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DISCOVERING HIDDEN TOTALLY LEONTIEF SUBSTITUTION SYSTEMS. (U)  
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BY  
GEORGE B. DANTZIG

TECHNICAL REPORT SOL 77-17

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N00014-75-C-0267,  
N00014-75-C-0865

SYSTEMS OPTIMIZATION LABORATORY  
DEPARTMENT OF OPERATIONS RESEARCH

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Research and reproduction of this report were partially supported by the U.S. Energy Research and Development Administration Contract E(04-3)-326 PA #18; the Office of Naval Research Contracts N00014-75-C-0267 and N00014-75-C-0865; the National Science Foundation Grants MCS76-81259 and MCS76-22019.

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DISCOVERING HIDDEN TOTALLY LEONTIEF SUBSTITUTION SYSTEMS

by

George B. Dantzig

A Totally Leontief Square System  $Lu = b$  is defined by the following properties:

(a) Every column of  $L$  has at most one positive element;  $b > 0$ ;

$L$  is  $m \times m$ .

(b)  $Lu = b$  has a solution  $u \geq 0$ .

It is easy to see that every row of  $L$  has at least one positive element -- indeed there are exactly  $m$  positive elements, each in a different column. Other properties are:  $L$  is non-singular,  $L^{-1} \geq 0$  and  $Lu = b, \pi L = c$  yields  $u > 0, \pi > 0$  for all  $b > 0, c > 0$ . See [4].

A Totally Leontief Substitution System  $Ax = b$  is defined by the following properties:

(c) Every column  $A(*,j)$  of  $A$  has at most one positive component;  $b > 0$ ;

$A$  is  $m \times n$ .

(d) The linear program; Find Max  $z$ :

$$(1) \quad Ax = b, \quad x \geq 0, \quad ex = z,$$

where  $e = (1, 1, \dots, 1)$ , is feasible and Max  $z$  is finite.

\* Arthur F. Veinott, Jr. has made several suggestions that are incorporated into the algorithm and proofs presented, see [7].

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Other properties are that every column of  $A$  has exactly one positive element; every row of  $A$  has one or more positive elements; every subset of  $m$  columns with a positive element in each row forms a feasible basis. All basic feasible solutions are non-degenerate. Main references are [3], [7], [8], and [9].

Substitution Classes: Associated with any feasible basis  $B$  to a general linear program is the set  $S_i$  of column indices  $j$  such that  $A(*,j)$  if substituted for  $B(*,i)$  forms a feasible basis.  $S_i$  is non-empty for each  $i$  since it includes  $j: A(*,j) = B(*,i)$ . In general  $S_i$  depends on  $B$ . For the Leontief substitution case, however, it is independent of the choice of  $B$ , since the substitution class  $S_i$  clearly consists of all  $j$  that have a positive element in the same row. It is convenient to have  $S_i$  correspond to all columns  $j$  whose positive element is in row  $i$ . We shall refer to the latter as "row-standardized" substitution classes.

A system

$$(1) \quad \bar{A}x = \bar{b}$$

where  $\bar{A}$  is  $m \times n$  is a "hidden" totally Leontief substitution system if there exists a nonsingular matrix  $R$  such that

$$(2) \quad (R\bar{A})x = (R\bar{b})$$

is a totally Leontief substitution system.

The Problem. Given a linear system  $\bar{A}x = \bar{b}$  find a nonsingular matrix  $R$  if it exists such that  $R\bar{A}x = R\bar{b}$  is a totally Leontief substitution system.

Constructive Solution.

Step 1: Solve the linear program: Find Max  $z$

$$(2) \quad \bar{A}x = \bar{b}, \quad x \geq 0, \quad ex = z \text{ (Max)} .$$

Terminate if infeasible or unbounded, or if the optimal basic feasible solution is degenerate. Let  $\hat{B}$  be the optimal feasible basis.

Step 2: Determine substitution classes  $S_1, S_2, \dots, S_m$  with respect to  $\hat{B}$ . Terminate if there is ambiguity about which substitution class a column belongs to.

Step 3: For each  $i$ , for  $i = (1, \dots, m)$  determine multipliers

$R(i, *)$  such that

$$\begin{aligned} R(i, *) \bar{A}(*, j) &\geq 1, & \text{if } j \in S_i \\ &\leq 0, & \text{if } j \notin S_i \end{aligned}$$

$$R(i, *) \bar{b} \geq 1 .$$

Terminate if for any  $i = (1, \dots, m)$  the above linear inequality system is infeasible. If not terminate, the matrix  $R$  whose rows are  $R(i, *)$  is the solution.

Proof Step 1: If the original system (2) is infeasible or unbounded, i.e., an optimal solution of the linear program (2) fails to exist, then the same is true for every linear transform using non-singular  $R$ , contradicting the existence of an  $R$  for (2) such that (d) holds.

Proof Step 2: The only way there can be difficulty in determining which columns an entering column replaces in the basis is the unbounded case (ruled out on Step 1) or there is a tie in the ratio test for selecting a pivot in the simplex method. The latter implies that the feasible basic feasible solution obtained after the substitution is degenerate. This is ruled out since all basic feasible solutions are non-degenerate in a Leontief substitution system.

Proof Step 3: Note that feasibility and the adjacency properties of feasible bases that differ by one column are invariant under non-singular linear transforms. After transformation by  $R(i, *)$ , assuming  $R$  exists, row  $i$  of  $\bar{R}A$  should have  $+$  components corresponding to substitution class  $S_k$ . We can permute the rows of  $R$  so that  $S_1$  corresponds to the "row standardized" substitution class associated with  $i$  of  $\bar{R}A$ , namely  $R(i, *) A(*, j) > 0$  for  $j \in S_1$  and  $R(i, *) A(*, j) \leq 0$  otherwise; we also have  $R(i, *) \cdot \bar{b} > 0$  for  $i = (1, \dots, m)$ . Step III can be solved by applying Phase I of the simplex method to a linear program.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER SOL 77-17 ✓	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Discovering Hidden Totally Leontief Substitution Systems		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) George B. Dantzig		8. CONTRACT OR GRANT NUMBER(s) NR-047-064 NR-047-143
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research -- SOL ✓ Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Operations Research Program, Code 434 Office of Naval Research Arlington, VA 22217		12. REPORT DATE March 1977
		13. NUMBER OF PAGES 5
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) This document has been approved for public release and sale; its distribution is unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Leontief Substitution Systems Linear Programs		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In earlier work by R. Saigal, necessary and sufficient conditions were stated for the equivalence of a given linear program with a totally Leontief substitution system. In this paper, we present a constructive procedure for determining the existence and evaluating (when it does exist) the non-singular matrix that transforms the L.P. into a member of the substitution class.		

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