GO TO 190
  180    IF (L .NE. M) H(K,K-1) = -H(K,K-1)
```

```
190    P = P + S
       X = P / S
       Y = Q / S
       ZZ = R / S
       Q = Q / P
       R = R / P
C    ********** ROW MODIFICATION **********
       DO 210 J = K, EN
          P = H(K,J) + Q * H(K+1,J)
          IF (.NOT. NOTLAS) GO TO 200
          P = P + R * H(K+2,J)
          H(K+2,J) = H(K+2,J) - P * ZZ
200       H(K+1,J) = H(K+1,J) - P * Y
          H(K,J) = H(K,J) - P * X
210    CONTINUE
C
       J = MINO(EN,K+3)
C    ********** COLUMN MODIFICATION *********
       DO 230 I = L, J
          P = X * H(I,K) + Y * H(I,K+1)
          IF (.NOT. NOTLAS) GO TO 220
          P = P + ZZ * H(I,K+2)
          H(I,K+2) = H(I,K+2) - P * R
220       H(I,K+1) = H(I,K+1) - P * Q
          H(I,K) = H(I,K) - P
230    CONTINUE
C
260 CONTINUE
C
    GO TO 70
C   ********** ONE ROOT FOUND *********
270 WR(EN) = X + T
    WI(EN) = 0.0
    EN = NA
    GO TO 60
C   ********** TWO ROOTS FOUND **********
```

```
  280 P = (Y - X) / 2.0
      Q = P * P + W
      ZZ = SQRT(ABS(Q))
      X = X + T
      IF (Q .LT. 0.0) GO TO 320
C     ********** REAL PAIR **********
      ZZ = P + SIGN(ZZ,P)
      WR(NA) = X + ZZ
      WR(EN) = WR(NA)
      IF (ZZ .NE. 0.0) WR(EN) = X - W / ZZ
      WI(NA) = 0.0
      WI(EN) = 0.0
      GO TO 330
C     ********** COMPLEX PAIR **********
  320 WR(NA) = X + P
      WR(EN) = X + P
      WI(NA) = ZZ
      WI(EN) = -ZZ
  330 EN = ENM2
      GO TO 60
C     ********** SET ERROR -- NO CONVERGENCE TO AN
C                EIGENVALUE AFTER 30 ITERATIONS **********
 1000 IERR = EN
 1001 RETURN
C     ********** LAST CARD OF HQR **********
      END
C
C     ------------------------------------------------------------------
C
      SUBROUTINE HQR2(NM,N,LOW,IGH,H,WR,WI,Z,IERR)
C
      INTEGER I,J,K,L,M,N,EN,II,JJ,LL,MM,NA,NM,NN,
     X        IGH,ITS,LOW,MP2,ENM2,IERR
      REAL H(NM,N),WR(N),WI(N),Z(NM,N)
      REAL P,Q,R,S,T,W,X,Y,RA,SA,VI,VR,ZZ,NORM,MACHEP
C     REAL SQRT,ABS,SIGN
```

```
C           INTEGER MINO
            LOGICAL NOTLAS
            COMPLEX Z3
C           COMPLEX CMPLX
            REAL T3(2)
            EQUIVALENCE (Z3,T3(1))
C
C           THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE HQR2,
C           NUM. MATH. 16, 181-204(1970) BY PETERS AND WILKINSON.
C           HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 372-395(1971).
C           (REFERENCE 7)
C           THIS SUBROUTINE FINDS THE EIGENVALUES AND EIGENVECTORS
C           OF A REAL UPPER HESSENBERG MATRIX BY THE QR METHOD.  THE
C           EIGENVECTORS OF A REAL GENERAL MATRIX CAN ALSO BE FOUND
C           IF  ELMHES  AND  ELTRAN  OR  ORTHES  AND  ORTRAN  HAVE
C           BEEN USED TO REDUCE THIS GENERAL MATRIX TO HESSENBERG FORM
C           AND TO ACCUMULATE THE SIMILARITY TRANSFORMATIONS.
C
C           ON INPUT-
C
C              NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C                ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C                DIMENSION STATEMENT.
C
C              N IS THE ORDER OF THE MATRIX.
C
C              LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING
C                SUBROUTINE  BALANC.  IF  BALANC  HAS NOT BEEN USED,
C                SET LOW=1, IGH=N.
C
C              H CONTAINS THE UPPER HESSENBERG MATRIX.
C
C              Z CONTAINS THE TRANSFORMATION MATRIX PRODUCED BY  ELTRAN
C                AFTER THE REDUCTION BY  ELMHES, OR BY  ORTRAN  AFTER THE
C                REDUCTION BY  ORTHES, IF PERFORMED.  IF THE EIGENVECTORS
C                OF THE HESSENBERG MATRIX ARE DESIRED, Z MUST CONTAIN THE
```

```
C           IDENTITY MATRIX.
C
C     ON OUTPUT-
C
C        H HAS BEEN DESTROYED.
C
C        WR AND WI CONTAIN THE REAL AND IMAGINARY PARTS,
C          RESPECTIVELY, OF THE EIGENVALUES.  THE EIGENVALUES
C          ARE UNORDERED EXCEPT THAT COMPLEX CONJUGATE PAIRS
C          OF VALUES APPEAR CONSECUTIVELY WITH THE EIGENVALUE
C          HAVING THE POSITIVE IMAGINARY PART FIRST.  IF AN
C          ERROR EXIT IS MADE, THE EIGENVALUES SHOULD BE CORRECT
C          FOR INDICES IERR+1,...,N.
C
C        Z CONTAINS THE REAL AND IMAGINARY PARTS OF THE EIGENVECTORS,
C          IF THE I-TH EIGENVALUE IS REAL, THE I-TH COLUMN OF Z
C          CONTAINS ITS EIGENVECTOR.  IF THE I-TH EIGENVALUE IS COMPLEX
C          WITH POSITIVE IMAGINARY PART, THE I-TH AND (I+1)-TH
C          COLUMNS OF Z CONTAIN THE REAL AND IMAGINARY PARTS OF ITS
C          EIGENVECTOR.  THE EIGENVECTORS ARE UNNORMALIZED.  IF AN
C          ERROR EXIT IS MADE, NONE OF THE EIGENVECTORS HAS BEEN FOUND.
C
C        IERR IS SET TO
C          ZERO       FOR NORMAL RETURN,
C          J          IF THE J-TH EIGENVALUE HAS NOT BEEN
C                     DETERMINED AFTER 30 ITERATIONS.
C
C     ARITHMETIC IS REAL EXCEPT FOR THE REPLACEMENT OF THE ALGOL
C     PROCEDURE CDIV BY COMPLEX DIVISION.
C
C
C
C
C     ------------------------------------------------------------------
C
C     ********** MACHEP IS A MACHINE DEPENDENT PARAMETER SPECIFYING
```

```
C                 THE RELATIVE PRECISION OF FLOATING POINT ARITHMETIC.
C
C                    **********
      MACHEP = 2.**(-47)
C
      IERR = 0
C     ********** STORE ROOTS ISOLATED BY BALANC **********
      DO 50 I = 1, N
         IF (I .GE. LOW .AND. I .LE. IGH) GO TO 50
         WR(I) = H(I,I)
         WI(I) = 0.0
   50 CONTINUE
C
      EN = IGH
      T = 0.0
C     ********** SEARCH FOR NEXT EIGENVALUES **********
   60 IF (EN .LT. LOW) GO TO 340
      ITS = 0
      NA = EN - 1
      ENM2 = NA - 1
C     ********** LOOK FOR SINGLE SMALL SUB-DIAGONAL ELEMENT
C                FOR L=EN STEP -1 UNTIL LOW DO -- **********
   70 DO 80 LL = LOW, EN
         L = EN + LOW - LL
         IF (L .EQ. LOW) GO TO 100
         IF (ABS(H(L,L-1)) .LE. MACHEP * (ABS(H(L-1,L-1))
     X      + ABS(H(L,L)))) GO TO 100
   80 CONTINUE
C     ********** FORM SHIFT **********
  100 X = H(EN,EN)
      IF (L .EQ. EN) GO TO 270
      Y = H(NA,NA)
      W = H(EN,NA) * H(NA,EN)
      IF (L .EQ. NA) GO TO 280
      IF (ITS .EQ. 30) GO TO 1000
      IF (ITS .NE. 10 .AND. ITS .NE. 20) GO TO 130
```

```
C       ********** FORM EXCEPTIONAL SHIFT **********
        T = T + X
C
        DO 120 I = LOW, EN
  120   H(I,I) = H(I,I) - X
C
        S = ABS(H(EN,NA)) + ABS(H(NA,ENM2))
        X = 0.75 * S
        Y = X
        W = -0.4375 * S * S
  130   ITS = ITS + 1
C       ********** LOOK FOR TWO CONSECUTIVE SMALL
C                  SUB-DIAGONAL ELEMENTS.
C                  FOR M=EN-2 STEP -1 UNTIL L DO -- **********
        DO 140 MM = L, ENM2
           M = ENM2 + L - MM
           ZZ = H(M,M)
           R = X - ZZ
           S = Y - ZZ
           P = (R * S - W) / H(M+1,M) + H(M,M+1)
           Q = H(M+1,M+1) - ZZ - R - S
           R = H(M+2,M+1)
           S = ABS(P) + ABS(Q) + ABS(R)
           P = P / S
           Q = Q / S
           R = R / S
           IF (M .EQ. L) GO TO 150
           IF (ABS(H(M,M-1)) * (ABS(Q) + ABS(R)) .LE. MACHEP * ABS(P)
       X     * (ABS(H(M-1,M-1)) + ABS(ZZ) + ABS(H(M+1,M+1)))) GO TO 150
  140   CONTINUE
C
  150   MP2 = M + 2
C
        DO 160 I = MP2, EN
           H(I,I-2) = 0.0
           IF (I .EQ. MP2) GO TO 160
```

```
      H(I,I-3) = 0.0
  160 CONTINUE
C     ********** DOUBLE QR STEP INVOLVING ROWS L TO EN AND
C                COLUMNS M TO EN **********
      DO 260 K = M, NA
         NOTLAS = K .NE. NA
         IF (K .EQ. M) GO TO 170
         P = H(K,K-1)
         Q = H(K+1,K-1)
         R = 0.0
         IF (NOTLAS) R = H(K+2,K-1)
         X = ABS(P) + ABS(Q) + ABS(R)
         IF (X .EQ. 0.0) GO TO 260
         P = P / X
         Q = Q / X
         R = R / X
  170    S = SIGN(SQRT(P*P+Q*Q+R*R),P)
         IF (K .EQ. M) GO TO 180
         H(K,K-1) = -S * X
         GO TO 190
  180    IF (L .NE. M) H(K,K-1) = -H(K,K-1)
  190    P = P + S
         X = P / S
         Y = Q / S
         ZZ = R / S
         Q = Q / P
         R = R / P
C     ********** ROW MODIFICATION **********
         DO 210 J = K, N
            P = H(K,J) + Q * H(K+1,J)
            IF (.NOT. NOTLAS) GO TO 200
            P = P + R * H(K+2,J)
            H(K+2,J) = H(K+2,J) - P * ZZ
  200       H(K+1,J) = H(K+1,J) - P * Y
            H(K,J) = H(K,J) - P * X
  210    CONTINUE
```

```
C
         J = MINO(EN,K+3)
C     ********** COLUMN MODIFICATION **********
         DO 230 I = 1, J
            P = X * H(I,K) + Y * H(I,K+1)
            IF (.NOT. NOTLAS) GO TO 220
            P = P + ZZ * H(I,K+2)
            H(I,K+2) = H(I,K+2) - P * R
  220       H(I,K+1) = H(I,K+1) - P * Q
            H(I,K) = H(I,K) - P
  230    CONTINUE
C     ********** ACCUMULATE TRANSFORMATIONS **********
         DO 250 I = LOW, IGH
            P = X * Z(I,K) + Y * Z(I,K+1)
            IF (.NOT. NOTLAS) GO TO 240
            P = P + ZZ * Z(I,K+2)
            Z(I,K+2) = Z(I,K+2) - P * R
  240       Z(I,K+1) = Z(I,K+1) - P * Q
            Z(I,K) = Z(I,K) - P
  250    CONTINUE
C
  260 CONTINUE
C
      GO TO 70
C     ********** ONE ROOT FOUND **********
  270 H(EN,EN) = X  +  T
      WR(EN) = H(EN,EN)
      WI(EN) = 0.0
      EN = NA
      GO TO 60
C     ********** TWO ROOTS FOUND **********
  280 P = (Y - X) / 2.0
      Q = P * P + W
      ZZ = SQRT(ABS(Q))
      H(EN,EN) = X + T
      X = H(EN,EN)
```

```
      H(NA,NA) = Y + T
      IF (Q .LT. 0.0) GO TO 320
C     ********** REAL PAIR **********
      ZZ = P + SIGN(ZZ,P)
      WR(NA) = X + ZZ
      WR(EN) = WR(NA)
      IF (ZZ .NE. 0.0) WR(EN) = X - W / ZZ
      WI(NA) = 0.0
      WI(EN) = 0.0
      X = H(EN,NA)
      R = SQRT(X*X+ZZ*ZZ)
      P = X / R
      Q = ZZ / R
C     ********** ROW MODIFICATION **********
      DO 290 J = NA, N
         ZZ = H(NA,J)
         H(NA,J) = Q * ZZ + P * H(EN,J)
         H(EN,J) = Q * H(EN,J) - P * ZZ
  290 CONTINUE
C     ********** COLUMN MODIFICATION **********
      DO 300 I = 1, EN
         ZZ = H(I,NA)
         H(I,NA) = Q * ZZ + P * H(I,EN)
         H(I,EN) = Q * H(I,EN) = P * ZZ
  300 CONTINUE
C     ********** ACCUMULATE TRANSFORMATIONS **********
      DO 310 I = LOW, IGH
         ZZ = Z(I,NA)
         Z(I,NA) = Q * ZZ + P * Z(I,EN)
         Z(I,EN) = Q * Z(I,EN) - P * ZZ
  310 CONTINUE
C
      GO TO 330
C     ********** COMPLEX PAIR **********
  320 WR(NA) = X + P
      WR(EN) = X + P
```

```
      WI(NA) = ZZ
      WI(EN) = -ZZ
  330 EN = ENM2
      GO TO 60
C     ********** ALL ROOTS FOUND, BACKSUBSTITUTE TO FIND
C                VECTORS OF UPPER TRIANGULAR FORM **********
  340 NORM = 0.0
      K = 1
C
      DO 360 I = 1, N
C
         DO 350 J = K, N
  350    NORM = NORM + ABS(H(I,J))
C
         K = I
  360 CONTINUE
C
      IF (NORM .EQ. 0.0) GO TO 1001
C     ********** FOR EN=N STEP -1 UNTIL 1 DO -- **********
      DO 800 NN = 1, N
         EN = N + 1 - NN
         P = WR(EN)
         Q = WI(EN)
         NA = EN - 1
         IF (Q) 710, 600, 800
C     ********** REAL VECTOR **********
  600    M = EN
         H(EN, EN) = 1.0
         IF (NA .EQ. 0) GO TO 800
C     ********** FOR I=EN-1 STEP -1 UNTIL 1 DO -- **********
         DO 700 II = 1, NA
            I = EN - II
            W = H(I,I) - P
            R = H(I,EN)
            IF (M .GT. NA) GO TO 620
C
```

```
      DO 610 J = M, NA
  610 R = R + H(I,J) * H(J,EN)
C
  620 IF (WI(I) .GE. 0.0) GO TO 630
      ZZ = W
      S = R
      GO TO 700
  630 M = I
      IF (WI(I) .NE. 0.0) GO TO 640
      T = W
      IF (W .EQ. 0.0) T = MACHEP * NORM
      H(I,EN) = -R / T
      GO TO 700
C     ********** SOLVE REAL EQUATIONS **********
  640 X = H(I,I+1)
      Y = H(I+1,I)
      Q = (WR(I) - P) * (WR(I) - P) + WI(I) * WI(I)
      T = (X * S - ZZ * R) / Q
      H(I,EN) = T
      IF (ABS(X) .LE. ABS(ZZ)) GO TO 650
      H(I+1,EN) = (-R - W * T) / X
      GO TO 700
  650 H(I+1,EN) = (-S - Y * T) / ZZ
  700 CONTINUE
C     ********** END REAL VECTOR **********
      GO TO 800
C     ********** COMPLEX VECTOR **********
  710 M = NA
C     ********** LAST VECTOR COMPONENT CHOSEN IMAGINARY SO THAT
C                EIGENVECTOR MATRIX IS TRIANGULAR **********
      IF (ABS(H(EN,NA)) .LE. ABS(H(NA,EN))) GO TO 720
      H(NA,NA) = Q / H(EN,NA)
      H(NA,EN) = -(H(EN,EN) -P) / H(EN,NA)
      GO TO 730
  720 Z3 = CMPLX(0.0,-H(NA,EN)) / CMPLX(H(NA,NA)-P,Q)
      H(NA,NA) = T3(1)
```

```
      H(NA,EN) = T3(2)
 730  H(EN,NA) = 0.0
      H(EN,EN) = 1.0
      ENM2 = NA - 1
      IF (ENM2 .EQ. 0) GO TO 800
C
      DO 790 II = 1, ENM2
         I = NA - II
         W = H(I,I) - P
         RA = 0.0
         SA = H(I,EN)
C
         DO 760 J = M, NA
            RA = RA + H(I,J) * H(J,NA)
            SA = SA + H(I,J) * H(J,EN)
 760     CONTINUE
C
         IF (WI(I) .GE. 0.0) GO TO 770
         ZZ = W
         R = RA
         S = SA
         GO TO 790
 770     M = I
         IF (WI(I) .NE. 0.0) GO TO 780
         Z3 = CMPLX(-RA,-SA) / CMPLX(W,Q)
         H(I,NA) = T3(1)
         H(I,EN) = T3(2)
         GO TO 790
C    ********** SOLVE COMPLEX EQUATIONS **********
 780     X = H(I,I+1)
         Y = H(I+1,I)
         VR = (WR(I) - P) * (WR(I) - P) + WI(I) * WI(I) - Q * Q
         VI = (WR(I) - P) * 2.0 * Q
         IF (VR .EQ. 0.0 .AND. VI .EQ. 0.0) VR = MACHEP * NORM
     X    * (ABS(W) + ABS(Q) + ABS(X) + ABS(Y) + ABS(ZZ))
         Z3 = CMPLX(X*R-ZZ*RA+Q*SA,X*S-ZZ*SA-Q*RA) / CMPLX(VR,VI)
```

```
            H(I,NA) = T3(1)
            H(I,EN) = T3(2)
            IF (ABS(X) .LE. ABS(ZZ) + ABS(Q)) GO TO 785
            H(I+1,NA) = (-RA - W * H(I,NA) + Q * H(I,EN)) / X
            H(I+1,EN) = (-SA - W * H(I,EN) - Q * H(I,NA)) / X
            GO TO 790
  785       Z3 = CMPLX(-R-Y*H(I,NA),-S-Y*H(I,EN)) /CMPLX(ZZ,Q)
            H(I+1,NA) = T3(1)
            H(I+1,EN) = T3(2)
  790    CONTINUE
C     ********** END COMPLEX VECTOR **********
  800 CONTINUE
C     ********** END BACK SUBSTITUTION.
C                VECTORS OF ISOLATED ROOTS **********
      DO 840 I = 1, N
         IF (I .GE. LOW .AND. I .LE. IGH) GO TO 840
C
         DO 820 J = I, N
  820    Z(I,J) = H(I,J)
C
  840 CONTINUE
C     ********** MULTIPLY BY TRANSFORMATION MATRIX TO GIVE
C                VECTORS OF ORIGINAL FULL MATRIX.
C                FOR J=N STEP -1 UNTIL LOW DO -- **********
      DO 880 JJ = LOW, N
         J = N + LOW - JJ
         M = MIN0(J,IGH)
C
         DO 880 I = LOW, IGH
            ZZ = 0.0
C
            DO 860 K = LOW, M
  860       ZZ = ZZ + Z(I,K) * H(K,J)
C
            Z(I,J) = ZZ
  880 CONTINUE
```

```
C
      GO TO 1001
C     ********** SET ERROR -- NO CONVERGENCE TO AN
C                EIGENVALUE AFTER 30 ITERATIONS **********
 1000 IERR = EN
 1001 RETURN
C     ********** LAST CARD OF HQR2 **********
      END
C
C     ------------------------------------------------------------------
C
      SUBROUTINE ELTRAN(NM,N,LOW,IGH,A,INT,Z)
C
      INTEGER I,J,N,KL,MM,MP,NM,IGH,LOW,MP1
      REAL A(NM,IGH),Z(NM,N)
      INTEGER INT(IGH)
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE ELMTRANS,
C     NUM, MATH, 16, 181-204(1970) BY PETERS AND WILKINSON,
C     HANDBOOK FOR AUTO. COMP,. VOL,II-LINEAR ALGEBRA, 372-395(1971).
C     (REFERENCE 7)
C     THIS SUBROUTINE ACCUMULATES THE STABILIZED ELEMENTARY
C     SIMILARITY TRANSFORMATIONS USED IN THE REDUCTION OF A
C     REAL GENERAL MATRIX TO UPPER HESSENBERG FORM BY  ELMHES,
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT,
C
C        N IS THE ORDER OF THE MATRIX,
C
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING
C          SUBROUTINE BALANC,  IF  BALANC  HAS NOT BEEN USED,
C          SET LOW=1, IGH=N,
```

```
C
C      A CONTAINS THE MULTIPLIERS WHICH WERE USED IN THE
C        REDUCTION BY  ELMHES  IN ITS LOWER TRIANGLE
C        BELOW THE SUBDIAGONAL,
C
C      INT CONTAINS INFORMATION ON THE ROWS AND COLUMNS
C        INTERCHANGED IN THE REDUCTION BY  ELMHES,
C        ONLY ELEMENTS LOW THROUGH IGH ARE USED.
C
C     ON OUTPUT -
C
C      Z CONTAINS THE TRANSFORMATION MATRIX PRODUCED IN THE
C        REDUCTION BY  ELMHES,
C
C
C
C
C     ------------------------------------------------------------------
C
C     ********** INITIALIZE Z TO IDENTITY MATRIX **********
      DO 80 I = 1, N
C
         DO 60 J = 1, N
   60    Z(I,J) = 0.0
C
         Z(I,I) = 1.0
   80 CONTINUE
C
      KL = IGH - LOW - 1
      IF (KL .LT. 1) GO TO 200
C     ********** FOR MP=IGH-1 STEP -1 UNTIL LOW+1 DO -- **********
      DO 140 MM = 1, KL
         MP = IGH - MM
         MP1 = MP + 1
C
         DO 100 I = MP1, IGH
```

```
 100    Z(I,MP) = A(I,MP-1)
C
        I = INT(MP)
        IF (I .EQ. MP) GO TO 140
C
        DO 130 J = MP, IGH
           Z(MP,J) = Z(I,J)
           Z(I,J) = 0.0
 130    CONTINUE
C
        Z(I,MP) = 1.0
 140 CONTINUE
C
 200 RETURN
C     ********** LAST CARD OF ELTRAN **********
      END
      SUBROUTINE RGEIG (M,N,A,IND,EIGVAL,EIGVEC,WORK)
C
C
C
C     EIGENVALUES AND (OPTIONALLY) EIGENVECTORS OF
C     REAL GENERAL MATRIX.
C
C     CALL RGEIG (M,N,A,IND,EIGVAL,EIGVEC,WORK)
C
C       INPUT PARAMETERS
C         M        ROW DIMENSION OF A
C         N        ORDER OF MATRIX A.  N MUST BE LESS THAN OR EQUAL
C                  TO M.
C         A        INPUT MATRIX, REAL,TWO-DIMENSIONAL
C                  A(M,K) WHERE K IS GREATER THAN OR EQUAL TO N,
C         IND      ZERO IF ONLY EIGENVALUES ARE TO BE COMPUTED
C                  NON ZERO OTHERWISE.
C         WORK     A WORKING STORAGE AREA OF DIMENSION 2*N
C
C       OUTPUT PARAMETERS
```

```
C       A         IS DESTROYED.
C       IND       IF MORE THAN 30 ITERATIONS ARE REQUIRED TO DETERMINE
C                 AN EIGENVALUE, THIS SUBROUTINE TERMINATES WITH IND
C                 SET TO THE INDIX OF THE EIGENVALUE FOR WHICH THE
C                 FAILURE OCCURS.  THE EIGENVALUES IN THE EIGVAL VECTOR
C                 SHOULD BE CORRECT FOR INDICES IND+1, IND+2, .... N
C                 BUT NO EIGENVECTORS ARE COMPUTED.  IF ALL THE EIGEN-
C                 VALUES ARE DETERMINED WITHIN 30 ITERATIONS, IND IS
C                 SET TO ZERO.  IF N IS GREATER THAN M, IND IS SET TO
C                 N+1 AS AN ERROR FLAG.
C       EIGVAL    COMPLEX N-VECTOR OF EIGENVALUES.
C       EIGVEC    COMPLEX TWO-DIMENSIONAL VARIABLE WITH ROW DIMENSION
C                 M AND COLUMN DIMENSION AT LEAST N.  EIGVEC CONTAINS
C                 THE N EIGENVECTORS.  IF IND IS ZERO, EIGVEC IS NOT
C                 USED.
C
C     REMARKS
C         THE EIGENVALUES ARE UNORDERED EXCEPT THAT COMPLEX
C         CONJUGATE PARIS OF EIGENVALUES APPEAR CONSECUTIVELY
C         WITH THE EIGENVALUE HAVING THE POSITIVE IMAGINARY
C         PART FIRST.
C         EIGENVECTORS ARE STORED COLUMNWISE IN EIGVEC SO THAT THE
C         EIGENVECTOR CORRESPONDING TO THE J-TH EIGENVALUE IS
C         STORED IN THE J-TH COLUMN OF THE COMPLEX MATRIX EIGVEC.
C         EACH EIGENVECTOR IS NORMALIZED SO THAT THE ELEMENT
C         WITH THE LARGEST MODULUS IS REDUCED TO ONE.
C
C     METHOD
C         THE INPUT MATRIX IS REDUCED TO UPPER HESSENBERG FORM
C         USING ELEMENTARY TRANSFORMATIONS, THE EIGENVALUES OF
C         THE REDUCED MATRIX ARE OBTAINED BY THE QR-ALGORITHM.
C         THE EIGENVECTORS ARE DETERMINED BY BACK-SUBSTITUTION AND
C         THEN TRANSFORMED TO THE EIGENVECTORS OF THE ORIGINAL
C         MATRIX USING THE SIMILARITY TRANSFORMATIONS OF THE
C         ITERATIONS.
C
```

```
C        REFERENCES
C             SEE DOCUMENTATION OF EISPACK ROUTINES BALANC, ELMHES.
C             ELTRAN, HQR2, HQR, AND BALBAK,
C
C
C              THIS PROGRAM USES SUBROUTINES BALANC, ELMHES, ELTRAN,
C              HQR2, HQR, AND BALBAK WHICH WERE OBTAINED FROM ARGONNE
C              NATIONAL LABORATORY AS PART OF THE EISPACK SUBROUTINE
C              PACKAGE. (REFERENCE 8)
C
      COMPLEX B,DENOM
      DIMENSION A(M,N), EIGVAL(2,N), EIGVEC(2,M,N), WORK (1)
C
      IF (N.GT.M) GO TO 20
      INT = N
      INTP1 = INT + 1
      CALL BALANC(M,N,A,LOW,IGH,WORK)
      CALL ELMHES(M,N,LOW,IGH,A,WORK(INTP1))
      IF (IND.EQ.0) GO TO 9
      CALL ELTRAN(M,N,LOW,IGH,A,WORK(INTP1),EIGVEC)
      IND = 0
      CALL HQR2(M,N,LOW,IGH,A,EIGVAL,EIGVAL(N+1),EIGVEC,IND)
      L = N
      IF (IND.NE.0) RETURN
      CALL BALBAK(M,N,LOW,IGH,WORK,L,EIGVEC)
      DO 1 I=1,N
      WORK(I) = EIGVAL(N+I)
    1 CONTINUE
      DO 2 I=1,N
      EIGVAL(2,N-I+1) = WORK(N-I+1)
      EIGVAL(1,N-I+1) = EIGVAL(N-I+1)
    2 CONTINUE
      MN = M*N
      DO 3 I=1,MN
      A(I) = EIGVEC(I)
    3 CONTINUE
```

```
      K = 1
    4 IF (EIGVAL(2,K).NE.0.0) GO TO 6
      DO 5 J=1,M
      EIGVEC(2,J,K) = 0.0
      EIGVEC(1,J,K) = A(J,K)
    5 CONTINUE
      GO TO 8
    6 DO 7 J=1,M
      EIGVEC(2,J,K) = A(J,K+1)
      EIGVEC(2,J,K+1) = -A(J,K+1)
      EIGVEC(1,J,K) = A(J,K)
      EIGVEC(1,J,K+1) = A(J,K)
    7 CONTINUE
      K = K+1
    8 K = K+1
      IF (K.LE.N) GO TO 4
      GO TO 12
    9 CALL HQR(M,N,LOW,IGH,A,EIGVAL,EIGVAL(N+1),IND)
      DO 10 I=1,N
      WORK(I) = EIGVAL(N+I)
   10 CONTINUE
      DO 11 I=1,N
      EIGVAL(2,N-I+1) = WORK(N-I+1)
      EIGVAL(1,N-I+1) = EIGVAL(N-I+1)
   11 CONTINUE
      RETURN
   12 DO 15 J=1,N
      CMOD = 0.0
      DO 13 I=1,N
      B = CMPLX(EIGVEC(1,I,J),EIGVEC(2,I,J))
      TEMP = CABS(B)
      IF(TEMP.LE.CMOD) GO TO 13
      K = I
      CMOD = TEMP
   13 CONTINUE
      DENOM = CMPLX(EIGVEC(1,K,J),EIGVEC(2,K,J))
```

```
      DO 14 I=1,N
      B = CMPLX(EIGVEC(1,I,J),EIGVEC(2,I,J))
      B = B/DENOM
      EIGVEC(1,I,J) = REAL(B)
      EIGVEC(2,I,J) = AIMAG(B)
   14 CONTINUE
   15 CONTINUE
      RETURN
   20 IND = N + 1
      RETURN
      END
```

APPENDIX C

INTERMEDIATE MATRICES (FOR CASE A)

VO MATRIX

5.00000000E-02	0.	0.	0.	
1.25000000E-01	1.25000000E-01	0.	0.	
0.	1.25000000E-01	1.25000000E-01	0.	
0.	0.	1.25000000E-01	1.25000000E-01	
0.	0.	0.	1.25000000E-01	

K-SUB-A DIAGONAL

1.00000000E+06	8.50000000E+04	8.50000000E+04	8.50000000E+04	8.50000000E+04

K-SUB-ETA DIAGONAL

2.92000000E+02	7.30000000E+02	7.30000000E+02	7.30000000E+02

K-SUB-I= 2.60416667E+03

SMALL K SUB I

8.33333333E+02	4.42708333E+02	4.42708333E+02	4.42708333E+02	4.42708333E+02

T-BAR

4.25000000E-01	3.50000000E-01	3.50000000E-01	3.50000000E-01

B MATRIX

2.92000000E+02	-2.92000000E+02	0.	0.	0.
-2.92000000E+02	1.02200000E+03	-7.30000000E+02	0.	0.
0.	-7.30000000E+02	1.46000000E+03	-7.30000000E+02	0.
0.	0.	-7.30000000E+02	1.46000000E+03	-7.30000000E+02
0.	0.	0.	-7.30000000F+02	7.30000000E+02

A MATRIX

1	0.	0.	0.	0.	0.	1.00000000E-06	0.
	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	1.17647059E-05
	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.
	1.17647059E-05	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.
	0.	1.17647059E-05	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.
	0.	0.	1.17647059E-05	0.	0.	0.	0.
6	2.92000000E+02	-2.92000000E+02	0.	0.	0.	0.	0.
	0.	0.	0.	0.	1.24100000E+02	0.	0.
7	-2.92000000E+02	1.02200000E+03	-7.30000000E+02	0.	0.	0.	0.
	0.	0.	0.	0.	1.31400000E+02	0.	0.
8	0.	-7.30000000E+02	1.46000000E+03	-7.30000000E+02	0.	0.	0.
	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	-7.30000000E+02	1.46000000E+03	-7.30000000E+02	0.	0.
	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	-7.30000000E+02	7.30000000E+02	0.	0.
	0.	0.	0.	0.	-2.55500000F+02	0.	0.
11	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.	1.00000000F+00	0.	0.
12	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.	0.	1.00000000E+00	0.
13	0.	0.	0.	0.	0.	0.	0.
	0.	0.	0.	0.	0.	0.	1.00000000E+00
14	0.	0.	0.	0.	0.	4.76544000E-08	5.93618824E-07
	0.	0.	-1.15425882E-06	0.	0.	1.23270720E-01	0.

```
                                H-SUB-L MATRIX
 1.22787019E-06    -4.87358838E-07     1.28767161E-06    -1.45002533E-05
 1.78079582E-05     3.11099190E-05    -8.74137929E-05     1.58323755E-04
-1.58825576E-06    -6.89160049E-05     5.77333181E-05     1.12942523E-04
 2.72271204E-06     6.48436234E-05     8.11147757E-05     5.01488380E-06
-3.30755802E-05    -2.14218260E-05    -6.65833787E-05    -1.05689947E-04
-4.38671809E-01     8.23427505E-02    -1.73303060E-01     1.00000000E+00
-5.20292252E-01    -4.52282571E-01     1.00000000E+00    -9.28088558E-01
 4.80180711E-02     1.00000000E+00    -6.60460051E-01    -6.62065297E-01
-8.23117869E-02    -9.40805757E-01    -9.27940247E-01    -2.93970812E-02
 1.00000000E+00     3.10436578E-01     7.61703354E-01     6.19550936E-01
-3.62833935E-03     2.27243407E-04     1.54267157E-04     2.16156621E-03
 1.24985808E-03    -3.87924965E-05    -2.07622581E-05    -1.49076928E-04
-4.58791952E-04     6.02221138E-06     2.79431714E-06     1.02805741E-05
 1.63188255E-04    -1.13046820E-06    -3.76077027E-07    -7.08992723E-07
                                F-SUB-L
 1.68868074E-01     4.25903357E-01     5.10210498E-01     7.08346738E-01
                                   T
 1.00000000E+06     8.50000000E+04     8.50000000E+04     8.50000000E+04     8.50000000E+04
                                   W
-1.07500000E+06    -8.92500000E+04    -5.95000000E+04    -2.97500000E+04     0.
                                U-BAR-SUB(2)
 8.25493809E+02     1.13427469E+03     1.38088070E+03     1.52456042E+03
                 ALPHA=     1.23395522E+00
                                U-SUB-(1)
-2.41084776E-01     1.83955224E-01     5.33955224E-01     8.83955224E-01     1.23395522E+00
```

B MATRIX

1	1.22767019E-06	-4.82358638E-07	1.23767161E-06	-1.45002533E-05	2.07351621E-07	-2.05438163E-07	6.56983576E-07
	-1.02712071E-05	1.00000000E+00	0.	0.	-2.41044776E-01	0.	0.
2	1.74870542E-05	3.11699150E-05	-8.74137929E-05	1.58323755E-04	2.95537776E-06	1.32753714E-05	-4.45994348E-05
	1.12148115E-04	1.00000000E+00	0.	0.	1.83955224E-01	0.	8.25495809E+02
3	-1.58825576E-06	-6.89169049E-05	5.77333181E-05	1.12042523E-04	-2.68253974E-07	-2.93519411E-05	2.94561450E-05
	8.00024680E-05	1.00000000E+00	0.	0.	5.33055224E-01	0.	1.13427469E+03
4	2.72271204E-06	6.48436234E-05	8.11147757E-05	5.01488380E-06	4.59861907E-07	2.76171169E-05	4.13856101E-05
	3.55227658E-06	1.00000000E+00	0.	0.	8.83955224E-01	0.	1.38088070E+03
5	-3.30755892E-05	-2.14218260E-05	-6.65833787E-05	-1.05689947E-04	-5.58641653E-06	-9.12362758E-06	-3.39715388E-05
	-7.48651291E-05	1.00000000E+00	0.	0.	1.23395522E+00	0.	1.52456042E+03
6	-3.62635935E-03	2.27243997E-04	1.54267157E-04	2.16156621E-03	6.12483181E-04	-9.67839812E-05	-7.87872229E-05
	-1.93113838E-03	0.	0.	1.00000000E+00	0.	0.	0.
7	1.28085808E-03	-3.87924065E-05	-2.07622581E-05	-1.49070928E-04	2.17855060E-04	-1.65218545E-05	-1.05931220E-05
	-1.05593906E-04	0.	0.	0.	1.00000000E+00	0.	0.
8	1.63198255E-04	-1.13046820E-06	-3.76077027E-07	-7.08992723E-07	2.75622471E-05	-4.81470199E-07	-1.91978447E-07
	-5.02212683E-07	0.	0.	0.	0.	0.	1.00000000E+00
9	2.07351621E-07	-2.05438163E-07	6.56983576E-07	-1.02712071E-05	1.22767019E-06	-4.82358638E-07	1.23767161E-06
	-1.45002533E-05	1.00000000E+00	5.00000000E+00	0.	-2.41044776E-01	-1.20522388E+00	-3.01305970E+00
10	2.95537776E-06	1.32753714E-05	-4.45994348E-05	1.12148115E-04	1.74870542E-05	3.11699150E-05	-8.74137924E-05
	1.58323755E-04	1.00000000E+00	5.00000000E+00	0.	1.83955224E-01	9.19776119E-01	8.27795249E+02
11	-2.68253974E-07	-2.93519411E-05	2.94561450E-05	8.00024680E-05	-1.58825576E-06	-6.87169049E-05	5.77333181E-05
	1.12042523E-04	1.00000000E+00	5.00000000E+00	0.	5.33055224E-01	2.66977612E+00	1.14094213E+03
12	4.59861907E-07	2.76171169E-05	4.13856101E-05	3.55227658E-06	2.72271204E-06	6.48436234E-05	8.11147757E-05
	5.01488380E-06	1.00000000E+00	5.00000000E+00	0.	8.83955224E-01	4.41977612E+00	1.39193014E+03
13	-5.58641653E-06	-9.12362758E-06	-3.39715388E-05	-7.48651291E-05	-3.30755892E-05	-2.14218260E-05	-6.65833787E-05
	-1.05689947E-04	1.00000000E+00	5.00000000E+00	0.	1.23395522E+00	6.16977612E+00	1.53998485E+03
14	2.17855060E-04	-1.65218545E-05	-1.05931220E-05	-1.05593906E-04	1.28085808E-03	-3.87924065E-05	-2.07622581E-05
	-1.49070928E-04	0.	0.	0.	1.00000000E+00	5.00000000E+00	1.25000000E+01

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0.	0.	0.	0.	0.	0.	0.
-1.92000000E-04	0.	0.	0.	0.	0.	0.

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-9.79123123E-01	5.26340271E-01	2.02753870E-01	2.06860961E+00	-9.79123123E-01	5.26340271E-01	2.02753870E-01
2.06860961E+00	5.43787583E-04	-4.87890978E-18	-4.33903147E-03	2.03851051E-03	4.40669642E-06	-1.76267857E-06

```
                              F-SUB-L ( 0)
1.00000000E+00     1.00000000E+00     1.00000000E+00     1.00000000E+00
                              F-SUB-G ( 0)
1.68898474E-01     4.25903357E-01     5.10210498E-01     7.08346738E-01
                              Y-SUB-AU ( 0)                                                 0.
1.00000000E+00     0.                 0.                -2.41044776E-01     0.              8.25495809E+02
1.00000000E+00     0.                 0.                 1.83955224E-01     0.              1.13427469E+03
1.00000000E+00     0.                 0.                 5.33955224E-01     0.              1.38088070E+03
1.00000000E+00     0.                 0.                 8.83955224E-01     0.              1.52456042E+03
1.00000000E+00     0.                 0.                 1.23395522E+00     0.
                              Y-SUB-AM ( 0)
0.                 1.00000000E+06     0.                 0.                -2.41044776E+05  0.
0.                 8.50000000E+04     0.                 0.                 1.56361940E+04  0.
0.                 8.50000000E+04     0.                 0.                 4.53861940E+04  0.
0.                 8.50000000E+04     0.                 0.                 7.51361940E+04  0.
0.                 8.50000000E+04     0.                 0.                 1.04886194E+05  0.
                              Y-SUB-AV ( 0)
0.                 0.                 1.00000000E+00     0.                 0.              0.
                              Y-SUB-A THETA( 0)
0.                 0.                 0.                 1.00000000E+00     0.              0.
                              Y-SUB-A BETA ( 0)
0.                 0.                 0.                 0.                 1.00000000E+00  0.
                              Y-SUB-A PHI  ( 0)
0.                 0.                 0.                 0.                 0.              1.00000000E+00
```

		P-SUB-L (1)			
8.37091932E-01	9.18186930E-01	9.34921052E-01	9.66105978E-01		
		P-SUB-G (1)			
2.01793047E-01	4.63852559E-01	5.45725756E-01	7.33190062E-01		
		V-SUB-AU (1)			
1.00000000E+00	5.00000000E-01	0.	-2.41044776E-01	-1.20522388E-01	-3.01305970E-02
1.00000000E+00	5.00000000E-01	0.	1.83955224E-01	9.19776119E-02	8.25518803E+02
1.00000000E+00	5.00000000E-01	0.	5.33955224E-01	2.66977612E-01	1.13434143E+03
1.00000000E+00	5.00000000E-01	0.	8.83955224E-01	4.41977612E-01	1.38099119E+03
1.00000000E+00	5.00000000E-01	0.	1.23395522E+00	6.16977612E-01	1.52471466E+03
		V-SUB-AH (1)			
0.	1.00000000E+06	0.	0.	-2.41044776E+05	-1.20522388E+05
0.	8.50000000E+04	0.	0.	1.56361940E+04	7.81809701E+03
0.	8.50000000E+04	0.	0.	4.53861940E+04	2.26930970E+04
0.	8.50000000E+04	0.	0.	7.51361940E+04	3.75680970E+04
0.	8.50000000E+04	0.	0.	1.04886194E+05	5.24430970E+04
		V-SUB-AV (1)			
0.	0.	1.00000000E+00	5.00000000E-01	1.25000000E-01	2.08333333E-02
		V-SUB-A THETA(1)			
0.	0.	0.	1.00000000E+00	5.00000000E-01	1.25000000E-01
		V-SUB-A BETA (1)			
0.	0.	0.	0.	1.00000000E+00	5.00000000E-01
		V-SUB-A PHI (1)			
0.	0.	0.	0.	0.	1.00000000E+00

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                          F-SUB-L ( 2)
7.00688790E-01      8.43067238E-01      8.74077373E-01      9.33390089E-01
                          F-SUB-G ( 2)
2.41006301E-01      5.05183103E-01      5.83713197E-01      7.58921259E-01
                          Y-SUB-AU ( 2)
1.00000000E+00      1.00000000E+00      0.                  -2.41044776E-01     -2.41044776E-01     -1.20522388E-01
1.00000000E+00      1.00000000E+00      0.                  1.83955224E-01      1.83955224E-01      8.25587746E+02
1.00000000E+00      1.00000000E+00      0.                  5.33955224E-01      5.33955224E-01      1.13454167E+03
1.00000000E+00      1.00000000E+00      0.                  8.83955224E-01      8.83955224E-01      1.38132268E+03
1.00000000E+00      1.00000000E+00      0.                  1.23395522E+00      1.23395522E+00      1.52517739E+03
                          Y-SUB-AH ( 2)
0.                  1.00000000E+06      0.                  0.                  -2.41044776E+05     -2.41044776E+05
0.                  8.50000000E+04      0.                  0.                  1.56361940E+04      1.56361940E+04
0.                  8.50000000E+04      0.                  0.                  4.53861940E+04      4.53861940E+04
0.                  8.50000000E+04      0.                  0.                  7.51361940E+04      7.51361940E+04
0.                  8.50000000E+04      0.                  0.                  1.04886194E+05      1.04886194E+05
                          Y-SUB-AV ( 2)
0.                  0.                  1.00000000E+00      1.00000000E+00      5.00000000E-01      1.66666667E-01
                          Y-SUB-A THETA( 2)
0.                  0.                  0.                  1.00000000E+00      1.00000000E+00      5.00000000E-01
                          Y-SUB-A BETA ( 2)
0.                  0.                  0.                  0.                  1.00000000E+00      1.00000000E+00
                          Y-SUB-A PHI  ( 2)
0.                  0.                  0.                  0.                  0.                  1.00000000E+00
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                                F-SUR-L ( 3)
5.86596606E-01   7.74093318E-01   8.17193337E-01   9.01724292E-01
                                F-SUR-G ( 3)
2.87963874E-01   5.50196399E-01   6.24344907E-01   7.85546917E-01
                                V-SUR-AU ( 3)
1.00000000E+00   1.50000000E+00   0.               -2.41044776E-01   -3.61567164E-01   -2.71175373E-01
1.00000000E+00   1.50000000E+00   0.               1.83955224E-01    2.75932836E-01    8.25702758E+02
1.00000000E+00   1.50000000E+00   0.               5.33955224E-01    8.00932836E-01    1.13487539E+03
1.00000000E+00   1.50000000E+00   0.               8.83955224E-01    1.32593284E+00    1.38187515E+03
1.00000000E+00   1.50000000E+00   0.               1.23395522E+00    1.85093284E+00    1.52594862E+03
                                V-SUR-AH ( 3)
0.               1.00000000E+06   0.               0.                -2.41044776E+05   -3.61567164E+05
0.               8.50000000E+04   0.               0.                1.56361940E+04    2.34542910E+04
0.               8.50000000E+04   0.               0.                4.53861940E+04    6.80792910E+04
0.               8.50000000E+04   0.               0.                7.51361940E+04    1.12704291E+05
0.               8.50000000E+04   0.               0.                1.04886194E+05    1.57329291E+05
                                V-SUR-AV ( 3)
0.               0.               1.00000000E+00   1.50000000E+00    1.12500000E+00    5.62500000E-01
                                V-SUR-A THETA( 3)
0.               0.               0.               1.00000000E+00    1.50000000E+00    1.12500000E+00
                                V-SUR-A BETA ( 3)
0.               0.               0.               0.                1.00000000E+00    1.50000000E+00
                                V-SUR-A PHI  ( 3)
0.               0.               0.               0.                0.                1.00000000E+00
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		F-SUB-L (4)			
8.90968725E-01	7.10762767E-01	7.64011255E-01	8.71160869E-01		
		F-SUB-G (4)			
3.48013460E-01	5.90220466E-0[illegible]	6.67804951E-01	8.13106498E-01		
		Y-SUB-AU (4)			
1.00000000E+00	2.00000000E+00	0.	-2.41044776E-01	-4.82089552E-01	-4.82089552E-01
1.00000000E+00	2.00000000E+00	0.	1.83955224E-01	3.67910448E-01	8.25863719E+02
1.00000000E+00	2.00000000E+00	0.	5.33955224E-01	1.06791045E+00	1.13534260E+03
1.00000000E+00	2.00000000E+00	0.	8.83955224E-01	1.76791045E+00	1.38264861E+03
1.00000000E+00	2.00000000E+00	0.	1.23395522E+00	2.46791045E+00	1.52702833E+03
		Y-SUB-AM (4)			
0.	1.00000000E+06	0.	0.	-2.41044776E+05	-4.82089552E+05
0.	8.50000000E+04	0.	0.	1.56361940E+04	3.12723881E+04
0.	8.50000000E+04	0.	0.	4.53861940E+04	9.07723881E+04
0.	8.50000000E+04	0.	0.	7.51361940E+04	1.50272388E+05
0.	8.50000000E+04	0.	0.	1.04886194E+05	2.09772388E+05
		Y-SUB-AV (4)			
0.	0.	1.00000000E+00	2.00000000E+00	2.00000000E+00	1.33333333E+00
		Y-SUB-A THETA(4)			
0.	0.	0.	1.00000000E+00	2.00000000E+00	2.00000000E+00
		Y-SUB-A BETA (4)			
0.	0.	0.	0.	1.00000000E+00	2.00000000E+00
		Y-SUB-A PHI (4)			
0.	0.	0.	0.	0.	1.00000000E+00

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                                        F-SUB-L ( 5)
4.10972905E-01          6.52012716E-01      7.14290206E-01          8.41033375E-01
                                        F-SUB-G ( 5)
4.10972595E-01          6.52012716E-01      7.14290206E-01          8.41031175E-01
                                        Y-SUB-AU ( 5)
1.00000000E+00          2.50000000E+00      0.                     -2.41044776E-01        -6.02611940E-01        -7.53264925E-01
1.00000000E+00          2.50000000E+00      0.                      1.83955224E-01         4.59880060E-01         8.26070609E+02
1.00000000E+00          2.50000000E+00      0.                      5.33955224E-01         1.33488806E+00         1.13594330E+03
1.00000000E+00          2.50000000E+00      0.                      8.83955224E-01         2.20988806E+00         1.38364306E+03
1.00000000E+00          2.50000000E+00      0.                      1.23395522E+00         3.08488806E+00         1.52841653E+03
                                        Y-SUB-AM ( 5)
0.                      1.00000000E+06      0.                      0.                    -2.41044776E+05        -6.02611940E+05
0.                      8.50000000E+04      0.                      0.                     1.56361940E+04         3.90904851E+04
0.                      8.50000000E+04      0.                      0.                     4.53861940E+04         1.13465485E+05
0.                      8.50000000E+04      0.                      0.                     7.51361940E+04         1.87840485E+05
0.                      8.50000000E+04      0.                      0.                     1.04886194E+05         2.62215485E+05
                                        Y-SUB-AV ( 5)
0.                      0.                  1.00000000E+00          2.50000000E+00         3.12500000E+00         2.60416657E+00
                                        Y-SUB-A THETA( 5)
0.                      0.                  0.                      1.00000000E+00         2.50000000E+00         3.12500000E+00
                                        Y-SUB-A BETA ( 5)
0.                      0.                  0.                      0.                     1.00000000E+00         2.50000000E+00
                                        Y-SUB-A PHI  ( 5)
0.                      0.                  0.                      0.                     0.                     1.00000000E+00
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                              P-SUB-L ( 6)
3.40013460E-01     5.99220466E-01     6.67804951E-01     8.13106698E-01
                              F-SUB-Q ( 6)
4.90964725E-01     7.10762367E-01     7.64011255E-01     8.71160869E-01
                              V-SUB-AU ( 6)
1.00000000E+00     3.00000000E+00     0.                -2.41044776E-01    -7.23134328E-01    -1.08470149E+00
1.00000000E+00     3.00000000E+00     0.                 1.83955224E-01     5.31865672E-01     8.26323607E+02
1.00000000E+00     3.00000000E+00     0.                 5.33955224E-01     1.60186567E+00     1.13667749E+03
1.00000000E+00     3.00000000E+00     0.                 8.83955224E-01     2.65186567E+00     1.38485850E+03
1.00000000E+00     3.00000000E+00     0.                 1.23395522E+00     3.70186567E+00     1.53011322E+03
                              V-SUB-AM ( 6)
0.                 1.00000000E+06     0.                 0.                -2.41044776E+05    -7.23134328E+05
0.                 8.50000000E+04     0.                 0.                 1.56361940E+04     4.69085871E+04
0.                 8.50000000E+04     0.                 0.                 4.53861940E+04     1.36158582E+05
0.                 8.50000000E+04     0.                 0.                 7.51361940E+04     2.25408582E+05
0.                 8.50000000E+04     0.                 0.                 1.04886194E+05     3.14658582E+05
                              V-SUB-AV ( 6)
0.                 0.                 1.00000000E+00     3.00000000E+00     4.50000000E+00     4.50000000E+00
                              V-SUB-A THETA( 6)
0.                 0.                 0.                 1.00000000E+00     3.00000000E+00     4.50000000E+00
                              V-SUB-A BETA ( 6)
0.                 0.                 0.                 0.                 1.00000000E+00     3.00000000E+00
                              V-SUB-A PHI  ( 6)
0.                 0.                 0.                 0.                 0.                 1.00000000E+00
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                              F-SUB-L ( 7)
2.87963874E-01      5.50196309E-01      6.24344907E-01      7.85546917E-01
                              F-SUB-G ( 7)
5.86596606E-01      7.74093318E-01      8.17193337E-01      9.01724292E-01
                              V-SUB-AU ( 7)
1.00000000E+00      3.50000000E+00      0.                 -2.41044776E-01     -8.43656716E-01     -1.47639925E+00
1.00000000E+00      3.50000000E+00      0.                  1.83955224E-01      6.43843284E-01      8.26622534E+02
1.00000000E+00      3.50000000E+00      0.                  5.33955224E-01      1.86884328E+00      1.13754516E+03
1.00000000E+00      3.50000000E+00      0.                  8.83955224E-01      3.09384328E+00      1.38629492E+03
1.00000000E+00      3.50000000E+00      0.                  1.23395522E+00      4.31884328E+00      1.53211839E+03
                              V-SUB-AH ( 7)
0.                  1.00000000E+06      0.                  0.                 -2.41044776E+05     -8.43656716E+05
0.                  8.50000000E+04      0.                  0.                  1.56361940E+04      5.47266791E+04
0.                  8.50000000E+04      0.                  0.                  4.53861940E+04      1.58851679E+05
0.                  8.50000000E+04      0.                  0.                  7.51361940E+04      2.62976679E+05
0.                  8.50000000E+04      0.                  0.                  1.04886194E+05      3.67101679E+05
                              V-SUB-AV ( 7)
0.                  0.                  1.00000000E+00      3.50000000E+00      6.12500000E+00      7.14583333E+00
                              V-SUB-A THETA( 7)
0.                  0.                  0.                  1.00000000E+00      3.50000000E+00      6.12500000E+00
                              V-SUB-A BETA ( 7)
0.                  0.                  0.                  0.                  1.00000000E+00      3.50000000E+00
                              V-SUB-A PHI  ( 7)
0.                  0.                  0.                  0.                  0.                  1.00000000E+00
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                                  F-SUB-L ( 8)
2.41086361E-01       5.05183143E-01    5.83713197E-01       7.58921259E-01
                                  F-SUB-G ( 8)
7.00648780E-01       8.43067238E-01    8.74077373E-01       9.33359989E-01
                                  Y-SUB-AU ( 8)
1.00000000E+00       4.00000000E+00    0.                   -2.41044776E-01    -9.64179104E-01    -1.92835821E+00
1.00000000E+00       4.00000000E+00    0.                   1.83955224E-01     7.35820896E-01     8.26967450E+02
1.00000000E+00       4.00000000E+00    0.                   5.33955224E-01     2.13582090E+00     1.13854633E+03
1.00000000E+00       4.00000000E+00    0.                   8.83955224E-01     3.53582090E+00     1.38795234E+03
1.00000000E+00       4.00000000E+00    0.                   1.23395522E+00     4.93582090E+00     1.55443206E+03
                                  Y-SUB-AH ( 8)
0.                   1.00000000E+06    0.                   0.                 -2.41044776E+05    -9.64179104E+05
0.                   8.50000000E+04    0.                   0.                 1.56361940E+04     6.25447761E+04
0.                   8.50000000E+04    0.                   0.                 4.53861940E+04     1.81544776E+05
0.                   8.50000000E+04    0.                   0.                 7.51361940E+04     3.00544776E+05
0.                   8.50000000E+04    0.                   0.                 1.04886194E+05     4.19544776E+05
                                  Y-SUB-AV ( 8)
0.                   0.                1.00000000E+00       4.00000000E+00     8.00000000E+00     1.06666667E+01
                                  Y-SUB-A THETA( 8)
0.                   0.                0.                   1.00000000E+00     4.00000000E+00     8.00000000E+00
                                  Y-SUB-A BETA ( 8)
0.                   0.                0.                   0.                 1.00000000E+00     4.00000000E+00
                                  Y-SUB-A PHI  ( 8)
0.                   0.                0.                   0.                 0.                 1.00000000E+00
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```
                                  F-SUB-L ( 9)
2.01773087E-01     4.03852559E-01     5.45725756E-01     7.33194062E-01
                                  F-SUB-G ( 9)
8.39071932E-01     9.18186930E-01     9.34921052E-01     9.66105578E-01
                                  V-SUB-AU ( 9)
1.00000000E+00     4.50000000E+00     0.                 -2.41044776E-01    -1.08470149E+00    -2.44057836E+00
1.00000000E+00     4.50000000E+00     0.                 1.83955224E-01     8.27798507E-01     8.27358355E+02
1.00000000E+00     4.50000000E+00     0.                 5.33955224E-01     2.40279851E+00     1.13968098E+03
1.00000000E+00     4.50000000E+00     0.                 8.83955224E-01     3.97779851E+00     1.38983075E+03
1.00000000E+00     4.50000000E+00     0.                 1.23395522E+00     5.55279851E+00     1.53705421E+03
                                  V-SUB-AH ( 9)
0.                 1.00000000E+06     0.                 0.                 -2.41044776E+05    -1.08470149E+06
0.                 8.50000000E+04     0.                 0.                 1.56361940E+04     7.03628731E+04
0.                 8.50000000E+04     0.                 0.                 4.53861940E+04     2.04237873E+04
0.                 8.50000000E+04     0.                 0.                 7.51361940E+04     3.38112873E+05
0.                 8.50000000E+04     0.                 0.                 1.04886194E+05     4.71987873E+05
                                  V-SUB-AV ( 9)
0.                 0.                 1.00000000E+00     4.50000000E+00     1.01250000E+01     1.51875000E+01
                                  V-SUB-A THETA( 9)
0.                 0.                 0.                 1.00000000E+00     4.50000000E+00     1.01250000E+01
                                  V-SUB-A BETA ( 9)
0.                 0.                 0.                 0.                 1.00000000E+00     4.50000000E+00
                                  V-SUB-A PHI  ( 9)
0.                 0.                 0.                 0.                 0.                 1.00000000E+00
```

```
                                F-SUB-L (10)
1.68898874E-01      4.25903357E-01      5.10210498E-01      7.08346738E-01
                                F-SUB-G (10)
1.00000000E+00      1.00000000E+00      1.00000000E+00      1.00000000E+00
                                V-SUB-AU (10)
1.00000000E+00      5.00000000E+00      0.                 -2.41044776E-01     -1.20522388E+00     -3.01305970E+00
1.00000000E+00      5.00000000E+00      0.                  1.83955224E-01      9.19776119E-01      8.27705249E+02
1.00000000E+00      5.00000000E+00      0.                  5.33955224E-01      2.66977612E+00      1.14094913E+03
1.00000000E+00      5.00000000E+00      0.                  8.83955224E-01      4.41977612E+00      1.[illegible]4193014E+03
1.00000000E+00      5.00000000E+00      0.                  1.23395522E+00      6.16977612E+00      1.53998486E+03
                                V-SUB-AH (10)
0.                  1.00000000E+06      0.                  0.                 -2.41044776E+05     -1.20522388E+06
0.                  6.50000000E+04      0.                  0.                  1.56361940E+04      7.81809701E+04
0.                  8.50000000E+04      0.                  0.                  4.53861940E+04      2.26930970E+05
0.                  8.50000000E+04      0.                  0.                  7.51361940E+04      3.75680970E+05
0.                  8.50000000E+04      0.                  0.                  1.04886194E+05      5.24430970E+05
                                V-SUB-AV (10)
0.                  0.                  1.00000000E+00      5.00000000E+00      1.25000000E+01      2.08333333E+01
                                V-SUB-A THETA(10)
0.                  0.                  0.                  1.00000000E+00      5.00000000E+00      1.25000000E+01
                                V-SUB-A BETA (10)
0.                  0.                  0.                  0.                  1.00000000E+00      5.00000000E+00
                                V-SUB-A PHI  (10)
0.                  0.                  0.                  0.                  0.                  1.00000000E+00
```

APPENDIX D

OUTPUT FOR ILLUSTRATIVE CASE A, B, C, AND D AND FOR THE TEST BEAM

CASE A - 4 PLY P.C., LENGTH = 10 INCHES

```
NY,F:L:BM    10   1.000000E+00    5.000000E+00    1.000000E+00
T     1.000000E-01    2.500000E-01    2.500000E-01    1.000000E-01    2.500000E-01
      1.000000E-01    2.500000E-01    1.000000E-01    2.500000E-01
E     1.000000E+07    1.830000E+02    3.400000E+05    1.830000E+02    3.400000E+05
      1.830000E+02    3.400000E+05    1.830000E+02    3.400000E+05
G     4.000000E+06    7.300000E+01    1.090000E+05    7.300000E+01    1.090000E+05
      7.300000E+01    1.090000E+05    7.300000E+01    1.090000E+05
Y     0.              5.000000E-01    1.000000E+00    1.500000E+00    2.000000E+00
      2.500000E+00    3.000000E+00    3.500000E+00    4.000000E+00    4.500000E+00    5.000000E+00
A     0.              0.              0.              0.              0.              2.920000E+02    -2.920000E+02
    0.              0.              0.              0.              0.              0.              0.
A     0.              0.              0.              0.              0.              -2.920000E+02   1.022000E+03
    -7.300000E+02   0.              0.              0.              0.              0.              0.
A     0.              0.              0.              0.              0.              0.              -7.300000E+02
    1.460000E+03    -7.300000E+02   0.              0.              0.              0.              0.
A     0.              0.              0.              0.              0.              0.              0.
    -7.300000E+02   1.460000E+03    -7.300000E+02   0.              0.              0.              0.
A     0.              0.              0.              0.              0.              0.              0.
    0.              -7.300000E+02   7.300000E+02    0.              0.              0.              0.
A     1.000000E-06    0.              0.              0.              0.              0.              4.765440E-08
    0.              0.              0.              0.              0.              0.              0.
A     0.              1.176471E-05    0.              0.              0.              0.              5.936188E-07
    0.              0.              0.              0.              0.              0.              0.
A     0.              0.              1.176471E-05    0.              0.              0.              0.
    0.              0.              0.              0.              0.              0.              0.
A     0.              0.              0.              1.176471E-05    0.              0.              0.
    0.              0.              0.              0.              0.              0.              0.
A     0.              0.              0.              0.              1.176471E-05    0.              -1.154259E-06
    0.              0.              0.              0.              0.              0.              0.
A     0.              0.              0.              0.              0.              1.241000E+02    1.314000E+02
    0.              0.              -2.555000E+02   1.000000E+00    0.              0.              0.
A     0.              0.              0.              0.              0.              0.              1.232707E-01
    0.              0.              0.              0.              1.000000E+00    0.              0.
A     0.              0.              0.              0.              0.              0.              0.
    0.              0.              0.              0.              0.              1.000000E+00    0.
EIGENVALUES    0.              -3.556915E-01   3.556915E-01    -1.707086E-01   1.707086E-01    -1.345860E-01   1.345860E-01
    6.896431E-02   -6.896431E-02   -4.213817E-09   4.213819E-09    -2.155976E-16   -5.682813E-19   -5.682813E-19
PIVOT      1.046820E+03
PIVOT      9.682427E+02
PIVOT      9.226709E+02
PIVOT      1.627621E+02
PIVOT     -3.626339E-03
PIVOT     -6.901603E-05
PIVOT      1.355524E-04
PIVOT      6.307198E-04
PIVOT      1.311034E-03
PIVOT     -5.401868E-05
PIVOT      1.035143E-04
PIVOT      4.303206E-04
PIVOT      2.576271E+00
PIVOT     -1.214413E+01
PIVOT      4.551242E-02
PIVOT     -1.176701E+00
PIVOT     -5.883053E+00
PIVOT      1.089251E+02

Y(    0)

U      5.904907E-17    3.469447E-17    2.775558E-17    -1.387779E-17    -2.359101E-17
```

CASE A

```
W     -4.988101E-02   -4.788906E-02   -1.925307E-03    3.891531E-03    1.918038E-01
VPE    0.             THETA   -5.551115E-17 BETA    3.862296E-04 PHI   -1.920000E-04
QXJ   -1.870272E-15   -4.117707E-15    1.620926E-14   -7.851358E-15
VXJ    1.600000E-01    8.500000E-02    8.500000E-02    8.500000E-02    8.500000E-02
WXJ    3.218880E-01    1.709871E-01    1.709871E-01    1.709871E-01    1.709871E-01
X(    1)
U     -4.699614E-08   -5.644097E-07   -1.113675E-08    2.251040E-08    1.105932E-06
W     -4.033333E-02   -4.219557E-02   -1.829517E-03    3.698023E-03    1.806604E-01
VPE    4.439591E-05 THETA    1.700406E-04 BETA    2.956672E-04 PHI   -1.711782E-04
QXJ    9.125513E-02    4.304150E-02    4.342082E-02    4.265449E-02
VXJ    1.437111E-01    8.381884E-02    8.658980E-02    8.654143E-02    8.111382E-02
WXJ    2.463893E-01    1.308943E-01    1.308943E-01    1.308943E-01    1.308943E-01
X(    2)
U     -4.479011E-08   -1.066100E-06   -2.116841E-08    4.278813E-08    2.089070E-06
W     -7.451720E-02   -7.702895E-02   -1.553135E-03    3.139585E-03    1.509507E-01
VPE    1.628973E-04 THETA    2.971771E-04 BETA    2.181735E-04 PHI   -1.556298E-04
QXJ    3.716505E-02    7.516594E-02    7.988205E-02    7.443495E-02
VXJ    1.315497E-01    8.293946E-02    8.777959E-02    8.768822E-02    7.820296E-02
WXJ    1.784779E-01    9.481638E-02    9.481638E-02    9.481638E-02    9.481638E-02
X(    3)
U     -1.213941E-07   -1.457251E-06   -2.911225E-08    5.884679E-08    2.855683E-06
W     -5.397921E-02   -5.502985E-02   -1.125889E-03    2.276101E-03    1.078588E-01
VPE    3.350784E-04 THETA    3.853066E-04 BETA    1.392418E-04 PHI   -1.448625E-04
QXJ    4.820662E-02    9.740324E-02    9.838162E-02    9.640814E-02
VXJ    1.231291E-01    8.233306E-02    8.860494E-02    8.848004E-02    7.618234E-02
WXJ    1.160345E-01    6.164333E-02    6.164333E-02    6.164333E-02    6.164333E-02
X(    4)
U     -1.420410E-07   -1.704838E-06   -3.420239E-08    6.913732E-08    3.340974E-06
W     -2.806722E-02   -2.860301E-02   -5.909254E-04    1.194984E-03    5.606647E-02
VPE    5.421987E-04 THETA    4.371281E-04 BETA    6.897172E-05 PHI   -1.185355E-04
QXJ    5.470393E-02    1.104667E-01    1.116108E-01    1.092978E-01
VXJ    1.181814E-01    8.197714E-02    8.909090E-02    8.894439E-02    7.090304E-02
WXJ    5.714310E-02    3.035727E-02    3.035727E-02    3.035727E-02    3.035727E-02
```

CASE A

```
X(    5)
U    -1.491028E-07   -1.789495E-06   -3.595431E-08    7.267924E-08    3.506917E-06
W    -6.771492E-12   -2.140510E-13   -2.078338E-13    3.161915E-13    3.446132E-13
VEE    7.664254E-04 THETA    4.582276E-04 BETA    1.566672E-17 PHI    -1.364489E-04
QXJ    5.684864E-02    1.147751E-01    1.159756E-01    1.135482E-01
VXJ    1.165495E-01    8.185985E-02    8.925076E-02    8.909739E-02    7.860041E-02
MXJ    1.305560E-14    6.935788E-15    6.935788E-15    6.935788E-15    6.935788E-15
X(    6)
U    -1.428410E-07   -1.708838E-06   -3.420239E-08    6.913732E-08    3.340974E-06
W    2.806722E-02    2.860301E-02    5.909254E-04   -1.194684E-03   -5.606607E-02
VEE    9.006922E-04 THETA    4.371281E-04 BETA    -6.857172E-05 PHI    -1.385355E-04
QXJ    5.870393E-02    1.104667E-01    1.116108E-01    1.092978E-01
VXJ    1.181818E-01    8.197714E-02    8.909050E-02    8.894439E-02    7.894304E-02
MXJ    -5.714310E-02   -3.035727E-02   -3.035727E-02   -3.035727E-02   -3.035727E-02
X(    7)
U    -1.213941E-07   -1.457251E-06   -2.911225E-08    5.884679E-08    2.859683E-06
W    5.394921E-02    5.502985E-02    1.125889E-03   -2.276101E-03   -1.078588E-01
VEE    1.197772E-03 THETA    3.853066E-04 BETA    -1.392414E-04 PHI    -1.488625E-04
QXJ    8.820662E-02    9.740329E-02    9.838162E-02    9.640414E-02
VXJ    1.231891E-01    8.233306E-02    8.860494E-02    8.848008E-02    7.618234E-02
MXJ    -1.160345E-01   -6.164333E-02   -6.164333E-02   -6.164333E-02   -6.164333E-02
X(    8)
U    -8.879011E-08   -1.066100E-06   -2.116841E-08    4.278813E-08    2.089070E-06
W    7.591728E-02    7.702895E-02    1.553135E-03   -3.139585E-03   -1.509507E-01
VEE    1.364954E-03 THETA    2.971771E-04 BETA    -2.141735E-04 PHI    -1.556298E-04
QXJ    3.716905E-02    7.516594E-02    7.588205E-02    7.443495E-02
VXJ    1.315497E-01    8.293996E-02    8.777959E-02    8.768822E-02    7.820296E-02
MXJ    -1.788779E-01   -9.481638E-02   -9.481638E-02   -9.481638E-02   -9.481638E-02
X(    9)
U    -4.696614E-08   -5.644097E-07   -1.113675E-08    2.251040E-08    1.109932E-06
W    9.033333E-02    9.219557E-02    1.829517E-03   -3.698023E-03   -1.806604E-01
VEE    1.488455E-03 THETA    1.700806E-04 BETA    -2.956672E-04 PHI    -1.711782E-04
QXJ    2.129313E-02    4.304190E-02    4.342082E-02    4.265449E-02
```

CASE A

```
VXJ     1,837111E-01   8,381884E-02   8,658980E-02   8,654143E-02   8,111382E-02
WXJ    -2,463893E-01  -1,308943E-01  -1,308943E-01  -1,308943E-01  -1,308943E-01
X(   10)
U      1,930567E-17   2,775558E-17   3,469447E-18  -5,204170E-18  -1,387779E-17
W      9,988101E-02   9,788906E-02   1,925307E-03  -3,891531E-03  -1,910038E-01
VEE     1,932851E-03 THETA     0.          BETA   -3,862296E-04 PHI    -1,920000E-04
BXJ    -3,609092E-15   1,772887E-14   6,331741E-15   6,331741E-15
VXJ     1,600000E-01   8,500000E-02   8,500000E-02   8,500000E-02   8,500000E-02
WXJ    -3,218580E-01  -1,709871E-01  -1,709871E-01  -1,709871E-01  -1,709871E-01
PI(EFF)      3,397808E+03
```

CASE A - 4 PLY P.C., LENGTH = 30 INCHES

```
NX,F:L:BB    10   1.300000E+00   1.500000E+01   1.000000E+00
T     1.000000E-01   2.500000E-01   2.500000E-01   1.000000E-01   2.500000E-01
      1.000000E-01   2.500000E-01   1.000000E-01   2.500000E-01
E     1.000000E+07   1.830000E+02   3.400000E+05   1.830000E+02   3.400000E+05
      1.830000E+02   3.400000E+05   1.830000E+02   3.400000E+05
G     4.000000E+06   7.300000E+01   1.090000E+05   7.300000E+01   1.090000E+05
      7.300000E+01   1.090000E+05   7.300000E+01   1.090000E+05
X     0.             1.500000E+00   3.000000E+00   4.500000E+00   6.000000E+00
      7.500000E+00   9.000000E+00   1.050000E+01   1.200000E+01   1.350000E+01   1.500000E+01
A     0.             0.             0.             0.             0.             2.920000E+02  -2.920000E+02
      0.             0.             0.             0.             0.             0.             0.
A     0.             0.             0.             0.             0.            -2.920000E+02   1.022000E+03
     -7.300000E+02   0.             0.             0.             0.             0.             0.
A     0.             0.             0.             0.             0.             0.            -7.300000E+02
      1.460000E+03  -7.300000E+02   0.             0.             0.             0.             0.
A     0.             0.             0.             0.             0.             0.             0.
     -7.300000E+02   1.460000E+03  -7.300000E+02   0.             0.             0.             0.
A     0.             0.             0.             0.             0.             0.             0.
      0.            -7.300000E+02   7.300000E+02   0.             0.             0.             0.
A     1.000000E-06   0.             0.             0.             0.             0.             0.
      0.             0.             0.             0.             0.             0.             4.765440E-08
A     0.             1.176471E-05   0.             0.             0.             0.             0.
      0.             0.             0.             0.             0.             0.            -[illegible].936188E-07
A     0.             0.             1.176471E-05   0.             0.             0.             0.
      0.             0.             0.             0.             0.             0.             0.
A     0.             0.             0.             1.176471E-05   0.             0.             0.
      0.             0.             0.             0.             0.             0.             0.
A     0.             0.             0.             0.             1.176471E-05   0.            -1.150259E-06
      0.             0.             0.             0.             0.             0.             0.
A     0.             0.             0.             0.             0.             0.             0.
      0.             0.            -2.555000E+02   1.000000E+00   0.             1.241000E+02   1.314000E+02
A     0.             0.             0.             0.             0.             0.             0.
      0.             0.             0.             0.             1.000000E+00   0.             1.232707E-01
A     0.             0.             0.             0.             0.             0.             0.
      0.             0.             0.             0.             0.             1.000000E+00   0.
EIGENVALUES    0.            -3.556915E-01   3.556915E-01  -1.707086E-01   1.707086E-01  -1.345864E-01   1.345864E-01
      6.896431E-02  -6.896431E-02  -4.213817E-09   4.213819E-09  -2.155976E-16  -5.682813E-19  -5.682813E-19
PIVOT     1.086820E+03
PIVOT     9.682427E+02
PIVOT     9.226709E+02
PIVOT     1.627621E+02
PIVOT    -3.626339E-03
PIVOT    -6.901643E-05
PIVOT     1.355524E-04
PIVOT     6.307198E-04
PIVOT     1.290846E-03
PIVOT    -6.800228E-05
PIVOT     1.340624E-04
PIVOT     3.092951E-04
PIVOT     2.576271E+00
PIVOT    -3.680610E+01
PIVOT     1.147004E-01
PIVOT    -1.176701E+00
PIVOT     1.765186E+01
PIVOT     1.089251E+02

X(    0)

U    -1.301043E-17   3.469447E-18   2.891416E-17   1.897075E-17  -2.081668E-17
```

CASE A

```
W     -9,989579E-01   -8,097114E-01   -7,902772E-02    2,148342E-01    1,672663E+00
VEE     -8,951115E-17 THETA     -8,163336E-17 BETA     5,554590E-04 PHI     -1,020000E-04
QXJ     -9,978823E-15   -2,456715E-14   -3,355823E-15    1,373988E-14
VXJ      1,600000E-01    8,500000E-02    8,500000E-02    8,500000E-02    8,500000E-02
MXJ      4,628825E-01    2,459063E-01    2,459063E-01    2,459063E-01    2,459063E-01
X(    1)
U     -1,462524E-06   -1,392023E-05   -1,370042E-06    3,725642E-06    2,877079E-05
W     -9,302907E-01   -7,498692E-01   -7,487957E-02    2,037555E-01    1,551288E+00
VEE      5,294404E-04 THETA      6,498113E-04 BETA     3,299932E-04 PHI     -1,154921E-04
QXJ      8,827923E-02    1,568651E-01    1,623069E-01    1,477438E-01
VXJ      1,004574E-01    8,127238E-02    9,102584E-02    8,988567E-02    6,959731E-02
MXJ      2,949943E-01    1,460907E-01    1,460907E-01    1,460907E-01    1,460907E-01
X(    2)
U     -2,741618E-06   -2,599116E-05   -2,598006E-06    7,070531E-06    5,377293E-05
W     -7,629132E-01   -6,083805E-01   -6,314398E-02    1,722098E-01    1,261828E+00
VEE      1,817719E-03 THETA      1,033795E-03 BETA     1,929854E-04 PHI     -7,128126E-05
QXJ      1,350828E-01    2,470575E-01    2,570765E-01    2,300418E-01
VXJ      6,615819E-02    7,932439E-02    9,457396E-02    9,244659E-02    6,031203E-02
MXJ      1,608212E-01    8,543627E-02    8,543627E-02    8,543627E-02    8,543627E-02
X(    3)
U     -3,720680E-06   -3,513893E-05   -3,563511E-06    9,706826E-06    7,276832E-05
W     -5,555129E-01   -4,232616E-01   -4,544356E-02    1,242404E-01    8,799777E-01
VEE      3,549556E-03 THETA      1,253797E-03 BETA     1,065091E-04 PHI     -4,653331E-05
QXJ      1,647703E-01    2,972951E-01    3,106578E-01    2,743103E-01
VXJ      4,701628E-02    7,835886E-02    9,659480E-02    9,372169E-02    5,488947E-02
MXJ      8,875758E-02    4,715247E-02    4,715247E-02    4,715247E-02    4,715247E-02
X(    4)
U     -4,331640E-06   -4,060222E-05   -4,178251E-06    1,138899E-05    8,459146E-05
W     -2,752593E-01   -2,161712E-01   -2,372479E-02    6,497938E-02    4,501560E-01
VEE      4,926080E-03 THETA      1,366828E-03 BETA     4,730022E-05 PHI     -5,407086E-05
QXJ      1,802728E-01    3,220891E-01    3,378605E-01    2,958157E-01
VXJ      3,740602E-02    7,792840E-02    9,762716E-02    9,420298E-02    5,206041E-02
MXJ      3,941685E-02    2,094020E-02    2,094020E-02    2,094020E-02    2,094020E-02
```

CASE A

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X(    5)
U    -8.938969E-06   -4.271581E-05   -4.389059E-06    1.196655E-05    8.853795E-05
W     1.115552E-12    2.030989E-14   -2.131628E-14    2.664535E-14    1.421085E-13
VEE     7.411232E-03 THETA     1.401596E-03 BETA     3.035766E-18 PHI    -3.028247E-05
QXJ     1.850858E-01    3.301294E-01    3.461683E-01    3.022108E-01
VXJ     5.488968E-02    7.780819E-02    9.794351E-02    9.445369E-02    5.118265E-02
WXJ     2.529805E-15    1.343959E-15    1.343959E-15    1.343959E-15    1.343959E-15
X(    6)
U    -4.331640E-06   -4.080222E-05   -4.178251E-06    1.138899E-05    8.455196E-05
W     2.752303E-01    2.161712E-01    2.372479E-02   -6.497938E-02   -4.501560E-01
VEE     9.696383E-03 THETA     1.366828E-03 BETA    -4.730022E-05 PHI    -5.407086E-05
QXJ     1.802728E-01    3.224891E-01    3.378605E-01    2.958157E-01
VXJ     3.740602E-02    7.792869E-02    9.762716E-02    9.429298E-02    5.206081E-02
WXJ    -5.041685E-08   -2.094020E-02   -2.094020E-02   -2.094020E-02   -2.094020E-02
X(    7)
U    -3.720680E-06   -3.513893E-05   -3.563511E-06    9.706826E-06    7.276832E-05
W     5.355129E-01    4.232610E-01    4.544356E-02   -1.242404E-01   -8.799777E-01
VEE     1.167291E-02 THETA     1.253797E-03 BETA    -1.065091E-04 PHI    -8.653331E-05
QXJ     1.687703E-01    2.972951E-01    3.106578E-01    2.743103E-01
VXJ     4.701628E-02    7.835886E-02    9.659880E-02    9.372169E-02    5.488947E-02
WXJ    -8.875758E-02   -4.715247E-02   -4.715247E-02   -4.715247E-02   -4.715247E-02
X(    8)
U    -2.741615E-06   -2.599116E-05   -2.598006E-06    7.070531E-06    5.377203E-05
W     7.625132E-01    6.083805E-01    6.314398E-02   -1.722098E-01   -1.261078E+00
VEE     1.540474E-02 THETA     1.033795E-03 BETA    -1.929850E-04 PHI    -7.128126E-05
QXJ     1.350828E-01    2.470575E-01    2.570765E-01    2.300018E-01
VXJ     6.615519E-02    7.932455E-02    9.457356E-02    9.240659E-02    6.031203E-02
WXJ    -1.408212E-01   -8.543627E-02   -8.543627E-02   -8.543627E-02   -8.543627E-02
X(    9)
U    -1.469528E-06   -1.392023E-05   -1.370042E-06    3.725642E-06    2.877079E-05
W     9.302907E-01    7.498692E-01    7.487957E-02   -2.037555E-01   -1.551284E+00
VEE     1.869302E-02 THETA     6.498113E-04 BETA    -3.299932E-04 PHI    -1.154921E-04
QXJ     8.427923E-02    1.568691E-01    1.623069E-01    1.477418E-01
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CASE A

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VXJ      1,004874E-01    8,127238E-02    9,102584E-02    8,988567E-02    6,959731E-02
MXJ     -2,989943E-01   -1,460907E-01   -1,460907E-01   -1,460907E-01   -1,460907E-01
V(   10)
U      2,818926E-18    2,081668E-17    1,734723E-17    1,778092E-17    8,673617E-18
W      9,087579E-01    8,097114E-01    7,902772E-02   -2,148342E-01   -1,672663E+00
VRE      1,522246E-02 THETA     2,775558E-17 BETA    -5,554590E-08 PHI     -1,920000E-04
QXJ     -1,810878E-15    9,628246E-15    6,774963E-15    1,373988E-14
VXJ      1,600000E-01    8,500000E-02    8,500000E-02    8,500000E-02    8,500000E-02
MXJ     -8,628825E-01   -2,459063E-01   -2,459063E-01   -2,459063E-01   -2,459063E-01
EI(EPF)      9,237992E+03
```

CASE B - 2 PLY GLASS, LENGTH = 20 INCHES

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YY.F:L,AA     10   1.000000E+00    1.000000E+01    1.000000E+00
T    1.100000E-01    1.880000E-01    5.000000E-01    1.200000E-01    5.000000E-01
     8.000000E-02    1.100000E-01    2.100000E-02    2.100000E-02
P    1.000000E+07    1.100000E+03    1.000000E+07    1.100000E+03    1.000000E+07
     1.100000E+03    1.000000E+07    1.000000E+00    1.000000E+00
B    8.000000E+06    8.000000E+02    4.000000E+06    4.400000E+02    4.000000E+06
     8.800000E+02    4.000000E+06    1.000000E+00    1.000000E+00
V    0.              1.000000E+00    2.000000E+00    3.000000E+00    4.000000E+00
     9.000000E+00    6.000000E+00    7.000000E+00    8.000000E+00    9.000000E+00    1.000000E+01
A    0.              0.              0.              0.              0.              2.340426E+03    -2.340426E+03
     0.              0.              0.              0.              0.              0.              0.
A    0.              0.              0.              0.              0.             -2.340426E+03     6.007092E+03
    -4.666667E+03    0.              0.              0.              0.              0.              0.
A    0.              0.              0.              0.              0.              0.             -3.666667E+03
     4.166667E+03   -5.300000E+03    0.              0.              0.              0.              0.
A    0.              0.              0.              0.              0.              0.              0.
    -4.900000E+03    5.547619E+03   -4.761905E+01    0.              0.              0.              0.
A    0.              0.              0.              0.              0.              0.              0.
     0.             -4.761905E+01    4.761905E+01    0.              0.              0.              0.
A    9.090909E-07    0.              0.              0.              0.              0.              0.
     0.              0.              0.              0.              0.              0.              4.981087E-09
A    0.              2.000000E-07    0.              0.              0.              0.              0.
     0.              0.              0.              0.              0.              0.              1.063400E-09
A    0.              0.              2.000000E-07    0.              0.              0.              0.
     0.              0.              0.              0.              0.              0.             -1.850238E-10
A    0.              0.              0.              9.090909E-07    0.              0.              0.
     0.              0.              0.              0.              0.              0.             -9.124864E-09
A    0.              0.              0.              0.              4.761905E+01    0.              0.
     0.              0.              0.              0.              0.              0.             -9.315772E-08
A    0.              0.              0.              0.              0.              0.              0.
     0.              0.              0.              0.              0.              1.153830E+03    0.
A    0.              0.              0.              0.              0.              0.              1.119508E+03
    -1.955333E+02   -2.115381E+03   -4.119088E+00    1.000000E+00    0.              0.              0.
A    0.              0.              0.              0.              0.              0.              0.
     0.              0.              0.              0.              1.000000E+00    0.              1.326942E-02
A    0.              0.              0.              0.              0.              0.              0.
     0.              0.              0.              0.              0.              1.000000E+00    0.
EIGENVALUES    0.             -4.761905E+01    4.761905E+01   -1.249802E-01    1.249802E-01   -7.101305E-02    7.101305E-02
    -5.260446E-02    5.260446E-02   -1.017225E-08    1.017225E-08   -8.669103E-22    3.526315E-22    3.526315E-22
PIVOT        1.664539E+04
PIVOT        1.070239E+04
PIVOT        2.986001E+03
PIVOT        8.685967E+01
PIVOT        9.999982E-01
PIVOT        3.113886E-04
PIVOT        9.897111E-06
PIVOT        1.006918E-05
PIVOT        9.999982E-01
PIVOT        8.204007E-05
PIVOT        7.338110E-06
PIVOT        6.581005E-06
PIVOT        1.259427E+00
PIVOT        1.286075E+01
PIVOT        8.009406E-02
PIVOT        1.598119E+00
PIVOT       -1.508119E+01
PIVOT        1.105003E+01

Y(    0)

U     7.868560E-18   -2.859984E-18    0.              9.797820E-19    1.317883E-18
```

CASE B

W	-9,830180E-02	-9,880324E-02	2,139297E-02	1,768817E-01	2,278349E-08
VEE	-1,734723E-17 THETA	6.071532E-18 BETA	1,073388E-05 PHI	-2,378718E-06	
QXJ	1,999955E-18	6,249341E-15	7,809669E-15	-8,090073E-19	
VXJ	2,633054E-03	2,473660E-01	2,473660E-01	2,633954E-03	1,832686E-12
MXJ	1,190566E-02	1,118112E+00	1,118112E+00	1,190566E-02	8,283868E-12
X(1)					
U	-8,860079E-08	-1,037944E-08	4,192286E-09	1,976393E-07	9,852286E-07
W	-9,373090E-02	-9,319360E-02	2,019494E-02	1,667295E-01	1,848533E-08
VEE	4,076734E-06 THETA	9,568588E-06 BETA	8,424092E-06 PHI	-2,249938E-06	
QXJ	1,087850E-02	2,166616E-02	1,941753E-02	4,857693E-09	
VXJ	3,093873E-03	2,425087E-01	2,446394E-01	3,963920E-03	4,894217E-11
MXJ	9,343722E-03	8,775096E-01	8,775096E-01	9,343722E-03	6,501293E-12
X(2)					
U	-1,679194E-07	-3,604349E-08	7,942098E-09	2,979803E-07	1,798417E-06
W	-7,844436E-02	-7,807088E-02	1,700066E-02	1,395646E-01	1,597049E-08
VEE	1,838670E-05 THETA	1.688477E-05 BETA	6,224123E-06 PHI	-2,154807E-06	
QXJ	1,917589E-02	3,822123E-02	3,415829E-02	4,960304E-09	
VXJ	3,444712E-03	2,388083E-01	2,425539E-01	4,268746E-03	4,960916E-11
MXJ	6,903590E-03	6,483461E-01	6,483461E-01	6,903590E-03	4,803467E-12
X(3)					
U	-2,291855E-07	-5,013893E-08	1,088740E-08	4,076015E-07	2,314292E-06
W	-5,610795E-02	-5,591200E-02	1,223053E-02	9,987982E-02	9,363695E-09
VEE	3,802754E-05 THETA	2.204380E-05 BETA	4,105084E-06 PHI	-2,087873E-06	
QXJ	2,501874E-02	4,988914E-02	4,449581E-02	4,642071E-09	
VXJ	3,691665E-03	2,362130E-01	2,410830E-01	4,763069E-03	5,039307E-11
MXJ	4,553222E-03	4,276129E-01	4,276129E-01	4,553222E-03	3,168098E-12
X(4)					
U	-2,682895E-07	-5,869899E-08	1,276269E-08	4,770908E-07	2,699240E-06
W	-2,929262E-02	-2,911017E-02	6,387011E-03	5,197577E-02	4,696682E-09
VEE	6,177789E-05 THETA	2.511269E-05 BETA	2,039323E-06 PHI	-2,048116E-06	
QXJ	2,848824E-02	5,682749E-02	5,062231E-02	4,698814E-09	
VXJ	3,838390E-03	2,346734E-01	2,402078E-01	5,035929E-03	5,079218E-11
MXJ	2,261949E-03	2,124295E-01	2,124295E-01	2,261949E-03	1,573884E-12

CASE B

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X(    5)
U    -2.816733E-07   -6.162920E-08    1.340594E-08    5.008698E-07    2.761124E-06
W    -4.618928E-14    6.110668E-13   -2.010836E-12    1.527667E-13    2.646978E-21
VEE     8.954976E-05 THETA     2.613125E-05 BETA    -1.807354E-19 PHI    -2.034930E-06
QXJ     2.463602E-02    5.912992E-02    5.265187E-02    4.701608E-09
VXJ     1.887098E-03    2.341634E-01    2.399173E-01    5.152930E-03    5.093735E-11
MXJ    -2.104482E-16   -1.976410E-14   -1.976410E-14   -2.104482E-16   -1.464283E-25
X(    6)
U    -2.682895E-07   -5.869899E-08    1.276269E-08    4.770908E-07    2.649240E-06
W     2.929262E-02    2.911017E-02   -6.387011E-03   -5.197577E-02   -4.696682E-09
VEE     1.135616E-04 THETA     2.511269E-05 BETA    -2.039323E-06 PHI    -2.048116E-06
QXJ     2.488924E-02    5.682789E-02    5.062231E-02    4.686814E-09
VXJ     1.838390E-03    2.346736E-01    2.402078E-01    5.055929E-03    5.079218E-11
MXJ    -2.261949E-03   -2.124295E-01   -2.124295E-01   -2.261949E-03   -1.573888E-12
X(    7)
U    -2.291855E-07   -5.013893E-08    1.088740E-08    4.076015E-07    2.314292E-06
W     5.610798E-02    5.591240E-02   -1.223053E-02   -9.987982E-02   -9.363655E-09
VEE     1.371120E-04 THETA     2.204380E-05 BETA    -4.105084E-06 PHI    -2.087873E-06
QXJ     2.501974E-02    4.989914E-02    4.409581E-02    4.642071E-09
VXJ     1.691665E-03    2.362130E-01    2.410830E-01    4.763069E-03    5.035307E-11
MXJ    -4.553222E-03   -4.276129E-01   -4.276129E-01   -4.553222E-03   -3.168098E-12
X(    8)
U    -1.675194E-07   -3.664349E-08    7.942098E-09    2.979803E-07    1.758417E-06
W     7.889836E-02    7.807088E-02   -1.700066E-02   -1.395646E-01   -1.397049E-08
VEE     1.567528E-04 THETA     1.688477E-05 BETA    -6.224123E-06 PHI    -2.154807E-06
QXJ     1.917585E-02    3.822123E-02    3.415829E-02    4.566308E-09
VXJ     5.444712E-03    2.388085E-01    2.425539E-01    4.208746E-03    4.960916E-11
MXJ    -6.903590E-03   -6.483461E-01   -6.483461E-01   -6.903590E-03   -4.803467E-12
X(    9)
U    -8.868679E-08   -1.937944E-08    4.192286E-09    1.576393E-07    9.852286E-07
W     9.573090E-02    9.319360E-02   -2.019094E-02   -1.667295E-01   -1.888533E-08
VEE     1.701628E-04 THETA     9.568988E-06 BETA    -8.424092E-06 PHI    -2.249938E-06
QXJ     [illegible]    [illegible]    1.081793E-02    4.497693E-09
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CASE B

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VXJ     9,093873E-03   2,425047E-01   2,446394E-01   3,503520E-03   4,850217E-11
MXJ    -9,583922E-03  -8,775096E-01  -8,775096E-01  -9,583722E-03  -6,901293E-12
V(   10)
U       1,192622E-18  -1,130412E-18  -3,794708E-18   6,336809E-19  -8,673617E-19
W       9,059140E-02   9,880324E-02  -2,139297E-02  -1,768417E-01  -2,278349E-08
VPE     1,751395E-04 THETA    2,602085E-18 BETA   -1,073388E-05 PHI    -2,378714E-06
QXJ     8,057976E-15   1,565516E-14  -1,774622E-14   7,267252E-17
VXJ     2,633954E-03   2,473660E-01   2,473660E-01   2,633954E-03   1,837686E-12
MXJ    -1,190566E-02  -1,118112E+00  -1,118112E+00  -1,190566E-02  -8,283868E-12
PI(EPP)    2,370056E+09
```

CASE C - 1 PLY P.C., LENGTH = 10 INCHES

```
NY.P.L:89    10    1.000000E+00    5.000000E+00    1.000000E+00
T    1.050000E-01    2.500000E-01    8.700000E-01    1.200000E-01    1.500000E-01
     3.879000E-02    3.875000E-02    3.875000E-02    3.875000E-02
E    1.000000E+07    1.830000E+02    3.400000E+05    1.830000E+02    3.400000E+05
     1.000000E+00    1.000000E+00    1.000000E+00    1.000000E+00
G    4.000000E+06    7.300000E+01    1.090000E+05    7.300000E+01    1.090000E+05
     1.000000E+00    1.000000E+00    1.000000E+00    1.000000E+00
X    0.              5.000000E-01    1.000000E+00    1.500000E+00    2.000000E+00
     2.500000E+00    3.000000E+00    3.500000E+00    4.000000E+00    4.500000E+00    5.000000E+00
A    0.    0.    0.    0.    0.    2.920000E+02    -2.920000E+02
   0.    0.    0.    0.    0.    0.    0.
A    0.    0.    0.    0.    0.    -2.920000E+02    9.003333E+02
   -6.083333E+02    0.    0.    0.    0.    0.    0.
A    0.    0.    0.    0.    0.    0.    -6.003333E+02
   6.341398E+02    -2.580645E+01    0.    0.    0.    0.    0.
A    0.    0.    0.    0.    0.    0.    0.
   -3.580645E+01    5.161290E+01    -2.580645E+01    0.    0.    0.    0.
A    0.    0.    0.    0.    0.    0.    0.
   0.    -2.580645E+01    2.580645E+01    0.    0.    0.    0.
A    9.523810E-07    0.    0.    0.    0.    0.    0.
   0.    0.    0.    0.    0.    0.    1.080148E-08
A    0.    3.380663E-06    0.    0.    0.    0.    0.
   0.    0.    0.    0.    0.    0.    2.870670E-08
A    0.    0.    1.960784E-05    0.    0.    0.    0.
   0.    0.    0.    0.    0.    0.    -3.776946E-07
A    0.    0.    0.    2.580645E+01    0.    0.    0.
   0.    0.    0.    0.    0.    0.    -1.878737E-03
A    0.    0.    0.    0.    2.580645E+01    0.    0.
   0.    0.    0.    0.    0.    0.    -2.617506E-03
A    0.    0.    0.    0.    0.    0.    0.
   0.    0.    0.    0.    0.    0.    0.
A    0.    0.    0.    0.    0.    2.153500E+02    1.679000E+02
   -4.798145E+02    -1.835484E+00    -2.000000E+00    1.000000E+00    0.    0.    0.
A    0.    0.    0.    0.    0.    0.    0.
   0.    0.    0.    0.    1.000000E+00    0.    2.033079E-02
A    0.    0.    0.    0.    0.    0.    0.
   0.    0.    0.    0.    0.    1.000000E+00    0.
EIGENVALUES    0.    -4.175572E+01    4.175572E+01    -1.594928E+01    1.594928E+01    -1.725351E-01    1.725351E-01
   -7.603471E-02    7.603471E-02    -9.044909E-16    -1.110223E-16    -1.110223E-16    4.614621E-16    4.614621E-16
PIVOT    9.825939E+02
PIVOT    2.662955E+02
PIVOT    4.911202E+01
PIVOT    1.224617E+01
PIVOT    -6.180340E-01
PIVOT    1.381919E+00
PIVOT    -1.835831E-03
PIVOT    2.526380E-04
PIVOT    -6.180340E-01
PIVOT    1.381919E+00
PIVOT    2.762031E-04
PIVOT    1.846959E-04
PIVOT    1.321014E+00
PIVOT    5.015604E+00
PIVOT    -5.220264E-02
PIVOT    -1.001036E+00
PIVOT    5.005180E+00
PIVOT    1.098698E+01

Y(    0)

U    2.609088E-18    0.    1.734723E-17    -1.040834E-17    3.469447E-18
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CASE C

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W     -2.712223E-02   -1.060120E-02   8.672257E-02   3.414436E-07   9.193776E-07
VEE    -1.110223E-16 THETA   6.938894E-18 BETA   6.088689E-05 PHI   -2.535767E-05
QXJ    2.254100E-15   -7.893570E-15   7.401114E-16   -3.442587E-16
VXJ    2.446223E-02   4.731129E-01   2.424827E-03   1.229542E-10   1.229542E-10
MXJ    9.873682E-02   1.136002E+00   5.822309E-03   2.952281E-10   2.952281E-10
X(    1)
U     -1.266844E-08   -3.249249E-08   4.492694E-07   4.069884E-06   6.177317E-06
W     -2.559488E-02   -1.848696E-02   8.408113E-02   2.828844E-07   4.280738E-07
VEE    7.085613E-06 THETA   2.729776E-05 BETA   4.834958E-05 PHI   -2.481268E-05
QXJ    9.884362E-03   1.016880E-02   3.458214E-07   2.101525E-07
VXJ    2.424581E-02   4.699278E-01   3.135398E-03   1.089231E-08   4.192016E-09
MXJ    8.664228E-02   9.020864E-01   4.623429E-03   2.344372E-10   2.344372E-10
X(    2)
U     -2.396236E-08   -6.143498E-08   8.496516E-07   7.277889E-06   1.101985E-05
W     -2.146067E-02   -1.548493E-02   3.698506E-02   2.140347E-07   3.223040E-07
VEE    2.626364E-05 THETA   6.838976E-05 BETA   3.605338E-05 PHI   -2.439267E-05
QXJ    1.043168E-02   1.799113E-02   3.522303E-07   2.127401E-07
VXJ    2.407497E-02   4.674722E-01   3.681910E-03   1.106458E-08   4.240114E-09
MXJ    3.478025E-02   6.726691E-01   3.447605E-03   1.748155E-10   1.748155E-10
X(    3)
U     -3.279863E-08   -8.405648E-08   1.162775E-06   9.586555E-06   1.449027E-05
W     -1.938232E-02   -1.108848E-02   2.647044E-02   1.435807E-07   2.154610E-07
VEE    5.445875E-05 THETA   6.338104E-05 BETA   2.393657E-05 PHI   -2.409473E-05
QXJ    1.366407E-02   2.353224E-02   3.566827E-07   2.145189E-07
VXJ    2.396124E-02   4.657294E-01   4.069007E-03   1.118386E-08   4.273135E-09
MXJ    2.309131E-02   4.465986E-01   2.288935E-03   1.160635E-10   1.160635E-10
X(    4)
U     -3.880732E-08   -9.840355E-08   1.361459E-06   1.097861E-05   1.657709E-05
W     -8.013099E-03   -5.772583E-03   1.378550E-02   7.205569E-08   1.079069E-07
VEE    8.864049E-05 THETA   7.238614E-05 BETA   1.193866E-05 PHI   -2.391678E-05
QXJ    1.559726E-02   2.683858E-02   3.593370E-07   2.155789E-07
VXJ    2.389108E-02   4.646890E-01   4.299962E-03   1.125496E-08   4.292810E-09
MXJ    1.191708E-02   2.227444E-01   1.141635E-03   5.788813E-11   5.788813E-11
```

CASE C

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X(   5)
U    -4.932797E-08   -1.033144E-07    1.429484E-06    1.144377E-05    1.727345E-05
W     2.259526E-12   -1.563194E-13   -3.037570E-13   -2.422514E-19   -2.490277E-19
VZE     1.258081E-04 THETA     7.932957E-05 BETA    -2.574980E-18 PHI    -2.385760E-05
QXJ     1.628062E-02    2.793761E-02    3.602190E-07    2.159311E-07
VXJ     2.386776E-02    4.643428E-01    4.376731E-03    1.127859E-08    4.299345E-09
MXJ    -2.484051E-18   -4.804291E-14   -2.462325E-16   -1.248555E-23   -1.248555E-23
X(   6)
U    -5.880732E-08   -9.840355E-08    1.361459E-06    1.097861E-05    1.657709E-05
W     8.013099E-03    5.772583E-03   -1.378550E-02   -7.205569E-08   -1.079069E-07
VZE     1.629757E-04 THETA     7.234614E-05 BETA    -1.193866E-05 PHI    -2.391678E-05
QXJ     1.559726E-02    2.683858E-02    3.593370E-07    2.155789E-07
VXJ     2.389108E-02    4.646890E-01    4.299962E-03    1.125496E-08    4.292810E-09
MXJ    -1.151708E-02   -2.227466E-01   -1.141635E-03   -5.788813E-11   -5.788813E-11
X(   7)
U    -5.370863E-08   -8.405648E-08    1.162775E-06    9.586555E-06    1.449027E-05
W     1.438232E-02    1.108848E-02   -2.647044E-02   -1.435807E-07   -2.154610E-07
VZE     1.071575E-04 THETA     6.338104E-05 BETA    -2.393657E-05 PHI    -2.409473E-05
QXJ     1.466407E-02    2.553224E-02    3.566827E-07    2.145189E-07
VXJ     2.396128E-02    4.657298E-01    4.069007E-03    1.118386E-08    4.273135E-09
MXJ    -2.309131E-02   -4.465986E-01   -2.288935E-03   -1.160635E-10   -1.160635E-10
X(   8)
U    -2.396230E-08   -6.143498E-08    8.496516E-07    7.277889E-06    1.101985E-05
W     2.186067E-02    1.548493E-02   -3.694506E-02   -2.140397E-07   -3.223060E-07
VZE     2.253526E-04 THETA     4.838976E-05 BETA    -3.605338E-05 PHI    -2.439267E-05
QXJ     1.043168E-02    1.799113E-02    3.522303E-07    2.127401E-07
VXJ     2.407897E-02    4.674722E-01    3.681910E-03    1.106458E-08    4.240114E-09
MXJ    -3.478025E-02   -6.726601E-01   -3.447605E-03   -1.748155E-10   -1.748155E-10
X(   9)
U    -1.266888E-08   -3.249209E-08    4.492694E-07    4.069884E-06    6.177317E-06
W     2.594488E-02    1.848696E-02   -4.408113E-02   -2.828844E-07   -4.280738E-07
VZE     2.445306E-04 THETA     2.729776E-05 BETA    -4.834958E-05 PHI    -2.481268E-05
QXJ     5.883402E-03    1.016880E-02    3.458214E-07    2.101525E-07
```

CASE C

VXJ	2,428541E-02	4,69927AE-01	3,135398E-03	1,089231E-08	4,192016E-09
MXJ	-8,66422AE-02	-9,020866E-01	-8,623429E-03	-2,344372E-10	-2,344372E-10
X(10)					
U	1,127570E-17	-6,938894E-18	-2,081668E-17	1,040834E-17	-2,081668E-17
W	2,712223E-02	1,960120E-02	-4,672257E-02	-3,014436E-07	-5,193776E-07
VEE	2,936162E-04 THETA	-6,938894E-18 BETA	-6,088689E-05 PHI	-2,535767E-05	
QXJ	3,824371E-15	5,782990E-15	-8,296455E-16	[illegible],919292E-16	
VXJ	2,846223E-02	8,731129E-01	2,424827E-03	1,229542E-10	1,229542E-10
MXJ	-9,873682E-02	-1,136002E+00	-5,822309E-0[illegible]	-2,932281E-10	-2,932281E-10
PI(EPF)	2,069951E+04				

CASE D - 2 PLY P.C., LENGTH = 10 INCHES

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NX,F,L,BR     10    1.000000E+00    5.000000E+00    1.090000E+00
T       1.000000E-01    2.500000E-01    5.000000E-01    1.000000E-01    5.000000E-01
        1.000000E-01    8.000000E-02    1.000000E-02    1.000000E-02
P       1.000000E+07    1.100000E+03    3.400000E+05    1.100000E+03    3.400000E+05
        1.100000E+03    4.900000E+05    1.000000E+00    1.000000E+00
R       8.000000E+06    4.400000E+02    1.090000E+05    4.400000E+02    1.090000E+05
        8.400000E+02    1.810000E+05    1.000000E+00    1.000000E+00
Y       0.              5.000000E-01    1.000000E+00    1.900000E+00    2.000000E+00
        2.500000E+00    3.000000E+00    3.500000E+00    4.000000E+00    4.500000E+00
                                                        5.000000E+00    1.760000E+03    -1.760000E+03
A       0.          0.          0.          0.          0.          0.                  0.
     0.          0.          0.          0.          0.          -1.760000E+03       6.160000E+03
A       0.          0.          0.          0.          0.          0.                  0.
     -8.400000E+03  0.          0.          0.          0.          0.              -8.400000E+03
A       0.          0.          0.          0.          0.          0.                  0.
     8.400000E+03   -8.400000E+03   0.      0.          0.          0.                  0.
A       0.          0.          0.          0.          0.          0.                  0.
     -8.400000E+03  4.500000E+03    -1.000000E+02   0.  0.          0.                  0.
A       0.          0.          0.          0.          0.          0.                  0.
     0.          -1.000000E+02  1.000000E+02    0.      0.          0.                  0.
A       1.000000E-06    0.      0.          0.          0.          0.                  1.219516E-07
     0.          0.          0.          0.          0.          0.                  0.
A       0.          5.882353E-06    0.      0.          0.          0.                  1.239081E-06
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          5.882353E-06    0.      0.          0.                  -6.807551E-07
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          0.          2.551020E-05    0.      0.                  -5.497298E-06
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          0.          0.          1.000000E+02    0.              -6.929070E-02
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          0.          0.          0.          9.680000E+02        1.672000E+03
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          0.          0.          0.          0.                  3.509816E-01
     -8.740000E+02  -1.710500E+03   -5.900000E+00   1.000000E+00    0.      0.          0.
A       0.          0.          0.          0.          1.000000E+00    0.              0.
     0.          0.          0.          0.          0.          0.                  0.
A       0.          0.          0.          0.          0.          1.000000E+00        0.
     0.          0.          0.          0.          0.          0.                  0.
EIGENVALUES   0.            -1.000000E+02   1.000000E+02    -6.255053E-01   6.255053E-01   -3.641687E-01   3.641687E-01
     -1.709073E-01  1.709073E-01    -9.564202E-08   9.564202E-08    0.      0.          2.338432E-31
PIVOT       6.459200E+03
PIVOT       6.006540E+03
PIVOT       1.311273E+03
PIVOT       9.237383E+01
PIVOT       9.999808E-01
PIVOT       -8.228884E-04
PIVOT       8.503510E-05
PIVOT       -9.538385E-05
PIVOT       9.909808E-01
PIVOT       2.670110E-04
PIVOT       8.238246E-05
PIVOT       -5.423669E-05
PIVOT       1.787049E+00
PIVOT       -7.140561E+00
PIVOT       -1.887467E-01
PIVOT       -1.139524E+00
PIVOT       5.697620E+00
PIVOT       3.636567E+01

Y(    0)

U     -1.730723E-18    1.301043E-18    -2.873136E-18    3.469447E-18    1.730723E-18
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CASE D

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W      -1.820321E-01   -2.681195E-01   1.959916E-01   2.501598E-01   1.185814E-07
VEE     0.             THETA   1.734723E-18 BETA   9.835109E-05 PHI   -6.299154E-05
QXJ    -1.663736E-15    2.298609E-14  -2.493098E-14   1.830133E-16
VXJ     5.249295E-02    2.230951E-01   2.230951E-01   1.316943E-03   5.249296E-12
WXJ     8.195924E-02    3.483268E-01   3.483268E-01   2.056194E-03   8.195924E-12
X(   1)
U      -8.913285E-08   -7.717303E-07   5.687649E-07   3.171349E-06   5.476809E-06
W      -1.710214E-01   -2.510146E-01   1.843579E-01   2.380781E-01   9.950982E-08
VEE     1.106103E-05 THETA   4.192681E-05 BETA   7.050342E-05 PHI   -4.920238E-05
QXJ     4.178653E-02    1.048062E-01   6.047744E-02   4.248814E-08
VXJ     8.309131E-02    2.109066E-01   2.155793E-01   3.447757E-03   2.165409E-10
WXJ     5.875285E-02    2.496996E-01   2.496996E-01   1.473992E-03   5.875285E-12
X(   2)
U      -1.680708E-07   -1.453210E-06   1.069824E-06   5.967510E-06   9.897617E-06
W      -1.422623E-01   -2.080045E-01   1.536823E-01   1.969844E-01   7.689479E-08
VEE     3.086793E-05 THETA   7.146514E-05 BETA   4.844449E-05 PHI   -3.965209E-05
QXJ     7.144010E-02    1.775842E-01   1.010668E-01   4.759021E-08
VXJ     3.661941E-02    2.026906E-01   2.100972E-01   4.871666E-03   2.412554E-10
WXJ     4.037041E-02    1.715742E-01   1.715742E-01   1.012813E-03   4.037041E-12
X(   3)
U      -2.295542E-07   -1.980459E-06   1.455624E-06   8.126940E-06   1.313262E-05
W      -1.012276E-01   -1.478085E-01   1.095650E-01   1.394710E-01   5.220394E-08
VEE     8.086715E-05 THETA   9.102166E-05 BETA   3.030013E-05 PHI   -3.341961E-05
QXJ     9.119091E-02    2.251784E-01   1.268394E-01   5.073719E-08
VXJ     3.240922E-02    1.974534E-01   2.063656E-01   5.772270E-03   2.560709E-10
WXJ     2.525010E-02    1.073129E-01   1.073129E-01   6.334746E-04   2.525010E-12
X(   4)
U      -2.690153E-07   -2.312163E-06   1.701930E-06   9.485536E-06   1.510219E-05
W      -5.247565E-02   -7.645433E-02   5.687070E-02   7.205925E-02   2.639760E-08
VEE     1.299915E-04 THETA   1.021669E-04 BETA   1.857330E-05 PHI   -2.990078E-05
QXJ     1.024953E-01    2.520587E-01   1.410794E-01   5.243680E-08
VXJ     3.004302E-02    1.955513E-01   2.041973E-01   6.268387E-03   2.644761E-10
WXJ     1.214449E-02    5.161406E-02   5.161406E-02   3.046808E-04   1.214449E-12
```

CASE D

X(	5)				
U	-2,812107E-07	-2,425208E-06	1,786070E-06	9,985510E-06	1,576325E-05
W	9,760963E-13	-1,492140E-13	-5,329071E-14	-2,486900E-14	-5,505714E-21
VPE	1,817831E-04 THETA	1,057867E-04 BETA	1,026303E-19 PHI	-2,876955E-05	
QXJ	1,061789E-01	2,607472E-01	1,456284E-01	5,297292E-08	
VXJ	2,928337E-02	1,936227E-01	2,038860E-01	6,426612E-03	2,672621E-10
MXJ	1,355253E-16	5,759824E-16	5,759824E-16	3,400058E-18	1,555253E-26
X(	6)				
U	-2,680153E-07	-2,312163E-06	1,701930E-06	9,483536E-06	1,510219E-05
W	5,287569E-02	7,645433E-02	-5,687070E-02	-7,205925E-02	-2,639760E-08
VPE	2,340747E-04 THETA	1,021669E-04 BETA	-1,457338E-05 PHI	-2,990478E-05	
QXJ	1,020953E-01	2,520587E-01	1,410794E-01	5,243680E-08	
VXJ	3,008942E-02	1,945513E-01	2,041973E-01	6,268387E-03	2,646761E-10
MXJ	-1,214449E-02	-5,161406E-02	-5,161406E-02	-3,046808E-04	-1,214449E-12
X(	7)				
U	-2,293542E-07	-1,980459E-06	1,455624E-06	8,126940E-06	1,313262E-05
W	1,012276E-01	1,478085E-01	-1,095650E-01	-1,394710E-01	-5,220394E-08
VPE	2,826991E-04 THETA	9,102166E-05 BETA	-3,030013E-05 PHI	-3,341961E-05	
QXJ	9,119091E-02	2,251788E-01	1,260394E-01	5,073719E-08	
VXJ	3,240922E-02	1,974534E-01	2,063656E-01	5,772270E-03	2,568709E-10
MXJ	-2,525010E-02	-1,073129E-01	-1,073129E-01	-6,334746E-04	-2,525010E-12
X(	8)				
U	-1,680708E-07	-1,453210E-06	1,065824E-06	5,967510E-06	9,897617E-06
W	1,422623E-01	2,084045E-01	-1,536823E-01	-1,969844E-01	-7,689479E-08
VPE	3,236083E-04 THETA	7,146514E-05 BETA	-4,844449E-05 PHI	-3,965209E-05	
QXJ	7,144010E-02	1,775842E-01	1,010668E-01	4,759021E-08	
VXJ	3,661341E-02	2,026906E-01	2,100972E-01	4,871666E-03	2,412558E-10
MXJ	-4,037041E-02	-1,715742E-01	-1,715742E-01	-1,012813E-03	-4,037041E-12
X(	9)				
U	-8,913283E-08	-7,717303E-07	5,647649E-07	3,171349E-06	5,476499E-06
W	1,710214E-01	2,518144E-01	-1,843979E-01	-2,380781E-01	-9,950982E-08
VPE	3,529052E-04 THETA	4,192681E-05 BETA	-7,050342E-05 PHI	-4,420238E-05	
QXJ	4,178653E-02	1,048062E-01	6,047744E-02	4,240810E-08	

CASE D

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VXJ     8,309131E-02    2,109066E-01    2,155793E-01    3,447757E-03    2,165409E-10
MXJ    -9,879805E-02   -2,496996E-01   -2,496996E-01   -1,473992E-03   -5,879285E-12
X(   10)
U     2,081556E-10   -2,168404E-18   -4,703385E-18    2,602085E-18   -8,673617E-19
W     1,820321E-01    2,681195E-01   -1,959916E-01   -2,541598E-01   -1,185814E-07
VEE     3,439602E-04 THETA     0.             BETA    -9,835109E-05 PHI    -6,299150E-05
QXJ     9,063030E-15    1,132901E-14   -3,232007E-14    3,469447E-16
VXJ     5,249295E-02    2,230951E-01    2,230951E-01    1,316943E-03    5,249297E-12
MXJ    -8,195924E-02   -3,403268E-01   -3,403268E-01   -2,056194E-03   -8,195924E-12
EI(EFF)     1,432568E+00
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TEST BEAM (Reference 1)

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NV,FIL,9A     10   1.000000E+00   1.735000E+01   1.000000E+00
     1.850000E-01   2.500000E-01   8.700000E-01   1.200000E-01   1.900000E-01
     3.875000E-02   3.875000E-02   3.875000E-02   3.875000E-02
     1.000000E+00   1.830000E+02   3.000000E+05   1.830000E+02   3.400000E+05
     1.000000E+00   1.000000E+00   1.000000E+00   1.000000E+00
     1.000000E+00   7.300000E+01   1.000000E+05   7.300000E+01   1.090000E+05
     1.000000E+00   1.000000E+00   1.000000E+00   1.000000E+00
     0.             1.735000E+00   3.470000E+00   5.205000E+00   6.940000E+00
     8.678000E+00   1.041000E+01   1.214500E+01   1.388000E+01   1.561500E+01   1.735000E+01
                                                                                2.920000E+02   -2.920000E+02
     0.        0.        0.        0.        0.        0.        0.
  0.        0.        0.        0.        0.        0.        -2.920000E+02   0.003333E+02
     0.        0.        0.        0.        0.        0.        0.
 -4.083333E+02   0.        0.        0.        0.        0.        -4.083333E+02
     0.        0.        0.        0.        0.        0.        0.
  4.141309E+02   -2.580645E+01   0.        0.        0.        0.        0.
     0.        0.        0.        0.        0.        0.        0.
 -2.580645E+01   5.161290E+01   -2.580645E+01   0.        0.        0.        0.
     0.        0.        0.        0.        0.        0.        0.
  0.        -2.580645E+01   2.580645E+01   0.        0.        0.        0.
     8.823810E+00   0.        0.        0.        0.        0.        1.003454E-01
  0.        0.        0.        0.        0.        0.        0.
     0.        3.180663E-06   0.        0.        0.        0.        3.094752E+00
  0.        0.        0.        0.        0.        0.        0.
     0.        0.        1.960784E-05   0.        0.        0.        -3.971237E-07
  0.        0.        0.        0.        0.        0.        0.
     0.        0.        0.        2.580645E+01   0.        0.        -1.974382E-03
  0.        0.        0.        0.        0.        0.        0.
     0.        0.        0.        0.        2.580645E+01   0.        -2.742217E-03
  0.        0.        0.        0.        0.        0.        0.
     0.        0.        0.        0.        0.        2.153300E+02   1.670000E+02
  0.        0.        0.        0.        0.        0.        0.
 -1.709105E+02   -1.834484E+00   -2.000000E+00   1.000000E+00   0.        0.        0.
     0.        0.        0.        0.        0.        0.        2.137663E-02
  0.        0.        0.        0.        1.000300E+00   0.        0.
     0.        0.        0.        0.        0.        0.        0.
  0.        0.        0.        0.        0.        1.000000E+00   0.
EIGEN VALUES   0.   -5.273483E+01   5.273483F+01   -8.175572E+01   8.175572E+01   -1.594928E+01   1.594928E+01
 -1.638891E-01   1.638891E-01   1.087672E-15   1.087672E-15   -1.510809E-14   -1.510809E-14   2.701181E-14
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[illegible]   8.220606E+08
[illegible]   7.100203E+02
[illegible]   8.071175E+01
[illegible]   1.267104E+01
[illegible]  -1.805081E-01
[illegible]  -6.180304E-01
[illegible]   1.381092E+00
[illegible]   1.290728F-03
[illegible]  -1.805081E-01
[illegible]  -6.180304E-01
[illegible]   1.381092E+00
[illegible]  -2.090743E-04
[illegible]   1.000001E+00
[illegible]   1.735002E+01
[illegible]   1.627814E-01
[illegible]  -1.314104E+00
[illegible]   9.280315E+01
[illegible]
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DIVDT     2.280315E+01
DIVDP     1.020496E+00

VI    0)
U     0.                   1.961291E-17   4.938894E-18   -5.673617E-19   8.336809E-18
W     -1.500859E-05   -1.230382E+00   1.230393E+00   1.908497E-06   2.873365E-06
VRE   -2.775558E-17 THETA   1.387779E-17 BETA   1.898293E-08 PHI   -2.466210E-05
RVJ   -1.970272E-15   1.059911E-14   2.891287E-16   -1.065456E-16
VVJ   2.472068E-09   4.974590E-01   2.589564E-03   1.292791E-10   1.292791E-10
WVJ   1.831213E-08   3.541064E+00   1.815196E-02   9.204200E-10   9.204200E-10

ARGUMENT TOO SMALL
ERROR NUMBER   115   DETECTED BY EXP

VI    1)
U     -2.218594E-08   -7.089348E-06   8.111853E-05   7.980288E-05   1.023233E-04
W     -1.172440E-05   -1.150237E+00   1.150245E+00   1.630356E-06   2.073054E-06
VRE   2.431597E-08 THETA   2.807155E-08 BETA   1.840908E-08 PHI   -2.390827E-05
RVJ   1.880232E-06   8.209017E-02   8.031249E-07   2.574533E-07
VVJ   1.019182E-07   4.817090E-01   8.842682E-03   1.368961E-08   5.108068E-09
WVJ   1.809316E-08   2.725690E+00   1.396990E-02   7.083626E-10   7.083624E-10

ARGUMENT TOO SMALL
ERROR NUMBER   115   DETECTED BY EXP

VI    2)
U     -9.889901E-08   -1.336383E-05   7.751077E-05   1.852877E-04   1.887490E-04
W     -8.881872E-06   -9.635427E-01   9.635484E-01   1.280970E-06   1.608901E-06
VRE   9.478901E-08 THETA   5.092695E-08 BETA   1.068061E-08 PHI   -2.196234E-05
RVJ   1.750372E-06   1.398955E-01   5.051053E-07   2.855281E-07
VVJ   9.801322E-08   4.706194E-01   1.259235E-02   1.502501E-08   5.635958E-09
WVJ   1.026084E-08   1.985281E+00   1.017508E-02   5.159414E-10   5.159414E-10

VI    3)
U     -5.061170E-04   -1.823122E-05   1.057418E-04   1.937902E-04   2.850873E-04
W     -5.828201E-06   -6.850984E-01   6.851020E-01   8.790868E-07   1.087797E-06
VRE   1.092033E-05 THETA   4.615078E-08 BETA   6.950170E-09 PHI   -2.067818E-05
RVJ   1.888898E-06   1.781219E-01   9.460921E-07   9.480938E-07
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TEST BEAM

VYJ	8.038094E-08	4.032884E-01	1.933649E-02	1.657261E-08	5.992078E-09
WYJ	6.904782E-09	1.296734E+00	6.646100E-03	3.369995E-10	3.369995E-10
VF	8)				
U	-8.798828E-04	-2.129986E-05	1.259400E-04	2.235537E-04	2.817782E-04
W	-2.788726E-06	-3.949659E-01	3.549677E-01	4.468064E-07	5.499088E-07
VPE	9.226420E-03 THETA	7.514420E-04 BETA	3.833372E-05 PHI	-1.098720E-05	
RYJ	1.419519E-06	1.998792E-01	5.694215E-07	3.146610E-07	
VYJ	8.674824E-08	4.591155E-01	1.689844E-02	1.722582E-08	6.193277E-09
WYJ	5.312131E-09	6.405803E-01	3.283162E-03	1.664772E-10	1.664772E-10
VF	9)				
U	-8.084521E-04	-2.238626E-05	1.296126E-04	2.339831E-04	2.981113E-04
W	6.685798E-20	-2.593481E-13	-6.790156E-18	-5.759824E-20	-8.573978E-20
VPE	8.960768E-03 THETA	7.811670E-04 BETA	-1.409463E-18 PHI	-1.971003E-05	
RYJ	1.400756E-06	2.009402E-01	5.769925E-07	3.180904E-07	
VYJ	8.888861E-08	4.577613E-01	1.740533E-02	1.743780E-08	6.258572E-09
WYJ	-1.150691E-22	-2.620717E-18	-1.347709E-16	-6.838197E-24	-6.838197E-24
VF	6)				
U	-8.798828E-04	-2.129986E-05	1.259400E-04	2.235537E-04	2.817782E-04
W	2.788726E-06	3.949659E-01	-3.549677E-01	-4.468064E-07	-5.499088E-07
VPE	9.002915E-03 THETA	7.514420E-04 BETA	-3.833372E-05 PHI	-1.098720E-05	
RYJ	1.415619E-06	1.998792E-01	5.694215E-07	3.146610E-07	
VYJ	8.674824E-09	4.591155E-01	1.689844E-02	1.722582E-08	6.193277E-09
WYJ	-5.312131E-09	-6.405803E-01	-3.283162E-03	-1.664772E-10	-1.664772E-10
VF	7)				
U	-8.061170E-04	-1.423122E-05	1.057418E-04	1.937892E-04	2.450473E-04
W	5.624201E-06	6.850984E-01	-6.851020E-01	-8.790468E-07	-1.087757E-06
VPE	7.137502E-03 THETA	6.615878E-04 BETA	-6.950170E-05 PHI	-2.067818E-05	
RYJ	1.866498E-06	1.781213E-01	5.460921E-07	3.040936E-07	
VYJ	8.038094E-08	4.032884E-01	1.933649E-02	1.657261E-08	5.992078E-09
WYJ	-6.704782E-09	-1.296734E+00	-6.646100E-03	-3.369995E-10	-3.369995E-10
VF	8)				
U	-5.880441E-04	-1.134583E-05	7.791077E-05	1.459877E-04	1.807480E-04

TEST BEAM

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U     -4.880441E-04   -1.136383E-05    7.791077E-05    1.452877E-04    1.887490E-04
W      8.481872E-06    9.635427E-01   -9.635488E-01   -1.280970E-06   -1.600501E-06
VPE      8.162840E-03 THETA     -3.002609E-08 BETA    -1.040061E-08 PHI    -2.196734E-05
NXJ      1.758372E-06    1.398999E-01    5.051053E-07    2.055281E-07
VXJ      9.401322E-08    8.706104E-01    1.259235E-02    1.542501E-08    5.638598E-09
MXJ     -1.026486E-08   -1.985281E+00   -1.017508E-02   -5.159414E-10   -5.159414E-10

VF     4)
U     -2.210454E-04   -7.089348E-06    4.111893E-05    7.980284E-05    1.023243E-04
W      1.172880E-05    1.159237E+00   -1.159205E+00   -1.630356E-06   -2.073094E-06
VPE      8.443370E-03 THETA      2.907153E-08 BETA    -1.860906E-08 PHI    -2.390427E-05
NXJ      1.880232E-06    8.209017E-02    4.631240E-07    2.570533E-07
VXJ      1.010187E-07    4.817090E-01    8.482642E-03    1.308961E-08    5.100848E-09
MXJ     -1.009316E-08   -2.725004E+00   -1.546940E-02   -7.083026E-10   -7.083026E-10

VF     10)
U      1.730723E-18    1.301043E-18   -1.214306E-17   -3.404447E-18   -9.540979E-18
W      1.400890E-05    1.230382E+00   -1.230393E+00   -1.908497E-06   -2.073365E-06
VPE      9.120536E-03 THETA      1.469447E-18 BETA    -1.098245E-08 PHI    -2.406210E-05
NXJ      8.737802E-16    9.508164E-15   -2.119161E-16    1.636236E-16
VXJ      9.972060E-09    4.974508E-01    2.549564E-03    1.292791E-10    1.292791E-10
MXJ     -1.831213E-08   -3.541066E+00   -1.815196E-02   -9.208200E-10   -9.208200E-10
PY(END)      2.383028E+00
```

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