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ON THE RATE DISTORTION FUNCTIONS OF MEMORYLESS SOURCES UNDER A MAGNITUDE-ERROR CRITERION*

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ABSTRACT

We consider the evaluation of and bounds for the rate distortion functions of independent and identically distributed (i.i.d.) sources under a magnitude-error criterion. By refining the ingeneous approach of Tan and Yao we evaluate explicitly the rate distortion functions of larger classes of i.i.d. sources and we obtain familieis of lower bounds for arbitrary i.i.d. sources

I. INTRODUCTION

The rate distortion function of an independent and identically distributed (i.i.d.) source is clearly equal to the rate distortion function of each random variable of the source. We will evaluate the rate distortion function of a random variable X with probability density p(x) satisfying certain conditions. A magnitude-error criterion will be used throughout without further reminder.

The procedure used was introduced by Tan and Yao (1975) and is based on the well-known analytical expression of R(D) which is stated for reference. (See for instance Berger, 1971).

THEOREM A. Let X *be a random variable with probability density function* p(x) *and rate distortion function* R(D). *For each* $s \leq 0$, *let* Λ_s *be the set of all non-negative functions* λ_s *satisfying*

$$C_s(y) = \int_{-\infty}^{\infty} \lambda_s(x)p(x)\exp(s|x-y|)dx \leq 1 \tag{1}$$

for all y. Then for all $D > 0$,

$$R(D) = \sup_{s\leq 0,\lambda_s\in\Lambda_s} [sD + \int_{-\infty}^{\infty} p(x)\ln\lambda_s(x)dx] . \tag{2}$$

For each $s\leq 0$, *a necessary and sufficient condition for* λ_s *to realize the supremum in (2) is the existence of a probability distribution* G_s *which is related to* λ_s *by*

$$[\lambda_s(x)]^{-1} = \int_{-\infty}^{\infty} \exp(s|x-y|)dG_s(y) \tag{3}$$

and is such that $C_s(y) = 1$ *a.e.* $[dG_s]$. *Moreoever, for such* λ_s *and* G_s, R(D) *is given parametrically in* s *by*

$$R(D_s) = sD_s + \int_{-\infty}^{\infty} p(x)\ln\lambda_s(x)dx \qquad s\leq 0 \tag{4}$$

$$D_s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda_s(x)p(x)|x-y|\exp(s|x-y|)dx\, dG_s(y). \tag{5}$$

An ingeneous procedure to search for λ_s satisfying (1) and (3) was given by Tan and Yao (1975) and is described in the following immediate corollary of Theorem A.

COROLLARY A.1 [Tan and Yao (1975)]. Let X *be a random variable with probability density function* $p(x)$ *which vanishes outside the interval* (a,b), $-\infty\le a<b\le\infty$. *For each* $s\le 0$, *let* V_s *be a subinterval of* (a,b) *and assume that the distribution function* $G_s(y)$, $y\in V_s$, *whose total probability is concentrated on* V_s, *and* $\lambda_s(x)$, $x\in[a,b]$, satisfy

$$[\lambda_s(x)]^{-1} = \int_{V_s} \exp(s|x-y|)dG_s(y), \qquad \text{for all } x\in[a,b] \tag{6}$$

and

$$\int_a^b \lambda_s(x)p(x)\exp(s|x-y|)dx = 1 , \qquad \text{for all } y\in V_s . \tag{7}$$

If λ_s *satisfies (1), then the rate distortion function of* X *is given by (4) and (5).*

The significance of this straightforward corollary lies in the fact that for some densities p, intelligent (or appropriate) choices of V_s can be made such that (6) and (7) can be solved and the solutions satisfy the properties stated in the Corollary. Using this procedure, the rate distortion functions of an i.i.d. Gaussian source, and of a certain class of i.i.d. sources were calculated explicitly [Tan and Yao, (1975)].

In this paper we make two uses of this procedure of Tan and Yao. First, in section II, we refine their results, by a substantial weakening of the conditions on the density, thus calculating explicitly the rate distortion functions of larger classes of i.i.d. sources. In Theorem 1 the density of the source has finite support, Theorem 2 treats concave source densities, and in Theorem 1' the support of the source density is the entire real line or a half line. Secondly, we develop a family of lower bounds for the rate distortion function of an arbitrary i.i.d. source (Theorem 3) and compare them with the Shannon lower bound in

Section III. We also indicate how Theorems 1 and 1' may be combined with a known approximate result (Theorem B) in evaluating the rate distortion functions of certain i.i.d. sources whose densities do not satisfy the assumptions of Theorems 1 and 1'.

II. RATE DISTORTION FUNCTIONS OF I.I.D. SOURCES

We first consider rate distortion functions of random variables with continuous densities which vanish outside a finite interval.

THEOREM 1. Let X *be a random variable with probability density function* $p(x)$ *which vanishes outside the interval* $[a,b]$, $-\infty < a < b < \infty$. *Assume the following:*

(A) p *is continuous with median* μ *and there is an at most finite set of points* $a = d_0 < d_1 < \ldots < d_m < d_{m+1} = b$ $(m \geq 0)$ *such that on each* $[d_j, d_{j+1}]$, $j=0,1,\ldots,m$, $p(x)$ *is differentiable and its derivative* $p'(x)$ *is absolutely continuous and satisfies* $p'_-(d_j) \geq p'_+(d_j)$, $j=1,\ldots,m$, *where* $p'_-(d_j)$ *and* $p'_+(d_j)$ *are the left and right limits of* p' *at* d_j *respectively. Also*

$$\int_a^x p(t)dt > 0 \quad \text{for } x > a; \qquad \int_x^b p(t)dt > 0 \quad \text{for } x < b .$$

(B) The function

$$K_1(x) = p(x) / \int_x^b p(t)dt \quad \text{for } x \in [\mu, b) \tag{8}$$

diverges to $+\infty$ *as* x *increases to* b*; and the function*

$$K_2(x) = p(x) / \int_a^x p(t)dt \quad \text{for } x \in (a, \mu] \tag{9}$$

diverges to $+\infty$ *as* x *decreases to* a.

Then for each $s \in (-\infty, -2p(\mu))$, *there exist unique* $a_s > 0$ *and* $b_s > 0$ *such that* $a_s \uparrow \mu - a$ *and* $b_s \uparrow b - \mu$ *as* $s \downarrow -\infty$ *and* a_s *and* b_s *are determined by*

$$\mu-a_s = \min\{y\in(a,\mu):\ K_2(y) = |s|\} \tag{10}$$

$$\mu+b_s = \max\{y\in(\mu,b):\ K_1(y) = |s|\}\ . \tag{11}$$

Suppose in addition that

(C) *for each* $s\in(-\infty,-2p(\mu))$,

$$p(x) - s^{-2}p''(x) \geq 0 \quad \text{a.e. [Leb.] on } [\mu-a_s,\mu+b_s]\ .$$

Then the rate distortion function $R(D)$, $0 < D < D_{max}$, *of* X *is given parametrically in* s *by*

$$R(D_s) = \ell n\frac{|s|}{2} - \int_{\mu-a_s}^{\mu+b_s} p(x)\ell n(ep(x))dx - \ell n(p(\mu-a_s))\int_a^{\mu-a_s} p(x)dx$$
$$- \ell n(p(\mu+b_s))\int_{\mu+b_s}^{b} p(x)dx \tag{12}$$

$$D_s = \frac{1}{|s|}\int_{\mu-a_s}^{\mu+b_s} p(x)dx + \int_a^{\mu-a_s}(\mu-a_s-x)p(x)dx + \int_{\mu+b_s}^{b}(x-\mu-b_s)p(x)dx \tag{13}$$

where $-\infty < s < -2p(\mu)$ *and* $D_{max} = \int_a^b |x-\mu|p(x)dx$.

Proof: We will show that all conditions of Corollary A.1 are satisfied with $V_s = [\mu-a_s,\mu+b_s]$ where the dependence of a_s, b_s on s will be specified later.

Substituting (6) into (7) and writing the integral from a to b as the sum of the integrals from a to $\mu-a_s$ to $\mu+b_s$ to b, we have

$$A_{1,s}\exp(sy) + A_{2,s}\exp(-sy) + \int_{V_s}\alpha_s(x)\exp(s|x-y|)dx = 1 \tag{14}$$

for $y\in V_s$, where $A_{1,s}$, $A_{2,s}$ and α_s are given by

$$A_{1,s} = \int_a^{\mu-a_s} p(t)dt/\int_{V_s}\exp(st)dG_s(t) \tag{15}$$

$$A_{2,s} = \int_{\mu+b_s}^{b} p(t)dt \Big/ \int_{V_s} \exp(-st)\,dG_s(t) \tag{16}$$

$$\alpha_s(x) = p(x) \Big/ \int_{V_s} \exp(s|x-t|)\,dG_s(t) \ . \tag{17}$$

By differentiating (14) with respect to y, we find that it is necessary that

$$\alpha_s(x) = |s|/2, \quad A_{1,s} = \tfrac{1}{2}\exp(-s(\mu-a_s)), \quad A_{2,s} = \tfrac{1}{2}\exp(s(\mu+b_s))$$

and these are also sufficient for (14) (which can be verified by substituting into (14)).

Substituting the solutions into (15), (16) and (17) we obtain

$$\int_{V_s} \exp(s(t-\mu+a_s))\,dG_s(t) = 2\int_{a}^{\mu-a_s} p(t)dt \tag{18}$$

$$\int_{V_s} \exp(s(\mu+b_s-t))\,dG_s(t) = 2\int_{\mu+b_s}^{b} p(t)dt \tag{19}$$

$$\int_{V_s} \exp(s|x-t|)\,dG_s(t) = (2/|s|)p(x) \quad x\in V_s \ . \tag{20}$$

Clearly (18) and (19) are consistent with (20) if and only if

$$\int_{a}^{\mu-a_s} p(t)dt = \frac{1}{|s|}\,p(\mu-a_s) \tag{21}$$

and

$$\int_{\mu+b_s}^{b} p(t)dt = \frac{1}{|s|}\,p(\mu+b_s) \ . \tag{22}$$

Now (21) is equivalent to $K_2(\mu-a_s) = |s|$. Note that conditions (A) and (B) imply that $K_2(x)$ is continuous on $(a,\mu]$, differentiable on $(a,\mu]$ except at those d_j's which belong to $(a,\mu]$ at which left and right derivatives exist, and satisfies $K_2(\mu) = 2p(\mu)$ and $\lim_{x\downarrow a} K_2(x) = +\infty$. It follows that given any $s\in(-\infty,-2p(\mu))$ the equation $K_2(\mu-a_s) = |s|$ has at least one solution. For reasons which will become clear later on in this proof we will choose the smallest solution:

$$\mu - a_s = \min\{y \in (a,\mu):\ K_2(y) = |s|\} \tag{23}$$

which is clearly such that $\mu - a_s \downarrow a$ as $s \downarrow -\infty$ and has the following properties (to be used later on):

$$K'_{2,+}(\mu - a_s) \le 0, \quad \text{and } K_2(y) > |s| \quad \text{for all } y \in (a, \mu - a_s)\ .$$

Similarly, by the properties of $K_1(x)$, b_s is uniquely determined by

$$\mu + b_s = \max\{y \in (\mu,b):\ K_1(y) = |s|\} \tag{24}$$

and has the following properties: $\mu + b_s \uparrow b$ as $s \downarrow -\infty$,

$$K'_{1,-}(\mu + b_s) \ge 0, \quad \text{and } K_1(y) > |s| \quad \text{for all } y \in (\mu + b_s, b)\ .$$

We next show that for each $s \in (-\infty, -2p(\mu))$, the distribution function $G_s(x)$ which has absolutely continuous part with density $p(x) - s^{-2}p''(x)$ on V_s and zero elsewhere, discrete part with atoms at the points $\mu - a_s$, $\mu + b_s$ and the d_j's which are in $[\mu - a_s,\ \mu + b_s]$ and masses to be determined, and zero continuous singular part, is a solution of (20). For notational convenience we will work with the "density" g_s of the above described distribution function G_s which is thus of the form

$$g_s(t) = p(t) - s^{-2}p''(t) + C_{1,s}\delta(t - \mu + a_s) + C_{2,s}\delta(t - \mu - b_s) + \sum_{j=\ell}^{\ell+n-1} D_{j,s}\delta(t - d_j) \tag{25}$$

for $t \in V_s$ and zero elsewhere, where

$$d_{\ell-1} < \mu - a_s \le d_\ell < \ldots < d_{\ell+n-1} \le \mu + b_s < d_{\ell+n}$$

and $\delta(\cdot)$ is the Dirac Delta function.

The masses $C_{1,s}$, $C_{2,s}$ and $D_{j,s}$ can now be determined so that (20) will be satisfied. We find (see Appendix 1)

$$\begin{aligned} C_{1,s} &= |s|^{-2}[|s|p(\mu - a_s) - p'_+(\mu - a_s)]\ , \\ C_{2,s} &= |s|^{-2}[|s|p(\mu + b_s) + p'_-(\mu + b_s)]\ , \\ D_{j,s} &= |s|^{-2}[p'_-(d_j) - p'_+(d_j)]\ , \qquad \ell \le j \le \ell + n - 1\ . \end{aligned} \tag{26}$$

Having determined g_s so as to satisfy (20) it remains to be shown that g_s is a probability "density" function, i.e. that G_s is a probability distribution function. Since $p(t) - s^{-2}p''(t) \geq 0$ a.e. [Leb] on V_s by assumption (C) and $p'_-(d_j) - p'_+(d_j) \geq 0$ by assumption (A), it is clear from the expressions (25) and (26) that G_s is a distribution function if and only if

$$|s|p(\mu-a_s) - p'_+(\mu-a_s) \geq 0 \tag{27}$$

$$|s|p(\mu+b_s) + p'_-(\mu+b_s) \geq 0 \tag{28}$$

and

$$\int_{V_s} g_s(t)dt = \int_{V_s} dG_s(t) = 1 . \tag{29}$$

To show (27), we proceed as follows. Since $K'_{2,+}(\mu-a_s) \leq 0$ we have from (9)

$$p'_+(\mu-a_s)\int_a^{\mu-a_s} p(t)dt - p^2(\mu-a_s) \leq 0$$

and using (21) obtain

$$\left[\int_a^{\mu-a_s} p(t)dt\right]\left[p'_+(\mu-a_s) - |s|p(\mu-a_s)\right] \leq 0 .$$

Now since $a < \mu-a_s$ we have $\int_a^{\mu-a_s} p(t)dt > 0$ by condition (A) and thus (27) follows. (28) can be proved similarly, and (29) can be verified easily.

Next we need to show that $C_s(y) \leq 1$ for $y \notin V_s$. $\lambda_s(x)$ is found by substituting g_s into (6) and we have (see Appendix 2)

$$\lambda_s(x) = \begin{cases} [|s|/2p(\mu-a_s)]\exp(-s(\mu-a_s)+sx) & x\in[a,\mu-a_s] \\ |s|/2p(x) & x\in V_s \\ [|s|/2p(\mu+b_s)]\exp(s(\mu+b_s)-sx) & x\in[\mu+b_s,b] . \end{cases} \tag{30}$$

Now suppose that $a \leq y < \mu-a_s$. Then

$$C_s(y) = \int_a^y \lambda_s(x)p(x)\exp(s(y-x))dx + \int_y^b \lambda_s(x)p(x)\exp(s(x-y))dx .$$

Differentiating $C_s(y)$ with respect to y, we have

$$C_s'(y) = -|s| \int_a^y \lambda_s(x)p(x)\exp(s(y-x))dx + |s| \int_y^b \lambda_s(x)p(x)\exp(s(x-y))dx$$
$$= |s|(C_s(y) - h_s(y)) \tag{31}$$

where

$$h_s(y) = 2\exp(sy)\int_a^y \lambda_s(x)p(x)\exp(-sx)dx \ . \tag{32}$$

Substituting (30) into (32) we have

$$h_s(y) = |s|\exp(-s(\mu-a_s)+sy)\int_a^y p(x)dx/p(\mu-a_s) \qquad a\le y<\mu-a_s$$

and finally, because of (21)

$$h_s(y) = \exp(sy)\int_a^y p(x)dx/\exp(s(\mu-a_s))\int_a^{\mu-a_s} p(x)dx \quad a\le y<\mu-a_s \ .$$

We now show that

$$f(y) = \exp(sy)\int_a^y p(t)dt \qquad\qquad y\in(a,\mu-a_s]$$

is increasing. Indeed we have

$$f'(y) = \exp(sy)(p(y) - |s|\int_a^y p(t)dt) \ .$$

Now (23) and (9) imply, as was remarked, that

$$K_2(y) \ge K_2(\mu-a_s) = |s| \quad \text{for } a<y\le\mu-a_s \ .$$

It then follows from (9) that $f'(y) \ge 0$, $a<y\le\mu-a_s$, and thus f is increasing on $(a,\mu-a_s]$. Hence $h_s(y) \le 1$ for $y\in(a,\mu-a_s]$ and since $h_s(a) = 0$, it follows that $h_s(y) \le 1$ for $y\in[a,\mu-a_s]$.

Now, as in Appendix A of [Tan and Yao, (1975)] it is shown that $C_s(y) \le 1$ for all $y\notin V_s$.

Thus, by Corollary A.1, R(D) is given parametrically by (4) and (5). (4) and (5) are shown in Appendix 3 to have the final expressions given in (12) and (13). This completes the proof of the Theorem. □

The following examples are applications of Theorem 1.

Example 1. Let $p(x)$ be a truncated double-exponential density function defined on $[-c,c]$, $c > 0$, i.e.

$$p(x) = \alpha\exp(-\alpha|x|)/2(1-\exp(-\alpha c)) \qquad |x|\le c, \quad \alpha>0 .$$

The assumptions of Theorem 1 are satisfied and are verified in the following:

(A) $p(x)$ is clearly continuous with median $\mu = 0$. $p(x)$ has continuous second derivative everywhere except at $x = 0$ at which the first derivative is discontinuous and

$$p'_-(0) = \alpha^2/2(1-\exp(-\alpha c)) > p'_+(0) = -\alpha^2/2(1-\exp(-\alpha c)) .$$

(B) $K_1(x) = \alpha \exp(-\alpha x)/[\exp(-\alpha x) - \exp(-\alpha c)]\uparrow+\infty$ as $x\uparrow c$. Similarly, $K_2(x)\uparrow+\infty$ as $x\downarrow-c$. In fact, $K_1(x)$ is easily seen to be monotonically increasing for $x\in[0,c]$ and $K_2(x)$ monotonically decreasing for $x\in[-c,0]$. Thus (10) and (11) give

$$a_s = c + \alpha^{-1}\ln(1-\alpha/|s|) .$$

Now $|s|$ takes on its minimum when $a_s = 0$. This implies that $|s| \ge \alpha/[1-\exp(-\alpha c)] > \alpha$. Thus

(C) $p(x) - s^{-2}p''(x) = (1-\alpha^2 s^{-2})p(x) > 0$ for $s\in(-\infty,-\alpha/[1-\exp(-\alpha c)]$ and for all $x\in[-a_s,a_s]$.

Therefore $R(D)$, $0 < D < D_{max}$, is given by (12) and (13). Calculation (routine and thus omitted) shows that

$$R(D_s) = \ln[|s|(1-\exp(-\alpha c))/\alpha] - \alpha \exp(-\alpha c)[c+\alpha^{-1}\ln(1-\alpha/|s|)]/[1-\exp(-\alpha c)] \qquad (33)$$

$$D_s = [|s|^{-1} + \alpha^{-1}\exp(-\alpha c)\ln(1-\alpha/|s|)]/[1-\exp(-\alpha c)] \qquad (34)$$

where $|s|\in[\alpha/(1-\exp(-\alpha c)),\infty)$ and

$$D_{max} = [\alpha^{-1} - (c+\alpha^{-1})\exp(-\alpha c)]/[1-\exp(-\alpha c)] . \qquad (35)$$

Example 2. Let X be a random variable with density

$$p(x) = 1/(x \,\ell n 100), \quad 0.01 \le x \le 1.$$

Then p(x) is continuous and differentiable with $\mu = 0.1$. Conditions (A) and (B) are clearly satisfied. Note that $K_1(x)$ decreases for $\mu \le x \le e^{-1}$ and then increases to $+\infty$ as $x \uparrow 1$. Condition (C) is not satisfied for all s but only for some s in $(-\infty, -2p(\mu))$. In this case, only a portion of R(D) can be obtained (corresponding to those D_s for which s satisfies (C)). We have

$$p(x) - p''(x)/s^2 = p(x)(1-2/s^2x^2) \ge 0$$

if and only if $x \ge \sqrt{2}/|s|$. Thus only for large $|s|$ (C) will be satisfied. For $s = -72.135$, we have $\sqrt{2}/|s| = .0196$ and from (10) $\mu - a_s = 0.02$. Thus for $s \le -72.135$ (C) is satisfied. This portion of s corresponds to a region of small distortion D (since s is the slope of R(D)) and for this region R(D) is given parametricaly by (12) and (13).

We now show that the class of continuous concave densities satisfies the assumptions of Theorem 1 and thus its rate distortion function can be obtained explicitly.

THEOREM 2. Let X *be a random variable with density* p(x) *which vanishes outside the interval* $[a,b]$, $-\infty < a < b < \infty$. *Suppose* p(x) *is a continuous concave function on* $[a,b]$ *and there is an at most finite set of points* $a = d_0 < d_1 < \dots < d_m < d_{m+1} = b (m \ge 0)$ *such that on each* $[d_j, d_{j+1}]$, $j=0,\dots,m$, p(x) *is differentiable and its derivative is absolutely continuous. Then the rate distortion function of* X *is given by* (12) *and* (13) .

Proof: Since p(x) is concave, $p''(x) \le 0$ and $p'_-(x) \ge p'_+(x)$. Also $\int_a^x p(t)dt > 0$ for $x > a$. For suppose $\int_a^{x_0} p(t)dt = 0$ for some $x_0 > a$. Then $p(t) = 0$ for each

$t \in [a, x_0]$ by continuity of p. Thus $p'(t) = 0$ for each $t \in [a, x_0]$. Since $p'_-(t) \geq p'_+(t)$, we have $p'(t) \leq 0$ for each $t \in [a, b]$. This implies $p(t) = 0$ for $t \in [a, b]$ which is a contradiction. Similarly $\int_x^b p(t)dt > 0$ for $x < b$. Thus the only assumption left to be verified in Theorem 1 is (B).

If $p(b) \neq 0$, then it is clear from (8) that $K_1(x) \to +\infty$ as $x \uparrow b$. Suppose now that $p(b) = 0$. Then by l'Hospital's rule

$$\lim_{x \uparrow b} K_1(x) = \lim_{x \uparrow b} - p'(x)/p(x) .$$

We will show that $p'(b) < 0$ and thus $K_1(x) \to +\infty$ as $x \uparrow b$. By the concavity of $p(x)$, $p'(x)$ is a non-increasing function. Suppose $p'(b) \geq 0$. Then $p'(x) \geq 0$ for all $x \in [a, b]$ for which the derivative exists. Thus $p(x)$ is non-decreasing. Since $p(b) = 0$, this implies that $p(x) = 0$ for all $x \in [a, b]$, which contradicts the fact that p is a density. Hence $p'(b) < 0$.

The proof of $K_2(x) \to +\infty$ as $x \downarrow a$ is of course similar.

It should be noted that it can also be shown that $K_1(x) \uparrow \infty$ as $x \uparrow b$ and $K_2(x) \uparrow \infty$ as $x \downarrow a$ monotonically. □

COROLLARY 2.1. Let X *be a random variable with continuous probability density function* p *consisting of line segments and vanishing outside a finite interval. Then the rate distortion function of* X *is given by* (12) *and* (13) *if and only if* p *is concave.*

Proof: It follows from Theorem 2 that if p is concave then its rate distortion function is given by (12) and (13).

Now suppose that p is not concave. Then there exist two adjacent line segments such that the left derivative at their common point is smaller than the right derivative. Hence for each s, $G_s(y)$ in the proof of Theorem 1 is not a probability distribution function and thus, by Theorem A, the parametric expressions (12) and (13) do not give the rate distortion function of X. □

<u>Example 3.</u> <u>Trapezoid density</u> (see Appendix 4 for the derivation). If $0 < c < a$ and

$$p(x) = \begin{cases} (a+c)^{-1} & |x| \le c \\ (a-|x|)/(a^2-c^2) & c \le |x| \le a \end{cases} \tag{36}$$

then for $0 < D \le (a-c)(a+2c)/3(a+c)$

$$\begin{aligned} R(D) &= 2\cos^2[\tfrac{4\pi}{3} + \tfrac{1}{3}\cos^{-1}(-3D/\sqrt{a^2-c^2})] - (a+3c)/2(a+c) \\ &\quad - \ln(2\sqrt{(a-c)/(a+c)}\,\cos[\tfrac{4\pi}{3} + \tfrac{1}{3}\cos^{-1}(-3D/\sqrt{a^2-c^2})]) \end{aligned} \tag{37}$$

and for $(a-c)(a+2c)/3(a+c) \le D \le D_{max} = (a^3-c^3)/3(a^2-c^2)$

$$R(D) = -\omega(D) - \ln(1-\omega(D)) \tag{38}$$

where $\omega(D) = [1-4D/(a+c)+(a-c)^2/3(a+c)^2]^{\frac{1}{2}}$.

Theorem 1 is also valid when the support of $p(x)$ is not finite. The result is stated in the following:

THEOREM 1'. Let X *be a random variable with density* $p(x)$*. If* $p(x)$ *satisfies all assumptions in Theorem 1 with* $-\infty \le a < b \le +\infty$*, then the rate distortion function of* X *is given by* (12) *and* (13) *with* $-\infty \le a < b \le +\infty$.

Theorem 1' is an improvement of Theorem 3 in [Tan and Yao, (1975)]. Here we no longer require the monotonicity of $K_i(x)$, $i=1,2$, and we allow $p'(x)$ to have a finite number of discontinuities instead of a single discontinuity at μ.

The following (known) result can be used along with Theorem 1 and 1' in evaluating the rate distortion functions of certain random variables.

THEOREM B. If the random variable X_i*, has distribution function* F_i*, and distortion rate function* $D_i(R)$, $i=1,2$, *then for all* $R > 0$,

$$|D_1(R)-D_2(R)| \le \int_{-\infty}^{\infty} |F_1(t)-F_2(t)|dt \ . \tag{39}$$

Thus if $F_n - F \to 0$ *in* L_1, *or if* $F_n \to F$ *weakly and* F_n, F *have finite means, then* $D_n(R) \to D(R)$ *uniformly.*

Proof: Corollary 1 of [Gray, Neuhoff and Schields (1975)] applied to i.i.d. sources with distributions F_1 and F_2 gives

$$|D_1(R)-D_2(R)| \le \bar{\rho}(F_1,F_2) \ .$$

But by [Vallender (1973)] $\bar{\rho}(F_1,F_2) = \int_{-\infty}^{\infty} |F_1(t)-F_2(t)|dt$ and thus (39) follows. Now if F_n converges to F weakly and all distributions involved have finite means, then by Theorem 2 of [Dobrushin (1970)], we have $\bar{\rho}(F_n,F) \to 0$ and hence $D_n(R) \to D(R)$ uniformly. □

The following well-known property is useful in connection with Theorem B: If a sequence of probability density functions p_n converges to a probability density function p almost everywhere, then the corresponding sequence of distributions F_n converges to the distribution F of p weakly. Thus by letting $c \to \infty$ in Example 1, we find the rate distortion function for the double exponential density on the entire real line (since all distributions involved have finite means), i.e. $R(D) = -\ln \alpha D$, $0 < D \le \alpha^{-1} = D_{max}$. Of course, this has been calculated by using the Shannon lower bound method [Berger (1971), p. 95]. Note that the double exponential density does not satisfy assumption (B) of Theorem 1', while the truncated double exponential densities satisfy all the assumptions of Theorem 1. This demonstrates how Theorem 1 and B can be used in evaluating the rate distortion function of certain random variables and following are some further examples.

Example 4. (a) Triangular density: Letting $c \downarrow 0$ in (36) we see that the trapezoid density converges to the triangular density

$$(a-|x|)/a^2, \quad |x| \le a \ .$$

Its rate distortion function is then found by letting $c \downarrow 0$ in (37): for

$0 < D \leq a/3 = D_{max}$,

$$R(D) = 2\cos^2[\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}(-\frac{3D}{a})] - \frac{1}{2} - \ln(2\cos[\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}(-\frac{3D}{a})]) . \qquad (40)$$

(b) Uniform density: Letting $c \uparrow a$ in (36) we see that the trapezoid density converges to the uniform density

$$1/(2a), \quad |x| \leq a ,$$

whose rate distortion function is thus found by letting $c \uparrow a$ in (38): for $0 < D \leq a/2 = D_{max}$,

$$R(D) = -\sqrt{1-\frac{2D}{a}} - \ln(1-\sqrt{1-\frac{2D}{a}}) . \qquad (41)$$

Note that the triangular and uniform densities satisfy all assumptions of Theorem 1 and thus (40) and (41) could be obtained directly from (12) and (13). (41) was first given by Tan and Yao in [Gray (editor) (1974)].

III. BOUNDS TO RATE DISTORTION FUNCTIONS

In this section, bounds are derived for rate distortion functions of random variables whose densities do not satisfy all the assumptions in the theorems of Section II. Examples are then given comparing these bounds with the Shannon lower bound.

THEOREM 3. Let X *be a random variable with probability density function* $p(x)$ *satisfying the assumptions in Theorem 1. Let* X_1 *be another random variable whose probability density function* $p_1(x)$ *vanishes outside the interval* $[a,b]$, *and* $p_1(x)$ *has at most a finite number of simple discontinuities. Then a lower bound for the rate distortion function of* X_1 *is given parametrically in* s *by*

$$R_L(D_s) = -H_p(p_1) + \ln\frac{|s|}{2} - \ln(p(\mu-a_s))\int_a^{\mu-a_s} p_1(x)dx$$
$$- \int_{\mu-a_s}^{\mu+b_s} p_1(x)\ln(ep(x))dx - \ln(p(\mu+b_s))\int_{\mu+b_s}^{b} p_1(x)dx \qquad (42)$$

$$D_s = \int_a^{\mu-a_s} (\mu-a_s-x)p_1(x)dx + \frac{1}{|s|}\int_{\mu-a_s}^{\mu+b_s} p_1(x)dx + \int_{\mu+b_s}^{b} (x-\mu-b_s)p_1(x)dx \tag{43}$$

where $H_p(p_1) = \int_a^b p_1(x)\ln \frac{p_1(x)}{p(x)} dx$ *is the generalized entropy of* p_1 *with respect to* p *[Pinsker (1964), p. 18];* μ *is the median of* p *and* a_s *and* b_s *are related to* s *by* (10) *and* (11).

Proof: Since $p(x)$ satisfies the assumptions of Theorem 1, $\lambda_s(x)$ given by (30) satisfies

$$C_s(y) = \int_a^b \lambda_s(x)p(x)\exp(s|x-y|)dx \le 1 \text{ , for all } y\epsilon[a,b].$$

Now define $\lambda_s^{(1)}(x)$ by

$$\lambda_s^{(1)}(x)p_1(x) = \lambda_s(x)p(x) \ . \tag{44}$$

Then $\lambda_s^{(1)}(x)$ also satisfies

$$C_s(y) = \int_a^b \lambda_s^{(1)}(x)p_1(x)\exp(s|x-y|)dx \le 1, \text{ for all } y\epsilon[a,b].$$

According to Theorem A, $\lambda_s^{(1)}(x)$ yields a lower bound to the rate distortion function of X_1, i.e.

$$\sup_{s\le 0}(sD + \int_a^b p_1(x)\ln \lambda_s^{(1)}(x)dx) \ .$$

Let $R_L(D,s) = sD + \int_a^b p_1(x)\ln \lambda_s^{(1)}(x)dx$ and let d_j, $j=1,\ldots,n$, $a = d_0 < d_1 < \ldots < d_n < d_{n+1} = b$ be the points where $p_1(d_j)$ has simple discontinuities. Then from (44), (30) and (A.5), $p_1(x)\ln \lambda_s^{(1)}(x)$ is continuous both in s and x and its partial derivative with respect to s exists for each $x\epsilon[a,b]$ and $s \le 0$ and is bounded by a constant. Thus

$$\frac{\partial R_L(D,s)}{\partial s} = D + \{\int_a^{d_1} + \sum_{j=1}^{n-1}\int_{d_j}^{d_{j+1}} + \int_{d_n}^{b}\} \frac{p_1(x)}{\lambda_s^{(1)}(x)} \frac{\partial \lambda_s^{(1)}(x)}{\partial s} dx \ .$$

Setting $\frac{\partial R_L(D,s)}{\partial s} = 0$ and substituting (44) in the above expression, we have

$$D_s = -\int_a^b \frac{p_1(x)}{\lambda_s(x)} \frac{\partial\lambda_s(x)}{\partial s}\, dx \ . \tag{45}$$

Thus for each fixed D, if

$$\partial^2 R_L(D,s)/\partial s^2 \le 0 \quad \text{for all } s \le 0 \tag{46}$$

then $R_L(D,s)$ as a function of s is concave and its supremum is achieved by the point s_D satisfying (45), i.e.

$$R_L(D,s_D) = \sup_{s\le 0} R_L(D,s) \ .$$

Whether or not (46) is satisfied, $R_L(D_s) = R_L(D_s,s)$ along with (45) provide the parametric expressions (in s) of a lower bound of the rate distortion function of X_1. Substituting (A.5) into (45), we obtain (43). Substituting (43) and (30) into (4), we obtain (42). Note that the generalized entropy of p_1 with respect to p has the property $H_p(p_1) \ge 0$ with equality if and only if $p(x) = p_1(x)$ a.s. [Pinsker (1964), p. 19]. a_s and b_s are related to s by (10) and (11). It should be clear from (42) that R_L is useful only when $H_p(p_1) < \infty$. □

Clearly Theorem 3 is also valid when the support of the densities is not finite.

THEOREM 4. For each fixed $s \le 0$, $R_L(D_s)$ *given in Theorem 3 is equal to* $R(D_s)$ *if and only if there exists a probability distribution function* Q_s *whose total probability is concentrated on (a subset of)* $[a,b]$ *and is such that*

$$p_1(x) = \begin{cases} \dfrac{|s|e^{-s(\mu-a_s-x)}p(x)}{2p(\mu-a_s)} \displaystyle\int_a^b e^{s|x-y|}dQ_s(y) \ , & x\in[a,\mu-a_s) \ , \\[2ex] \dfrac{|s|}{2} \displaystyle\int_a^b e^{s|x-y|}dQ_s(y) \ , & x\in[\mu-a_s,\mu+b_s] \ , \\[2ex] \dfrac{|s|e^{s(\mu+b_s-x)}p(x)}{2p(\mu+b_s)} \displaystyle\int_a^b e^{s|x-y|}dQ_s(y) \ , & x\in(\mu+b_s,b] \ , \end{cases} \tag{47}$$

where a_s *and* b_s *are given by* (10) *and* (11), *and*

$$\int_a^b \lambda_s(x)p(x)\exp(s|x-y|)dx = 1 \quad \text{a.e. } [dQ_s] \ . \tag{48}$$

Proof: Suppose the assumptions are satisfied for a given $s \le 0$. Substituting (30) into (47) and using (44), we have

$$p_1(x) = \lambda_s(x)p(x)\int_a^b \exp(s|x-y|)dQ_s(y) = \lambda_s^{(1)}(x)p_1(x)\int_a^b \exp(s|x-y|)dQ_s(y) .$$

If $p_1(x) \neq 0$, then we have

$$[\lambda_s^{(1)}(x)]^{-1} = \int_a^b \exp(s|x-y|)dQ_s(y) \ . \tag{49}$$

If $p_1(x_0) = 0$ for some $x_0 \in [a,b]$, then from (47), $p_1(x_0) = p(x_0) = 0$ for some $x_0 \in [a,\mu-a_s)\cup(\mu+b_s,b]$. In this case, we can define $\lambda_s^{(1)}(x)$ by (49). Therefore $R_L(D_s) = R(D_s)$ by Theorem A. Conversely, for a given $s \le 0$, suppose $R_L(D_s) = R(D_s)$. Then $\lambda_s^{(1)}(x)$ achieves the supremum in (2) and hence it satisfies (3), i.e. (49), and is such that $C_s(y) = 1$ a.e. $[dQ_s(y)]$ for some probability distribution Q_s. Substituting (44) and (30) into (49), we obtain (47). □

It would be of interest to compare the lower bound of Theorem 3 with the Shannon lower bound which is now computed for densities which vanish outside the interval $[a,b]$, $-\infty < a < b < \infty$.

THEOREM C. Let X *be a random variable with probability density function* $p(x)$ *which vanishes outside* $[a,b]$, $-\infty < a < b < \infty$. *Then the Shannon lower bound to* $R(D)$ *of* X *is given parametrically in* $s<0$ *by*

$$R_{sL}(D_s) = h(p) - |s|(b-a)/2k(s) + \ln|s|/2ek(s) \tag{50}$$

$$D_s = |s|^{-1} + (b-a)/2k(s) \tag{51}$$

where $h(p) = -\int_a^b p(x)\ln p(x)dx$ *and* $k(s) = 1 - \exp(|s|(b-a)/2)$. *Moreover,*

$R_{SL}(D) < R(D)$ *for all* D *with* $R(D) > 0$ *unless*

$$p(x) = |s|\exp(s|x-(a+b)/2|)/2k(s), \quad x\epsilon[a,b] \ , \tag{52}$$

in which case $R_{SL}(D_0) = R(D_0)$ *at the point* D_0 *with slope* s.

Proof: Let

$$\lambda_s(x)p(x) = \begin{cases} K_s & \text{if } p(x) > 0 \\ 0 & \text{otherwise} . \end{cases} \tag{53}$$

From condition (1), since

$$C_s(y) = K_s \int_a^b \exp(s|x-y|)dx \ , \qquad y\epsilon[a,b] \ ,$$

one can take

$$K_s = s/2[\exp(s/2(b-a)) - 1] \ .$$

Theorem A and a simple calculation yield (50) and (51).

Now from Theorem A, $R_{SL}(D) = R(D)$ at a point D with slope s, if and only if $C_s(y) = 1$ a.e. $[dG_s]$. For the Shannon lower bound, it can be shown easily that for each $s \leq 0$, $C_s(y) = 1$ if and only if $y = \frac{1}{2}(a+b)$. Hence $R_{SL}(D) = R(D)$ at D with slope s if and only if G_s puts its total probability mass at $\frac{1}{2}(a+b)$ in which case

$$[\lambda_s(x)]^{-1} = \int_a^b \exp(s|x-y|)dG_s(y) = \exp(s|x-\tfrac{1}{2}(a+b)|) \ . \tag{54}$$

(52) now follows from (53) and (54) and can be verified as a density. Hence there is one $s < 0$ such that at the point D_0 with slope s, we have $R_{SL}(D_0) = R(D_0)$. For all other densities p which vanish outside $[a,b]$, we have $R_{SL}(D) < R(D)$ for all $D > 0$ such that $R(D) > 0$. $\square$

Example 5. Let $p(x)$ be the uniform density on $[a,b]$. Then $p(x)$ satisfies all assumptions in Theorem 1. Let $p_1(x)$ be a piecewise continuous density defined on $[a,b]$. Then by applying Theorem 3, a lower bound for $R(D)$ of $p_1(x)$ can be found. Evaluation of (42) and (43) shows that (see Appendix 5) for $s < -(b-a)^{-1}$

$$R_L(D_s) = h(p_1) + \ell n|s|/2e + \int_0^{|s|^{-1}} [p_1(a+t) + p_1(b-t)]dt \tag{55}$$

$$D_s = |s|^{-1} - \int_0^{|s|^{-1}} t[p_1(a+t) + p_1(b-t)]dt \; . \tag{56}$$

In (56), if for a given D, $|s|$ is not single-valued, and if a branch of $|s|$ can be chosen such that (46) is satisfied, then for this branch of $|s|$, $R_L(D)$ is the best possible lower bound achieved by the method of Theorem 3. Note that condition (46) is equivalent to

$$p_1(a+|s|^{-1}) + p_1(b-|s|^{-1}) < |s| \tag{57}$$

in this example (see Appendix 5).

The lower bound of Theorem 3 is of course useful when Theorem 1 is not applicable to $p_1(x)$. As an illustration we now calculate the lower bound of Theorem 3 when p_1 is the truncated double exponential density of Example 1. In this case the rate distortion function of p_1 has been calculated in Section II and therefore we can see how tight is the lower bound determined by (55) and (56). Calculations show the following:

$$h(p) = 1 - \frac{\alpha}{2} + \ell n\; 2u(\alpha) - \tfrac{1}{2}u(\alpha)^{-1}$$

$$R_L(D_s) = -\frac{\alpha}{2} + \ell n|s|u(\alpha) + [2\exp(\alpha/|s|) - \alpha - 2]/2\alpha u(\alpha) \qquad |s| > 2 \tag{58}$$

$$D_s = |s|^{-1} - [\alpha u(\alpha)]^{-1}[(|s|^{-1} - \alpha^{-1})\exp(\alpha/|s|) + \alpha^{-1}]$$

$$D_{max} = [\frac{1}{\alpha} - (\tfrac{1}{2} + \alpha^{-1})\exp(\alpha/2)]/[1-\exp(-\alpha/2)]$$

where $u(\alpha) = [\exp(\alpha/2) - 1]/\alpha$. For the Shannon lower bound we have

$$R_{SL}(D_s) = (|s|-\alpha)/2 - \tfrac{1}{2}u(\alpha)^{-1} + \ell n\ 2u(\alpha) + \ell n\ v(s) + v(s)$$
$$D_s = |s|^{-1}(1 - v(s)) \ . \qquad |s| > 0 \tag{59}$$

where $v(s) = |s|/2(\exp(|s|/2) - 1)$. Curves are plotted for $\alpha = 0.1, 2$ in Figures 1 and 2. In general, R_L is a better lower bound than R_{SL}, except in a small neighborhood of D_{max} where R_{SL} is better. As $\alpha \to 0$, the difference between R_L and R_{SL} becomes larger and as $\alpha \to \infty$, the difference becomes smaller. It can also be seen that R_L is a very good approximation to R. If, for $\alpha > 0$ fixed, we plot D as a function of $|s|$ as given by (58), we obtain a curve as shown in Figure 3. Note that at $|s| = 2$, $D_s = D_{max}$. Also D_s achieves its maximum at some point $|s_0| > 2$, and D_s is a decreasing function for all $|s| \geq |s_0|$. It follows (as it is easily checked analytically) that for all $|s| \geq |s_{max}|$, condition (57) is satisfied and thus the branch of $|s|$: $|s| > |s_{max}|$ gives the tightest possible lower bound $R_L(D_s)$.

Another lower bound for the rate distortion function of the truncated *double* exponential density can be obtained by using the truncated Gaussian density instead of the uniform density, i.e.,

$$p(x) = K \exp(-x^2/2\sigma^2) \ , \quad |x| \leq \tfrac{1}{2}$$

where $K^{-1} = \sqrt{2\pi}\ [2\Phi(1/2\sigma) - 1]$ and Φ is the standard normal distribution. Since p(x) satisfies all assumptions in Theorem 1, the following lower bound for the rate distortion function of $p_1(x)$ is obtained by Theorem 3:

$$\begin{aligned} R_L(D_s) = {} & \ell n[|s|K(1-\exp(-\alpha/2))/\alpha] \\ & + [\{(8\sigma^2)^{-1} - (1-\alpha^{-2}\sigma^{-2})(1+\alpha/2) + \ell n\ K - a_s^2/2\sigma^2\}\exp(-\alpha/2) \\ & - \ell n\ K + (1-(a_s+\alpha^{-1})/\alpha\sigma^2)\exp(-\alpha a_s)]/[1-\exp(-\alpha/2)] \end{aligned}$$
$$D_s = [|s|^{-1} + (a_s-\tfrac{1}{2}-\alpha^{-1})\exp(-\alpha/2) - (|s|^{-1}-\alpha^{-1})\exp(-\alpha a_s)]/[1-\exp(-\alpha/2)] \ .$$

The relationship between a_s and $s < 0$ is given by

$$|s| = \exp[-a_s^2/(2\sigma^2)]/\{[\Phi((2\sigma)^{-1}) - \Phi(a_s/\sigma)]\sqrt{2\pi}\,\sigma\}.$$

Numerical calculations show that this lower bound is slightly better than the one obtained via the *uniform distribution.*

If the method used in Theorem 1 is applied to a discontinuous probability density function, a $\lambda_s(x) \geq 0$ may be found satisfying condition (1) whereas a distribution function $G_s(y)$ satisfying (6) and (7) may not exist. In this case, using the above mentioned λ_s, a lower bound for the rate distortion function of the discontinuous density can be obtained by (2). The following example illustrates this point.

Example 6. (For the derivation, see Appendix 6.) Let

$$p(x) = \begin{cases} 1/4 & -1 \leq x < 0 \\ 3/8 & 0 \leq x \leq 2 . \end{cases}$$

Then for $0 < D \leq 9/16$, we have

$$R_L(D) = -\tfrac{1}{2}\sqrt{4-5D} - \ln(2-\sqrt{4-5D}) - \tfrac{1}{4}\ln(54/625) \tag{60}$$

and for $13/24 \leq D \leq 17/24 = D_{max}$, R(D) itself can be found and is given by

$$R(D) = -\sqrt{1-(24D-1)/16} - \ln(1-\sqrt{1-(24D-1)/16}) . \tag{61}$$

Theorem B may yet be used in another way to find bounds for distortion rate functions of discontinuous densities.

Example 7. Let p(x) be the density of Example 6 and, for $0 < \varepsilon < 1$, consider the continuous approximating density

$$p_\varepsilon(x) = \begin{cases} 1/4 & -1 \le x \le -\varepsilon \\ x^2/16\varepsilon^2 + x/8\varepsilon + 5/16 & -\varepsilon \le x \le 0 \\ -x^2/16\varepsilon^2 + x/8\varepsilon + 5/16 & 0 \le x \le \varepsilon \\ 3/8 & \varepsilon \le x \le 2 \end{cases} .$$

Note that $p_\varepsilon(x)$ converges to $p(x)$ almost everywhere as $\varepsilon \to 0$. In this case the $R_\varepsilon(D)$ of p_ε can be evaluated by Theorem 1 and for $\varepsilon = 0.59367$, we have

$$D_\varepsilon(R) - 0.00367 \le D_L(R) \le D(R) \le D_\varepsilon(R) + 0.00367$$

where $D_L(R)$ is the inverse function of $R_L(D)$ in Example 6. (For the evaluation of $D_\varepsilon(R)$, see [Leung (1976)].)

IV. APPENDIX

Appendix 1. Derivation of (26)

Substituting (25) into (20) we have

$$\int_{V_s} [p(t)-s^{-2}p''(t)+C_{1,s}\delta(t-\mu+a_s)+C_{2,s}\delta(t-\mu-b_s) + \sum_{j=\ell}^{\ell+n-1} D_{j,s}\delta(t-d_j)]e^{s|x-t|}dt = \frac{2}{|s|} p(x),$$
$$x \in V_s .$$

Suppose $x \in (d_k, d_{k+1}]$, where $\mu - a_s \le d_k < \mu + b_s$. Then

$$\int_{\mu-a_s}^{\mu+b_s} (p(t)-s^{-2}p''(t))e^{s|x-t|}dt + C_{1,s}e^{s(x-\mu+a_s)} + C_{2,s}e^{s(\mu+b_s-x)}$$
$$+ \sum_{j=\ell}^{k} D_{j,s}e^{s(x-d_j)} + \sum_{j=k+1}^{\ell+n-1} D_{j,s}e^{s(d_j-x)} = \frac{2}{|s|} p(x) . \tag{A.1}$$

Now

$$\int_{\mu-a_s}^{\mu+b_s} (p(t)-s^{-2}p''(t))e^{s|x-t|}dt = \int_{\mu-a_s}^{d_\ell} + \sum_{j=\ell}^{\ell+n-2} \int_{d_j}^{d_{j+1}} + \int_{d_{\ell+n-1}}^{\mu+b_s}$$
$$(p(t)-s^{-2}p''(t))e^{s|x-t|}dt . \tag{A.2}$$

Since for each $s \in (-\infty, -2p(\mu))$, $(p(t)-s^{-2}p''(t))e^{s(x-t)}$ is a.e. [Leb.] on $[\mu-a_s, d_\ell]$ equal to the derivative of $-s^{-2}(p'(t) + sp(t))e^{s(x-t)}$ which by assumption (A) is absolutely continuous, we have

$$\int_{\mu-a_s}^{d_\ell} (p(t)-s^{-2}p''(t))e^{s(x-t)}dt = -\frac{e^{sx}}{s^2}[e^{-st}p'(t) + se^{-st}p(t)]\Big|_{\mu-a_s}^{d_\ell}$$

$$= -\frac{e^{sx}}{s^2}[e^{-sd_\ell}p'_-(d_\ell) + se^{-sd_\ell}p(d_\ell) - e^{-s(\mu-a_s)}p'_+(\mu-a_s) - se^{-s(\mu-a_s)}p(\mu-a_s)] .$$

Similarly, for $\ell \le j \le k-1$,

$$\int_{d_j}^{d_{j+1}} (p(t)-s^{-2}p''(t))e^{s(x-t)}dt$$

$$= -\frac{e^{sx}}{s^2}[e^{-sd_{j+1}}p'_-(d_{j+1}) + se^{-sd_{j+1}}p(d_{j+1}) - e^{sd_j}p'_+(d_j) - se^{-sd_j}p(d_j)] ;$$

$$\int_{d_k}^{x} (p(t) - \frac{p''(t)}{s^2})e^{s(x-t)}dt$$

$$= -\frac{1}{s^2}p'(x) + \frac{1}{|s|}p(x) + s^{-2}[p'_+(d_k) - |s|p(d_k)]e^{s(x-d_k)} ;$$

for $k + 1 \le j \le \ell+n-2$,

$$\int_{x}^{d_{k+1}} (p(t) - s^{-2}p''(t))e^{s(t-x)}dt$$

$$= s^{-2}p'(x) + \frac{1}{|s|}p(x) - s^{-2}[|s|p(d_{k+1}) + p'_-(d_{k+1})]e^{s(d_{k+1}-x)}$$

$$\int_{d_j}^{d_{j+1}} (p(t) - s^{-2}p''(t))e^{s(t-x)}dt$$

$$= -\frac{e^{-sx}}{s^2}[e^{sd_{j+1}}p'_-(d_{j+1}) - se^{sd_{j+1}}p(d_{j+1}) - e^{sd_j}p'_+(d_j) + se^{sd_j}p(d_j)] ;$$

$$\int_{d_{\ell+n-1}}^{\mu+b_s} (p(t) - s^{-2}p''(t))e^{s(t-x)}dt$$

$$= -\frac{e^{-sx}}{s^2}[e^{s(\mu+b_s)}p'_-(\mu+b_s) - se^{s(\mu+b_s)}p(\mu+b_s) - e^{sd_{\ell+n-1}}p'_+(d_{\ell+n-1})$$
$$+ se^{sd_{\ell+n-1}}p(d_{\ell+n-1})] .$$

Substituting the above expressions in (A.2) and combining similar terms, we have

$$\int_{\mu-a_s}^{\mu+b_s} (p(t) - s^{-2}p''(t))e^{s|x-t|}dt$$

$$= \frac{2}{|s|}p(x) - \frac{1}{s^2}[|s|p(\mu-a_s) - p'_+(\mu-a_s)]e^{s(x-\mu+a_s)} - \frac{1}{s^2}[|s|p(\mu+b_s) + p'_-(\mu+b_s)]e^{s(\mu+b_s-x)} -$$

$$-\sum_{j=\ell}^{k} s^{-2}[p'_-(d_j) - p'_+(d_j)]e^{s(x-d_j)} - \sum_{j=k+1}^{\ell+n-1} s^{-2}[p'_-(d_j) - p'_+(d_j)]e^{s(d_j-x)} .$$

Equating coefficients in (A.1) we obtain (26).

Appendix 2. Calculation of $\lambda_s(x)$ (i.e. (30)).

From (6), we have, for $x\in[a,\mu-a_s]$

$$[\lambda_s(x)]^{-1} = \int_{\mu-a_s}^{\mu+b_s} \exp(s(t-x))dG_s(t)$$

and substituting (25) and (26), we obtain

$$[\lambda(x)]^{-1} = \{ \int_{\mu-a_s}^{d_\ell} + \sum_{j=\ell}^{\ell+n-2} \int_{d_j}^{d_{j+1}} + \int_{d_{\ell+n-1}}^{\mu+b_s} \}[p(t)-s^{-2}p''(t)]\exp(s(t-x))dt$$

$$+ \exp(\mu-a_s-x)[|s|p(\mu-a_s) - p'_+(\mu-a_s)]/s^2$$

$$+ \sum_{j=\ell}^{\ell+n-1} s^{-2}[p'_-(d_j)-p'_+(d_j)]\exp(s(d_j-x))$$

$$+ s^{-2}[|s|p(\mu+b_s) + p'_-(\mu+b_s)]\exp(s(\mu+b_s-x)) .$$

The integrals in the above expression have been calculated in Appendix 1 (except for minor adjustments in the exponents). Thus, using results similar to those in Appendix 1, we have

$$\lambda_s(x)^{-1} = 2\exp[s(\mu-a_s-x)]p(\mu-a_s)/|s|$$

and the expression for $\lambda_s(x)$, $x\in[a,\mu-a_s]$, given in (30). The calculation of $\lambda_s(x)$ for $x\in[\mu+b_s,b]$ is similar. For $x\in[\mu-a_s, \mu+b_s]$ we have

$$\lambda_s(x) = \alpha_s(x)/p(x) = |s|/2p(x)$$

from the solution of (17).

Appendix 3. Derivation of (12) and (13).

The relation

$$D_s = -\int_a^b (p(x)/\lambda_s(x))(\partial\lambda_s(x)/\partial s)dx$$

is proved as in [Tan and Yao (1975)] and thus its proof is omitted here. The calculation of $\partial\lambda_s(x)/\partial s$ is given in the following: From (30), we have for all $x\epsilon(a,\mu-a_s)$

$$\frac{\partial\lambda_s(x)}{\partial s} = [-\frac{1}{2p(\mu-a_s)} + \frac{s}{2p^2(\mu-a_s)} \cdot \frac{\partial p(\mu-a_s)}{\partial s}]\exp(-s(\mu-a_s-x)) \tag{A.3}$$

$$-\frac{s}{2p(\mu-a_s)}[s\frac{\partial a_s}{\partial s} - (\mu-a_s-x)]\exp(-s(\mu-a_s-x)) \ .$$

Differentiating (21) with respect to s, we have

$$-p(\mu-a_s)\frac{da_s}{ds} = -s^{-2}[s\frac{\partial p(\mu-a_s)}{\partial s} - p(\mu-a_s)]$$

and thus

$$\frac{da_s}{ds} = \frac{1}{sp(\mu-a_s)} \cdot \frac{\partial p(\mu-a_s)}{\partial s} - s^{-2} \ . \tag{A.4}$$

Substituting (A.4) into (A.3), we obtain

$$\frac{\partial\lambda_s(x)}{\partial s} = -(\mu-a_s-x)\lambda_s(x) \quad \text{for all } x\epsilon(a,\mu-a_s) \ .$$

The calculation for $x\epsilon(\mu+b_s,b)$ is similar. Thus

$$\frac{\partial\lambda_s(x)}{\partial s} = \begin{cases} -(\mu-a_s-x)\lambda_s(x) & x\epsilon(a,\mu-a_s) \\ -\frac{1}{|s|}\lambda_s(x) & x\epsilon(\mu-a_s,\mu+b_s) \\ -(x-\mu-b_s)\lambda_s(x) & x\epsilon(\mu+b_s,b) \ . \end{cases} \tag{A.5}$$

Substituting (A.5) in the expression for D_s we obtain (13). From (4) and (A.5) we have

$$R(D_s) = sD_s + \ln\frac{|s|}{2} + \int_a^{\mu-a_s} s(x-\mu+a_s)p(x)dx - \ln(p(\mu-a_s))\int_a^{\mu-a_s} p(x)dx$$

$$-\int_{\mu-a_s}^{\mu+b_s} p(x)\ln p(x)dx + \int_{\mu+b_s}^{b} s(\mu+b_s-x)p(x)dx - \ln(p(\mu+b_s))\int_{\mu+b_s}^{b} p(x)dx .$$

Substituting (A.6) into the above expression, we obtain (12). Finally, since $D_{max} = \inf_y \int_a^b |x-y|p(x)dx$, it is easy to verify that the infimum is achieved when $y = \mu$, the median of $p(x)$, and thus $D_{max} = \int_a^b |x-\mu|p(x)dx$.

Appendix 4. Derivation of Example 3.

Here we only sketch the calculations in Example 3 and omit the lengthy details. Since $p(x)$ is symmetric about $x = 0$, we have $a_s = b_s$ and (11) gives the relationship between $|s|$ and a_s:

$$|s| = 2/(a-a_s) \qquad \text{for } a_s \in [c,a] \tag{A.7}$$

$$|s| = 2/(a+c-2a_s) \qquad \text{for } a_s \in [0,c] . \tag{A.8}$$

We now distinguish two cases (i) and (ii).

(i) If $|s| \geq 2/(a-c)$, using (A.7), (12) and (13), we obtain

$$\begin{aligned} R(D_s) &= \ln\frac{|s|(a+c)}{2} + 2/s^2(a^2-c^2) - (a+3c)/2(a+c) \\ D_s &= \frac{1}{|s|} - 4/3|s|^3(a^2-c^2) . \end{aligned} \tag{A.9}$$

The parameter s in (A.9) can be eliminated as follows: Substituting $x = 2/|s|(a+c)$ into (A.9), we obtain the following cubic equation:

$$x^3 - \frac{3(a-c)}{a+c}x + \frac{6(a-c)D}{(a+c)^2} = 0.$$

The solutions of this cubic equation consist of three unequal real roots

$$x_k = 2\sqrt{(a-c)/(a+c)}\,\cos[\tfrac{1}{3}\cos^{-1}(-3D/\sqrt{a^2-c^2} + 2k\pi/3)], \qquad k=0,1,2 .$$

It can be shown that only x_2 satisfies the condition

$$|s| \geq 2/(a-c) .$$

Therefore

$$R(D) = -\ln x_2 + x_2^2(a+c)/2(a-c) - (a+3c)/2(a+c)$$

which gives the expression in (37) for $0 < D \leq \dfrac{(a-c)(a+2c)}{3(a+c)}$.

(ii) If $|s| \leq 2/(a-c)$, using (A.8), (12) and (13), we obtain

$$R(D_s) = \ln|s|(a+c)/2 + 2/|s|(a+c) - 1$$
$$D_s = |s|^{-1} - 1/s^2(a+c) + (a-c)^2/12(a+c). \tag{A.10}$$

Eliminating s in (A.10), we obtain the expression in (38) for $(a-c)(a+2c)/3(a+c) \leq D \leq (a^3-c^3)/3(a^2-c^2) = D_{max}$.

Appendix 5. Derivation of Example 5.

In Theorem 3, let $p(x) = (b-a)^{-1}$ for $x \in [a,b]$. Then from (10) and (11), we have

$$a_s = b_s = \tfrac{1}{2}(b-a) - |s|^{-1}, \ \mu = \tfrac{1}{2}(b-a) \ .$$

Also

$$-H_p(p_1) = -\int_a^b p_1(x)\ln(p_1(x)(b-a))dx = h(p_1) - \ln(b-a) \ .$$

Thus (42) becomes

$$R_L(D_s) = h(p_1) - \ln(b-a) + \ln\frac{|s|}{2} + \ln(b-a)\int_a^{a+|s|^{-1}} p_1(x)dx$$
$$-\int_{a+|s|^{-1}}^{b-|s|^{-1}} p_1(x)\ln\frac{e}{b-a}\,dx + \ln(b-a)\int_{b-|s|^{-1}}^{b} p_1(x)dx$$
$$= h(p_1) + \ln\frac{|s|}{2} + \int_{a+|s|^{-1}}^{b-|s|^{-1}} p_1(x)dx$$
$$= h(p_1) + \ln\frac{|s|}{2} - 1 + \int_a^{a+|s|^{-1}} p_1(x)dx + \int_{b-|s|^{-1}}^{b} p_1(x)dx$$
$$= h(p_1) + \ln\frac{|s|}{2e} + \int_0^{|s|^{-1}} [p_1(a+t) + p_1(b-t)]dt \ .$$

From (43) we have

$$D_s = \int_a^{a+|s|^{-1}} (a + \frac{1}{|s|} - x) p_1(x)dx + \frac{1}{|s|} \int_{a+|s|^{-1}}^{b-|s|^{-1}} p_1(x)dx + \int_{b-|s|^{-1}}^{b} (x - b + \frac{1}{|s|}) p_1(x)dx$$

$$= \frac{1}{|s|} - \int_0^{|s|^{-1}} t[p_1(a+t) + p_1(b-t)]dt \; .$$

This proves (55) and (56).

Now condition (46) is equivalent to

$$\int_a^b \frac{p_1(x)}{\lambda_s(x)} [\frac{\partial^2 \lambda_s(x)}{\partial s^2} - \frac{1}{\lambda_s(x)} (\frac{\partial \lambda_s(x)}{\partial s})^2]dx < 0$$

which is equivslent to (57) when $\lambda_s(x)$ is replaced by (30) for $p(x) = (b-a)^{-1}$.

Appendix 6. Derivation of Example 6.

Calculation shows that $\mu = 2/3$. Let $V_s = [\frac{2}{3} - a_s, \frac{2}{3} + b_s]$ be the support of the distribution $G_s(y)$.

(i) Suppose $\frac{2}{3} - a_s \geq 0$. This case is similar to the case of the uniform distribution. One can apply Theorem 1 directly to get:

$$R(D_s) = \frac{3}{4|s|} - 1 + \ell n \frac{4|s|}{3},$$

$$D_s = \frac{1}{|s|} - \frac{3}{8|s|^2} + \frac{1}{24} \; , \qquad \frac{3}{4} \leq |s| \leq \frac{3}{2} \; .$$

Eliminating $|s|$, one obtaines the desired expression (61) for $\frac{13}{24} \leq D \leq \frac{17}{24} = D_{max}$.

(ii) Suppose $-1 \leq \frac{2}{3} - a_s < 0$. Then $g_s(y)$ has the form for all $y \in V_s$

$$g_s(y) = p(y) + C_{1,s}\delta(y - \frac{2}{3} + a_s) + C_{2,s}\delta'(y-0) + C_{3,s}\delta(y - \frac{2}{3} - b_s)$$

where

$$C_{1,s} = \frac{1}{4|s|} \; , \quad C_{2,s} = -\frac{1}{8|s|^2} \; , \quad C_{3,s} = \frac{3}{8|s|} \; .$$

Also

$$a_s = \frac{5}{3} - \frac{1}{|s|} \quad , \quad b_s = \frac{4}{3} - \frac{1}{|s|} \; .$$

From these expressions, $\lambda_s(x)$ can be calculated and has the following form:

$$\lambda_s(x) = \begin{cases} 2|s|e^{|s|(\frac{2}{3} - a_s) - |s|x} & -1 \le x \le \frac{2}{3} - a_s \\ 2|s| & \frac{2}{3} - a_s \le x < 0 \\ \frac{4}{3}|s| & 0 \le x \le \frac{2}{3} + b_s \\ \frac{4}{3}|s|e^{-|s|(\frac{2}{3} + b_s) + |s|x} & \frac{2}{3} + b_s \le x \le 2. \end{cases}$$

It remains to show that $C_s(y) \le 1$ for $y \notin V_s$. We will show this when $y \ge \frac{2}{3} + b_s$. The case $y \le \frac{2}{3} - a_s$ is similar. Now

$$C_s(y) = \int_{-1}^{2} \lambda_s(x)p(x)\exp(s|x-y|)dx = \int_{-1}^{(2/3)-a_s} \frac{1}{2}|s|\exp(|s|(\frac{2}{3} - a_s - y)$$

$$+ \int_{(2/3)-a_s}^{0} \frac{1}{2}|s|\exp(-|s|(y-x))dx + \int_{0}^{(2/3)+b_s} \frac{1}{2}|s|\exp(-|s|(y-x))dx$$

$$+ \int_{(2/3)+b_s}^{y} \frac{1}{2}|s|\exp(-|s|(\frac{2}{3} + b_s + 2x - y)dx + \int_{y}^{2} \frac{1}{2}|s|\exp(-|s|(\frac{2}{3} + b_s) + y)dx$$

$$= \frac{1}{4}\exp[|s|(\frac{2}{3} + b_s - y)] + (\frac{1}{4} + |s| - \frac{1}{2}|s|y)\exp[|s|(y - \frac{2}{3} - b_s)] .$$

Differentiating $C_s(y)$ with respect to y, we have

$$C_s'(y) = |s|\exp[|s|(y - \frac{2}{3} - b_s)]f(y)$$

where $f(y) = |s| - \frac{1}{4} - \frac{1}{2}|s|y - \frac{1}{4}\exp[2|s|(\frac{2}{3} + b_s - y)]$. Now $f'(y) =$

$\frac{1}{2}|s|\{\exp[2|s|(\frac{2}{3} + b_s - y)] - 1\}$, and $f'(y_0) = 0$ if and only if $y_0 = \frac{2}{3} + b_s$. Since $f''(y_0) = -s^2 < 0$, $f(y_0)$ is a maximum. But $f(y_0) = 0$ and thus $f(y) \le 0$ and $C_s'(y) \le 0$ which implies that $C_s(y)$ is non-increasing. Since $y_0 = \frac{2}{3} + b_s = 2 - |s|^{-1}$, we have $C_s(y_0) = 1$. Thus

$$C_s(y) \le 1 \quad \text{for all} \quad y \ge y_0 = \frac{2}{3} + b_s .$$

Since $g_s(y)$ is not a probability density function, the $\lambda_s(x)$ found above can be used only to provide a lower bound to the R(D) of p(x). Thus by Theorem A

$$R_L(D,s) = -|s|D + 5/(16|s|) + \frac{1}{4}\ln 2|s| + (3/4)\ln(4|s|/3) .$$

Differentiating $R_L(D,s)$ with respect to $|s|$ and setting $R_L'(D,s) = 0$, we obtain

$$D_s = |s|^{-1} - 5/(16|s|^2) , \quad |s| > 1 .$$

Also, $R_L''(D,s) = 5/(8|s|^3) - |s|^{-1} < 0$. Eliminating $|s|$ we obtain (60).

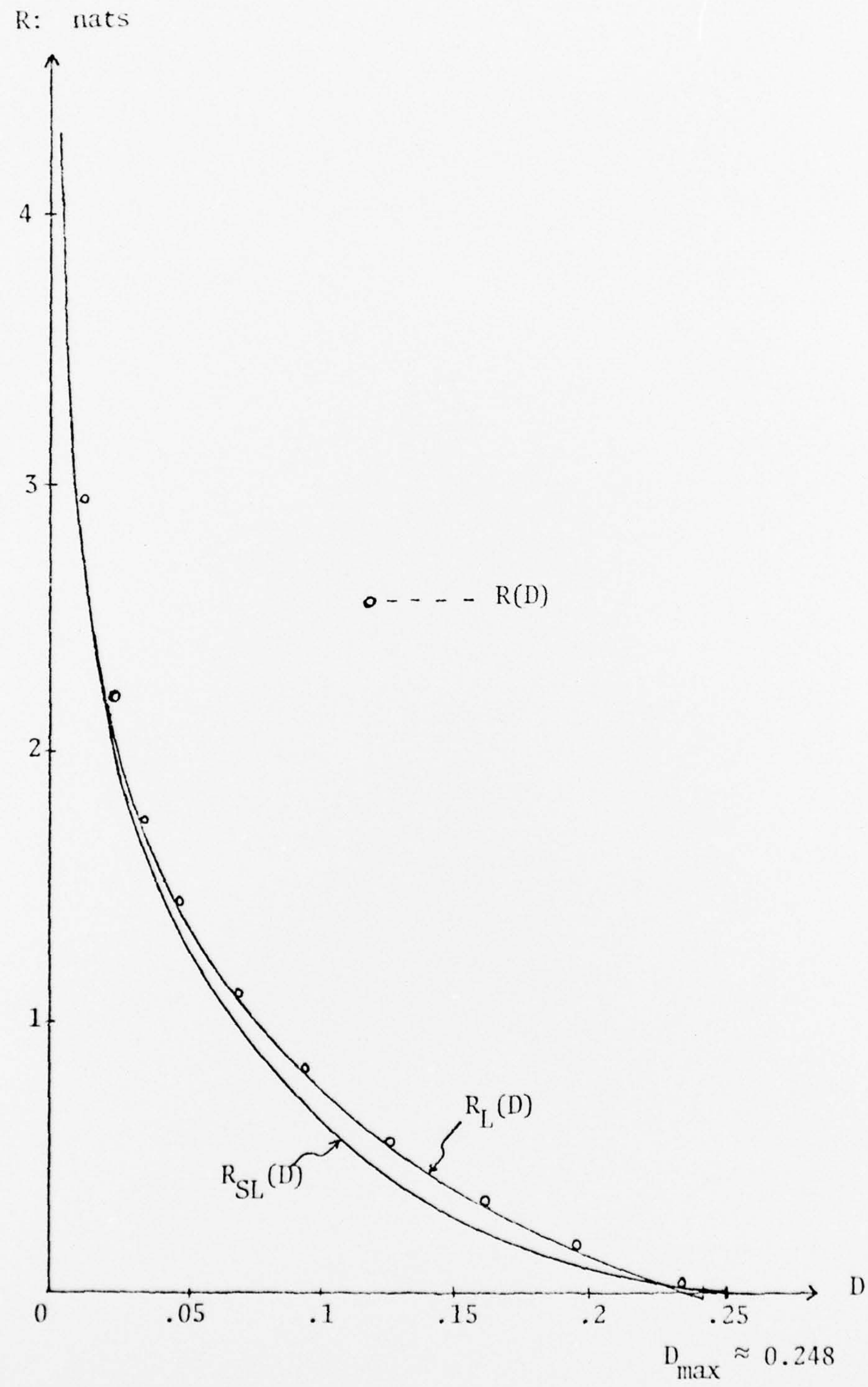

Figure 1 (α=0.1)

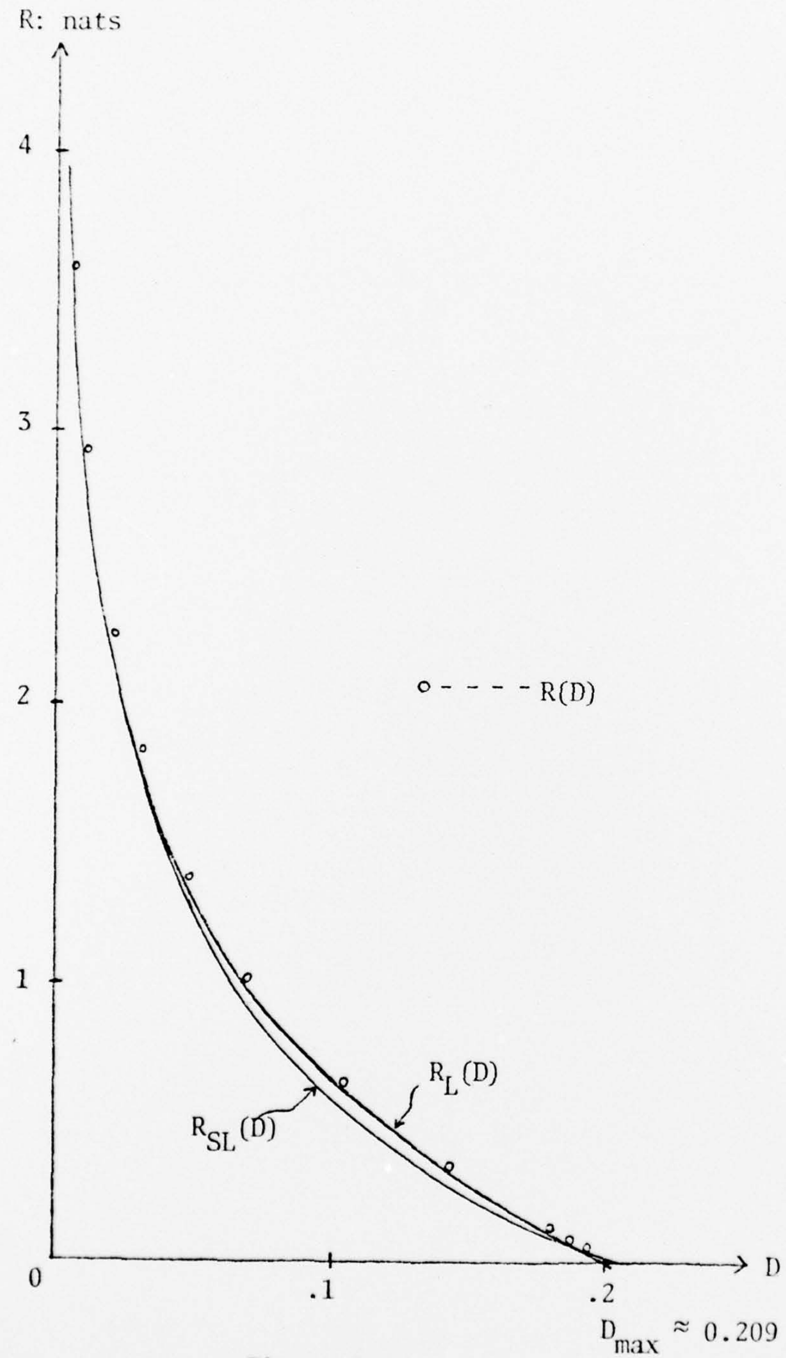

Figure 2 ($\alpha = 2$)

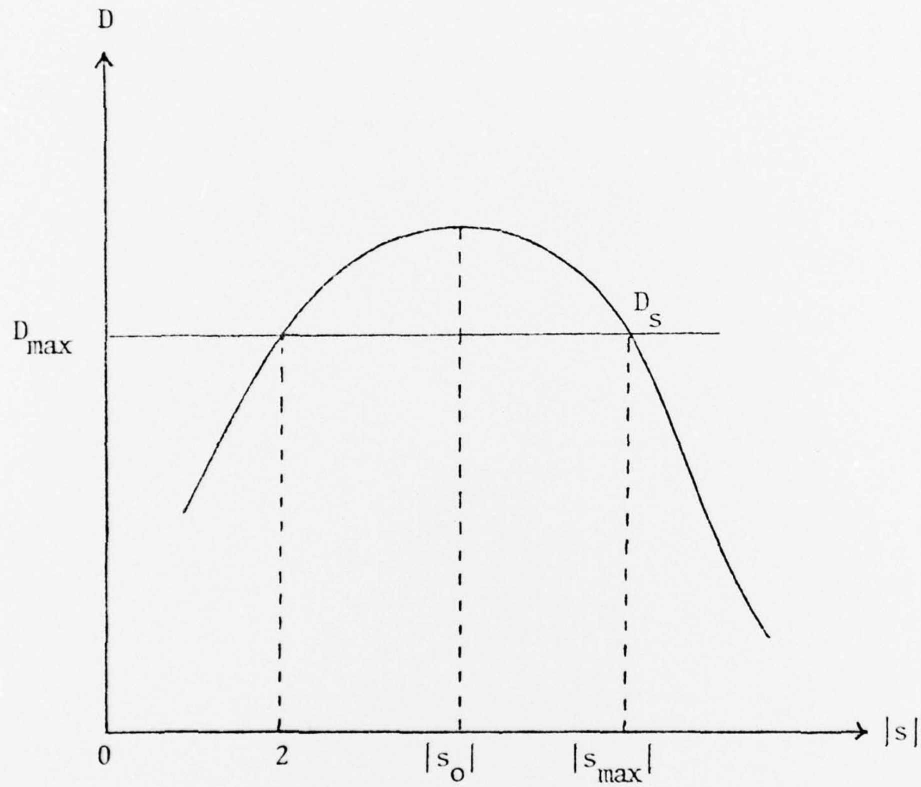

Figure 3

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