AD-A044 769 OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE F/G 8/5
ON THE COMPUTATION OF A GLOBAL COVARIANCE MODEL.(U)
JUL 77 H MORITZ F19628-76-C-0010
UNCLASSIFIED DGS-255 AFGL-TR-77-0163 NL

1 OF 1
ADA044769

END
DATE
FILMED
10-77
DDC

AFGL-TR-77-0163

ON THE COMPUTATION OF A GLOBAL COVARIANCE MODEL

Helmut Moritz

The Ohio State University
Research Foundation
Columbus, Ohio 43212

July 1977

Scientific Report No. 10

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	AFGL-TR-77-0163
2. GOVT ACCESSION NO.	
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE *(and Subtitle)*	ON THE COMPUTATION OF A GLOBAL COVARIANCE MODEL
5. TYPE OF REPORT & PERIOD COVERED	Scientific Interim Scientific Report No. 10
6. PERFORMING ORG. REPORT NUMBER	Dept. Geodetic Science No. 255
7. AUTHOR(s)	Helmut Moritz
8. CONTRACT OR GRANT NUMBER(s)	F19628-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS	Department of Geodetic Science The Ohio State University - 1958 Neil Avenue Columbus, Ohio 43210
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	62101F 7600-03-02
11. CONTROLLING OFFICE NAME AND ADDRESS	Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor: Bela Szabo/LW
12. REPORT DATE	July 1977
13. NUMBER OF PAGES	39 pages
14. MONITORING AGENCY NAME & ADDRESS *(if different from Controlling Office)*	DGS-255, Scientific-10
15. SECURITY CLASS. *(of this report)*	Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

TECH, OTHER

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Geodesy, Gravity, and Covariance Functions

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The report treats the problem of determining a global covariance function if the following data are given: the variances of the gravity anomalies, the variance of second-order gradients, the correlation length, and the lower degree variances.

The proposed covariance model is a linear combination of the reciprocal distance covariance function and a covariance function of logarithmic type.

It is found that the correlation length is already fixed, within rather narrow limits, by the remaining data.

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technische Hochschule in Graz and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19728-76-C-0010, The Ohio State University Research Foundation, Project No. 4214A1, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, L.G. Hanscom Field, Bedford, Massachusetts, with Mr. Bela Szabo, Project Scientist.

CONTENTS

1. Introduction

In the report (Moritz, 1976) we have seen that the local behavior of a covariance function (c.f.) of the gravity anomaly Δg can be characterized quite well by means of three constants:

C_o .... variance of Δg ,

ξ .... correlation length (argument for which the c.f. has the value $C(\xi) = \frac{1}{2} C_o$) ,

G_o .... variance of the horizontal gradient of Δg .

Instead of G_o we may also use the curvature parameter χ related to G_o by (*ibid*., p.25):

$$\chi = G_o \xi^2 / C_o \quad . \tag{1-1}$$

As regards the global behavior of a covariance function, the principal requirement is to fit the lower degree variances c_n , say from c_3 to c_{20} , as determined from satellite and gravimetric data.

As an example of a set of local parameters for a global c.f. we may take the rounded values

$$\begin{aligned} C_o &= 1800 \text{ mgal}^2 , \\ \xi &= 80 \text{ km} , \\ G_o &= 200 \text{ E}^2 , \end{aligned} \tag{1-2}$$

where

$$1 \text{ mgal} = 10^{-5} \text{ms}^{-2} \ ,$$
$$1 \text{ E} = 1 \text{ Eötvös} = 0.1 \text{ mgal/km} = 10^{-9} \text{s}^{-2} \ . \qquad (1\text{-}3)$$

This implies a curvature parameter

$$\chi = 7.1 \ . \qquad (1\text{-}4)$$

The value of C_o is taken from (Tscherning and Rapp, 1974). The correlation length ξ corresponds approximately to Kaula's (1959) c.f.; it would be highly desirable to have a reliable new determination using the presently available data.

The value of G_o is probably the weakest in the set (1-2). It seems to be in rough agreement with the scarce information on gravity gradients found in various publication (*cf.* Schwarz, 1976, p.15) and also with the values of gradients of deflections of the vertical given by Burša (1974). Further determinations of G_o are urgently necessary.

In their fundamental work, Tscherning and Rapp (1974) have given a very complete discussion of possible analytical models for global covariance functions and have fitted a model to given degree variances c_3 through c_{20} and to the variance C_o. However, the correlation length ξ comes out too small and G_o, or χ, is far too large.

In the present report we shall try to fit a global c.f. model to both the local parameters C_o, ξ, G_o and to the lower degree variances. The numerical values (1-2) will serve only for the purpose of an example; the method given is applicable also to other numerical values.

The reason why χ for the Tscherning-Rapp model is too high is that this model is essentially a logarithmic c.f. (cf. Moritz, 1976, especially p.48). Since the reciprocal distance c.f. (*ibid*.,p.35) has a χ of only 3, we might try a linear combination of a reciprocal distance c.f. and a logarithmic c.f.:

$$C(P,Q) = \alpha_1 \sum_{n=3}^{\infty} \sigma_1^{n+2} P_n(\cos\psi) + \alpha_2 \sum_{n=3}^{\infty} \frac{1}{n} \sigma_2^{n+2} P_n(\cos\psi) \ . \qquad (1\text{-}5)$$

Here $C(P,Q)$ expresses the covariance between the gravity anomalies Δg at the points P and Q in space; ψ is the angle between the radius vectors r_P and r_Q leading from the earth's center of mass to P and to Q ;

$$\sigma_1 = \frac{R_1^2}{r_P r_Q} \quad \text{and} \quad \sigma_2 = \frac{R_2^2}{r_P r_Q} \ , \qquad (1\text{-}6)$$

R_1 and R_2 denoting suitable constants, as are α_1 and α_2 ; and the $P_n(\cos\psi)$ are the Legendre polynomials. The notation follows that of (Moritz, 1976, pp.34-51), which may also be consulted for mathematical details.

The disadvantage of the form (1-5) rests in the fact that this model cannot be used for the covariance function $K(P,Q)$ of the anomalous potential T because it implies

$$K(P,Q) = \alpha_1 R_1^2 \sum_3^{\infty} \frac{\sigma_1^{n+1}}{(n-1)^2} P_n(\cos\psi) + \alpha_2 R_2^2 \sum_3^{\infty} \frac{\sigma_2^{n+1}}{n(n-1)^2} P_n(\cos\psi) \ , \qquad (1\text{-}7)$$

and these series cannot be summed.

So we shall use the form

$$C(P,Q) = \alpha_1 \sum_3^{\infty} \frac{n-1}{n+A} \sigma_1^{n+2} P_n(\cos\psi) + \alpha_2 \sum_3^{\infty} \frac{n-1}{(n-2)(n+B)} \sigma_2^{n+2} P_n(\cos\psi) \qquad (1\text{-}8)$$

which has the same local behavior since this local behavior is mainly determined by the higher degree terms for which

$$\frac{n-1}{n+A} \to 1 , \qquad \frac{n-1}{(n-2)(n+B)} \to \frac{1}{n}$$

as $n \to \infty$. The corresponding $K(P,Q)$ can be summed, as well as $C(P,Q)$ itself, as we shall see in the next section.

2. The Model

Thus our covariance model will be

$$C(P,Q) = \alpha_1 C_1(P,Q) + \alpha_2 C_2(P,Q) \tag{2-1}$$

with (A and B always denote integers)

$$C_1(P,Q) = \sum_3^\infty \frac{n-1}{n+A} \sigma_1^{n+2} P_n(\cos\psi) , \tag{2-2}$$

$$C_2(P,Q) = \sum_3^\infty \frac{n-1}{(n-2)(n+B)} \sigma_2^{n+2} P_n(\cos\psi) . \tag{2-3}$$

At sea level we have in the spherical approximation $r_P = r_Q = R$, where $R = 6371$ km is a mean radius of the earth, and σ_1 and σ_2 as defined by (1-6) become constants. Thus our model contains six parameters

$$\alpha_1 , \alpha_2 , \sigma_1 , \sigma_2 , A , B \tag{2-4}$$

which may be used to fix the data.

The corresponding c.f. for T is given by

$$K(P,Q) = \alpha_1 K_1(P,Q) + \alpha_2 K_2(P,Q) \qquad (2\text{-}5)$$

with

$$K_1(P,Q) = R_1^2 \sum_3^{\infty} \frac{1}{(n-1)(n+A)} \sigma_1^{n+1} P_n(\cos\psi) \;, \qquad (2\text{-}6)$$

$$K_2(P,Q) = R_2^2 \sum_3^{\infty} \frac{1}{(n-1)(n-2)(n+B)} \sigma_2^{n+1} P_n(\cos\psi) \;. \qquad (2\text{-}7)$$

If $A \neq -1$, $B \neq -1$ and $\neq -2$, then all these series can be summed in closed form by a decomposition into partial fractions, as shown by Tscherning and Rapp (1974); in fact, $C_2(P,Q)$ is their Model 4 and $C_1(P,Q)$ reduces for $A = -2$ to their Model 3.

We have

$$\frac{n-1}{n+A} = 1 - \frac{A+1}{n+A} \;, \qquad (2\text{-}8)$$

$$\frac{n-1}{(n-2)(n+B)} = \frac{1}{B+2}\left(\frac{1}{n-2} + \frac{B+1}{n+B}\right) \;, \qquad (2\text{-}9)$$

$$\frac{1}{(n-1)(n+A)} = \frac{1}{A+1}\left(\frac{1}{n-1} - \frac{1}{n+A}\right) \;, \qquad (2\text{-}10)$$

$$\frac{1}{(n-1)(n-2)(n+B)} = \frac{1}{(B+1)(B+2)}\left(\frac{B+1}{n-2} - \frac{B+2}{n-1} + \frac{1}{n+B}\right) . \qquad (2\text{-}11)$$

With Tscherning and Rapp we put (ibid.,p.31)

$$F(\sigma,\psi) = \sum_{0}^{\infty}\sigma^{n+1}P_n(t) , \tag{2-12}$$

$$F_i(\sigma,\psi) = \sum_{0}^{\infty}\frac{1}{n+i}\sigma^{n+1}P_n(t) \quad \text{for} \quad i > 0 , \tag{2-13}$$

$$F_i(\sigma,\psi) = \sum_{1-i}^{\infty}\frac{1}{n+i}\sigma^{n+1}P_n(t) \quad \text{for} \quad i \leqq 0 \tag{2-14}$$

with

$$t = \cos\psi . \tag{2-15}$$

We then have for A and B > 0

$$C_1(P,Q) = \sigma_1\left[F(\sigma_1,\psi) - \sigma_1 - \sigma_1^2 t - \sigma_1^3 P_2(t)\right] -$$
$$- (A+1)\sigma_1\left[F_A(\sigma_1,\psi) - \frac{\sigma_1}{A} - \frac{\sigma_1^2 t}{A+1} - \frac{\sigma_1^3}{A+2}P_2(t)\right] , \tag{2-16}$$

$$C_2(P,Q) = \frac{1}{B+2}\sigma_2 F_{-2}(\sigma_2,\psi) +$$
$$+ \frac{B+1}{B+2}\sigma_2\left[F_B(\sigma_2,\psi) - \frac{\sigma_2}{B} - \frac{\sigma_2^2 t}{B+1} - \frac{\sigma_2^3}{B+2}P_2(t)\right] ; \tag{2-17}$$

$$K_1(P,Q) = \frac{R_1^2}{A+1}\left[F_{-1}(\sigma_1,\psi) - \sigma_1^3 P_2(t)\right] -$$
$$- \frac{R_1^2}{A+1}\left[F_A(\sigma_1,\psi) - \frac{\sigma_1}{A} - \frac{\sigma_1^2 t}{A+1} - \frac{\sigma_1^3}{A+2}P_2(t)\right] , \tag{2-18}$$

$$K_2(P,Q) = \frac{R_2^2}{B+2} F_{-2}(\sigma_2, \psi) -$$

$$- \frac{R_2^2}{B+1} \left[F_{-1}(\sigma_2, \psi) - \sigma_2^3 P_2(t) \right] +$$

$$+ \frac{R_2^2}{(B+1)(B+2)} \left[F_B(\sigma_2, \psi) - \frac{\sigma_2}{B} - \frac{\sigma_2^2 t}{B+1} - \frac{\sigma_2^3}{B+2} P_2(t) \right] . \tag{2-19}$$

Values of A and $B \leqq -3$ are impossible because otherwise a denominator in (2-2), (2-3), (2-6) or (2-7) would be zero. The values $B = -1$ and -2 are excluded because in this case the series (2-3) or (2-7) cannot be summed. This excludes all negative values for B ; for A only the negative value -2 is possible.

Thus we have for $\underline{A = -2}$:

$$C_1(P,Q) = \sigma_1 \left[F(\sigma_1, \psi) - \sigma_1 - \sigma_1^2 t - \sigma_1^3 P_2(t) \right] +$$

$$+ \sigma_1 F_{-2}(\sigma_1, \psi) ; \tag{2-20}$$

$$K_1(P,Q) = - R_1^2 \left[F_{-1}(\sigma_1, \psi) - \sigma_1^3 P_2(t) \right] +$$

$$+ R_1^2 F_{-2}(\sigma_1, \psi) . \tag{2-21}$$

Of particular importance for the present report will be the case $\underline{A = B = 0}$, for which

$$C_1(P,Q) = \sigma_1\left[F(\sigma_1,\psi) - \sigma_1 - \sigma_1^2 t - \sigma_1^3 P_2(t)\right] -$$

$$- \sigma_1\left[F_o(\sigma_1,\psi) - \sigma_1^2 t - \tfrac{1}{2}\sigma_1^3 P_2(t)\right] , \qquad (2\text{-}22)$$

$$C_2(P,Q) = \tfrac{1}{2}\sigma_2 F_{-2}(\sigma_2,\psi) +$$

$$+ \tfrac{1}{2}\sigma_2\left[F_o(\sigma_2,\psi) - \sigma_2^2 t - \tfrac{1}{2}\sigma_2^3 P_2(t)\right] ; \qquad (2\text{-}23)$$

$$K_1(P,Q) = R_1^2\left[F_{-1}(\sigma_1,\psi) - \sigma_1^3 P_2(t)\right] -$$

$$- R_1^2\left[F_o(\sigma_1,\psi) - \sigma_1^2 t - \tfrac{1}{2}\sigma_1^3 P_2(t)\right] , \qquad (2\text{-}24)$$

$$K_2(P,Q) = \tfrac{1}{2}R_2^2 F_{-2}(\sigma_2,\psi) -$$

$$- R_2^2\left[F_{-1}(\sigma_2,\psi) - \sigma_2^3 P_2(t)\right] +$$

$$+ \tfrac{1}{2}R_2^2\left[F_o(\sigma_2,\psi) - \sigma_2^2 t - \tfrac{1}{2}\sigma_2^3 P_2(t)\right] . \qquad (2\text{-}25)$$

Closed expressions and a recursion formula for F and the F_i for $i = -2, -1, 0, 1, 2, \ldots$ may be found in (Tscherning and Rapp, 1974). From this report we take:

$$F(\sigma,\psi) = \frac{\sigma}{L} , \qquad (2\text{-}26)$$

$$F_o(\sigma,\psi) = \sigma \ln\frac{2}{N} , \qquad (2\text{-}27)$$

$$F_{-1}(\sigma,\psi) = \sigma(M + \sigma t \cdot \ln\frac{2}{N}) , \qquad (2\text{-}28)$$

$$F_{-2}(\sigma,\psi) = \frac{1}{2}\sigma(1 + 3\sigma t)M + \sigma^3 P_2(t) \ln\frac{2}{N} + \frac{1}{4}\sigma^3(1-t^2) , \qquad (2\text{-}29)$$

where

$$L = \sqrt{1 - 2\sigma t + \sigma^2} , \qquad (2\text{-}30)$$

$$M = 1 - L - \sigma t , \qquad (2\text{-}31)$$

$$N = 1 + L - \sigma t . \qquad (2\text{-}32)$$

For the F_i with $i = 1, 2, 3, \ldots$ we have the recursion formula

$$F_{i+1} = \frac{1}{i\sigma}\left[L + (2i-1)tF_i - (i-1)\sigma^{-1}F_{i-1} \right] \qquad (2\text{-}33)$$

which is valid for $i \geqq 3$, starting with

$$F_1 = \ln(1 + \frac{2\sigma}{1-\sigma+L}) , \qquad (2\text{-}34)$$

$$F_2 = \sigma^{-1}(L - 1 + tF_1) . \qquad (2\text{-}35)$$

In the same manner, closed expressions can also be found for the corresponding covariance functions for other quantities of the anomalous gravity field (Tscherning and Rapp, 1974), especially for second-order gradients (Tscherning, 1976). We shall not give them here, because for the present report we need only G_o , for which an approximate expression will be derived in the following section and exact formulas in the Appendix.

3. Variances

The variance C_o represents the value of the covariance function $C(P,Q)$ for the case that the points P and Q coincide with a point at sea level, that is, for

$$r_P = r_Q = R , \quad \psi = 0 .$$

From (2-1) we get

$$C_o = \alpha_1 C_{1o} + \alpha_2 C_{2o} \tag{3-1}$$

where C_{1o} and C_{2o} are the variances corresponding to $C_1(P,Q)$ and $C_2(P,Q)$, respectively.

By substituting $\psi = 0$ in (2-22) and (2-23) we get for the case $A = B = 0$:

$$C_{1o} = \sigma_1\left[F(\sigma_1, 0) - \sigma_1 - \sigma_1^2 - \sigma_1^3\right] - $$

$$- \sigma_1\left[F_o(\sigma_1, 0) - \sigma_1^2 - \frac{1}{2}\sigma_1^3\right] , \tag{3-2}$$

$$C_{2o} = \frac{1}{2}\sigma_2 F_{-2}(\sigma_2, 0) + \frac{1}{2}\sigma_2\left[F_o(\sigma_2, 0) - \sigma_2^2 - \frac{1}{2}\sigma_2^3\right] . \tag{3-3}$$

The expressions (2-30) through (2-31) become for $\psi = 0$

$$L_o = \zeta ,$$

$$M_o = 0 , \tag{3-4}$$

$$N_o = 2\zeta , .$$

where we have put

$$\zeta = 1 - \sigma . \tag{3-5}$$

Using these values in (2-26) through (2-29) and substituting in (3-2) we get

$$C_{1o} = \frac{\sigma_1^2}{\zeta_1} - \sigma_1^2 \ln\frac{1}{\zeta_1} - \sigma_1^2 - \frac{1}{2}\sigma_1^4$$

or, with

$$\frac{\sigma_1^2}{\zeta_1} - \sigma_1^2 = \frac{\sigma_1^2}{\zeta_1}(1-\zeta_1) = \frac{\sigma_1^3}{\zeta_1} ,$$

finally

$$C_{1o} = \frac{\sigma_1^3}{\zeta_1} - \sigma_1^2 \ln\frac{1}{\zeta_1} - \frac{1}{2}\sigma_1^4 . \qquad (3\text{-}6)$$

Eq. (3-3) becomes similarly

$$C_{2o} = \frac{1}{2}\sigma_2^2(1+\sigma_2^2)\ln\frac{1}{\zeta_2} - \frac{1}{2}\sigma_2^3 - \frac{1}{4}\sigma_2^4 . \qquad (3\text{-}7)$$

Similar we get for the variance K_o of T, again for the case $A = B = 0$:

$$K_o = \alpha_1 K_{1o} + \alpha_2 K_{2o} \qquad (3\text{-}8)$$

where

$$K_{1o} = R_1^2\sigma_1(-\zeta_1 \ln\frac{1}{\zeta_1} + \sigma_1 - \frac{1}{2}\sigma_1^2) , \qquad (3\text{-}9)$$

$$K_{2o} = \frac{1}{2}R_2^2\sigma_2(\zeta_2^2 \ln\frac{1}{\zeta_2} - \sigma_2 + \frac{3}{2}\sigma_2^2) . \qquad (3\text{-}10)$$

For A or $B \neq 0$ we can easily evaluate directly the $F_i(\sigma, 0)$ that enter in (2-16) through (2-21). Since all $P_n(0) = 1$, we get from (2-12)

$$F(\sigma,0) = \sum_{0}^{\infty} \sigma^{n+1} = \frac{\sigma}{1-\sigma} = \frac{\sigma}{\zeta} \qquad (3\text{-}11)$$

and

$$F(\sigma,0) - \sigma - \sigma^2 - \sigma^3 = \sum_{4}^{\infty} \sigma^n = \sigma^4 \sum_{0}^{\infty} \sigma^n = \frac{\sigma^4}{\zeta} \,. \qquad (3\text{-}12)$$

By integrating

$$\sum_{1}^{\infty} \sigma^{n-1} = \frac{1}{1-\sigma} \qquad (3\text{-}13)$$

with respect to σ we get

$$\sum_{n=1}^{\infty} \frac{1}{n} \sigma^n = \ln \frac{1}{1-\sigma} = \ln \frac{1}{\zeta} \,. \qquad (3\text{-}14)$$

On changing the summation index by putting

$$m = n - i$$

we have

$$\sum_{m=1-i}^{\infty} \frac{1}{m+i} \sigma^{m+i} = \sigma^i \sum_{m=1-i}^{\infty} \frac{1}{m+i} \sigma^m = \ln \frac{1}{\zeta} \,.$$

Therefore (we write again n in the place of m)

$$\sum_{n=1-i}^{\infty} \frac{1}{n+i} \sigma^{n+i} = \sigma^{1-i} \ln \frac{1}{\zeta} \,. \qquad (3\text{-}15)$$

In this way, (2-14) becomes

$$F_i(\sigma,0) = \sigma^{1-i}\ln\frac{1}{\zeta} \quad \text{for} \quad i \leqq 0 , \tag{3-16}$$

whereas for (2-13), with $i > 0$, we have

$$\sum_{o}^{\infty} = \sum_{1-i}^{\infty} - \sum_{1-i}^{-1} \quad (\sum_{o}^{-1} = 0 \text{ by definition}) ,$$

so that

$$F_i(\sigma,0) = \sigma^{1-i}\ln\frac{1}{\zeta} - \sum_{1-i}^{-1}\frac{1}{n+i}\sigma^{n+1} \quad \text{for } i > 0 . \tag{3-17}$$

Thus, for $A = -2$, we get from (2-20)

$$C_{1o} = \frac{\sigma_1^5}{\zeta_1} + \sigma_1^4 \ln\frac{1}{\zeta_1} , \tag{3-18}$$

wheras for A and $B \geqq 0$ we find

$$C_{1o} = \frac{\sigma_1^5}{\zeta_1} - (A+1)\sigma_1^{2-A}\ln\frac{1}{\zeta_1} + (A+1)\sum_{1-A}^{2}\frac{\sigma_1^{n+2}}{n+A} , \tag{3-19}$$

$$C_{2o} = \frac{\sigma_2^4 + (B+1)\sigma_2^{2-B}}{B+2}\ln\frac{1}{\zeta_2} - \frac{B+1}{B+2}\sum_{1-B}^{2}\frac{\sigma_2^{n+2}}{n+B} . \tag{3-20}$$

For $A = 0$ this reduces to (3-6) and (3-7), as it should be.

In a similar way we could readily compute the variances K_{1o} and K_{2o} for $A \neq 0$ and $B \neq 0$, but we shall not need them in this report.

We finally consider the gradient variance G_o . It is the variance of the mixed horizontal-vertical gradient $\partial\Delta g/\partial x$, or half the variance of the vertical gradient $\partial\Delta g/\partial z$; cf. (Moritz, 1976). In fact, expanding $C(s)$, the c.f. of Δg , in powers of the horizontal distance s ,

$$C(s) = a_o - a_1 s^2 + a_2 s^4 - + \ldots , \tag{3-21}$$

and differentiating:

$$C'(s) = -2a_1 s + \ldots$$

$$C''(s) = -2a_1 + \ldots$$

we have

$$C''(0) = \left[\frac{1}{s}C'(s)\right]_{s=0} = -2a_1 ,$$

so that by eq. (1-31) (*ibid*., p.10):

$$\mathrm{var}\left(\frac{\partial\Delta g}{\partial z}\right) = -C''(0) - \left[\frac{1}{s}C'(s)\right]_{s=0}$$

$$= 4a_1 = 2G_o$$

since

$$G_o = 2a_1 \tag{3-22}$$

by eq. (3-9), *ibid*., p.24.

Expanding Δg in spatial spherical harmonics,

$$\Delta g = \sum \left(\frac{R}{r}\right)^{n+2} \Delta g_n$$

and differentiating along the radius vector r (with which the local z-axis coincides) we get

$$-\frac{\partial \Delta g}{\partial r} = \sum \frac{n+2}{R} \left(\frac{R}{r}\right)^{n+3} \Delta g_n ,$$

so that the degree variances for $\partial \Delta g / \partial r$ arise from those of Δg by multiplication by the factor

$$\left(\frac{n+2}{R}\right)^2 .$$

Hence we have corresponding to (2-1):

$$G_o = \alpha_1 G_{1o} + \alpha_2 G_{2o} \tag{3-23}$$

where by (2-2) and (2-3)

$$G_{1o} = \frac{\sigma_1^2}{2R^2} \sum_3^{\infty} \frac{n-1}{n+A} (n+2)^2 \sigma_1^{n+1} , \tag{3-24}$$

$$G_{2o} = \frac{\sigma_2^2}{2R^2} \sum_3^{\infty} \frac{n-1}{(n-2)(n+B)} (n+2)^2 \sigma_2^{n+1} ; \tag{3-25}$$

the factor 1/2 expresses the fact that G_o is half the variance of $\partial \Delta g / \partial r$.

Since the gradient is a very local quantity, only the higher-degree terms have much influence, so that only the asymptotic behavior of the terms in the sums is essential. Hence we may approximately replace

$$\frac{n-1}{n+A}(n+2)^2 \quad \text{by} \quad n(n-1) \ ,$$

$$\frac{n-1}{(n-2)(n+B)}(n+2)^2 \quad \text{by} \quad n \ ,$$

which behave asymptotically (for $n \to \infty$) in the same way.

We thus get, to a very good approximation,

$$G_{10} = \frac{\sigma_1^2}{2R^2}\sum_3^\infty n(n-1)\sigma_1^{n+1} \ , \tag{3-26}$$

$$G_{20} = \frac{\sigma_2^2}{2R^2}\sum_3^\infty n\sigma_2^{n+1} \ . \tag{3-27}$$

These series can be readily summed. By differentiating

$$\sum_0^\infty \sigma^n = \frac{1}{1-\sigma} \tag{3-28}$$

twice with respect to σ we find

$$\sum_1^\infty n\sigma^{n-1} = \frac{1}{(1-\sigma)^2} \ , \tag{3-29}$$

$$\sum_2^\infty n(n-1)\sigma^{n-2} = \frac{2}{(1-\sigma)^3} \ . \tag{3-30}$$

We thus have

$$\sum_3^\infty n\sigma_2^{n+1} = \frac{\sigma_2^2}{(1-\sigma_2)^2} - \sigma_2^2 - 2\sigma_2^3 \doteq \frac{1}{\xi_2^2} \ ,$$

$$\sum_3^\infty n(n-1)\sigma_1^{n+1} = \frac{2\sigma_1^3}{(1-\sigma_1)^3} - 2\sigma_1^3 \doteq \frac{2}{\zeta_1^3} ;$$

the approximations are justified since ζ_1 and ζ_2 will be very small, so that σ_1 and σ_2 are very close to 1 and the terms $1/\zeta_2^2$ and $2/\zeta_1^3 >> 1$.

In this manner we finally obtain

$$G_{10} = \frac{1}{R^2\zeta_1^3} , \tag{3-31}$$

$$G_{20} = \frac{1}{2R^2\zeta_2^2} . \tag{3-32}$$

These expressions are very simple but quite accurate: it is not difficult ot verify that they have a relative error of order ζ . Since ζ will be seen in the next section to be around 0.001 to 0.006, these expressions are accurate to within a few percent.[1)]

A thorough check of (3-31) and (3-32) is provided by expanding the c.f. into a series (3-21) and using (3-22). In doing this, we shall follow the procedure and the notations of (Moritz, 1976, sec.4).

By means of (3-5) and on putting

$$\lambda = 2\sin\frac{\psi}{2} , \tag{3-33}$$

$$\rho^2 = \sigma\lambda^2 , \tag{3-34}$$

the expressions (2-30), (2-31), and (2-32) are readily transformed into

---

[1)] Rigorous expressions for G_{10} and G_{20} will be found in the Appendix.

$$L = \sqrt{\zeta^2+\rho^2} \ , \tag{3-35}$$

$$N = \zeta + \sqrt{\zeta^2+\rho^2} + \frac{1}{2}\rho^2 \ , \tag{3-36}$$

$$M = \zeta - \sqrt{\zeta^2+\rho^2} + \frac{1}{2}\rho^2 \ . \tag{3-37}$$

These expressions are still rigorous.

We now substitute these expressions into (2-26) through (2-29), and these into (2-22) and (2-23). We now expand into a power series with respect to ρ, neglecting again a relative error of ζ. After somewhat lengthy calculation we obtain

$$C_1(\rho) = C_{10} - \frac{1}{2\zeta_1^3}\rho^2 + O(\rho^4) \ , \tag{3-38}$$

$$C_2(\rho) = C_{20} - \frac{1}{4\zeta_2^2}\rho^2 + O(\rho^4) \ . \tag{3-39}$$

The plane horizontal distance s is related to ρ by

$$s = 2R\sin\frac{\psi}{2} = R\lambda = R(1+\zeta)^{-\frac{1}{2}}\rho \ ,$$

so that, with a relative error of ζ,

$$\rho = \frac{s}{R} \ . \tag{3-40}$$

This is substituted into (3-38) and (3-39), so that

$$C_1(s) = C_{10} - \frac{1}{2R^2\zeta_1^3}s^2 + \dots \ ,$$

$$C_2(s) = C_{20} - \frac{1}{4R^2\zeta_2^2}s^2 + \dots ,$$

whence (3-31) and (3-32) follow in view of (3-22).

4. Numerical Model Fit

We shall now show how the present model can be fitted to the data by an appropriate choice of parameters.

The degree variances corresponding to the model (2-1) through (2-3) are given by

$$c_n = \frac{n-1}{n+A}\sigma_1^{n+2}\alpha_1 + \frac{n-1}{(n-2)(n+B)}\sigma_2^{n+2}\alpha_2 \ . \qquad (4\text{-}1)$$

Since estimates of c_3 through c_{20} (or even higher) are available and since each n from 3 to 20 gives one equation (4-1), it might be thought possible to determine all six parameters α_1, α_2, σ_1, σ_2, A, B in this way. However, it turns out very soon that this is impossible since the equation system for this purpose is very poorly conditioned.

In fact, it is reasonably feasible only to determine α_1 and α_2 in this manner. The values of σ_1, σ_2, A, B must be found by other considerations and given in advance. Then α_1 and α_2 can be obtained by a least-squares fit to the degree variances.

Such a least-squares fit by means of a two-parameter model has already been done by Tscherning and Rapp (1974, p.23). Since the "fitted" degree variances are very smooth, it is possible only to take two of them, say c_3 and c_{20}, to determine α_1 and α_2. Eq. (4-1) gives

$$\frac{2}{3+A}\sigma_1^5\alpha_1 + \frac{2}{3+B}\sigma_2^5\alpha_2 = c_3 \; ,$$

$$\frac{19}{20+A}\sigma_1^{22}\alpha_1 + \frac{19}{18(20+B)}\sigma_2^{22}\alpha_2 = c_{20} \; . \qquad (4\text{-}2)$$

If these two equations are solved for α_1 and α_2 and the values so obtained are used in (4-1) to compute the intermediate c_n , it will be seen that also these c_n will agree well with the "fitted" Tscherning-Rapp values.

From Tscherning-Rapp (1974, p.23) we thus take

$$c_3 = 31.5 \; ,$$

$$c_{20} = 10.2 \; . \qquad (4\text{-}3)$$

These are the "global" parameters to be used. For the "local" parameters we take C_o and G_o from (1-2):

$$C_o = 1800 \text{ mgal}^2 \; ,$$

$$G_o = 200 \text{ E}^2 \; ; \qquad (4\text{-}4)$$

the correlation length will not be considered at present.

To these four constants (4-3) and (4-4) we shall try to fit the model (2-1) through (2-3) with $A = B = 0$, which comprises four parameters $\alpha_1, \alpha_2, \sigma_1, \sigma_2$. These four parameters will be determined from the four given values c_3, c_{20}, C_o, G_o by an iterative procedure.

With $A = B = 0$, the system (4-2) becomes

$$\frac{2}{3}\sigma_1^5\alpha_1 + \frac{2}{3}\sigma_2^5\alpha_2 = c_3 \; , \qquad (4\text{-}5a)$$

$$\frac{19}{20}\sigma_1^{22}\alpha_1 + \frac{19}{18\cdot 20}\sigma_2^{22}\alpha_2 = c_{20} . \tag{4-5b}$$

It may be given the simpler form, putting $\sigma_2/\sigma_1 = q$,

$$\alpha_1 + q^5\alpha_2 = f_1 ,$$

$$\alpha_1 + \frac{1}{18}q^{22}\alpha_2 = f_2 \tag{4-6}$$

where

$$f_1 = \frac{3}{2\sigma_1^5}c_3 ,$$

$$f_2 = \frac{20}{19\sigma_1^{22}}c_{20} . \tag{4-7}$$

The solution for α_1 and α_2 may be expressed as

$$\alpha_2 = (q^5 - \frac{1}{18}q^{22})^{-1}(f_1 - f_2) ,$$

$$\alpha_1 = f_1 - q^5\alpha_2 . \tag{4-8}$$

Further equations are furnished by (3-6) and (3-7):

$$C_{10} = \frac{\sigma_1^3}{\zeta_1} - \sigma_1^2 \ln\frac{1}{\zeta_1} - \frac{1}{2}\sigma_1^4 , \tag{4-9}$$

$$C_{20} = \frac{1}{2}\sigma_2^2(1+\sigma_2^2)\ln\frac{1}{\zeta_2} - \frac{1}{2}\sigma_2^3 - \frac{1}{4}\sigma_2^4 , \tag{4-10}$$

where

$$\zeta_1 = 1 - \sigma_1 \ , \quad \zeta_2 = 1 - \sigma_2 \ . \tag{4-11}$$

Then the variance of the gravity anomalies is given by

$$C_o = \alpha_1 C_{1o} + \alpha_2 C_{2o} \ . \tag{4-12}$$

For the gradient variance we have (3-31) and (3-32):

$$G_{1o} = \frac{1}{R^2 \zeta_1^3} \ , \tag{4-13}$$

$$G_{2o} = \frac{1}{2R^2 \zeta_2^2} \tag{4-14}$$

and

$$G_o = \alpha_1 G_{1o} + \alpha_2 G_{2o} \ . \tag{4-15}$$

These are the basic formulas for our iteration, using the numerical values (4-3) and (4-4).

First we note that α_1 and α_2 as resulting from the solution of (4-5a,b) are not very sensitive with respect to σ_1 and σ_2 . At any rate, σ_1 and σ_2 will be close to (and slightly smaller than) unity. We therefore start with some approximate values for σ_1 and σ_2 ; if they are not available, we may use $\sigma_1 = \sigma_2 = 1$ as our starting values. With σ_1 and σ_2 we get f_1 and f_2 from (4-7) and α_1 and α_2 from (4-8).

Now we can try to determine better values of σ_1 and σ_2 in such a way as to get the deseired $C_o = 1800$ and $G_o = 200$. This may be done in the following manner. Start with some σ_1 , compute

$$\zeta_1 = 1 - \sigma_1$$

and G_{1o} by (4-13). Now ζ_2 can be calculated in such a way that the sum

$$\alpha_1 G_{1o} + \alpha_2 G_{2o}$$

gives the correct value $G_o = 200$. In fact, then G_{2o} must be

$$G_{2o} = \alpha_2^{-1}(200 - \alpha_1 G_{1o}) , \qquad (4\text{-}16)$$

and $\zeta_2 = 1 - \sigma_2$ can be found by solving (4-14) for ζ_2 .

With these values σ_1 and σ_2 we can compute C_o by (4-9), (4-10) and (4-12). In general, the C_o thus obtained will not be equal to the desired value 1800. We can, however, repeat the computation of C_o for various assumed σ_1 , always finding the corresponding σ_2 by (4-16). If in this way we get values for C_o that are partly < 1800 and partly $>$ 1800, we may interpolate the argument σ_1 for the desired value $C_o = 1800$. Again the corresponding σ_2 can be computed using (4-16).

Thus both G_o and C_o are fitted, but the values α_1 and α_2 are not yet completely correct. We thus repeat the computation of α_1 and α_2 on the basis of the values σ_1 and σ_2 last obtained. Using these values α_1 and α_2 we compute better σ_1 and σ_2 by again assuming a suitable σ_1 (usually it is best to take the last value), computing the corresponding σ_2 by (4-16) and evaluating the corresponding C_o . For a slightly different assumed σ_1 the process is repeated, and the σ_1 corresponding to the desired $C_o = 1800$ is interpolated.

Now we can anew compute α_1 and α_2 and repeat the iteration as long as necessary. The principle is always the same: assume σ_1 , compute the corresponding σ_2 in such a way as to ensure the correct $G_o = 200$, calculate C_o for different σ_1 and interpolate σ_1 corresponding to the desired $C_o = 1800$. The iteration is necessary because α_1 and α_2 depend on σ_1 and σ_2 , but fortunately this dependence is rather slight, so that, in general, the iteration will converge well.

For the given values (4-3) and (4-4) the result is

$$\sigma_1 = 0.993\,963 \,, \qquad (4\text{-}17)$$

$$\sigma_2 = 0.999\,281 \,;$$

$$\alpha_1 = 9.907 \,, \qquad (4\text{-}18)$$

$$\alpha_2 = 37.77 \,.$$

The correlation length ξ of the c.f. with these parameters can be computed from the functional representation (2-22) and (2-23), using eqs. (2-26) through (2-32). It is the horizontal distance ξ for which

$$C(\xi) = \frac{1}{2} C_o = 900 \text{ mgal}^2 \,. \qquad (4\text{-}19)$$

The computation may again be done by interpolation: calculate $C(\xi_1)$ and $C(\xi_2)$ for two assumed arguments ξ_1 and ξ_2 , and interpolate ξ for which $C(\xi) = 900$. If ξ_1 and ξ_2

have not yet been sufficiently close to ξ , the process may be repeated. The result is

$$\xi = 68.91 \text{ km} , \qquad (4\text{-}20)$$

corresponding to an angular distance

$$\psi = 0.6197^\circ . \qquad (4\text{-}21)$$

The variance K_o of the anomalous potential T is obtained from (3-9) and (3-10):

$$K_o = \alpha_1 K_{1o} + \alpha_2 K_{2o} = 186.3 + 381.8$$

or

$$K_o = 568.1 \text{ (kgal}\cdot\text{m)}^2 . \qquad (4\text{-}22)$$

The corresponding variance of geoidal heights N is

$$\text{var}(N) = \frac{K_o}{\bar{g}^2} = 591.5 \text{ m}^2 , \qquad (4\text{-}23)$$

$\bar{g} = 0.98$ kgal being a mean value of gravity.

5. Conclusions

The numerical model discussed in the preceding section does not fit all parameters (1-2); it only fits C_o and G_o . Its correlation length ξ has been found to be 68.9 km, which is smaller than the value 80 km corresponding to (1-2).

It is possible to get a different value of ξ by selcting A or B in a different way? Rather conclusively, the answer appears to be NO.

To see this, let us fix the value

$$A = -2 \tag{5-1}$$

and vary B . Then the system (4-2) becomes

$$\sigma_1^5\alpha_1 + \frac{1}{3+B}\sigma_2^5\alpha_2 = d_3 ,$$
$$\sigma_1^{22}\alpha_1 + \frac{1}{20+B}\sigma_2^{22}\alpha_2 = d_{20} , \tag{5-2}$$

where we have put

$$d_3 = \frac{1}{2}c_3 , \quad d_{20} = \frac{18}{19}c_{20} . \tag{5-3}$$

As a first, very crude, approximation we put $\sigma_1 \doteq \sigma_2 \doteq 1$, obtaining as a solution

$$\alpha_1 \doteq d_{20} - \frac{1}{17}(d_3-d_{20})(3+B) ,$$
$$\alpha_2 \doteq \frac{1}{17}(d_3-d_{20})(3+B)(20+B) . \tag{5-4}$$

With the values (4-3) this becomes

$$\alpha_1 \doteq 8.6 - 0.36\ B ,$$
$$\alpha_2 \doteq 0.36(3+B)(20+B) . \tag{5-5}$$

What are the possible limits between which B can vary? A lower limit is

$$B = 0 , \qquad (5\text{-}6)$$

as we have seen on p.7. An upper limit may be obtained by the condition

$$\alpha_1 \geqq 0 ; \qquad (5\text{-}7)$$

for negative α_1 the positive definiteness of the covariance function $C(P,Q)$ can no longer be guaranteed. The limiting case $\alpha_1 = 0$ corresponds, by (5-5), to

$$B = 8.6/0.36 \doteq 24 . \qquad (5\text{-}8)$$

This corresponds to the case of Tscherning-Rapp (1974, p.22), in which a model with only the function $C_2(P,Q)$ is used. However, for this model $G_o >> 200\ E^2$. This indicates that if $G_o = 200\ E^2$ is imposed, the upper possible limit should be smaller. It is found to be

$$B = 10 ; \qquad (5\text{-}9)$$

for larger B the conditions $C_o = 1800$ and $G_o = 200$ turn out to be incompatible.

In fact. for each B we can apply the iterative procedure described in the preceding section, to find α_1 , α_2 and σ_1 , σ_2 corresponding to the values C_o and G_o given in advance. It is found that this is possible for all B from 0 to 10, whereas for $B = 11$ and higher, the values of C_o corresponding to $G_o = 200$ are all < 1800 , so that an interpolation of σ_1 for $C_o = 1800$ can no longer be performed.

For <u>B = 0</u> we thus find

$$\sigma_1 = 0.994\,032 , \qquad \alpha_1 = 10.000 , \qquad (5\text{-}10)$$
$$\sigma_2 = 0.999\,484 ; \qquad \alpha_2 = 18.18 ;$$

and the correlation length

$$\xi = 68.7 \text{ km} . \qquad (5\text{-}11)$$

For <u>B = 10</u> we obtain

$$\sigma_1 = 0.995\,455 , \qquad \alpha_1 = 6.199 , \qquad (5\text{-}12)$$
$$\sigma_2 = 0.997\,953 ; \qquad \alpha_2 = 127.28 ;$$

$$\xi = 60.7 \text{ km} . \qquad (5\text{-}13)$$

(The comparison between these values of α_1 and α_2 with the approximate values given by (5-5) shows that the latter is only a rough approximation.)

It is thus seen that, for $A = -2$ and B variable, the possible values of ξ lie between rather narrow bounds:

$$60.7 \leqq \xi \leqq 68.7 \text{ km} , \qquad (5\text{-}14)$$

the value $\xi = 30$ km cannot be reached by this model (it can be expected that for $B = 1, 2, \ldots, 9$ the corresponding values ξ lie indeed within the limits (5-14)).

It is seen that even by selecting a different A, the situation will not change essentially. The model retains its general character, the coefficients of C_1 decreasing asymptotically as

$$\sigma_1^{n+2} \quad , \tag{5-15}$$

corresponding to the reciprocal distance covariance function, and those of C_2 as

$$\frac{1}{n}\sigma_2^{n+2} \quad , \tag{5-16}$$

corresponding to the logarithmic covariance function. What may change are the numerical values for the lower and upper bounds in (5-14); in fact, for $A = B = 0$ we had, by (4-20)

$$\xi = 68.9 \text{ km} .$$

It can be expected, however, that these changes are small. At any rate, we shall again have

$$\xi_1 \leqq \xi \leqq \xi_2 \quad , \tag{5-17}$$

ξ_1 and ξ_2 differening perhaps by 10 km.

It thus appears that given values of the lower degree variances (say, from 3 to 20), together with C_o and G_o, rather narrowly determine the correlation distance ξ. For instance, choosing in the present case

$$\xi = \frac{1}{2}(\xi_1+\xi_2) \quad , \tag{5-18}$$

taking $A = -2$ and fitting B to this ξ, the correlation distance of the model will be wrong at most by, say, 5 km.

The exact upper and lower bounds ξ_1 and ξ_2 if both A and B vary may probably best be found by a computer experiment since, on the one hand, the formulas for general A and B are not difficult to program but, on the other hand, the required hand calculations are very laborious. If this is done, then the exact expressions for G_{10} and G_{20} given in the Appendix might be employed rather than the approximate ones used above.

A condition of form (5-14) seems to be quite generally true, not only for the special model (1-8) used in this report. Even the "general" ξ_1 and ξ_2 should not differ very much from those for the model (1-8) with variable A and B, and even from those for $A = -2$ and B variable.

For this we may adduce the following argument. Other possible covariance models have coefficients

$$n\sigma^{n+2} \; , \quad n^2\sigma^{n+2} \; , \quad n^3\sigma^{n+2} \; , \quad \text{etc.},$$

but these are all ruled out because they would have increasing, rather than decreasing, degree variances.

The simplest model with decreasing degree variances is (5-16), corresponding to a logarithmic covariance function, but such a model alone will give too high a G_o ; therefore the reciprocal distance model (5-15) is also necessary. More general would be an expression with degree variances

$$\alpha_1 \sigma_1^{n+2} + \alpha_2 \frac{1}{n} \sigma_2^{n+2} + \alpha_3 \frac{1}{n^2} \sigma_3^{n+2} + \ldots, \tag{5-19}$$

but it can be expected that the first two terms, corresponding to the model used in the present report, will be dominant.

The expression (5-19) represents a linear combination of degree variances

$$n^{-k} \sigma^{n+2} \quad \text{with} \quad k = 0, 1, 2, \ldots \qquad (5\text{-}20)$$

Even non-integer values, say $k = 0.5$, do not seem to have an essentially different behavior.

Such a different behavior could only be expected if, in an essential way, the degree variances behave capriciously, for instance decreasing from $n = 3$ to 30, then increasing to $n = 100$, etc. If they decrease monotonically, at least on the average, then the behavior can be expected to be similar to our present covariance model.

This fact, if it is true, is of very great importance. Already C_o, G_o and the lower degree variances together determine to correlation length ξ rather well. Some values, such as $\xi = 80$ km with the presently assumed numerical data, are excluded. This emphasizes the importance of determining a good value for the gradient variance G_o. This quantity, which is a very local one, is the same for local and global covariance functions; this is not true for ξ. This G_o, rather than ξ, provides a basis for calculating a global c.f. from local covariance functions.

APPENDIX

Gradient Variances

Let us calculate the difference between the exact expressions for the gradient variances G_{10} and G_{20}, given by (3-24) and (3-25), and the approximate formulas (3-26) and (3-27).

We use the following auxiliary formulas which can be directly verified:

$$\frac{n-1}{n+A}(n+2)^2 - n(n-1) = (4-A)n + A(A-3) - \frac{(A+1)(A-2)^2}{n+A} ,$$

$$\frac{n-1}{(n-2)(n+B)}(n+2)^2 - n = 5 - B + \frac{16}{(B+2)(n-2)} + \frac{(B+1)(B-2)^2}{(B+2)(n+B)} . \qquad \text{(A-1)}$$

The difference between (3-24) and (3-26) is therefore:

$$\Delta G_{10} = G_{10}^{exact} - G_{10}^{approximate} =$$

$$= \frac{\sigma_1^2}{2R^2}\sum_3^\infty \left[\frac{n-1}{n+A}(n+2)^2 - n(n-1)\right] \sigma_1^{n+1} =$$

$$= \frac{\sigma_1^2}{2R^2}\left[(4-A)\sum_3^\infty n\sigma_1^{n+1} + A(A-3)\sum_3^\infty \sigma_1^{n+1} - \right.$$

$$\left. - (A+1)(A-2)^2\sum_3^\infty \frac{\sigma_1^{n+1}}{n+A}\right] . \qquad \text{(A-2)}$$

In view of (3-28) and (3-29) we have

$$\sum_{3}^{\infty} n\sigma_1^{n+1} = \frac{\sigma_1^2}{(1-\sigma_1)^2} - \sigma_1^2 - 2\sigma_1^3 \ , \qquad (A-3)$$

$$\sum_{3}^{\infty} \sigma_1^{n+1} = \sigma_1^4(1 + \sigma_1 + \sigma_1^2 + \ldots) = \frac{\sigma_1^4}{1-\sigma_1} \ , \qquad (A-4)$$

and by (3-15),

$$\sum_{1-A}^{\infty} \frac{1}{n+A}\sigma_1^{n+1} = \sigma_1^{1-A} \ln\frac{1}{1-\sigma_1} \ ,$$

so that

$$\sum_{3}^{\infty} \frac{1}{n+A}\sigma_1^{n+1} = \sigma_1^{1-A} \ln\frac{1}{1-\sigma_1} - \sum_{1-A}^{2} \frac{\sigma_1^{n+1}}{n+A} \ ; \qquad (A-5)$$

for $A = -2$, the value of the last sum is taken to be zero. Thus (A-2) becomes, with $1 - \sigma_1 = \zeta_1$,

$$\Delta G_{10} = \frac{\sigma_1^2}{2R^2}\left[(4-A)\left(\frac{\sigma_1^2}{\zeta_1^2} - \sigma_1^2 - 2\sigma_1^3\right) + \right.$$

$$+ A(A-3)\frac{\sigma_1^4}{\zeta_1} -$$

$$\left. - (A+1)(A-2)^2\left(\sigma_1^{1-A}\ln\frac{1}{\zeta_1} - \sum_{1-A}^{2}\frac{\sigma_1^{n+1}}{n+A}\right)\right] . \qquad (A-6)$$

The difference between (3-25) and (3-27) is transformed similarly. We have

$$\Delta G_{20} = G_{20}^{exact} - G_{20}^{approximate} =$$

$$= \frac{\sigma_2^2}{2R^2}\sum_3^\infty \left[\frac{(n-1)(n+2)^2}{(n-2)(n+B)} - n\right]\sigma_2^{n+1} =$$

$$= \frac{\sigma_2^2}{2R^2}\left[(5-B)\sum_3^\infty \sigma_2^{n+1} + \frac{16}{B+2}\sum_3^\infty \frac{\sigma_2^{n+1}}{n-2} + \right.$$

$$\left. + \frac{B+1}{B+2}(B-2)^2 \sum_3^\infty \frac{\sigma_2^{n+1}}{n+B}\right] .$$

By (A-4) and (A-5) this becomes, with $1 - \sigma_2 = \zeta_2$,

$$\Delta G_{20} = \frac{\sigma_2^2}{2R^2}\left[(5-B)\frac{\sigma_2^4}{\zeta_2} + \frac{16}{B+2}\sigma_2^3 \ln\frac{1}{\zeta_2} + \right.$$

$$\left. + \frac{B+1}{B+2}(B-2)^2\left(\sigma_2^{1-B}\ln\frac{1}{\zeta_2} - \sum_{1-B}^{2} \frac{\sigma_2^{n+1}}{n+B}\right)\right] . \quad \text{(A-7)}$$

The exact sums of the approximate expressions (3-26) and (3-27), are, by the last formula on p.17 and the first on p. 18,

$$G_{10}^{approximate} = \frac{\sigma_1^2}{2R^2}\left[\frac{2\sigma_1^3}{\zeta_1^3} - 2\sigma_1^3\right] , \quad \text{(A-8)}$$

$$G_{20}^{approximate} = \frac{\sigma_2^2}{2R^2}\left[\frac{\sigma_2^2}{\zeta_2^2} - \sigma_2^2 - 2\sigma_2^3\right] . \quad \text{(A-9)}$$

We thus have the exact expressions

$$G_{10}^{exact} = G_{10}^{approximate} + \Delta G_{10} , \quad \text{(A-10)}$$

$$G_{20}^{exact} = G_{20}^{approximate} + \Delta G_{20} , \quad \text{(A-11)}$$

the summands on the right-hand side being given by (A-8) and (A-6), and by (A-9) and (A-7), respectively.

The corrections ΔG_{1o} and ΔG_{2o} are of order ζ, so that the formulas (3-31) and (3-32) are, in fact, accurate to that order.

REFERENCES

Burša, M. (1974) The first derivatives of the components of the deflections of the vertical on the territory of the ČSSR. Geofysikālnī Sbornīk XXII, No. 416, pp.9-64, Academia, Prague.

Kaula, W.M (1959) Statistical and harmonic analysis of gravity. J.Geophys. Research, v.64, pp.2401-2421.

Moritz, H. (1976) Covariance functions in least-squares collocation. Rept. No. 240, Dept. of Geodetic Sci., Ohio State Univ.

Schwarz, K.P. (1976) Geodetic accuracies obtainable from measurements of first and second order gravitational gradients. Ibid., Rept. No. 242.

Tscherning, C.C. (1976) Covariance expressions for second and lower order derivatives of the anomalous potential. Ibid., Rept. No. 225.

Tscherning, C.C., and R.H. Rapp (1974) Closed covariance expressions for gravity anomalies, geoid undulations, and deflections of the vertical, implied by anomaly degree variance models. Ibid., Rept. No. 208.