AD-A044 886 BALLISTIC RESEARCH LABS ABERDEEN PROVING GROUND MD F/G 12/1
ON THE DISTRIBUTION OF THE PRODUCT OF TWO GAMMA VARIATES. (U)
SEP 77 P R SCHLEGEL
UNCLASSIFIED BRL-MR-2783 NL

1 OF 1
ADA
044886

END
DATE
FILMED
10-77
DDC

BRL MR 2783

AD A044886

BRL

(12)

AD

MEMORANDUM REPORT NO. 2783 ✓

ON THE DISTRIBUTION OF THE PRODUCT OF TWO GAMMA VARIATES

Palmer R. Schlegel

September 1977

Approved for public release; distribution unlimited.

USA ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
USA BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER BRL ~~MEMORANDUM REPORT NO.~~ MR- 2783	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE DISTRIBUTION OF THE PRODUCT OF TWO GAMMA VARIATES		5. TYPE OF REPORT & PERIOD COVERED Final rept. 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Palmer R. Schlegel		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U.S. Army Ballistic Research Laboratory Aberdeen Proving Ground, MD 21005		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RDTE Project 1L161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Material Development & Readiness Command 5001 Eisenhower Avenue Alexandria, VA 22333		12. REPORT DATE SEPTEMBER 1977 13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

050 750

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Mellin transorm, Bessel function, gamma function, distribution, density function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper Mellin tranforms are used to determine the distribution of the product of two random variables, each of which is distributed by gamma functions and parameterized by an integer n. Furthermore, a recursive relationship is developed with respect to the parameter n to facilitate the evaluation of the resulting distribution.

TABLE OF CONTENTS

I. INTRODUCTION

If one is interested in the distribution of the sum of two random variables, an immediate approach would be to use Fourier transforms. It is well known that the Fourier transform of the density function of the sum of two random variables is equal to the product of the Fourier transforms of the density functions of the respective random variables.

To determine the distribution of the product of two random variables, an analogous approach, namely, the application of Mellin transforms, will be given in this paper. In addition, a recursive relationship will be developed to evaluate the resulting distribution.

II. DEVELOPMENT OF DISTRIBUTION

If the density function of a random variable vanishes in the interval $(-\infty, 0)$, then the Mellin transform, which is defined by

$$M(f)(s) = \int_0^\infty f(x) x^{s-1} dx, \qquad (1)$$

exists. Furthermore, it is known and easy to show that the Mellin transform of the density function of the product of two random variables is the product of the Mellin transforms of each denity function.

Let X, Y and XY be random variables and f_X, f_Y and f_{XY} denote the respective density functions, where

$$f_X(x) = \frac{e^{-x} x^{n-2}}{(n-2)!} \qquad (2)$$

and

$$f_Y(y) = \frac{e^{-y} y^{n-1}}{(n-1)!} \quad . \qquad (3)$$

Then

$$M(f_X)(s) = \int_0^\infty \frac{e^{-x} x^{n-2} x^{s-1}}{(n-2)!} dx = \frac{\Gamma(s+n-2)}{(n-2)!} \qquad (4)$$

and

$$M(f_Y)(s) = \int_0^\infty \frac{e^{-y} y^{n-1} y^{s-1}}{(n-1)!} dy = \frac{\Gamma(s+n-1)}{(n-1)!} , \quad (5)$$

where $\Gamma(u)$ is the gamma function (see [1]). Therefore, from the relationship of Mellin transforms, we have

$$M(f_{XY})(s) = \frac{\Gamma(s+n-2)\Gamma(s+n-1)}{(n-2)!(n-1)!} . \quad (6)$$

From the inverse Mellin transform (see [2]),

$$f_{XY}(z) = \frac{2(\sqrt{z})^{2n-3} K_1(2\sqrt{z})}{(n-1)!(n-2)!} , \quad (7)$$

where $K_1(u)$ is a first order modified Bessel function of the third kind.

The distribution function of the random variable $Z = XY$ is given by

$$P(Z < z) = \int_0^z f_{XY}(u)du = \frac{[2]^{3-2n}}{(n-1)!(n-2)!} \int_0^{2\sqrt{z}} u^{2n-2} K_1(u)du. \quad (8)$$

III. EVALUATION OF THE DISTRIBUTION

The distribution (8) is parameterized by an integer n. We will now proceed to develop a recursive relationship in terms of the parameter that will facilitate the evaluation of the distribution.

Define

$$P_n(z) = \frac{[2]^{3-2n}}{(n-1)!(n-2)!} \int_0^{2\sqrt{z}} u^{2n-2} K_1(u)du. \quad (9)$$

To simplify notation, we will consider

$$I_n = \int_0^a u^{2n-2} K_1(u)du. \quad (10)$$

---

[1] This problem was suggested to the author by D. Clark, AMSAA.

[2] Erdelyi, A., et al, Editor, "Table of Integral Transforms", Bateman Manuscript Project, Vol. I, McGraw-Hill Book Co., Inc., New York, 1954

From the identity (see [3])

$$K_m'(x) = -K_{m+1}(x) + \frac{m}{x} K_m(x), \tag{11}$$

we have, where from (11) $K_1(u) = -K_0'(u)$, by integration by parts

$$I_n = -\int_0^a u^{2n-2} K_0'(u)du = -a^{2n-2} K_0(a) + (2n-2)\int_0^a u^{2n-3} K_0(u)du. \tag{12}$$

From (11), where m = -1 and the fact that $K_m(x) = K_{-m}(x)$, we have

$$K_0(x) = -K_1'(x) - x^{-1} K_1(x). \tag{13}$$

If we substitute (13) into (12) and integrate by parts, we obtain

$$\begin{aligned} I_n &= -a^{2n-2} K_0(a) - (2n-2)\int_0^a [u^{2n-3} K_1'(u) + u^{2n-4} K_1(u)]du \\ &= -a^{2n-2} K_0(a) - (2n-2)a^{2n-3} K_1(a) + \\ &\qquad + (2n-2)(2n-4)\int_0^a u^{2n-4} K_1(u)du. \end{aligned} \tag{14}$$

Therefore, from (9)

$$\begin{aligned} P_n(z) &= \frac{[2]^{3-2n}}{(n-1)!(n-2)!}\Big[-(2\sqrt{z})^{2n-2} K_0(2\sqrt{z}) - (2n-2)(2\sqrt{z})^{2n-3} K_1(2\sqrt{z}) + \\ &\qquad + (2n-2)(2n-4)\int_0^{2\sqrt{z}} u^{2n-4} K_1(u)du\Big] \\ &= P_{n-1}(z) - \frac{(\sqrt{z})^{2n-3}}{(n-1)!(n-2)!}\Big[2\sqrt{z}\, K_0(2\sqrt{z}) + (2n-2)K_1(2\sqrt{z})\Big], \end{aligned} \tag{15}$$

for n > 2, where

$$P_2 = 1 - 2zK_2(2\sqrt{z}). \tag{16}$$

---

[3]Abramowitz, Milton and Stegun, Irene A., Editors, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", National Bureau of Standards, Applied Mathematics Series-55, 1964.

and $K_m(x)$, m = 0, 1, 2, are mth order modified Bessel functions of the third kind.

IV. CONCLUSION

The original problem, as presented to the author, was to construct a double-entry table of the distribution over the variable z and the parameter n. A numerical scheme involved an approximation of a double integration over an infinite limit had been programmed for BRLESC. This procedure was estimated to take an average of five minutes of computer time to evaluate a single point in the double-entry table. Equations (15) and (16) reduced this to a table look-up and a simple calculation with a hand calculator.

ACKNOWLEDGEMENT

The author gratefully acknowledges Dr. S. S. Wolff for suggesting the Mellin transform and Dr. M. S. Taylor for his many useful editorial comments which were incorporated in this paper.

DISTRIBUTION LIST

No. of Copies	Organization
12	Commander Defense Documentation Center ATTN: DDC-TCA Cameron Station Alexandria, VA 22314
1	Commander US Army Materiel Development and Readiness Command ATTN: DRCDMA-ST 5001 Eisenhower Avenue Alexandria, VA 22333
1	Commander US Army Aviation Research and Development Command ATTN: DRSAV-E 12th and Spruce Streets St. Louis, MO 63166
	Director US Army Air Mobility Research and Development Laboratory Ames Research Center Moffett Field, CA 94035
1	Commander US Army Electronics Command ATTN: DRSEL-RD Fort Monmouth, NJ 07703
2	Commander US Army Missile Research & Development Command ATTN: DRDMI-R Redstone Arsenal, AL 35809
1	Commander US Army Tank-Automotive Development Command ATTN: DRDTA-RWL Warren, MI 48090
1	Commander US Army Armament Materiel Readiness Command Rock Island, IL 61201
1	Commander US Army Armament Research & Development Command ATTN: DRDAR-SCF - Mr. S. Goodman Dover, NJ 07801
1	Commander US Army Harry Diamond Labs ATTN: DRXDO-TI 2800 Powder Mill Road Adelphi, MD 20783
1	Director US Army TRADOC Systems Analysis Activity ATTN: ATAA-SA White Sands Missile Range, NM 88002

Aberdeen Proving Ground

Marine Corps Ln Ofc
Dir, USAMSAA
Attn: Mr. D. Clark