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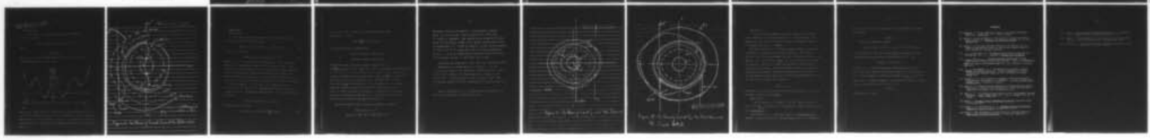

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A THEOREM ON HOMOTOPY PATHS

By

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C. B. Garcia and F. J. Gould

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A THEOREM ON HOMOTOPY PATHS

by

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ABSTRACT

to the nth power   a member of   to the n+1st power

We consider the set of points $x \in R^{n+1}$ satisfying $H(x) = 0$, where $H: R^{n+1} \to R^n$ is a C^2 function and 0 is a regular value. This set, $H^{-1}(0)$, is a C^1 one-dimensional manifold, and each component can be described by a curve $x(\theta)$. In this note a theorem is proved which is directly related to and motivated by a result due to Eaves and Scarf on piecewise linear functions. This theorem relates the signs of the derivatives $\dot{x}_i(\theta)$ to the signs of the determinants of submatrices of the Jacobian matrix H'. Applications to solving nonlinear equations are given.

y olds   squared   to the first power   theta   x dot(i)

1. Introduction and Notation

A well known technique for solving nonlinear equations $f(x) = 0$, where $f: R^n \to R^n$ is to imbed f into a one-parameter family of homotopy equations $H(x, t) = 0$ where $x \in R^n$, $t \in [0, 1]$, $H(x, 0) \equiv f(x)$, and $H(x^0, 1) = 0$ for some known $x^0 \in R^n$. Examples of such a function H are

$$H(x, t) = f(x) - t\, f(x^0)$$

$$H(x, t) = t(x - x^0) + (1 - t)\, f(x) \quad .$$

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Many previous studies have analyzed conditions under which the equation $H(x, t) = 0$ has a solution $x(t)$ which is a differentiable path for which $H_x(x(t), t)$ is nonsingular for all $t \in [0, 1]$, where H_x denotes the derivative of H with respect to the x variables. This requirement is equivalent to stating that the differential equation

$$\dot{x} = -H_x(x, t)^{-1} H_t(x, t) , \quad x(1) = x^0$$

has a solution $x(t)$ for $t \in [0, 1]$, where $\dot{x}$ denotes the derivative $\frac{dx}{dt}$. Indeed, in this latter case $\frac{d}{dt} H(x(t), t) \equiv 0$ for $t \in [0, 1]$ and hence $H(x(1), 1) = 0 = H(x(0), 0) = f(x(0))$ so that $x^* = x(0)$ is a root of $f(x) = 0$. One can then solve the equations $f(x) = 0$ by integrating the above differential equation (assuming $x(t)$ is a differentiable path). However, the requirement that $x(t)$ be a differentiable path is quite severe. For appropriate discussions, see the papers of Meyer [9], Davidenko [1], and Jacovlev [7].

Alternative methods for solving $f(x) = 0$ involve tracking, in a limiting sense, points (x, t) which satisfy $H(x, t) = 0$ by the so called method of complementary pivoting on a triangulation of R^n. All of these methods are extensions of the seminal work of Scarf [12], [13] on the application of complementary pivoting to general nonlinear problems. For detailed discussions see for example works of Merrill [8], Garcia [3], and Garcia and Gould [4], [5], as well as numerous other papers noted in these bibliographies.

In general, these latter complementary pivoting algorithms, although slow by nature, converge under assumptions much weaker than the existence of a differentiable path $x(t)$. The basic requirement is that the set of points $(x, t) \in R^{n+1}$, such that $H(x, t) = 0$ be a one-dimensional differentiable manifold for which x^0 and x^* are in the same component, where $f(x^*) = 0$. Thus one is lead to study differentiable objects such as $x(\theta)$, $t(\theta)$, $\theta \in [0, 1]$, using methods of differential and combinatorial topology. Such methods, although hardly new, have been rarely applied to the problem of solving $f(x) = 0$. Notable exceptions are the papers of Eaves and Scarf [2] and Smale [14]. We are indebted to Herbert Scarf for referring us to the latter paper.

In this note a theorem is proved which lends insight to the behavior of the set of points (x, t) for which $H(x, t) = 0$. This theorem relates the signs of the derivatives $\dot{x}(\theta)$, $\dot{t}(\theta)$ to the signs of the determinants of submatrices of the Jacobian matrix H'.

Let us generally consider a C^2 (twice continuously differentiable) function $H: R^{n+1} \to R^n$. An example of interest is

$$H(x, t) = f(x) - tf(x^0), \quad x \in R^n, \quad t \in R \quad .$$

Given $y \in R^n$, let

$$H^{-1}(y) = \{x \in R^{n+1} | H(x) = y\}$$

and

$$C = \{x \in R^{n+1} \mid \text{rank } H'(x) < n\}$$

where H' is the Jacobian matrix $\left(\frac{\partial H_i}{\partial x_j}\right)$ of H with respect to $x \in R^{n+1}$. The set C is said to be the set of critical points of H, and $H(C)$ the set of critical values. $R^n \sim H(C)$ is the set of regular values. Sard's Theorem [15] states that:

> Theorem 1. Let H: $R^{n+1} \to R^n$ be a C^2 map. Then $H(C)$ has measure zero.

Thus, as a corollary, the set of regular values is dense in R^n. Let us henceforth throughout this paper assume that 0 is a regular value of H. The following lemma will be used [10]:

> Lemma 1. Let H: $R^{n+1} \to R^n$ be a C^2 map and let 0 be a regular value of H. Then $H^{-1}(0)$ is a C^1 one-dimensional manifold.

We now recall that any connected C^1 one-dimensional manifold is diffeomorphic to a circle or an interval (open, closed, or half-open). Thus, each (connected) component of $H^{-1}(0)$ can be described by a curve $x(\theta)$ which is diffeomorphic to a circle or an interval. Furthermore, for any $x(\bar{\theta}) \in H^{-1}(0)$, we have

$$\text{rank } H'(x(\bar{\theta})) = n \tag{1}$$

and $\dot{x}(\bar{\theta})$ is a unique nonzero vector ($\dot{x}(\bar{\theta})$ denoting the derivative of x

with respect to θ at $\theta = \overline{\theta}$). Consequently, we can differentiate $H(x(\theta)) \equiv 0$ with respect to θ to obtain

$$H'(x(\theta))\ \dot{x}(\theta) = 0 \quad . \tag{2}$$

For a particular $\overline{\theta}$, $\dot{x}(\theta)$ is a vector tangent to the curve at $\theta = \overline{\theta}$ and spans the kernel of $H'(x(\overline{\theta}))$.

For any $i = 1, 2, \ldots, n + 1$, let $\dot{x}_i(\theta)$ and $H_i(x(\theta))$ denote the i^{th} component of $\dot{x}(\theta)$ and the i^{th} column of $H'(x(\theta))$, respectively, and let $\dot{x}^i(\theta)$, $H^i(x(\theta))$ be the remaining components of $\dot{x}(\theta)$ and columns of $H'(x(\theta))$, respectively.

2. The Main Theorem

Our following theorem for C^2 maps is related to and motivated by a theorem of Eaves and Scarf for piecewise linear maps [2].

Theorem 2. Let $H: R^{n+1} \to R^n$ be a C^2 map and 0 a regular value of H. Then for any component $x(\theta)$ of $H^{-1}(0)$ we have for all $i = 1, 2, \ldots, n + 1$:

$$\operatorname{sgn} \dot{x}_i(\theta) = \operatorname{sgn} \det H^i(x(\theta)) \quad \text{all } \theta$$

or

$$\operatorname{sgn} \dot{x}_i(\theta) = -\operatorname{sgn} \det H^i(x(\theta)) \quad \text{all } \theta$$

(where $\operatorname{sgn} 0 \triangleq 0$).

We prove the theorem in three parts.

<u>Lemma 2</u>. If $\dot{x}_i(\theta) = 0$ then $\det H^i(x(\theta)) = 0$.

Proof: By (1) and (2) we have $H'(x(\theta))\dot{x}(\theta) = 0 = H_i\dot{x}_i + H^i\dot{x}^i$, where rank $H'(x(\theta)) = n$ and $\dot{x}(\theta) \neq 0$. If $\dot{x}_i(\theta) = 0$, then $\dot{x}^i(\theta) \neq 0$ so that $\det H^i(x(\theta))$ must be zero. #

<u>Lemma 3</u>. If $\det H^i(x(\theta)) = 0$ then $\dot{x}_i(\theta) = 0$.

Proof: If $\det H^i(x(\theta)) = 0$, then for some $j \neq i$, we have $\det H^j(x(\theta)) \neq 0$ because by (1) rank $H'(x(\theta)) = n$. For simplicity, take $i = n$ and $j = n + 1$. Then, $H^{n+1}\dot{x}^{n+1} + H_{n+1}\dot{x}_{n+1} = 0$ implies $\dot{x}^{n+1} = -(H^{n+1})^{-1} H_{n+1}\dot{x}_{n+1}$. Note that the last component of $-(H^{n+1})^{-1}H_{n+1}$ is zero, otherwise

$$\det H^n = \det H^{n+1} \det [(H^{n+1})^{-1}H^n]$$

$$= \det H^{n+1} \det [H_1, H_2, \ldots, H_{n-1}, H_n]^{-1} [H_1, H_2, \ldots, H_{n-1}, H_{n+1}]$$

$= \det H^{n+1} \det M$, where M is the following $n \times n$ matrix. The first $n - 1$ columns are given by $\begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix} = [H^{n+1}]^{-1}[H_1, H_2 \ldots, H_{n-1}]$ and the last column is $[H^{n+1}]^{-1} H_{n+1}$. If the last component of the last column is not zero then $\det H^n \neq 0$, a contradiction.

Hence, $\dot{x}_n = \dot{x}^{n+1}_n = 0$. #

Lemma 4.

$$\dot{x}_i(\theta) \det H^i(x(\theta)) > 0 \quad \text{all } \theta \text{ such that } \dot{x}_i(\theta) \neq 0$$

or

$$\dot{x}_i(\theta) \det H^i(x(\theta)) < 0 \quad \text{all } \theta \text{ such that } \dot{x}_i(\theta) \neq 0 \ .$$

Proof: Let $$A(\theta) = \begin{bmatrix} H^i(x(\theta)) \ , & H_i(x(\theta)) \\ \dot{x}^i(\theta)^t \ , & \dot{x}_i(\theta) \end{bmatrix}$$

$$B(\theta) = \begin{bmatrix} H^i(x(\theta)) \ , & \dot{x}^i(\theta) \\ 0 \ , & \dot{x}_i(\theta) \end{bmatrix}$$

where $\dot{x}^i(\theta)^t$ is the transpose of $\dot{x}^i(\theta)$. Since $\dot{x}(\theta)$ is orthogonal to $H'(x(\theta))$, we have rank $A(\theta) = n + 1$ for all θ. Since $A(\theta)$ is continuous in θ, we have $\det A(\theta) > 0$ all θ or $\det A(\theta) < 0$ for all θ.

Now

$$\det A(\theta)\, B(\theta) = \det \begin{bmatrix} H^i H^i \ , & 0 \\ \dot{x}^i(\theta)^t H^i \ , & \dot{x}(\theta)^t\, \dot{x}(\theta) \end{bmatrix}$$

$$= \dot{x}(\theta)^t\, \dot{x}(\theta)(\det H^i)^2 > 0$$

since $\dot{x}_i(\theta) \neq 0$ implies $\det H^i \neq 0$ by Lemma 3. Thus the determinants of $A(\theta)$ and $B(\theta)$ have the same nonzero sign for all θ such that $\dot{x}_i(\theta) \neq 0$. But

$$\det B(\theta) = \dot{x}_i(\theta) \ \det H^i(x(\theta))$$

which proves the claim. #

From the previous 3 lemmas, we get Theorem 2 directly.

Note that the theorem holds if H is restricted to, say, $R^n \times [0, 1]$. In most applications to $R^n \times [0, 1]$ a further restriction would be required on the boundaries $R^n \times \{0\}$ and $R^n \times \{1\}$--namely, nonsingularity of the $n \times n$ submatrix $H_x(x, t)$ at points (x, t) in the boundary for which $H(x, t) = 0$. This condition assures that all loops which occur are contained in $R^n \times (0, 1)$. An interesting corollary to the theorem is the following monotonicity theorem.

> <u>Corollary</u> Let $H: R^{n+1} \to R^n$ be a C^2 map and 0 a regular value of H. Suppose for some i, $H^i(x)$ is nonsingular for all x in a particular <u>component</u> $x(\theta)$ of $H^{-1}(0)$. Then, on that component of $H^{-1}(0)$, $x_i(\theta)$ is either monotone increasing or monotone decreasing as a function of θ.

Observe that under the assumptions of the Corollary the distinguished component of $H^{-1}(0)$ cannot be diffeomorphic to a circle.

3. <u>Illustrations</u>

Illustration 1:

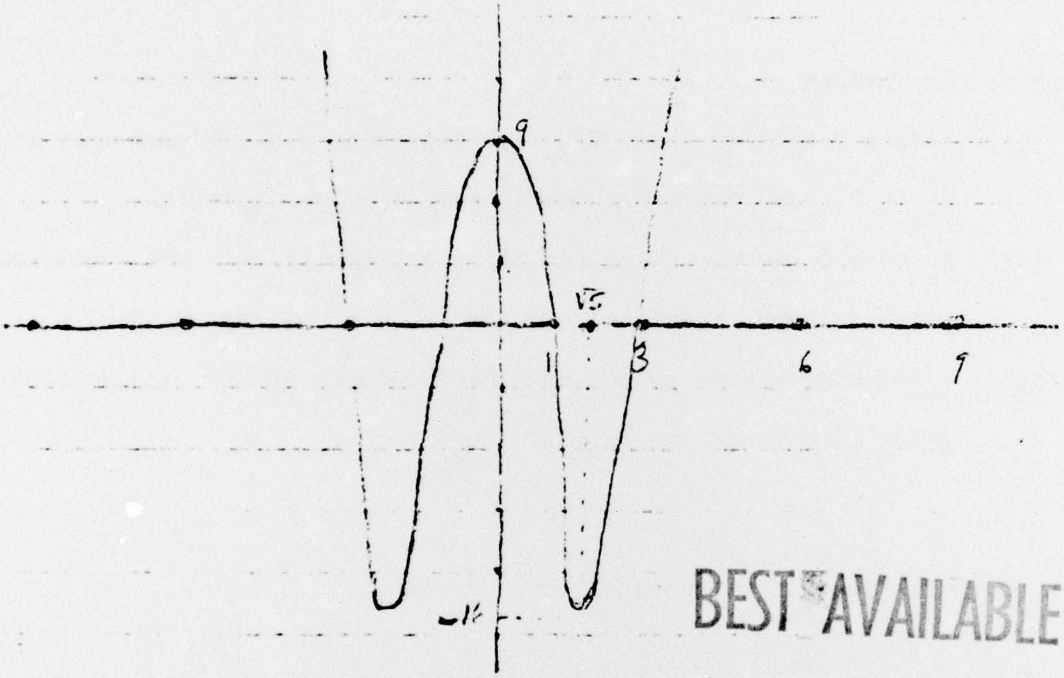
The theorem is illustrated in Figure 1 for the function $H: R^3 \to R^2$ given by

$$H(x, t) = f(x) - tf(x^0), \quad x \in R^2, \quad t \in R$$

where

$$f_1(x_1, x_2) = x_1$$
$$f_2(x_1, x_2) = (x_1^2 + x_2^2)^2 - 10(x_1^2 + x_2^2) + 9$$

and $f(x^0) = (-4, 9109)$. The function f_2 is a rotation about the vertical axis of $F(x) = (x - 1)(x + 1)(x - 3)(x + 3)$ whose graph is



For this example $H^{-1}(0)$ is a single component $(x(\theta), t(\theta))$, $\theta \in [0, 1]$, and the projection of $H^{-1}(0)$ into R^2 is a loop. The matrix H' of Theorem 2 is

$$\begin{bmatrix} \nabla f_1 & - f_1(x^0) \\ \nabla f_2 & - f_2(x^0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 4x_1(x_1^2 + x_2^2) - 20x_1 & 4x_2(x_1^2 + x_2^2) - 20x_2 & -9109 \end{bmatrix}$$

According to the Theorem,

(i) $\dot{t}(\theta) = 0 \iff \nabla f_1$ and ∇f_2 are dependent i.e., if and only if $\frac{\partial f}{\partial x_2} = 0$, and therefore $\iff x_1^2 + x_2^2 = 5$ or $x_2 = 0$.

(ii) $\dot{x}_2(\theta) = 0 \iff$ the first and third columns of H' are dependent, which is true if and only if $4x_1(x_1^2 + x_2^2) - 20x_1 = -9109/4$.

(iii) $\dot{x}_1(\theta) = 0 \iff$ the second and third columns of H' are dependent, which is true if and only if $\frac{\partial f_2}{\partial x_2} = 0$.

The path shown in Figure 1 illustrates these properties.

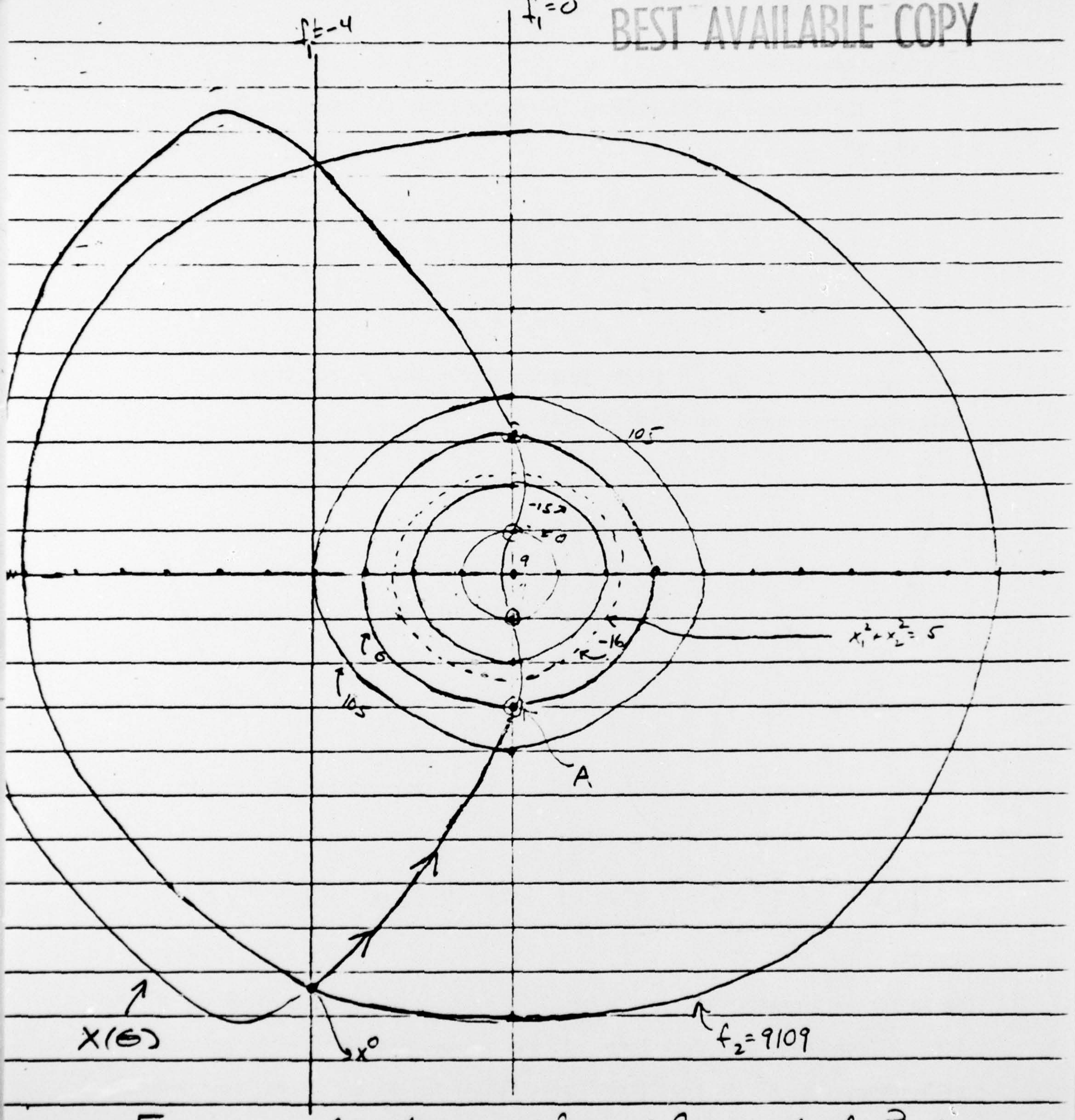


Figure 1: Contours of f_1 and f_2 and the Path $x(\theta)$.

Illustration 2:

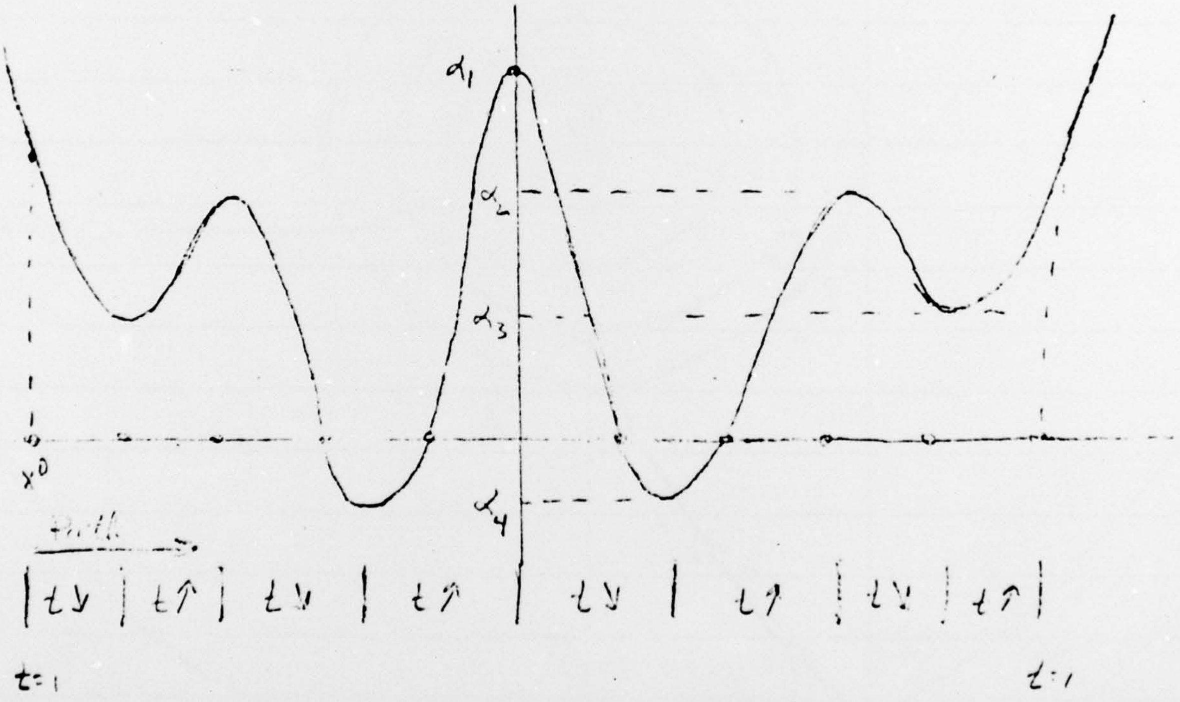
The theorem is illustrated in Figure 2 for the function H: $R^3 \rightarrow R^2$ given by

$$H(x, t) = f(x) - t\, f(x^0)\ , \quad x \in R^2, \quad t \in R$$

where

$$f_1(x_1, x_2) = x_1$$

and $f_2(x_1, x_2)$ is an 8th degree polynomial obtained by rotating the following graph about the vertical axis.



Again for this example there is a single component in the projection of $H^{-1}(0)$ into R^2 and this path is a loop. It can be verified that on the portion of the path connecting x^0 to the "first" zero (point A) t decreases, then increases, then decreases to 0. Hence this segment cannot be represented as a differentiable path $x(t)$.

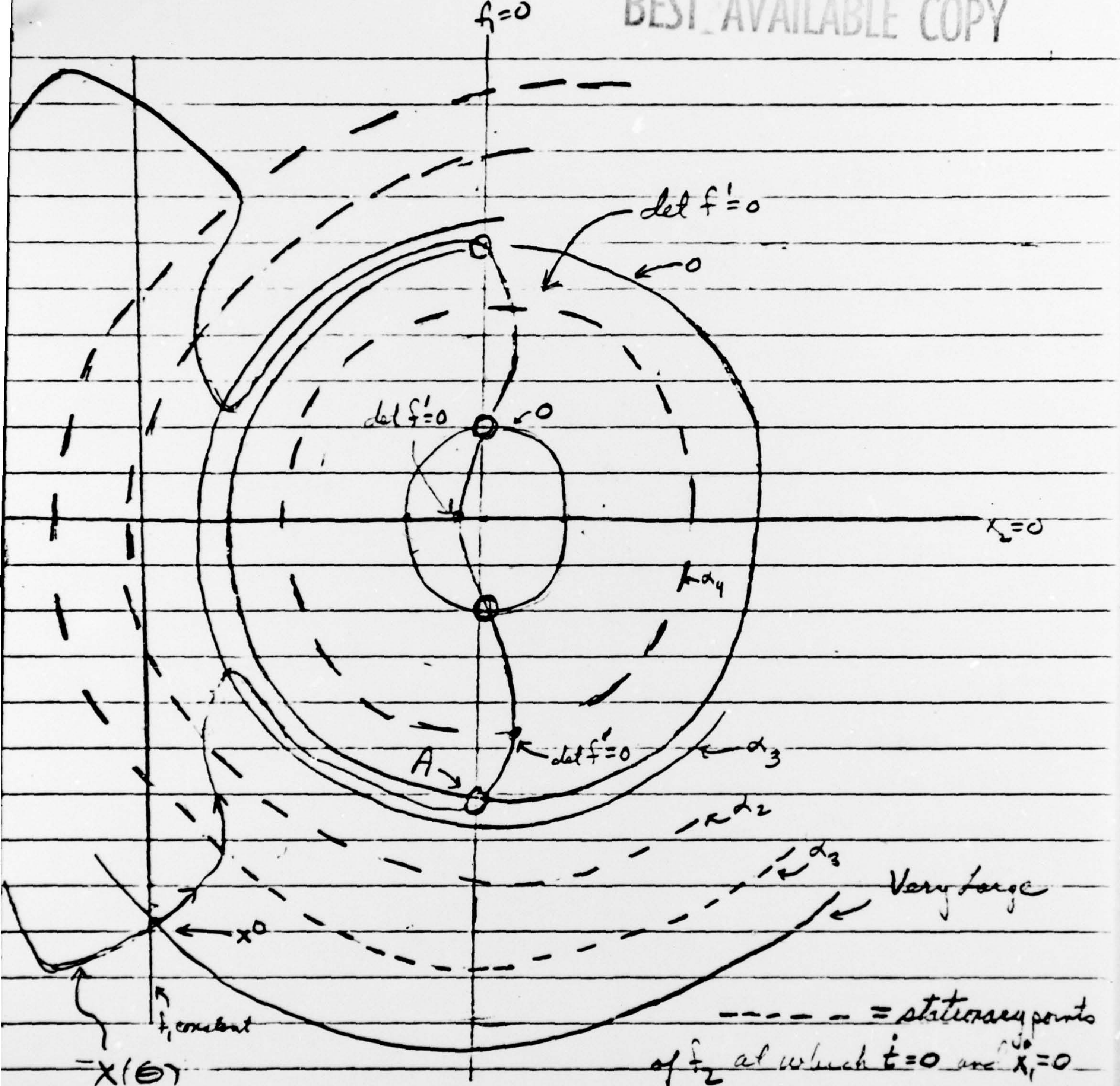


Figure 2: Contours of f_1 and f_2 and the Path $x(\Theta)$

4. Applications

Application 1:

As a first application we summarize some of the comments in [5] on relations between the set of solutions to

$$H(x, t) = f(x) - t\, f(x^0) = 0, \quad x \in R^n, \quad t \in R \tag{3}$$

to the set of solutions to

$$f'(x(\theta)\, \dot{x}(\theta) = - \lambda(\theta)\, f(x(\theta)), \quad x(0) = x^0 \tag{4}$$

where f is a C^2 map with $f'(x^0)$ nonsingular and λ is a real valued function of θ satisfying the condition $\operatorname{sgn} \lambda = \operatorname{sgn} \det f'(x(\theta))$. The set of points satisfying equation (3) can be followed (in a precise limiting sense) by a scalar labeling simplicial pivoting algorithm similar to the method introduced by the authors in [4]. The differential equation (4) was introduced by Smale in [14] and has the credentials of a "global Newton method."

Since by assumption f is C^2 and 0 is a regular value of H, the solutions to (3) comprise a C^1 one-dimensional manifold, so that the component of $H^{-1}(0)$ containing the "initial point" $(x^0, 1)$ may be described by a curve

$$(x(\theta), t(\theta)), \ (x(0), t(0)) = (x^0, 1) \ .$$

Differentiating (3) we obtain

$$f'(x(\theta))\, \dot{x}(\theta) = \dot{t}(\theta)\, f(x^0) = \frac{\dot{t}(\theta)}{t(\theta)}\, f(x(\theta)) \tag{5}$$

if $t(\theta) \neq 0$. Thus if $\dot{t}(\theta) \neq 0$ it follows from Theorem 2 that $\det f(x(\theta)) \neq 0$ so that

$$\dot{x}(\theta) = \frac{\dot{t}(\theta)}{t(\theta)} f'(x(\theta))^{-1} f(x(\theta)) .$$

In this particular case our theorem says

$$\operatorname{sgn} \dot{t}(\theta) = \operatorname{sgn} \det f'(x(\theta)) \quad \text{all } \theta$$

or

$$\operatorname{sgn} \dot{t}(\theta) = -\operatorname{sgn} \det f'(x(\theta)) \quad \text{all } \theta \quad .$$

We now observe that (5) provides a special instance of (4) if $\operatorname{sgn} \frac{\dot{t}(\theta)}{t(\theta)} = -\operatorname{sgn} \det f'(x(\theta))$. Recall that $t(0) = 1$, and adopt the convention that if $\det f'(x^0) > 0$ (< 0) we initially move on the path in such a direction that t decreases (increases) from its initial value 1. Then $\operatorname{sgn} \frac{\dot{t}(0)}{t(0)} = -\operatorname{sgn} \det f'(x^0)$ and by Theorem 2 this will be true for all θ such that $t(\theta) > 0$. This proves that a <u>piece</u> of the solution to (3) is a solution to (4), for Smale's algorithm terminates the moment the first zero is encountered $(t(\theta) = 0)$ whereas (3) can be "continued." In Figure 1, the path from x^0 to the point labeled A is the Smale path (solution to (4)). The solution to (3) is the entire closed loop which contains <u>all</u> of the zeros of f.

Now consider Figures 3 and 4. Here we are solving the equations

$$f_1(x_1, x_2) = x_1 - 3 = 0$$

$$f_2(x_1, x_2) = (x_1^2 + x_2^2)^2 - 10(x_1^2 + x_2^2) - 3 = 0 \quad .$$

The obvious relation to the function f in Illustration 1 should be noted. It is seen in Figure 3 that the projection of $H^{-1}(0)$ into R^2 contains two components. That component passing through the "starting point" x^0 does not pass through a root. However, consider Figure 4 where the starting point x^0 is chosen "at infinity." In this case the projection of $H^{-1}(0)$ into R^2 is a single path $x(\theta)$ which passes through both roots of f. It can be verified that this interesting behavior will occur, for this example, with any x^0 sufficiently large in norm.

In [14] Smale has demonstrated a result which can be restated as follows: Suppose there is a bounded open set C such that $f: \overline{C} \to R^n$, C and ∂C are connected, ∂C smooth, and $x \in \partial C \Rightarrow \det f'(x) > 0$ and $(f'(x))^{-1} f(x)$ intersects ∂C transversally at x. Suppose $f \in C^2$, $x^0 \in \partial C$ and 0 is a regular value of $H(x, t) = f(x) - t f(x^0)$. Then the connected component of $H^{-1}(0)$ containing $(x^0, 1)$ will contain a zero of f.

Figure 4 demonstrates a set C satisfying Smale's assumptions. Note that only 1 of the roots is contained in this set.

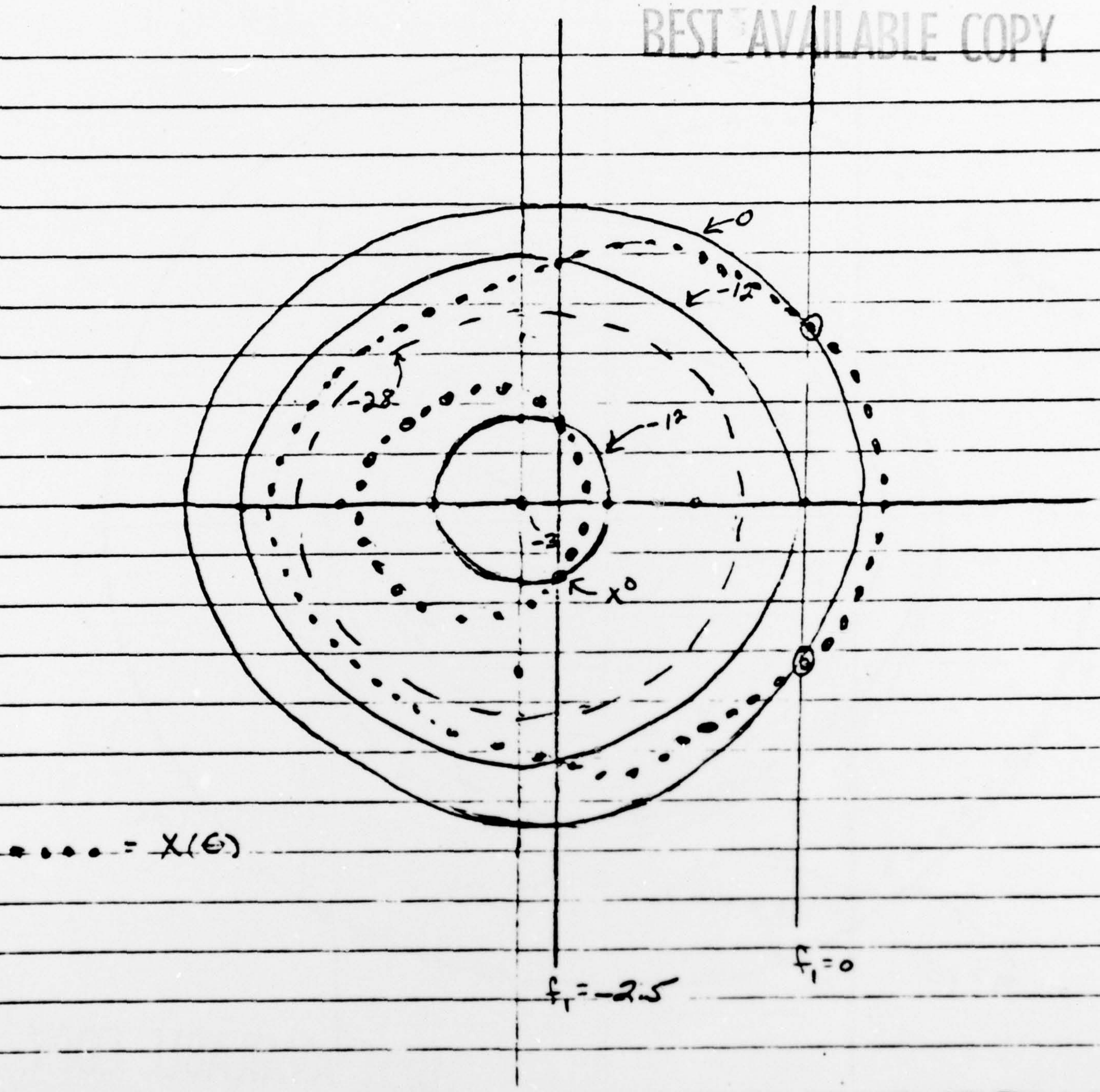


Figure 3: Contours of f_1 and f_2 and the Paths $X(\epsilon)$

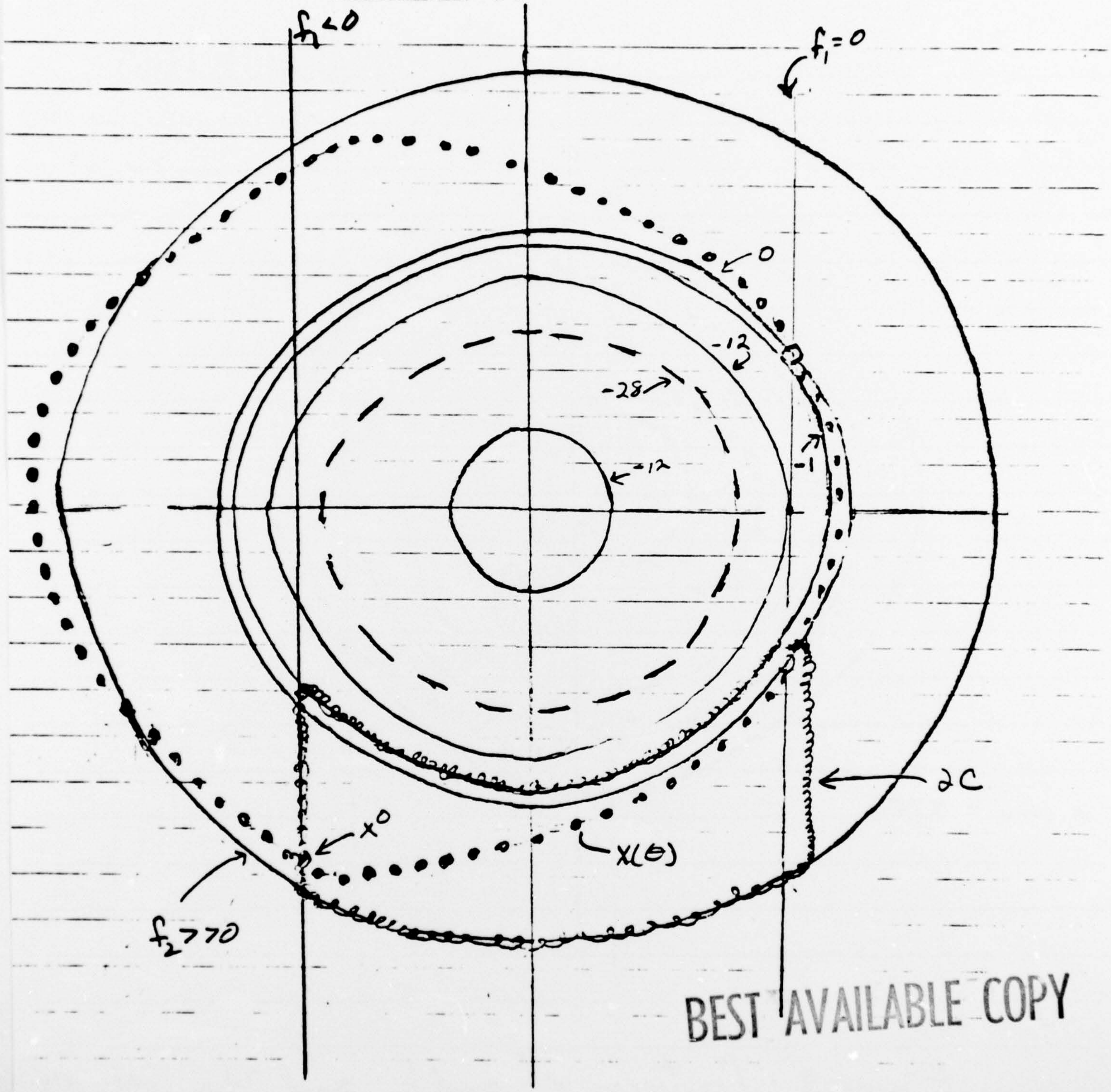


Figure 4: Contours of f_1 and f_2, The Path $x(\theta)$, and The Smale Set, C.

Application 2:

In [6], Garcia and Zangwill showed how to find all solutions to certain systems of n nonlinear equations in n complex variables. For example, they show how to construct all solutions to an arbitrary system of n polynomial equations in n unknowns.

The underlying theorem in [6] proves a monotonic behavior of the paths of solutions to a particular set of homotopy equations. This result is achieved by use of the Cauchy-Reimann conditions to show that $\det H_x(x, t) \geq 0$ for any (x, t) in $C^n \times [0, 1]$, (where H_x is the Jacobian of H written as a function in $R^{2n} \times [0, 1]$). The result is therefore a special case of our corollary to Theorem 2, which in this instance states that the path must be monotonic nondecreasing (or monotonic nonincreasing) in the variable t. This key result may then be used in [6] to show that starting from any solution to $H(x, 0) = 0$, the path $x(t)$ satisfying

$$H(x(t), t) = 0$$

must yield an $x(1)$ which solves the given system of equations. See [6] for a complete treatment of this problem.

Application 3:

Let us consider a C^2 function $f\colon R^n \to R^n$ where $\det f'(x) > 0$ all x and $\lim_{||x|| \to \infty} ||f(x)|| = \infty$ (f satisfying the latter condition is said to be <u>norm-coercive</u>).

This condition on f is essentially that for the <u>Hadamard theorem</u> (see Theorems 5.3.9 and 5.3.10 of [11]. See also Theorem 10.4.3). It is known that for any $y \in R^n$, the above assumptions assure the existence of a unique x^* satisfying

$$f(x^*) = y \quad .$$

To find x^*, consider the homotopy

$$H(x, t) = f(x) - [t\, f(x^0) + (1 - t)\, y] = 0 \; (x \in R^n, t \in R)$$

for an arbitrary $x^0 \in R^n$ (This is the limiting path generated by the complementary pivoting algorithm of Garcia-Gould in [4]). Then

$$\det H_x(x, t) = \det f'(x) > 0$$

for any (x, t), so that the algorithm of Garcia-Gould will trace a path $(x(\theta), t(\theta))$ which by the corollary to Theorem 2 will be monotonic in t. This monotonicity, along with the assumption $\lim_{||x|| \to \infty} ||f(x)|| = \infty$ implies that t cannot be asymptotic. One is therefore assured of finding the unique solution $(x(0), t(0)) = (x^*, 0)$ of

$$f(x) = y$$

for an arbitrary $y \in R^n$.

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