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ASYMPTOTIC FAILURE DISTRIBUTIONS.

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## 0. Non-Technical Summary

In this paper, a single device shock model is considered. The model we study consists of a single device which is subject to shocks from the outside environment. An example is an electrical device which occasionally experiences a large electrical surge due to a malfunction in the electrical system.

These shocks will eventually render the device inoperable. We consider the class of devices which will almost certainly be able to endure large numbers of shocks before failing. We find conditions on the shocking processes and on the ability of the devices to survive shocks so that the time to failure distribution of the device is asymptotically an Increasing Failure Rate Distribution.

## 1. Introduction

The shock model discussed in this paper is rather simple. It consists of a single device and a shocking process which acts independently of the device. As time goes on, the cumulative damage done to the device by the shocking process increases and the probability that the device is still surviving decreases.

We let  $Z_t$  represent the cumulative damage at time  $t$ , we let  $T$  be the lifetime of the device and we let  $\bar{h}(t) = P(T > t)$ . Then  $P(T > t | Z_t) = f(Z_t)$ , where  $f$  is some non-increasing Borel measurable function mapping the reals into  $[0,1]$ . So the probability of the event  $(T > t)$  is conditionally independent of  $t$  given  $Z_t$ .

In [6], [1] and [8],  $Z_t$  was taken to be a renewal process. Symbolically,  $Z_t = N_t$  where  $N_t$  is an ordinary renewal process. Letting  $P_k = f(k)$ ,  $k \geq 0$ , the problem is now fully described if we know the sequence  $\{P_k\}$  and the interarrival distribution of  $N_t$ . In this paper, we will consider the cases where  $Z_t$  is an ordinary renewal process, a generalized renewal process or the sum of Brownian motion with drift and an ordinary renewal process.

## 2. Distribution Classes in Reliability Theory

In reliability theory it is often important to classify  $H$ , where  $H$  is some failure distribution. Knowing which class  $H$  belongs to can tell us about its shape or about the form of the optimal maintenance policy.

Definition: A probability distribution  $H$  on  $(0, \infty)$  is Increasing Failure Rate (IFR) if  $\frac{\bar{H}(t+x)}{\bar{H}(t)}$  is non-increasing in  $t$  for  $x > 0$ , where  $\bar{H}(t) = 1 - H(t)$ .

If  $H$  has a density, the above definition is equivalent to  $\frac{h(t)}{\bar{H}(t)}$  non-decreasing in  $t$  for  $h$  some version of the density. A distribution  $H$  is IFR if and only if  $\bar{H}$  is log concave.

Definition: A sequence  $\{P_k\}$  is a probability sequence if  $P_k \geq 0$ , all  $k \geq 0$ ,  $P_0 \leq 1$  and  $P_k$  non-increasing.

Definition: A probability sequence  $\{P_k\}$  is discrete IFR if  $P_{k+1}/P_k$  is non-increasing in  $k$ .

Definition: A distribution  $H$  on  $[0, \infty)$  is Increasing Failure Rate on the Average (IFRA) if  $\bar{H}(t)^{1/t}$  is non-increasing in  $t$ . A probability sequence  $\{P_k\}$  is discrete IFRA if  $P_k^{1/k}$  is non-increasing in  $k$ .

In [6] and [1],  $N_t$  was assumed to be a Poisson Process. The authors showed that if  $\{P_k\}$  was a discrete IFR(IFRA) probability sequence, then for

$$\begin{aligned} \bar{H}(t) = P(T > t) &= \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} P_k \\ &= \sum_{k=0}^{\infty} P(N_t = k) P_k = EP_{N_t}, \end{aligned}$$

$H$  is IFR(IFRA).

In [8], weaker conditions on  $N_t$  were found so that if  $\{P_k\}$  is a discrete IFR(IFRA) probability sequence, then  $H$  is IFR(IFRA).

### 3. Summary of Results

The conditions on  $N_t$  in [8], while more general than those of [6] or [1], are still restrictive. The reader will note that for any ordinary renewal process  $N_t$  with interarrival times  $\{X_k\}$ , with  $X_1 \sim F_1$  and  $EX_1 = \mu < \infty$  and  $\text{var } X_1 = \sigma^2 < \infty$ , that

$$\frac{N_t - t/\mu}{\sqrt{\sigma^2 t/\mu^3}} \xrightarrow{d} N(0,1) .$$

$N(0,1)$  is a random variable with the standard normal distribution and  $\xrightarrow{d}$  represents convergence in distribution.

This result, the Central Limit Theorem for renewal processes, tells us that the one-dimensional distributions of many renewal processes, including the Poisson, begun in some sense to resemble each other as  $t$  gets large. As the pertinent theorems of [6], [1] and [8] make use of only the one-dimensional distributions of  $N_t$ , it seems possible that limiting results, of the nature of the results which made assertions about  $H$  given the form of the  $\{P_k\}$  sequence, can be derived.

Now, in practice, it is often the case that we will be considering a device which will almost certainly sustain tens or hundreds of shocks before it will even be remotely possible that the device will fail. If this is the case, we would be justified in considering the shape of  $H$  for values of  $t$  which are large compared to the expected interarrival times. Indeed, this is primarily what we do in this paper.

Definition: A function  $\phi$  is asymptotically log concave if there exists a log concave function  $\psi$  with the property that  $\lim_{t \rightarrow \infty} \frac{\phi(t)}{\psi(t)} = 1$ . A distribution  $H$  is asymptotically IFR if  $\tilde{H}$  is asymptotically log concave.

We will find conditions on  $N_t$  and  $\{P_k\}$  so that  $\tilde{H}(t) = EP_{N_t}$  is asymptotically log concave. We will more generally find conditions

on  $f$  and  $Z_t$ , where  $Z_t$  is either a generalized renewal process or the sum of a renewal process and Brownian motion with drift so that  $\bar{H}(t) = Ef(Z_t)$  is asymptotically log concave. In all cases, we will find explicitly, the asymptotic form of  $\bar{H}(t)$ . The major results of this paper are found in Section 11. We will first consider how different functions  $f$  arise naturally, where we assume that  $P_{n+1}/P_n \rightarrow e^{-\gamma}$ ,  $\gamma \geq 0$  and  $f(n) = P_n$ . In order to do this, we must first study regularly varying functions.

#### 4. Regularly Varying Functions

Definition: A positive function  $L$  defined on  $[0, \infty]$  varies slowly at infinity if and only if  $\frac{L(tx)}{L(t)} \rightarrow 1$  as  $t \rightarrow \infty$  for every  $x > 0$ . A function  $U$  varies regularly at infinity if and only if it is of the form  $U(x) = x^p L(x)$  with  $L$  varying slowly at infinity and  $-\infty < p < \infty$ .

Lemma 4.1: If  $L$  varies slowly at infinity, then  $x^{-\epsilon} < L(x) < x^\epsilon$ , for any fixed  $\epsilon > 0$  and all  $x$  sufficiently large.

Proof: The proof is in [7], p. 277.

Lemma 4.2:  $\frac{L(tx)}{L(t)} \rightarrow 1$  as  $t \rightarrow \infty$  uniformly in finite intervals  $0 < a < x < b$ .

Proof. The proof is in [7], p. 277.

To show that  $L$  is slowly varying at infinity, it is enough to show  $\frac{L(t)}{L(xt)} \rightarrow 1$  as  $t \rightarrow \infty$ , for  $0 < x < 1$ .

We will be considering functions which are slowly or regularly varying at infinity and functions of the form  $f(x) = U(x) e^{-\gamma x}$ , where  $U$  is regularly varying at infinity. A reference for regularly varying functions is [7].

When do such functions arise in the context of our model? Specifically, when is  $f(n) = P_n$  of the form,  $f(n) = U(n) e^{-\gamma n}$  with  $U$  regularly varying at infinity? We deal with this question in the next section.

### 5. The $\{P_k\}$ Sequence and Regularly Varying Functions

If any function  $\tau$  that we are considering is only defined on the non-negative integers, extend it to  $\mathbb{R}^+$  in the following manner. Let  $\phi(n) = \log \tau(n)$ . Extend  $\phi$  to  $\mathbb{R}^+$  by linear interpolation. Let  $\tau(t) = e^{\phi(t)}$ . So if  $\tau$  is log concave on the non-negative integers, then the extension of  $\tau$  to  $\mathbb{R}^+$  is log concave on  $\mathbb{R}^+$ .

Theorem 1: Suppose  $P_{n+1}/P_n = e^{-\gamma}(1 + a_n)$ , and  $a_n = o(n^{-1-c})$ , some  $c > 0$ . If  $a_n > -1$ , all  $n$ , then  $g(n) = e^{\gamma n} P_n = e^{\gamma n} f(n) = \prod_{k=1}^n (1 + a_k)$  is slowly varying at infinity and

$$0 < \underline{\lim}_{n \rightarrow \infty} g(n) \leq \overline{\lim}_{n \rightarrow \infty} g(n) < \infty.$$

If  $a_n$  is decreasing,  $g(n)$  is log concave.

Proof: Note that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{x^k (-1)^{k+1}}{k}, \quad \text{for } x > -1.$$

So for  $M$  sufficiently large,  $n \geq M$ ,

$$\begin{aligned} \sum_{k=1}^M \log(1+a_k) + \sum_{k=M+1}^n a_k - \frac{1}{2} \sum_{k=M+1}^n a_k^2 - \frac{2}{3} \sum_{k=M+1}^n |a_k^3| \\ \leq \log g(n) \leq \sum_{k=1}^M \log(1+a_k) + \sum_{k=M+1}^n a_k. \end{aligned}$$

As

$$\sum_{k=M+1}^{\infty} |a_k| < \infty, \quad \sum_{k=M+1}^{\infty} |a_k|^2 < \infty, \quad \sum_{k=M+1}^{\infty} |a_k|^3 < \infty,$$

it follows immediately that

$$\overline{\lim}_{n \rightarrow \infty} g(n) < \infty, \quad \underline{\lim}_{n \rightarrow \infty} g(n) > 0.$$

Letting  $M = xn$ ,  $0 < x < 1$ , we have

$$\begin{aligned} |\log g(n) - \log g(xn)| &\leq \sum_{k=[xn]}^n |a_k| + \frac{1}{2} |a_k^2| + \frac{1}{3} |a_k^3| + \dots \\ &\leq 2 \sum_{k=[xn]}^{\infty} |a_k| \quad \text{for } [xn] \text{ sufficiently large.} \end{aligned}$$

So,

$$\lim_{n \rightarrow \infty} |\log g(n) - \log g(xn)| \leq 2 \lim_{n \rightarrow \infty} \sum_{k=[xn]}^{\infty} |a_k| = 0,$$

we have that

$$\lim_{n \rightarrow \infty} |\log g(n) - \log g(xn)| = 0,$$

or

$$\lim_{n \rightarrow \infty} \frac{g(xn)}{g(n)} = 1, \quad \text{for any } x \in (0,1).$$

So,  $g$  is slowly varying at infinity. If  $a_n$  is decreasing, so is  $\frac{g(n+1)}{g(n)}$ , so  $g$  is log concave.

Theorem 5.2: Suppose that  $P_{n+1}/P_n = e^{-\gamma}(1 + a_n/n)$  where  $\sum_{n=0}^{\infty} \left| \frac{a_n - \rho}{n} \right| < \infty$ ,  $\rho \geq 0$ , and  $a_n/n > -1$ , all  $n$ . Then  $g(n) = e^{\gamma n}$ ,  $f(n) = e^{\gamma n} P_n = n^{\rho} L(n)$  where  $L(n)$  is slowly varying at infinity and

$$0 < \lim_{n \rightarrow \infty} L(n) \leq \lim_{n \rightarrow \infty} L(n) < \infty.$$

If  $\rho > 0$ ,  $g$  is asymptotically log concave, and if  $a_n/n$  is non-increasing,  $g$  is log concave.

Proof. Let

$$\phi(n) = \prod_{k=1}^n \left(1 + \frac{\rho}{k}\right).$$

We will first show that  $g(n) = \phi(n) f(n)$ , where  $f$  is slowly varying at infinity. For  $0 < x < 1$ ,

$$\begin{aligned}
 &= \sum_{k=[xn]+1}^{\infty} \left| \frac{\rho - a_k}{k} \right| - \frac{|\rho|}{[xn]} - \left| \frac{a_{[xn]}}{[xn]} \right| - \frac{1}{2} \sum_{k=[xn]}^{\infty} \frac{\rho^2}{k^2} - \frac{1}{3} \sum_{k=[xn]}^{\infty} \frac{|\rho|^3}{k^3} \\
 &\leq [\log \phi(n) - \log \phi(xn)] - [\log g(n) - \log g(xn)] \\
 &\leq \sum_{k=[xn]+1}^{\infty} \left| \frac{\rho - a_k}{k} \right| + \frac{1}{2} \sum_{k=[xn]}^{\infty} \frac{a_k^2}{k^2} + \frac{1}{3} \sum_{k=[xn]}^{\infty} \left| \frac{a_k}{k} \right|^3 + \left| \frac{a_{[xn]}}{[xn]} \right|.
 \end{aligned}$$

Letting  $n \rightarrow \infty$ , each of the terms on the left and right-most sides of the string of inequalities goes to zero. So,

$$\lim_{n \rightarrow \infty} [\log \phi(n) - \log \phi(xn)] - [\log g(n) - \log g(xn)] = 0,$$

or

$$\lim_{n \rightarrow \infty} \frac{\phi(n)/g(n)}{\phi(xn)/g(xn)} = 1, \quad \text{for } 0 < x < 1.$$

Letting  $f(n) = \phi(n)/g(n)$ , we see that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{f(xn)} = 1, \quad 0 < x < 1.$$

This suffices to show that  $f$  is slowly varying at infinity.

Replacing  $xn$  by  $N$  in the above string of inequalities gives

us

$$-K_1 \leq \log \frac{f(n)}{f(M)} \leq K_2, \quad \text{with } 0 < K_1 < \infty, 0 < K_2 < \infty.$$

So,

$$f(M) e^{-K_1} \leq f(n) \leq f(M) e^{-K_2},$$

so,

$$0 < \underline{\lim}_{n \rightarrow \infty} f(n) \leq \overline{\lim}_{n \rightarrow \infty} f(n) < \infty.$$

We will now show that  $\phi(n) = n^\rho Q(u)$ , where  $Q$  is slowly varying at infinity. For  $0 < x < 1$ ,  $n$  sufficiently large,

$$\begin{aligned} \rho \cdot \sum_{k=[nx]+1}^n \frac{1}{k} - \frac{\rho^2}{2} \sum_{k=[nx]}^n \frac{1}{k^2} - \frac{|\rho^3|}{3} \sum_{k=[nx]}^n \frac{1}{k^3} - \frac{|\rho|}{[nx]} \\ \leq \log \phi(n) - \log \phi(nx) \leq \sum_{k=[nx]+1}^n \frac{\rho}{k} + \frac{|\rho|}{[nx]} \leq \rho \log \frac{n}{[nx]} + \frac{|\rho|}{[nx]}, \end{aligned}$$

or,

$$\begin{aligned} \rho \log \frac{n+1}{[nx]+1} - \frac{|\rho|}{[nx]} - \frac{\rho^2}{2} \sum_{k=[nx]}^n \frac{1}{k^2} - \frac{|\rho^3|}{3} \sum_{k=[nx]}^n \frac{1}{k^3} \\ \leq [\log \phi(n) - \log \phi(nx)] - [\log n^\rho - \log (nx)^\rho] \leq \frac{|\rho|}{[nx]}. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we see that

$$\frac{\phi(n)/n^\rho}{\phi(nx)/(nx)^\rho} \rightarrow 1, \quad \text{for } 0 < x \leq 1.$$

Letting  $Q(n) = \phi(n)/n^\mu$ , we see that  $Q$  is slowly varying at infinity and it is easy to see that

$$0 < \liminf_{n \rightarrow \infty} Q(n) \leq \overline{\lim}_{n \rightarrow \infty} Q(n) < \infty .$$

So  $\phi(n) = Q(n) n^\mu$  and  $g(n) = \phi(n) \mathfrak{L}(n) = Q(n) \mathfrak{L}(n) n^\mu = L(n) n^\mu$ , where  $L$  is slowly varying at infinity and

$$0 < \liminf_{n \rightarrow \infty} L(n) \leq \overline{\lim}_{n \rightarrow \infty} L(n) < \infty .$$

If  $a_n/n$  is decreasing,  $g(t)$  is trivially log concave. For the case of  $\mu > 0$ , we omit the proof that  $g$  is asymptotically log concave.

## 6. Preliminaries

In this section, we review the notation and list the assumptions we will make. Especially in the next section, we may not make specific note of the assumptions that are being made.

- a)  $\{N_t, t \geq 0\}$  is an ordinary renewal process with inter-arrival times  $\{X_k, X_1 \sim F, EX_1 = \mu < \infty, \text{Var } X_1 = \sigma^2 < \infty, F \text{ is non-lattice}.$
- b)  $E[e^{sX_1}]$  exists in some neighborhood of the origin.
- c) When we consider  $E[e^{-rN_t}]$ , then we will assume there exists a number  $\rho(r)$  with the property that  $e^{-r} \int_0^\infty e^{\rho(r)s} F(ds) = 1.$

d)  $S_{N_t}$  is a generalized renewal process with

$$S_{N_t} = \begin{cases} \sum_{n=1}^{N_t} Y_n, & N_t \geq 1 \\ 0, & N_t = 0. \end{cases}$$

The  $\{Y_k\}$  sequence is i.i.d., with  $Y_1 \sim K$ ,  $K(0-) = 0$ , and  $E(e^{sY_1})$  exists in some open neighborhood of the origin.

e) When we consider  $E[e^{-\gamma S_{N_t}}]$ , we will assume there exists a number  $\rho(\gamma^*)$  such that

$$e^{-\gamma^*} \int_{s=0}^{\infty} e^{\rho(\gamma^*)s} F(ds) = 1, \quad \text{where } e^{-\gamma^*} \int_{v=0}^{\infty} e^{-\gamma^*v} K(dv) = 1.$$

f) Our probability space  $(\Omega, \mathcal{G}, P)$  will be of the following form,

$\Omega = \mathbb{R}^1 \times \dots \times \mathbb{R}^1 \times \dots$ , where  $\mathbb{R}^1$  is the set of non-negative real numbers.  $\mathcal{G} = \mathcal{G} \times \dots \times \mathcal{G} \times \dots$ , where  $\mathcal{G}$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}^1$ .

For  $A_k \in \mathcal{G}$ ,  $k = 1, \dots, n$ ,  $P(A_1 \times A_2 \times \dots \times A_n) = \prod_{k=1}^n P_0(A_k)$ , where  $P_0$  is the probability measure on  $\mathbb{R}^1$  associated with  $F$ .  $P$  is then uniquely extended to  $\mathcal{G}$ . Any  $\omega \in \Omega$  is of the form,

$$\omega = (\omega_1, \omega_2, \dots, \omega_n, \dots), \quad \text{where each } \omega_k \in \mathbb{R}^1 \text{ and } X_n(\omega) = \omega_n.$$

Finally, define a shift operator  $\theta : \Omega \rightarrow \Omega$  so that

$\theta\omega = (\omega_2, \omega_3, \dots, \omega_n, \dots)$ . So in particular,

$$N_t(\omega) = N_{t-X_1(\omega)}(\theta\omega) + 1 \quad \text{on } \{X_1(\omega) \leq t\}.$$

- g)  $B_t$  is Brownian motion with  $B_1 \sim N(0, \sigma^2)$ .
- h) If we consider  $N_t + B_t + ct$ , then we always assume that  $1/\mu_0 = 1/\mu + c > 0$ .
- i)  $L$  will always be a slowly varying function at infinity on  $\mathbb{R}^+$ .  $L$  is extended to all of  $\mathbb{R}$  by letting  $L(x) = 1$  for  $x < 0$ .

#### 7. Large Deviations

Let  $S_n = \sum_{i=1}^n X_i$ ,  $\{X_i\}$  i.i.d.,  $EX_1 = \mu < \infty$ ,  $\text{Var } X_1 = \sigma^2 < \infty$ . Let  $F_n$  be the distribution of  $(S_n - n\mu)/\sqrt{n\sigma^2}$ .

Theorem 7.1: If  $\pi(\xi) = \int_{-\infty}^{\xi} e^{t_0 x} F(dx)$  exists for all  $\xi$  in some interval  $|\xi| < \xi_0$ , and if  $x$  varies with  $n$  in such a way that as  $n \rightarrow \infty$ ,  $x = o(n^{1/6})$ , then  $(1 - F_n(x))/(1 - R(x)) \rightarrow 1$ , where

$$R(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-u^2/2} du.$$

Proof: The proof is in [7], p. 460.

### 8. Technical Lemmas

Lemma 8.1:  $\lim_{t \rightarrow \infty} E[L(N_t)/L(t)] = 1.$

Proof: Choose  $M > 1/\mu + 1$ ,  $S_n = \sum_{i=1}^n X_i.$

$$A) \quad \lim_{t \rightarrow \infty} E \left[ \frac{L(N_t)}{L(t)}; N_t \geq Mt \right]$$

$$\leq \overline{\lim}_{t \rightarrow \infty} E[N_t^\epsilon t^\epsilon; N_t \geq Mt], \quad \text{any } \epsilon > 0, \text{ by Lemma 4.1}$$

$$\leq \overline{\lim}_{t \rightarrow \infty} E[N_t; N_t \geq Mt]$$

$$\leq \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} E((n+1)t; 1_{N_t \in [nt, (n+1)t]})$$

$$\leq \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(N_t \geq nt)$$

$$\leq \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(S_{[nt]} \leq t) .$$

Let  $\hat{X}_i = -X_{i+1}$ ,  $i = 1, 2, \dots$ . So  $E\hat{X}_i = 0$ , all  $i$ ,  $\text{Var } \hat{X}_i = \sigma^2$ , all  $i$ , and  $\{\hat{X}_i\}$  are i.i.d. Also,  $E|e^{s\hat{X}_1}| < \infty$ , for all  $s$  in some neighborhood containing the origin. Let  $S_n = \sum_{i=1}^n \hat{X}_i$ . Then

$$\begin{aligned}
& \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(S_{[nt]} \leq t) \\
& \leq \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(\hat{S}_{[nt]} \geq (n\mu-1)t - \mu) \\
& = \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P\left(\frac{\hat{S}_{[nt]}}{\sigma\sqrt{[nt]}} \geq \frac{(n\mu-1)t - \mu}{\sigma\sqrt{nt}}\right) \\
& \leq \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P\left(\frac{\hat{S}_{[nt]}}{\sigma\sqrt{[nt]}} \geq cn^{1/7} t^{1/7}\right), \quad c > 0.
\end{aligned}$$

We can now apply Theorem 7.1. For  $nt > N_0(\delta)$ ,

$$1 - \delta \leq \frac{P\left(\frac{\hat{S}_{[nt]}}{\sigma\sqrt{[nt]}} \geq cn^{1/7} t^{1/7}\right)}{P(N(0,1) \geq cn^{1/7} t^{1/7})} \leq 1 + \delta,$$

$\delta$  arbitrarily small. So

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P\left(\frac{\hat{S}_{[nt]}}{\sigma\sqrt{[nt]}} \geq cn^{1/7} t^{1/7}\right) \\
& \leq (1+\delta) \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(N(0,1) \geq cn^{1/7} t^{1/7}).
\end{aligned}$$

For  $x$  sufficiently large,  $P(N(0,1) \geq x) \leq e^{-x}$ . So,

$$\begin{aligned}
(1+\delta) \overline{\lim}_{t \rightarrow \infty} \sum_{n=M}^{\infty} (n+1)t P(N(0,1) \geq cn^{1/7} t^{1/7}) \\
\leq (1+\delta) \overline{\lim}_{t \rightarrow \infty} t \sum_{n=M}^{\infty} \exp(-cn^{1/7} t^{1/7}) \\
\leq (1+\delta) \overline{\lim}_{t \rightarrow \infty} t \int_{\mu=M-1}^{\infty} \exp(-cn^{1/7} t^{1/7}) du \\
= (1+\delta) \overline{\lim}_{t \rightarrow \infty} t \int_{v=(M-1)^{1/7} t^{1/7}}^{\infty} \exp(-cv \cdot 7t^{-1} v^6) dv \\
\leq 8 \overline{\lim}_{t \rightarrow \infty} \int_{v=(M-1)^{1/7} t^{1/7}}^{\infty} e^{-cv} v^6 dv \\
= 0,
\end{aligned}$$

as  $\int_{v=0}^{\infty} v^6 e^{-cv} dv < \infty$ .

B) For  $\delta < 1/\mu$ ,

$$\overline{\lim}_{t \rightarrow \infty} E \left[ \frac{L(N_t)}{L(t)} ; N_t \leq \delta t \right] \leq \overline{\lim}_{t \rightarrow \infty} t^{\mu} P(N_t \leq \delta t) \quad \text{by Lemma 4.1.}$$

Clearly,

$$\overline{\lim}_{t \rightarrow \infty} t^{\mu} P(N_t \leq \delta t) \leq \overline{\lim}_{t \rightarrow \infty} t^{\mu} P(S_{\lfloor \delta t \rfloor} \geq t)$$

Let  $\hat{X}_i = X_i - \mu$ ,  $\hat{S}_n = \sum_{i=1}^n \hat{X}_i$ . Then,

$$\begin{aligned}
& \overline{\lim}_{t \rightarrow \infty} t^\epsilon P(S_{[\delta t]} \geq t) \\
&= \overline{\lim}_{t \rightarrow \infty} t^\epsilon P(\hat{S}_{[\delta t]} \geq t - \mu[\delta t]) \\
&\leq \overline{\lim}_{t \rightarrow \infty} t^\epsilon P(\hat{S}_{[\delta t]} \geq ct), \quad \text{For some } c > 0 \\
&\leq \overline{\lim}_{t \rightarrow \infty} t^\epsilon P\left(\frac{\hat{S}_{[\delta t]}}{\sigma\sqrt{[\delta t]}} \geq ct^{1/7}\right) \\
&\leq (1+\delta) \overline{\lim}_{t \rightarrow \infty} t^\epsilon P(N(0,1) \geq ct^{1/7}), \quad \text{by Theorem 7.1} \\
&\leq (1+\delta) \overline{\lim}_{t \rightarrow \infty} t^\epsilon e^{-ct^{1/7}} \\
&= 0.
\end{aligned}$$

So,

$$\lim_{t \rightarrow \infty} E\left[\frac{L(N_t)}{L(t)}; (N_t \leq \delta t) \cup (N_t \geq Mt)\right] = 0.$$

C) On  $(\delta t < N_t < Mt)$ ,  $L(N_t)/L(t) \rightarrow 1$  uniformly by Lemma 4.2. So,

$$\lim_{t \rightarrow \infty} E\left[\frac{L(N_t)}{L(t)}; \{\delta t < N_t < Mt\}\right] = \lim_{t \rightarrow \infty} P(\delta t < N_t < Mt) = 1.$$

Combining A, B, and C gives the desired result.

Lemma 8.2.

$$\lim_{t \rightarrow \infty} E \left[ \frac{L(N_t + B_t + ct)}{L(t)} \right] = 1.$$

Proof:

A) Choose  $M > 0$  so that  $\frac{M-c}{2} > \frac{1}{\mu}$ ,

$$\begin{aligned} & \overline{\lim}_{t \rightarrow \infty} E \left[ \frac{L(N_t + B_t + ct)}{L(t)}; N_t + B_t + ct > Mt \right] \\ & \leq \overline{\lim}_{t \rightarrow \infty} [E(N_t + |B_t| + ct); N_t + B_t > M_0 t], \quad M_0 = M - c \\ & \leq \overline{\lim}_{t \rightarrow \infty} \left[ E(N_t + |B_t| + c); \left\{ N_t > \frac{M_0 t}{2} \right\} \cup \left\{ B_t > \frac{M_0 t}{2} \right\} \right] \\ & \leq \overline{\lim}_{t \rightarrow \infty} E \left( N_t; N_t > \frac{M_0 t}{2} \right) + \overline{\lim}_{t \rightarrow \infty} E \left( N_t; B_t > \frac{M_0 t}{2} \right) \\ & \quad + \overline{\lim}_{t \rightarrow \infty} E \left( |B_t|; N_t > \frac{M_0 t}{2} \right) + \overline{\lim}_{t \rightarrow \infty} E \left( |B_t|; B_t > \frac{M_0 t}{2} \right) \\ & \quad + \overline{\lim}_{t \rightarrow \infty} ct P \left( N_t > \frac{M_0 t}{2} \right) + \overline{\lim}_{t \rightarrow \infty} ct P \left( B_t > \frac{M_0 t}{2} \right). \end{aligned}$$

From Lemma 8.1,

$$\overline{\lim}_{t \rightarrow \infty} E \left( N_t; N_t > \frac{M_0 t}{2} \right) = 0, \quad \text{as } \frac{M_0}{2} > \frac{1}{\mu}.$$

Then

$$\begin{aligned}
& \overline{\lim}_{t \rightarrow \infty} E\left(N_t; B_t > \frac{M_0 t}{2}\right) \\
&= \overline{\lim}_{t \rightarrow \infty} E(N_t) P\left(B_t > \frac{M_0 t}{2}\right) \leq \overline{\lim}_{t \rightarrow \infty} \left(\frac{t}{\mu} + 1\right) P\left(B_t > \frac{M_0 t}{2}\right) \\
&= \overline{\lim}_{t \rightarrow \infty} \left(\frac{t}{\mu+1}\right) P\left(N(0,1) \geq \frac{M_0 \sqrt{t}}{2\sigma}\right) \leq \overline{\lim}_{t \rightarrow \infty} \left(\frac{t}{\mu} + 1\right) \exp(-M_0 \sqrt{t}/2\sigma) \\
&= 0.
\end{aligned}$$

$$\overline{\lim}_{t \rightarrow \infty} E\left(|B_t|; B_t > \frac{M_0 t}{2}\right) = \overline{\lim}_{t \rightarrow \infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} \sigma^2 t \exp(-M_0^2 t/8\sigma^2) = 0.$$

$$\overline{\lim}_{t \rightarrow \infty} ctP\left(N_t > \frac{M_0 t}{2}\right) = 0 \quad \text{from Lemma 8.1}$$

$$\overline{\lim}_{t \rightarrow \infty} ctP\left(B_t > \frac{M_0 t}{2}\right) = 0 \quad \text{from above}$$

So,

$$\lim_{t \rightarrow \infty} E\left[\frac{L(N_t + B_t + ct)}{L(t)}; N_t + B_t + ct > Mt\right] = 0.$$

B) For  $0 < \delta = c + 1/\mu - \epsilon$ ,

$$\begin{aligned}
& \overline{\lim}_{t \rightarrow \infty} E\left[\frac{L(N_t + B_t + ct)}{L(t)}; N_t + B_t + ct \leq \delta t\right] \leq \overline{\lim}_{t \rightarrow \infty} tP(N_t + B_t + ct \leq \delta t) \\
&= \overline{\lim}_{t \rightarrow \infty} tP(N_t + B_t \leq (\delta - c)t) \leq \overline{\lim}_{t \rightarrow \infty} P(B_t \leq -\frac{c}{2}t) + tP(N_t \leq (\frac{1}{\mu} - \frac{\delta}{2})t).
\end{aligned}$$

Each of the above limits is zero from A) or the second part of Lemma 8.1.

C) So,

$$\begin{aligned} & \lim_{t \rightarrow \infty} E \left[ \frac{L(N_t + B_t + ct)}{L(t)} \right] \\ &= \lim_{t \rightarrow \infty} E \left[ \frac{L(N_t + B_t + ct)}{L(t)} ; \delta t \leq N_t + B_t + ct \leq Mt \right] \\ &= \lim_{t \rightarrow \infty} P(\delta t \leq N_t + B_t + ct \leq Mt) , \end{aligned}$$

as  $L(N_t + B_t + ct)/L(t) \rightarrow 1$  uniformly in  $t$  on  $\{\delta t \leq N_t + B_t + ct \leq Mt\}$ .

Now,

$$\frac{N_t}{t} \rightarrow \frac{1}{\mu} \text{ a.s.}, \quad \frac{ct}{t} = c, \quad \text{and} \quad \frac{B_t}{t} \rightarrow 0 \text{ a.s.}$$

So,

$$\frac{N_t + B_t + ct}{t} \rightarrow \frac{1}{\mu} + c \quad \text{a.s.}$$

Now

$$\delta < \frac{1}{\mu} + c < M ,$$

so,

$$\lim_{t \rightarrow \infty} P(\delta t \leq N_t + B_t + ct \leq Mt) = 1$$

proving the lemma.

Lemma 8.3: For  $k \geq 0$ ,  $k$  an integer;

$$\lim_{t \rightarrow \infty} E \left( \frac{\binom{N_t^k}{t/\mu}^k}{\binom{N_t^k}{t/\mu}^k} \right) = 1 .$$

Proof. Using the notation of Section 6, on  $\{X_1(\omega) \leq t\}$ ,

$$N_t^n(\omega) = (N_{t-X_1(\omega)}(\omega) + 1)^n = \sum_{k=0}^n N_{t-X_1(\omega)}^k(\omega) \cdot \binom{n}{k} ,$$

on  $\{X_1(\omega) > t\}$ ,  $N_t^n(\omega) = 0$ .

The proof will be by induction. It is well known that the lemma is true for  $k = 1$ . Suppose it is true for  $k \leq n$ .

Let  $k = n+1$ . Let  $\phi_j(t) = E[N_t^j]$ ,

$$\begin{aligned} E[N_t^{n+1}] &= E[N_t^{n+1}; X_1 > t] + E[N_t^{n+1}; X_1 \leq t] \\ &= 0 + \sum_{k=0}^{n+1} E \left[ E \left[ \sum_{k=0}^{n+1} N_{t-X_1(\omega)}^k(\omega) \binom{n+1}{k} 1_{X_1(\omega) \leq t} \middle| X_1 \right] \right] \\ &= \sum_{k=0}^{n+1} E \left[ 1_{X_1(\omega) \leq t} E \left[ \sum_{k=0}^{n+1} N_{t-X_1(\omega)}^k(\omega) \binom{n+1}{k} \middle| X_1 \right] \right] \\ &= \sum_{k=0}^{n+1} \int_{s=0}^t E[N_{t-s}^k] \binom{n+1}{k} F(ds) , \end{aligned}$$

or

$$(*) \quad \phi_{n+1}(t) = \sum_{k=0}^n \int_{s=0}^t \binom{n+1}{k} \phi_k(t-s) dF(s) + \int_{s=0}^t \phi_{n+1}(t-s) dF(s) .$$

Recognizing that this is a renewal equation, and letting  $R(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$ , where  $F^{(n)}$  is the  $n^{\text{th}}$  fold convolution of  $F$ , we have,

$$\phi_{n+1}(t) = \int_{v=0}^t \int_{s=0}^v \sum_{k=0}^n \binom{n+1}{k} \phi_k(v-s) F(ds) dR(t-v)$$

$$\frac{\phi_{n+1}(t)}{t^{n+1}} \leq \sum_{k=0}^n \frac{\binom{n+1}{k} t}{t^{n+1}} \int_{v=0}^t \phi_k(v) dR(t-v)$$

$$\lim_{t \rightarrow \infty} \frac{\phi_{n+1}(t)}{t^{n+1}} \leq \overline{\lim}_{t \rightarrow \infty} \sum_{k=0}^{n-2} \frac{1}{t} \int_{v=0}^t \binom{n+1}{k} \frac{\phi_k(v)}{t^n} dR(t-v)$$

$$+ \overline{\lim}_{t \rightarrow \infty} \sum_{k=n-1}^n \frac{1}{t^{n+1}} \int_{v=\epsilon}^t \binom{n+1}{k} \phi_k(v) dR(t-v), \quad \text{for any } \epsilon > 0.$$

Now,

$$\int_{v=\epsilon}^{\infty} \frac{\phi_k(v)}{v^n} dv < \infty \quad \text{for } k \leq n-2$$

by the inductive hypothesis. Therefore, for  $k \leq n-2$ ,

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t^{n+1}} \int_{v=0}^t \phi_k(v) dR(t-v) = 0.$$

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t^{n+1}} \int_{v=0}^t \phi_{n-1}(v) dR(t-v)$$

$$= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t^{n+1}} \int_{v=M}^{\infty} \frac{\phi_{n-1}(v)}{v^{n-1}} v^{n-1} dR(t-v) \quad \text{for any } M > 0$$

$$\leq \overline{\lim}_{t \rightarrow \infty} \frac{1}{t^{n+1}} \int_{v=M}^t \frac{(1+\delta)}{\mu^{n-1}} v^{n-1} dR(t-v), \quad \text{some } \delta > 0$$

$$\leq \overline{\lim}_{t \rightarrow \infty} \frac{(1+\delta)}{\mu^{n-1}} \frac{1}{t^2} \frac{t(1+\epsilon)}{\mu}, \quad \text{some } \epsilon > 0$$

= 0 .

Finally,

$$\begin{aligned} & \overline{\lim}_{t \rightarrow \infty} \frac{1}{t^{n+1}} \int_{v=0}^t \rho_n(v) dR(t-v) \\ &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{v=0}^t \frac{\rho_n(v)}{v^n} \frac{v^n}{t^n} dR(t-v) \\ &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{v=M}^t \frac{\rho_n(v)}{v^n} \frac{v^n}{t^n} dR(t-v) \\ &\leq \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{v=M}^t \frac{(1+\delta)}{\mu^n} (1+\epsilon) \frac{v^n}{t^n} \frac{dv}{\mu} ; \end{aligned}$$

where M is chosen arbitrarily large and  $\delta$  and  $\epsilon$  are arbitrarily close to 0. We can bound  $\int_{v=M}^t v^n/t^n dR(t-v)$  by  $(1+\epsilon) \int_{v=M}^t v^n/t^n dv/\mu$  because  $R(t+h) - R(t) \rightarrow h/\mu$  as  $t \rightarrow \infty$ . So,

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{v=M}^t \frac{(1+\delta)}{\mu^n} (1+\epsilon) \frac{v^n}{t^n} \frac{dv}{\mu} &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_{v=0}^t \frac{(1+\delta)}{\mu^n} (1+\epsilon) \frac{v^n}{t^n} \frac{dv}{\mu} \\ &= \frac{(1+\delta)(1+\epsilon)}{\binom{n+1}{1} \mu^{n+1}} . \end{aligned}$$

So,

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_{n+1}^{(t)}}{t^{n+1}} \leq \binom{n+1}{1} \cdot \frac{1}{\mu^{n+1}} \binom{n+1}{1} = \frac{1}{\mu^{n+1}} .$$

As  $N_t/t \rightarrow 1/\mu$  a.s.,  $N_t^n/t^n \rightarrow (1/\mu)^n$  a.s. So by Fatou's Lemma

$$\underline{\lim}_{t \rightarrow \infty} \frac{E(N_t^n)}{t^n} \geq \left(\frac{1}{\mu}\right)^n .$$

So

$$\lim_{t \rightarrow \infty} \frac{E(N_t^n)}{t^n} = \left(\frac{1}{\mu}\right)^n .$$

Lemma 8.4: For any  $\mu \geq 0$ ,

$$\lim_{t \rightarrow \infty} \frac{E(N_t^\mu)}{(t/\mu)^\mu} = 1 .$$

Proof:

$$\underline{\lim}_{t \rightarrow \infty} \frac{E(N_t^\mu)}{(t/\mu)^\mu} \geq E \left( \underline{\lim}_{t \rightarrow \infty} \frac{(N_t)^\mu}{(t/\mu)^\mu} \right) = 1 ,$$

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \frac{E(N_t^\mu)}{(t/\mu)^\mu} &\leq (1+\epsilon)^\mu \overline{\lim}_{t \rightarrow \infty} P(N_t \leq t/\mu (1+\epsilon)) \\ &\quad + \overline{\lim}_{t \rightarrow \infty} E \left( \frac{N_t^\mu}{(t/\mu)^\mu}; N_t \geq \frac{t}{\mu} (1+\epsilon) \right) , \end{aligned}$$

for any  $\epsilon > 0$  and  $n > \mu$ ,  $n$  an integer,

$$\begin{aligned} &\leq (1+\epsilon)^\mu + 1 - (1-\epsilon)^\mu \underline{\lim}_{t \rightarrow \infty} P \left( \left| \frac{N_t}{t} - \frac{1}{\mu} \right| < \epsilon \right) \\ &= 1 + (1+\epsilon)^\mu - (1-\epsilon)^\mu . \end{aligned}$$

Since  $\epsilon > 0$  is arbitrarily small the lemma is proved.

Lemma 8.5: Let  $f$  be regularly varying at infinity with exponent  $\rho$ .

Then

$$\lim_{t \rightarrow \infty} E \left( \frac{f(N_t)}{f(t)} \right) = \left( \frac{1}{\mu} \right)^\rho .$$

Proof: By Fatou's Lemma,

$$\underline{\lim}_{t \rightarrow \infty} E \left( \frac{f(N_t)}{f(t)} \right) \geq E \left( \underline{\lim}_{t \rightarrow \infty} \frac{N_t^\rho L(N_t)}{t^\rho L(t)} \right) = \left( \frac{1}{\mu} \right)^\rho .$$

By Hölders Inequality,

$$\underline{\lim}_{t \rightarrow \infty} E \left( \frac{f(N_t)}{f(t)} \right) \leq \underline{\lim}_{t \rightarrow \infty} \left( \frac{E(N_t^{2\rho})}{t^{2\rho}} \right)^{1/2} \underline{\lim}_{t \rightarrow \infty} E \left( \frac{L^2(N_t)}{L^2(t)} \right)^{1/2} .$$

From Lemma 8.4,

$$\underline{\lim}_{t \rightarrow \infty} \left( \frac{E(N_t^{2\rho})}{t^{2\rho}} \right)^{1/2} = \left( \frac{1}{\mu} \right)^\rho ,$$

and from Lemma 8.1,

$$\underline{\lim}_{t \rightarrow \infty} \frac{E(L^2(N_t))}{L^2(t)} = 1 .$$

Lemma 8.6.

$$\lim_{t \rightarrow \infty} \frac{E(N_t + B_t + ct)^n}{(t/\mu_0)^n} = 1$$

where  $\mu_0 = 1/(\frac{1}{\mu} + c) > 0$ .

Proof:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{E(N_t + B_t + ct)^n}{(t/\mu_0)^n} &= \lim_{t \rightarrow \infty} \sum_{k=0}^n \frac{E(N_t^k)}{(t/\mu_0)^k} \frac{E(B_t + ct)^{n-k}}{(t/\mu_0)^{n-k}} \binom{n}{k} \\ &= \sum_{k=0}^n \left(\frac{\mu_0}{\mu}\right)^k (c\mu_0)^{n-k} \binom{n}{k} \\ &= \left(\frac{\mu_0}{\mu} + c\mu_0\right)^n = 1 . \end{aligned}$$

Lemma 8.7:

$$\lim_{t \rightarrow \infty} \frac{E((N_t + B_t + ct) \vee 0)^\mu}{(t/\mu_0)^\mu} = 1 , \quad \mu \geq 0 .$$

Proof: The proof is a simple extension of the previous lemma in the same way that Lemma 8.4 is used to prove Lemma 8.5. If we are willing to consider complex numbers, we could just as well show that

$$\lim_{t \rightarrow \infty} \frac{E(N_t + B_t + ct)^\mu}{(t/\mu_0)^\mu} = 1 .$$

9. Asymptotic Limits of  $E[e^{-\gamma N_t}]$

We will now consider the behavior of  $E[e^{-\gamma N_t}]$ .

Lemma 9.1: If  $E[e^{sX}] < \infty$  for  $|s| \leq \gamma/\mu$ ,  $\gamma > 0$ , then there exists a number  $\varphi(\gamma)$  with the property that

$$e^{-\gamma} \int_{s=0}^{\infty} e^{\varphi(\gamma)s} F(ds) = 1 .$$

If  $\gamma \leq 0$ ,  $\varphi(\gamma)$  always exists.

Proof: Let  $a(\gamma, \nu) = e^{-\gamma} \int_{s=0}^{\infty} e^{\nu s} F(ds)$ ,  $\gamma > 0$ . For  $\gamma \geq 0$ ,  $|\nu| \leq \gamma/\mu$ ,  $a(\gamma, \nu)$  is increasing in  $\nu$  and continuous (possibly only left-continuous at  $\nu = \gamma/\mu$ ),

$$\begin{aligned} a(\gamma, \nu) &= e^{-\gamma} E[e^{\nu X}] \geq e^{-\gamma} e^{\nu EX} && \text{by Jensen's inequality} \\ &= e^{-\gamma + \nu \mu} . \end{aligned}$$

So  $a(\gamma, \gamma/\mu) \geq 1$ . Also,  $a(\gamma, 0) = e^{-\gamma} < 1$ . So there must exist a number  $\varphi(\gamma)$  with  $0 < \varphi(\gamma) \leq \gamma/\mu$  with the property that  $a(\gamma, \varphi(\gamma)) = 1$ .

If  $F$  is non-degenerate,  $e^{-\gamma} E[e^{\nu X}] > e^{-\gamma} e^{\nu EX}$ , so in fact  $\varphi(\gamma) < \gamma/\mu$ .

If  $\gamma \leq 0$ ,  $\varphi(\gamma)$  always exists unless  $F(0) = 1$ .

Theorem 9.2: Under the conditions of Lemma 9.1,  $\lim_{t \rightarrow \infty} E[e^{-\gamma N_t}] e^{\varphi(\gamma)t}$  exists and is some positive constant  $c_\gamma$ .

Proof: On  $\{X_1 \leq t\}$ ,

$$e^{-\gamma N_t(\omega)} = \exp[-\gamma(N_{t-X_1(\omega)}(\omega) + 1)] ,$$

On  $\{X_1 > t\}$ ,

$$e^{-\gamma N_t(\omega)} = 1 .$$

Letting  $\psi(t) = E[e^{-\gamma N_t}]$ , we have

$$\begin{aligned} \psi(t) &= P(X_1 > t) + e^{-\gamma} E[E[\exp(-\gamma N_{t-X_1(\omega)}(\omega)) 1_{\{X_1(\omega) \leq t\}} | X_1]] \\ &= \bar{F}(t) + e^{-\gamma} E[1_{\{X_1(\omega) \leq t\}} E[\exp(-\gamma N_{t-X_1(\omega)}(\omega)) | X_1]] \\ &= \bar{F}(t) + e^{-\gamma} \int_0^t \psi(t-s) F(ds) . \end{aligned}$$

Letting  $G(ds) = \gamma F(ds) e^{\varphi(\gamma)s}$ , and noting that  $G$  is a probability measure, we get

$$\psi(t) e^{\varphi(\gamma)t} = \bar{F}(t) e^{\varphi(\gamma)t} + \int_0^t \psi(t-s) e^{\varphi(\gamma)(t-s)} G(ds) .$$

Let

$$\psi_\gamma(t) = \psi(t) e^{\varphi(\gamma)t} .$$

Assuming for the moment that  $\bar{F}(t) e^{\varphi(\gamma)t}$  is Directly Riemann Integrable (PRI), then by the Basic Renewal Theorem [9], p. 191,

$$c_{\gamma} = \lim_{t \rightarrow \infty} \psi_0(t) = \frac{\int_{s=0}^{\infty} \bar{F}(s) e^{\varphi(\gamma)s} ds}{\int_{s=0}^{\infty} sG(ds)} .$$

Now,

$$\begin{aligned} \int_0^{\infty} sG(ds) &= e^{-\gamma} E[X e^{\varphi(\gamma)X}] \\ &\leq \text{const.} + e^{-\gamma} E[e^{(\varphi(\gamma)+\epsilon)X}] . \end{aligned}$$

Now,  $\varphi(\gamma) < \gamma/\mu$ , so we can choose  $\epsilon$  sufficiently small so that  $\varphi(\gamma) + \epsilon < \gamma/\mu$ . Therefore,  $e^{-\gamma} E[e^{(\varphi(\gamma)+\epsilon)X}] < \infty$ , so  $\int_{s=0}^{\infty} sG(ds) < \infty$ .

This sort of renewal argument is found in for example [2] or [7] or [9].

Showing that  $\bar{F}(t) e^{\varphi(\gamma)t}$  is DRI is not hard. If  $\varphi(\gamma) \leq 0$  ( $\gamma \leq 0$ ), it is trivial.

Suppose that  $\varphi(\gamma) > 0$ ,

$$\begin{aligned} 0 \leq \int_0^{\infty} \bar{F}(t) e^{\varphi(\gamma)t} dt &= \int_{t=0}^{\infty} e^{\varphi(\gamma)t} \int_{u=t}^{\infty} dF(u) dt \\ &= \int_{u=0}^{\infty} \int_{t=0}^u e^{\varphi(\gamma)t} dt dF(u) \\ &= \frac{1}{\varphi(\gamma)} \int_{u=0}^{\infty} (e^{\varphi(\gamma)u} - 1) dF(u) \\ &= \frac{1}{\varphi(\gamma)} (e^{\gamma} - 1) < \infty . \end{aligned}$$

Using Feller's notation [7], p. 361, it suffices to show that  $\bar{\sigma}(h) - \underline{\sigma}(h) \rightarrow 0$  as  $h \rightarrow 0$ , where  $\underline{\sigma}(h) = h \sum_{k=0}^{\infty} \underline{m}_k$ ,  $\bar{\sigma}(h) = h \sum_{k=0}^{\infty} \bar{m}_k$ , where  $\underline{m}_k$  ( $\bar{m}_k$ ) is the inf (sup) of  $\tilde{F}(t) e^{\varphi(\gamma)t}$  on  $[(k-1)h, kh)$ . So

$$\bar{\sigma}(h) = h \sum_{k=0}^{\infty} \tilde{F}((k-1)h) e^{\varphi(\gamma)kh},$$

$$\underline{\sigma}(h) = h \sum_{k=0}^{\infty} \tilde{F}(kh) e^{\varphi(\gamma)(k-1)h},$$

$$\lim_{h \downarrow 0} |\bar{\sigma}(h) - \underline{\sigma}(h)| = \lim_{h \downarrow 0} |\bar{\sigma}(h) - \underline{\sigma}(h) e^{\varphi(\gamma)h}|$$

$$\leq \lim_{h \downarrow 0} h \sum_{k=0}^{\infty} |\tilde{F}(kh) - \tilde{F}((k-1)h)| e^{\varphi(\gamma)kh}$$

$$= \lim_{h \downarrow 0} h \sum_{k=0}^{\infty} \tilde{F}(kh) e^{\varphi(\gamma)kh} (e^h - 1)$$

$$= \lim_{h \downarrow 0} (e^h - 1) \int_{t=0}^{\infty} \tilde{F}(t) e^{\varphi(\gamma)t} dt$$

$$= 0$$

as

$$\lim_{h \downarrow 0} e^h - 1 = 0 \quad \text{and} \quad \int_{t=0}^{\infty} \tilde{F}(t) e^{\varphi(\gamma)t} dt < \infty.$$

So  $\tilde{F}(t) e^{\varphi(\gamma)t}$  is DRI.

Lemma 9.3: Let  $\mu_G = \int_{s=0}^{\infty} s \, dG(s)$ . For  $\gamma > 0$ ,

$$\frac{1}{\mu_G} < \frac{1}{\mu}, \quad \frac{\varphi(\gamma)}{\gamma} < \frac{1}{\mu}.$$

For  $\gamma < 0$ ,

$$\frac{1}{\mu_G} > \frac{1}{\mu}, \quad \frac{\varphi(\gamma)}{\gamma} > \frac{1}{\mu}.$$

Proof: Choose  $\gamma > 0$ ,  $e^{-\gamma} E[e^{\varphi(\gamma)X}] = 1$ . By Jensen's inequality, and noting that  $g(\cdot) = e^{\cdot}$  is strictly convex, and that  $X$  is non-degenerate, we have

$$e^{-\gamma} e^{\varphi(\gamma)EX} < 1 \quad \text{or} \quad e^{-\gamma + \varphi(\gamma)\mu} < 1,$$

or  $\varphi(\gamma)/\gamma < 1/\mu$ ,

$$\mu_G = e^{-\gamma} E[X e^{\varphi(\gamma)X}] \geq e^{-\gamma} EX E[e^{\varphi(\gamma)X}] \geq \mu.$$

The above inequality follows as  $X$  and  $e^{\varphi(\gamma)X}$  are associated. See [3] for a proof and definition of associated random variables. Actually, the inequality is strict but we will omit the proof.

#### 10. $\{e^{\varphi(\gamma)t} E[e^{-\gamma N_t}]\}$ and Uniform Integrability

We know that  $e^{\varphi(\gamma)t} E[e^{-\gamma N_t}] \rightarrow c_\gamma$  as  $t \rightarrow \infty$ . We will show that the above family of random variables are not uniformly integrable. Further, we will find a sequence of sets  $(A_t)$ , with  $P(A_t) \rightarrow 0$  as

$t \rightarrow \infty$  with the property that

$$e^{\varphi(\gamma)t} E[e^{-\gamma N_t}; A_t] \rightarrow c_\gamma \quad \text{as } t \rightarrow \infty .$$

Lemma 10.1:

$$\lim_{t \rightarrow \infty} E[e^{-\gamma N_t} e^{\varphi(\gamma)t} \frac{N_t^n}{(t/\mu)^n}; \frac{N_t}{t} \geq \frac{1}{\mu}] = 0 .$$

Proof:

$$\begin{aligned} & E[e^{-\gamma N_t} e^{\varphi(\gamma)t} \frac{N_t^n}{(t/\mu)^n}; \frac{N_t}{t} \geq \frac{1}{\mu}] \\ & \leq e^{-\gamma t/\mu + \varphi(\gamma)t} E\left[\frac{N_t^n}{(t/\mu)^n}\right] . \end{aligned}$$

As  $\varphi(\gamma) - \gamma/\mu < 0$ ,  $\lim_{t \rightarrow \infty} e^{-\gamma t/\mu + \varphi(\gamma)t} = 0$ . Since  $\lim_{t \rightarrow \infty} E[N_t^n / (t/\mu)^n] = 1$ , the lemma follows.

Theorem 10.2:

$$\lim_{t \rightarrow \infty} E\left(\frac{N_t^n e^{-\gamma N_t}}{t^n}\right) e^{\varphi(\gamma)t} = \frac{c_\gamma}{\mu^n} , \quad \text{for } n \geq 0 .$$

Proof: Note that the constant term on the right-hand side is  $c_\gamma/\mu^n$ , not  $c_\gamma/\mu^n$ .

On  $\{X_1(\omega) \leq t\}$ ,

$$N_t^n(\omega) e^{-\gamma N_t(\omega)} = e^{-\gamma(N_{t-X_1(\omega)}(\omega) + 1)^n} \exp\{-\gamma N_{t-X_1(\omega)}(\omega)\} .$$

On  $\{X_1(\omega) > t\}$ ,

$$N_t^n(\omega) e^{-\gamma N_t(\omega)} = \begin{cases} 0, & n \geq 1 \\ 1, & n = 0 \end{cases}.$$

So for  $n \geq 1$ ,

$$N_t^n(\omega) e^{-\gamma N_t(\omega)} = e^{-\gamma} \sum_{k=0}^n \binom{n}{k} N_{t-X_1(\omega)}^k(\omega) \exp\{-\gamma N_{t-X_1(\omega)}(\omega)\}$$

on  $\{X_1(\omega) \leq t\}$

= 0

on  $\{X_1(\omega) > t\}$ .

Letting  $\phi_k(t) = E[N_t^k e^{-\gamma N_t}]$ ,  $k \geq 0$ ,

$$\phi_n(t) = e^{-\gamma} \sum_{k=0}^n \binom{n}{k} E\left[E\left[N_{t-X_1(\omega)}^k(\omega) \exp\{-\gamma N_{t-X_1(\omega)}(\omega)\} 1_{\{X_1(\omega) \leq t\}} \middle| X_1\right]\right]$$

$$= e^{-\gamma} \sum_{k=0}^n \binom{n}{k} E\left[1_{\{X_1(\omega) \leq t\}}\right]$$

$$\cdot E\left[N_{t-X_1(\omega)}^k(\omega) \exp\{-\gamma N_{t-X_1(\omega)}(\omega)\} \middle| X_1\right]$$

$$= e^{-\gamma} \sum_{k=0}^n \binom{n}{k} \int_{s=0}^t \phi_k(t-s) F(ds)$$

Letting  $\phi_k^r(t) = e^{\varphi(\gamma)t} \phi_k(t)$  for  $k \geq 0$ , the above equation reduces to

$$\phi_n^r(t) = \sum_{k=0}^n \binom{n}{k} \int_{s=0}^t \phi_k^r(t-s) G(ds)$$

This renewal equation is very similar to equation (\*) found in the proof of Lemma 8.3. We can mimic that proof as the result we wish to prove for  $n = 0$  is true by Lemma 9.3.

Theorem 10.3: Let  $A_t \subset \Omega$ ,

$$A_t = \left\{ \omega : \left| \frac{N_t}{t} - \frac{1}{\mu_G} \right| < \epsilon \right\}$$

where  $0 < \epsilon < |1/\mu - 1/\mu_G|$ . Then

$$\lim_{t \rightarrow \infty} e^{\varphi(\gamma)t} E[e^{-\gamma N_t}; A_t] = c_\gamma.$$

Proof: First note that  $P(A_t) \rightarrow 0$  as  $t \rightarrow \infty$ . For  $t > 0$ ,  $a \geq 0$ , let

$$\begin{aligned} H_t(a) &= \frac{1}{c_\gamma} e^{\varphi(\gamma)t} E[e^{-\gamma N_t}; N_t \leq a] \\ &= \frac{1}{c_\gamma} e^{\varphi(\gamma)t} \int_{s=0}^a e^{-\gamma s} P(N_t \in ds). \end{aligned}$$

We know that  $\lim_{t \rightarrow \infty} H_t(\infty) = 1$  by Theorem 9.2. Think of  $H_t(\cdot)$  as a measure on  $\mathbb{R}^+$  just as we can interpret a probability distribution as a measure on  $\mathbb{R}$ .

Specifically, for  $A$  a Borel measurable subset of  $\mathbb{R}^+$ , we have

$$H_t(A) = \frac{1}{c_\gamma} e^{\varphi(\gamma)t} \int_{s \in A} e^{-\gamma s} P(N_t \in ds).$$

Define a new sequence of measures  $\{M_t\}$  on  $\mathbb{R}^+$  with

$$M_t(A) = H_t(tA) ,$$

where  $A$  is any Borel measurable subset of  $\mathbb{R}^+$ , and  $tA = \{x \in \mathbb{R}^+ : x/t \in A\}$ ,  $t > 0$ . It is also the case that  $\lim_{t \rightarrow \infty} M_t[0, \infty) = 1$ .

From Lemma 10.2,

$$\frac{1}{t^n} E[N_t^n e^{-\gamma N_t}] e^{\varphi(\gamma)t} \rightarrow \frac{1}{n} \mu_G \quad \text{for any } n \geq 0 \text{ as } t \rightarrow \infty .$$

This is equivalent in our new notation to

$$\frac{1}{t^n} \int_{s=0}^{\infty} s^n H_t(ds) \rightarrow \frac{1}{n} \mu_G \quad \text{for any } n \geq 0 \text{ as } t \rightarrow \infty .$$

By Lemma 10.1, we can assert that

$$\frac{1}{t^n} \int_{s=t/\mu}^{\infty} s^n H_t(ds) \rightarrow 0 , \quad \text{as } t \rightarrow \infty , n \geq 1 .$$

Now,

$$\begin{aligned} \frac{1}{t^n} \int_0^{t/\mu} s^n H_t(ds) &= \frac{1}{t^n} \int_{v=0}^{1/\mu} v^n t^n dH_t(tv) \cdot t \\ &= \int_{v=0}^{1/\mu} v^n t dH_t(tv) \\ &= \int_{v=0}^{1/\mu} v^n dM_t(v) . \end{aligned}$$

So,

$$\lim_{t \rightarrow \infty} \int_{v=0}^{1/\mu} v^n dM_t(v) = \frac{1}{\mu_G^n}, \quad n \geq 1,$$

and

$$\lim_{t \rightarrow \infty} \int_{v=1/\mu}^{\infty} v^n dM_t(v) = 0, \quad n \geq 1.$$

Now, the sequence of measures  $\{M_t\}$  has a subsequential limit measure  $M$  from for instance [5], p. 83, and  $M$  must be a probability measure as  $M_t[0, 1/\mu] \rightarrow 1$  as  $t \rightarrow \infty$ .

So,  $M_{t_k} \Rightarrow M$  for  $\{t_k\}$  some increasing sequence going to infinity, where  $\Rightarrow$  represents weak convergence. So,

$$\begin{aligned} \frac{1}{\mu_G^n} &= \lim_{t_k \rightarrow \infty} \int_{v=0}^{\infty} v^n dM_{t_k}(v) \\ &= \lim_{t_k \rightarrow \infty} \int_{v=0}^{1/\mu} v^n dM_{t_k}(v) \\ &= \int_{v=0}^{1/\mu} v^n dM(v) \\ &= \int_{v=0}^{\infty} v^n dM(v), \end{aligned}$$

where the next to last equality follows from [4], p. 11, the second equality follows from Lemma 10.1, and the last equality follows as  $M(1/\mu, \infty) = 0$ .

Let  $\{Y_t\}$  be a sequence of random variables with probability measure  $c_t M_t$ , where  $c_t = (M_t(0, \infty))^{-1}$ . Let  $Y$  be a random variable

with measure  $M$ . Now,  $M_{t_k} \Rightarrow M$  and  $c_t \rightarrow 1$  as  $t \rightarrow \infty$ , so it is easy to see that  $c_{t_k} M_{t_k} \Rightarrow M$ . So,  $Y_{t_k} \xrightarrow{d} Y$ ,

$$\begin{aligned} EY &= E[Y; Y \leq \frac{1}{\mu}] \\ &= \lim_{t_k \rightarrow \infty} E[Y_{t_k}; Y_{t_k} \leq \frac{1}{\mu}] \\ &= \frac{1}{\mu_G} . \end{aligned}$$

Similarly,  $EY^2 = 1/\mu_G^2$ . Therefore,

$$\text{Var } Y = EY^2 - (EY)^2 = \frac{1}{\mu_G^2} - \left(\frac{1}{\mu_G}\right)^2 = 0 .$$

So,  $Y = 1/\mu_G$  a.s., or  $M((1/\mu_G)) = 1$ . That is, any subsequential limit measure of the sequence of measures  $(M_t)$  is simply a unit measure at  $1/\mu_G$ . This implies that

$$H_t\left(t\left(\frac{1}{\mu_G} - \epsilon\right), t\left(\frac{1}{\mu_G} + \epsilon\right)\right) \rightarrow 1 \quad \text{as } t \rightarrow \infty ,$$

but this is identical to

$$\lim_{t \rightarrow \infty} e^{q(r)t} E[e^{-rN_t}; A_t] = c_r .$$

Theorem 10.4:

$$\lim_{t \rightarrow \infty} e^{q(r)t} E\left[\frac{N_t^{(r)}}{t^\nu} e^{-rN_t}\right] = c_r \left(\frac{1}{\mu_G}\right)^\nu , \quad \nu \geq 0 .$$

Proof:

$$\begin{aligned}
 & \underline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ \frac{N_t^\rho}{t^\rho} e^{-\gamma N_t} \right] \\
 & \geq \underline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ \frac{N_t^\rho}{t^\rho} e^{-\gamma N_t}; A_t \right] \\
 & \geq \frac{(1-\epsilon)^\rho}{\mu_G^\rho} \underline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E[e^{-\gamma N_t}; A_t] \\
 & = \frac{c}{\mu_G^\rho} (1-\epsilon)^\rho,
 \end{aligned}$$

$$\begin{aligned}
 & \overline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ \frac{N_t^\rho}{t^\rho} e^{-\gamma N_t} \right] \\
 & \leq \overline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ \frac{N_t^\rho}{t^\rho} e^{-\gamma N_t}; A_t \right] \\
 & \quad + \frac{1}{\mu_G^\rho} \overline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ e^{-\gamma N_t}; \frac{N_t}{t} < \frac{1}{\mu_G} - \epsilon \right] \\
 & \quad + \overline{\lim}_{t \rightarrow \infty} e^{\varphi(\gamma)t} E \left[ \frac{N_t^n}{t^n}; \frac{N_t}{t} \geq \frac{1}{\mu_G} + \epsilon \right],
 \end{aligned}$$

where  $n \geq \rho$ ,  $n$  an integer.

The first term is bounded above by  $((1-\epsilon)/\mu_G)^\rho c_\gamma$ . The second and third terms are zero by Theorem 10.5. So the result is proved.

Theorem 10.5:

$$\lim_{t \rightarrow \infty} e^{\varphi(\gamma)t} \mathbb{E} \left[ \frac{N_t^\rho L(N_t)}{t^\rho L(t)} e^{-\gamma N_t} \right] = c_\gamma \left( \frac{1}{\mu_G} \right)^\rho, \quad \text{for } \rho \geq 0,$$

where  $L$  is slowly varying at infinity and  $\overline{\lim}_{t \rightarrow \infty} L(t) < \infty$ ,  $\underline{\lim}_{t \rightarrow \infty} L(t) > 0$ .

Proof: The proof is almost identical to the above proof.

Theorem 10.6:

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[ \frac{e^{\varphi(\gamma)t}}{t^n} (N_t + B_t + ct)^n e^{-\gamma N_t} \right] = \left( \frac{1}{\mu_G} + c \right)^n \cdot c_\gamma,$$

where  $n \geq 0$ , and  $\frac{1}{\mu} + c > 0$ .

Proof:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E} \left[ \frac{e^{\varphi(\gamma)t}}{t^n} (N_t + B_t + ct)^n e^{-\gamma N_t} \right] \\ &= \sum_{k=0}^n \binom{n}{k} \lim_{t \rightarrow \infty} \mathbb{E} \left( e^{\varphi(\gamma)t} e^{-\gamma N_t} \frac{N_t^k}{t^k} \right) \cdot \lim_{t \rightarrow \infty} \frac{\mathbb{E}(B_t + ct)^{n-k}}{t^{n-k}} \\ &= \sum_{k=0}^n \binom{n}{k} c_\gamma \left( \frac{1}{\mu_G} \right)^k c^{n-k} \\ &= c_\gamma \left( \frac{1}{\mu_G} + c \right)^n. \end{aligned}$$

Lemma 10.7:

$$\lim_{t \rightarrow \infty} E \left[ \frac{e^{\varphi(\gamma)t}}{t^\rho} ((N_t + B_t + ct) \vee 0)^\rho e^{-\gamma N_t} \right] = \left( \frac{1}{\mu_G} + c \right)^\rho c_\gamma,$$

for  $\rho \geq 0$  and  $1/\mu_G + c > 0$ .

Proof: We omit the proof.

Theorem 10.8:

$$\begin{aligned} \lim_{t \rightarrow \infty} E \left[ e^{\varphi(\gamma)t} \frac{(N_t + B_t + ct)^n}{t^n} \frac{L(N_t + B_t + ct)}{L(t)} e^{-\gamma N_t} \right] \\ = c_\gamma \left( \frac{1}{\mu_G} + c \right)^n \quad \text{for } \frac{1}{\mu_G} + c > 0, n \geq 0, \end{aligned}$$

and

$$0 < \underline{\lim}_{t \rightarrow \infty} L(t) \leq \overline{\lim}_{t \rightarrow \infty} L(t) < \infty.$$

Also,

$$\begin{aligned} \lim_{t \rightarrow \infty} E \left[ e^{\varphi(\gamma)t} \frac{((N_t + B_t + ct) \vee 0)^\rho}{t^\rho} \frac{L(N_t + B_t + ct)}{L(t)} e^{-\gamma N_t} \right] \\ = \left( \frac{1}{\mu_G} + c \right)^\rho, \quad \rho \geq 0. \end{aligned}$$

Proof: We omit the proof.

## 11. Asymptotic IFR Distributions

Theorem 11.1: Let  $P_{k+1}/P_k \rightarrow e^{-\gamma}$  as in either Theorem 5.1 or 5.2. If  $E[e^{\gamma/\mu X_1}] < \infty$ , then for  $\bar{H}(t) = P(T > t) = EP_{N(t)}$ ,  $H$  is asymptotically IFR.

Further, if  $P_{k+1}/P_k \rightarrow e^{-\gamma}$  as in Theorem 5.2 and  $\rho > 0$  (see Theorem 5.2),  $H$  is asymptotically IRF.

Proof: In the case of Theorem 5.1,  $P_n = f(n) = e^{-n\gamma} L(n)$ . Extend  $f$  and  $L$  to  $\mathbf{R}^+$  in the previously specified manner,

$$\bar{H}(t) = EP(N_t) = E[e^{-\gamma N_t} L(N_t)] .$$

From Theorem 10.2, with  $n = 0$ ,

$$\bar{H}(t) = E[e^{-\gamma N_t} L(N_t)] \sim L(t) e^{-\phi(\gamma)t} c_\gamma ,$$

which is log concave, so  $H$  is asymptotically IFR.

In the case of Theorem 5.2,  $f(n) = P_n = e^{-n\gamma} n^\rho L(n)$  with  $\rho \geq 0$ . Extend  $f$  to  $\mathbf{R}^+$ . By Theorem 10.5,

$$E[N_t^\rho L(N_t) e^{-\phi(\gamma)N_t}] \sim e^{-\phi(\gamma)t} t^\rho L(t) c_\gamma / \mu_G^\rho$$

and  $\bar{H}(t) = EP_{N_t} = E[N_t^\rho L(N_t) e^{-\gamma N_t}]$ . So,  $\bar{H}(t) \sim e^{-\phi(\gamma)t} t^\rho L(t) c_\gamma / \mu_G^\rho$ .

If  $P_{k+1}/P_k \downarrow e^{-\gamma}$ ,  $L$  is log concave, and  $e^{-\phi(\gamma)t}$  and  $t^\rho$  are log concave, so  $\bar{H}$  is asymptotically log concave.

If we only assume that  $P_{k+1}/P_k \rightarrow e^{-\gamma}$ , if  $\rho > 0$ ,  $\bar{H}$  is still asymptotically log concave. In either case, we have  $H$  asymptotically IFR.

Theorem 11.2. Let  $P_{k+1}/P_k \downarrow e^{-\gamma}$  as in either Theorem 5.1 or 5.2.

Let  $f(n) = P_n$  and extend  $f$  to  $\mathbb{R}^+$  in the prescribed manner. If  $E[e^{(\gamma/\mu)X_1}] < \infty$ , then  $\bar{H}(t) = Ef((B_t + N_t + ct) \vee 0)$  is asymptotically log concave for  $c + 1/\mu > 0$ .

Further, if  $P_{k+1}/P_k \rightarrow e^{-\gamma}$  as in Theorem 5.2 and  $\rho > 0$ ,  $\bar{H}$  is asymptotically log concave.

Proof. The proof is almost identical to the proof of Theorem 11.1.

Simply use Lemma 10.7 instead of Theorem 10.4.

## 12. Generalized Renewal Processes

We can prove the same sort of results found in Section 11 for the case where  $\bar{H}(t) = Ef(S_{N_t})$  and  $f$  is as in Theorem 11.1.

We will list some of the lemmas and theorems needed to get the same sort of results. The details are messy and closely resemble those of previous sections, so we omit all the proofs.

Recall that,

$$S_{N_t} = \begin{cases} \sum_{k=1}^{N_t} Y_k, & N_t \geq 1 \\ 0, & N_t = 0 \end{cases}$$

where the  $\{Y_k\}$  sequence is i.i.d.,  $Y_1 \sim K$ ,  $K(0-) = 0$ , and  $E[e^{sY_1}]$  exists for all  $s$  in some neighborhood of the origin. Let  $\mu_Y = EY_1$ .

Lemma 12.1:

$$\lim_{t \rightarrow \infty} E \left[ \frac{S_{N_t}^\rho}{t^\rho} \right] = \left( \frac{\mu_Y}{\mu} \right)^\rho \quad \text{for } \rho \geq 0.$$

Proof: Omitted.

Lemma 12.2:

$$\lim_{t \rightarrow \infty} E \left[ \frac{L(S_{N_t})}{L(t)} \right] = 1.$$

Proof: Omitted.

Let  $e^{-\gamma^*} = \int_{s=0}^{\infty} e^{-\gamma s} K(ds)$ . We will assume that  $\varphi(\gamma^*)$  exists.

Lemma 12.3:

$$\lim_{t \rightarrow \infty} e^{\varphi(\gamma^*)t} E[e^{-\gamma S_{N_t}}] = k_\gamma > 0.$$

Proof: Omitted.

Theorem 12.4: For  $f$  as in Theorem 11.1,  $\bar{H}(t) = Ef(S_{N_t})$  is asymptotically log concave.

Proof: Omitted.

So, in the case of our shock model, where the total damage can be represented as a generalized renewal process, the time to failure distribution  $\bar{H}$  is asymptotically IFR under the appropriate conditions. We have not considered the case of  $X(t) = S_{N_t} + ct + B_t$ , with  $\bar{H}(t) = Ef(X(t))$ , but it is not hard to see that similar results hold for  $X$  of this form.

### 13. The Sub-Exponential Class

This section is based on [2], Chapter 4.

Definition: The sub-exponential class  $\{$  consists of all distribution functions  $F$  with  $F(0-) = 0$  and

$$\lim_{t \rightarrow \infty} \frac{1 - F^{(2)}(t)}{1 - F(t)} = 2.$$

$F^{(2)}$  is the convolution of  $F$  with its density.

Examples of  $F$  include

$$1 - F(t) \sim t^{-k}, \quad k > 0,$$

$$1 - F(t) \sim e^{-t^B}, \quad 0 < B < 1,$$

$$1 - F(t) \sim e^{-t/\log^2 t}.$$

Roughly speaking, a distribution  $F$  is sub-exponential if  $1 - F(t)$  goes to zero slower than any exponential distribution. Let

$$R_{\xi}(t) = \sum_{n=0}^{\infty} R^{(n)}(t) \xi^n.$$

If we let  $F \in \zeta$ , we get the following theorem.

Theorem 13.1: If  $F \in \zeta$ , and  $0 < \xi < 1$ , then

$$\lim_{t \rightarrow \infty} \frac{(1-\xi)^{-1} - R_{\xi}(t)}{1 - F(t)} = \frac{\xi}{(1-\xi)^2}.$$

Proof: In [2], p. 150.

Theorem 13.2: Suppose  $P_k = \xi^k$ ,  $k \geq 0$ . Suppose  $N_t$  is a renewal process with  $F \in \zeta$ . Then  $\bar{h}(t) = EP_{N(t)} \sim (1-F(t))/(1-\xi)$ .

Proof:  $\bar{h}(t) = R_{\xi}(t)$ . Simple arithmetic and Theorem 13.1 yield the theorem.

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