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OPTIMIZED METHOD FOR THE DERIVATION OF THE DEFLECTION OF THE VE--ETC(U)

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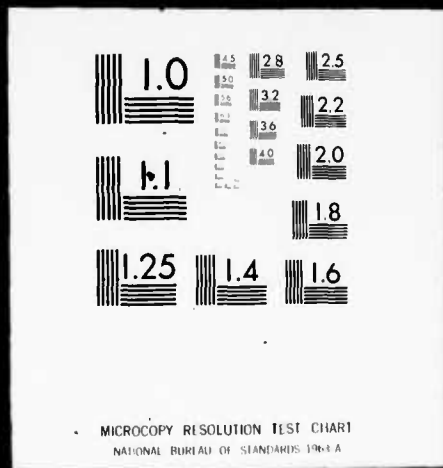
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ETL-0122
OPTIMIZED METHOD FOR THE DERIVATION
OF THE DEFLECTION OF THE VERTICAL
FROM RGSS DATA

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prepared for

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I. Introduction

The Rapid Gravity Survey System (RGGS) provides a means of quickly measuring precisely non-astrogeodetic values of the deflections of the vertical. A test vehicle carries an Inertial Positioning System (IPS) which at each of the vehicle's stops produces an error velocity which can be related to the inertial platform tilt errors and the deflections of the vertical. An optimal determination of the gyro drifts and the deflections of the vertical can only be obtained by a post-mission smoothing of the data. In this case, accurate data are available a priori for the deflections of the vertical at the start and stop of the vehicle's mission -- as well as the data on the IPS velocity errors at each stop.

The purpose of this report is to develop the equations for the position, velocity, and tilt angle errors into a useable algorithm for the optimal estimation of the deflections of the vertical. This entails three major analytic tasks that will be presented in the succeeding section: one, the development of an expression for the velocity errors at some time, t_n , in terms of the known initial conditions at time, $t = 0$, and the unknown variables to be estimated -- i.e. the gyro drift biases and the values of ξ_i and η_i , the vertical deflections at previous time intervals; two, the replacement of ξ_i and η_i by a smaller number of values $\hat{\xi}_k$ and $\hat{\eta}_k$ -- the needed values of ξ_i and η_i being produced by statistical collocation from the $\hat{\xi}_k, \hat{\eta}_k$ basis set; three, a least squares solution for the gyro biases and the $\hat{\xi}_k, \hat{\eta}_k$ that minimizes the

mean square deviation of the velocity errors from their estimates.

This mathematical development allows the production of a machine algorithm for use with actual data. A coded version of the algorithm with explanatory comments will make up the final section of this report.

II. Mathematical Development of the Error Adjustment Technique

The differential equations for the horizontal velocity errors are

$$\frac{du}{dt} = \frac{dx}{dt} = -S_E \dot{\phi}_Z + g \phi_E + g \xi \quad (1)$$

$$u = \frac{dx}{dt} \quad (2)$$

$$\frac{dv}{dt} = \frac{dy}{dt} = S_N \dot{\phi}_Z - g \phi_N + g \eta \quad (3)$$

$$\frac{d\phi_Z}{dt} = \tan \phi \frac{v}{R} - (\Omega_N + \zeta_N \sec^2 \phi) \frac{x}{R} + \omega_N \phi_E - \omega_E \phi_N + \alpha \quad (4)$$

$$\frac{d\phi_N}{dt} = \frac{v}{R} + \omega_E \phi_Z - \omega_Z \phi_E + \beta \quad (5)$$

$$\frac{d\phi_E}{dt} = -\frac{u}{R} + \omega_Z \phi_N - \omega_N \phi_Z + \gamma \quad (6)$$

where the meaning of the symbols is as follows:

x north position error
u north velocity error
X north vehicle position
 V_N north vehicle velocity
 S_N north vehicle acceleration
y east position error
v east velocity error
Y east vehicle position
 V_E east vehicle velocity
 S_E east vehicle acceleration
 ϕ_Z azimuth platform attitude error
 ϕ_N platform tilt error about north axis
 ϕ_E platform tilt error about east axis
 ϕ geographic latitude
g normal gravity
R mean earth radius
 Ω earth's angular rotation rate
 Ω_S Schuler rate $\equiv (g/R)^{1/2}$
 ω_N $\Omega \cos \phi + V_E/R$ north spatial rate
 ω_E $\equiv -V_N/R$ east spatial rate
 ω_Z $\equiv \Omega \sin \phi + \tan \phi V_E/R$ vertical spatial rate
 α azimuth axis error angular drift rate

β north axis error angular drift rate
 γ east axis error angular drift rate
 ξ north deflection of the vertical
 η east deflection of the vertical

Except for the terms in (1) and (3) involving S_E and S_N , this set may be solved in a straightforward manner. These restrictions are not really limiting, as we will show below. The reduced set of equations is

$$\frac{du}{dt} = g\phi_E - g\xi \quad (7)$$

$$\frac{dx}{dt} = u \quad (8)$$

$$\frac{dv}{dt} = -g\phi_N + g\eta \quad (9)$$

$$\frac{d\phi_Z}{dt} = \tan \varphi \frac{v}{R} - \Omega \cos \varphi \frac{x}{R} + \Omega \cos \varphi \phi_E + \alpha \quad (10)$$

$$\frac{d\phi_N}{dt} = \frac{v}{R} - \Omega \sin \varphi \phi_E + \beta \quad (11)$$

$$\frac{d\phi_E}{dt} = -\frac{v}{R} + \Omega \sin \varphi \phi_N - \Omega \cos \varphi \phi_Z + \gamma, \quad (12)$$

where we have neglected terms in v/R relative to the earth rotation rate, Ω . This is also the rationale for dropping -- at least initially -- the terms containing S_N and S_E which are of order vehicle acceleration/ g relative to the leading terms.

Equations (7) - (12) are of the form

$$\frac{d\underline{\mu}}{dt} = \underline{A} \underline{\mu} = \underline{\gamma} \quad (13)$$

where A is the coefficient matrix,

$$\underline{\mu} = \begin{pmatrix} \theta \\ v \\ z \\ \phi_Z \\ \phi_N \\ \phi_E \end{pmatrix} \quad \text{and} \quad \underline{\gamma} = \begin{pmatrix} -g\xi \\ -g\eta \\ 0 \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Assume that the solution to the homogeneous part of (13) has a functional dependence, $e^{\lambda t}$. Then, the homogeneous part of (13) is

$$(\lambda - \underline{A}) \underline{\mu} = 0 \quad (14)$$

a standard eigenvalue problem. The secular equation, $\text{Det}(\lambda - \underline{A}) = 0$ gives, as its roots, the values of λ for which (14) is satisfied.

The secular equation is

$$\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & -g \\ 0 & \lambda & 0 & 0 & +g & 0 \\ -1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & -\frac{\tan \varphi}{R} & \Omega \cos \varphi & \lambda & 0 & -\Omega \cos \varphi \\ 0 & -\frac{1}{R} & 0 & 0 & \lambda & \Omega \sin \varphi \\ \frac{1}{R} & 0 & 0 & +\Omega \cos \varphi & -\Omega \sin \varphi & \lambda \end{vmatrix} = 0 \quad (15a)$$

$$\lambda^6 + \lambda^4(2\Omega_s^2 + \Omega^2) + \lambda^2(\Omega_s^4 + \Omega_s^2\Omega^2 \sin^2 \varphi) - \Omega^2\Omega_s^4 \cos^2 \varphi = 0 \quad (15b)$$

which has roots

$$\lambda = \pm i\Omega_s, \pm i\Omega', \pm K$$

where $\Omega_s^2 = \left(\frac{g}{R}\right)^{1/2}$ the Schuler frequency,

$$\Omega' = \left[\frac{1}{2} (\Omega_s^2 + \Omega^2) + \frac{1}{2} \{ \Omega_s^4 + 2\Omega_s^2\Omega^2 (1 + 2\cos^2 \varphi) + \Omega^4 \}^{1/2} \right]^{1/2} \quad (16)$$

$$= \{ \Omega_s^2 + \Omega^2 (1 + \cos^2 \varphi) \}^{1/2}$$

and

$$K = \left[\frac{1}{2} \{ \Omega_s^4 + 2\Omega_s^2\Omega^2 (1 + 2\cos^2 \varphi) + \Omega^4 \}^{1/2} - \frac{1}{2}(\Omega_s^2 + \Omega^2) \right]^{1/2} \quad (17)$$

$$= \Omega \cos \varphi .$$

The eigenvectors may be found by solving (14) for all the other elements of $\underline{\mu}$ in terms of one element, using a specific eigenvalue. The eigenvectors are

$$\underline{\mu}^{(1)} = \begin{bmatrix} \frac{g_1}{\lambda_1} (\lambda_1^2 + \Omega_s^2) \\ g_1 \Omega \sin \varphi \\ \frac{g_2}{\lambda_1^2} (\lambda_1^2 + \Omega_s^2) \\ - \{ (\lambda_1^2 + \Omega_s^2)^2 + \lambda_1^2 \Omega^2 \sin^2 \varphi \} / \lambda_1 \Omega \cos \varphi \\ - \lambda_1 \Omega \sin \varphi \\ \lambda_1^2 + \Omega_s^2 \end{bmatrix} \quad (18)$$

$\underline{\mu}^{(1)}$ refers to the eigenvector associated with the 1 - th eigenvalue. It is also convenient to have the eigenvectors of the adjoint problem, i.e.,

$$\underline{\mu}^T (\lambda - \underline{\Lambda}) = 0$$

which has the same eigenvalues as (14). The eigenvectors here are

$$\begin{aligned} \underline{\mu}^T(1) = & \left[a_1^{-1} [(\lambda_1^2 + \Omega_s^2) (\Omega^2 \cos^2 \varphi - \lambda_1^2)] / R, \right. \\ & 2\lambda_1^3 \Omega \sin \varphi / R, \\ & \Omega^2 \cos^2 \varphi \lambda_1 (\lambda_1^2 + \Omega_s^2) / R, \\ & -\Omega \cos \varphi \lambda_1^2 (\lambda_1^2 + \Omega_s^2) \\ & \left. \Omega \sin \varphi \lambda_1^2 (\lambda_1^2 - \Omega_s^2) \right] \end{aligned} \quad (19a)$$

$$\lambda_1^3 (\lambda_1^2 + \Omega_1^2)]$$

where

$$a_1 = 4\lambda_1^3 \Omega_1^2 \Omega_s^2 \sin^2 \varphi + 2(\lambda_1^2 + \Omega_1^2) (\lambda_1^4 + \Omega_1^2 \Omega_s^2 \cos^2 \varphi) / \lambda_1. \quad (19b)$$

Note that $\underline{\mu}^{T(1)}$ is a row vector, and the μ 's are normalized such that $\underline{\mu}^{T(1)} \cdot \underline{\mu}^{(j)} = \delta_{1j}$. We now diagonalize (13) by applying the transformation

$$\underline{q} = \underline{S} \underline{\mu}, \quad q_1 = S_{1j} \mu_j$$

where $S_{1j} = \mu_j^{T(1)}$ and $S_{1j}^{-1} = \mu_1^{(j)}$ since $\mu_j^{T(1)} \mu_j^{(k)} = \delta_{1k}$.

Equation (13) becomes

$$\frac{dq_1}{dt} - \lambda_1 q_1 = r_1 \quad (20)$$

where $\underline{r} = \underline{S} \underline{v}$ with a solution, $q_1 = c_1 e^{\lambda_1 t} - r_1 / \lambda_1$ (21)

where c_1 is a constant. The boundary conditions at $t = 0$ imply

$c_1 = q_1(t=0) + r_1 / \lambda_1$, and

$$q_1(t) = q_1(t=0) e^{\lambda_1 t} + r_1 / \lambda_1 (e^{\lambda_1 t} - 1) \quad (22)$$

To get back into the original basis, we apply the inverse transform to

(22) and have

$$\begin{aligned} \mu_1(t) &= S_{1j}^{-1} (e^{\lambda_j t} q_j(t=0) + \frac{1}{\lambda_j} (e^{\lambda_j t} - 1) r_j) \\ &= S_{1j}^{-1} e^{\lambda_j t} S_{jk} \mu_k(t=0) + S_{1j}^{-1} \frac{1}{\lambda_j} (e^{\lambda_j t} - 1) S_{jk} r_k, \end{aligned} \quad (23)$$

which completes the solution of the reduced set (7) - (12).

Returning now to the neglected terms involving S_N and S_E , we try a perturbation addition to the solution (23). Thus, for example,

$$u(t) = u_0(t) + u_1(t) + \dots$$

where u_0 is found from (22), and evidently

$$\frac{du_1}{dt} = -S_E \phi_{z_0}(t), \text{ or}$$

$$u(t) = u_0(t) - \int_0^t S_E \phi_{z_0}(t') dt' \quad (24)$$

Equation (24) could be integrated numerically given the functional form of $S_E(t)$. However, the integration time is expected to be a few minutes (3 - 5) and the typical time scale, $1/\Omega_E$, for variation of the solution is $\sim 1/2$ hour. This suggests trying a Taylor series for ϕ_{z_0} , or

$$u(t) = u_0(t) - \int_0^t S_E \sum_{n=0}^{\infty} \phi_{z_0}^{(n)} \frac{t'^n}{n!} dt'$$

Keeping the first two terms in the expansion and integrating by parts

gives

$$\begin{aligned} u(t) = u_0(t) - \phi_{z_0}(t=0) v_E \Big|_0^t - \frac{d\phi_{z_0}}{dt} (t=0) \left\{ v_E t \Big|_0^t - \int_0^t v_E(t') dt' \right\} \\ = u_0(t) + (Y(t) - Y(0)) \frac{d\phi_{z_0}(t=0)}{dt} \end{aligned} \quad (25)$$

where $Y(t)$ is the east position of the vehicle and we assume that the vehicle is stopped at time t . If we differentiate and approximate $\frac{d\phi_{z_0}}{dt}$ by $\Omega_s \phi_{z_0}$ since $\frac{d\phi_{z_0}}{dt} \leq \Omega_s \phi_{z_0}$, we can compare the

size of the two terms in (25), and find

$$\frac{\frac{du_0}{dt}}{\frac{du_1}{dt}} = \frac{g\phi_z}{\Omega_s v_E \phi_z} = \frac{g/R}{\Omega_s v_E/R} = \frac{\Omega_s}{v_E/R}$$

or the same order as terms thrown out earlier. However, we wish the equations for u and v to be as accurate as possible, since they are the ones that enter directly into the determination of the deflections of the vertical. A further improvement on (25) can be gained by expanding the Taylor series for ϕ_{z_0} about the mid-point of the integration interval. Then, the $\frac{t}{2} \frac{d^2 \phi_{z_0}}{dt^2}$ term will integrate to near zero, because the acceleration of the vehicle will be close to an odd function about the midpoint. In this case (25) is

$$u(t) = u_0(t) + (Y(t) - Y(0)) \left\{ s_{\phi_z j}^{-1} \lambda_j e^{\lambda_j t/2} s_{jk} u_k(t=0) + s_{\phi_z j}^{-1} e^{\lambda_j t/2} s_{jk} v_k \right\} \quad (26)$$

where the subscript ϕ_z indicates the index required to couple to the variable, ϕ_z . A similar expression can be derived for $v(t)$. The full solution can be written as

$$\begin{aligned}
u_i(t) &= [S_{ij}^{-1} e^{\lambda_j t} S_{jk} + u_{ij} S_{j1} \lambda_1 e^{\lambda_1 t/2} S_{1k}] u_k(t=0) \\
&\quad + [S_{ij}^{-1} \frac{e^{\lambda_j t} - 1}{\lambda_j} S_{jk} + u_{ij} S_{j1} e^{\lambda_1 t/2} S_{1k}] v_k \\
&= \Phi_{ik}(t) u_k(t=0) + \Lambda_{ik}(t) v_k
\end{aligned} \tag{27}$$

where $u_{ij} = \begin{cases} 0 & \text{if } u, v \text{ and } j \neq \phi_z \\ (Y(t) - Y(0)) & \text{if } i=u, j=\phi_z \\ -(X(t) - X(0)) & \text{if } i=v, j=\phi_z \end{cases}$

So far we have been concerned with the time evolution of the variables while the vehicle is moving. The vehicle makes frequent stops, however. During these stops the gyro angular errors, ϕ_z , ϕ_N , ϕ_E keep evolving, while the other variables are held constant by fiat. To describe the evolution of angle errors, we extract the relevant equations from (7) - (12) and find

$$\frac{d\phi_z}{dt} = \Omega \cos \varphi \phi_E + [\alpha - \Omega \cos \varphi \frac{x}{R}] \tag{28}$$

$$\frac{d\phi_N}{dt} = -\Omega \sin \varphi \phi_E + \beta \tag{29}$$

$$\frac{d\phi_E}{dt} = \Omega \sin \phi_N - \Omega \cos \phi \phi_x + \gamma \quad (30)$$

where $u = v = 0$ identically and since x is constant, we have lumped it with α in (28). The secular equation of (28) - (30) is

$$v (v^2 + \Omega^2) = 0$$

with roots $v = 0, v = \pm i\Omega$

and transformation matrices

$$S^{-1} = \begin{bmatrix} -\frac{\cos \phi}{\sqrt{2}} & \frac{\sin \phi}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\cos \phi}{\sqrt{2}} & \frac{\sin \phi}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \sin \phi & \cos \phi & 0 \end{bmatrix} \quad (31a)$$

and

$$S^{-1} = \begin{bmatrix} -\frac{\cos \phi}{\sqrt{2}} & -\frac{\cos \phi}{\sqrt{2}} & \frac{\sin \phi}{\sqrt{2}} \\ \frac{\sin \phi}{\sqrt{2}} & \frac{\sin \phi}{\sqrt{2}} & \cos \phi \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad (31b)$$

where the rows in \underline{S}' and the columns in \underline{S}'^{-1} are the eigenvectors corresponding to eigenvalues $i\Omega$, $-i\Omega$, 0 , respectively. Referring to (23), we see that the solution of (28) - (30) is

$$\begin{aligned} \mu'_i(t) = & S'^{-1}_{ij} e^{v_j t} S'_{jk} \mu'_k(t=0) \\ & + S'^{-1}_{ij} \frac{1}{v_j} (e^{v_j t} - 1) S'_{jk} \gamma'_k \end{aligned} \quad (32)$$

where $\mu'_i = \begin{bmatrix} \phi_Z \\ \phi_N \\ \phi_E \end{bmatrix}$ and $\gamma'_i = \begin{bmatrix} \alpha - \Omega \cos \varphi \frac{x}{R} \\ \beta \\ \gamma \end{bmatrix}$

and $\frac{1}{v_j} (e^{v_j t} - 1)$ can be understood for $v = 0$ as $\lim_{v \rightarrow 0} = t$.

We rewrite solution (32) in terms of all of the variables, i.e.

$$\mu_i = \Gamma_{ij}(t) \mu_j(t=0) + \Delta_{ij}(t=0) \gamma_j \quad (33a)$$

with

$$\Gamma_{ij} = 1 \quad i = k = u, v, x$$

$$S_{ij}^{-1} e^{v_j t} S_{jk} \quad i, k = \phi_Z, \phi_N, \phi_E \quad (33b)$$

$$S_{ij}^{-1} v_j^{-1} (e^{v_j t} - 1) S_{j\phi_Z} \left(-\frac{\Omega \cos \varphi}{R} \right) \quad \begin{array}{l} i = \phi_Z, \phi_N, \phi_E \\ k = x \end{array}$$

0 elsewhere

$$\Delta_{ik} = S_{ij}^{-1} v_j^{-1} (e^{v_j t} - 1) S_{jk} \quad i, k = \phi_Z, \phi_N, \phi_E$$

0 elsewhere

(33c)

During the course of time during which the vehicle is first stopped for a time $\delta\tau$ and then in motion for a time τ , the evolution of $\underline{\mu}$ is given by

$$\underline{\mu}_i(\tau + \delta\tau) = \nabla_{ij}(\tau) \Gamma_{jk}(\delta\tau) \mu_k(t=0) + [\phi_{ij}(\tau) \Delta_{jk}(\delta\tau) + \Lambda_{ik}(\tau)] \nabla_k$$

$$= \hat{\phi}_{ik}(\tau, \delta\tau) \mu_k(t=0) + \hat{\lambda}_{ik}(\tau, \delta\tau) \nu_k \quad (34)$$

where $\hat{\phi}_{ik}$ and $\hat{\lambda}_{ik}$ are defined by (34). We will refer to $\hat{\phi}(\tau, \delta\tau)$ and $\hat{\lambda}(\tau, \delta\tau)$ as time shift operators since they have the effect of updating $\underline{\mu}(t)$ to $\underline{\mu}(\tau, \tau, \delta\tau)$ for any t .

What we wish to do is to express the observed quantities at some time to the unknown gyro biases, α , β , and γ and unknown vertical deflections. Let us denote the value of $\underline{\mu}$ at time t_n as $\underline{\mu}^n$, and let $\hat{\phi}(\tau_{n-1}, \delta\tau_{n-1}) \equiv \hat{\phi}^{n-1}$ where $t_n - t_{n-1} = \tau_{n-1} + \delta\tau_{n-1}$.

Then, for example

$$\begin{aligned} \underline{\mu}^1 &= \hat{\phi}^0 \underline{\mu}^0 + \hat{\lambda}^0 \underline{\nu}^0 \\ \underline{\mu}^2 &= \hat{\phi}^1 \underline{\mu}^1 + \hat{\lambda}^1 \underline{\nu}^1 \\ &= \hat{\phi}^1 \hat{\phi}^0 \underline{\mu}^0 + \hat{\phi}^1 \hat{\lambda}^0 \underline{\nu}^0 + \hat{\lambda}^1 \underline{\nu}^1 \end{aligned}$$

and in general

$$\underline{\mu}^{n+1} = \prod_{i=0}^n \hat{\phi}^i \underline{\mu}^0 + \sum_{j=0}^{n-1} \prod_{i=j+1}^n \hat{\phi}^i \hat{\lambda}^j \underline{\nu}^j + \hat{\lambda}^n \underline{\nu}^n \quad (35)$$

In (35) the value of $\underline{\mu}$ and, in particular u and v are seen to depend only on the quantities for which we wish to solve and on the

initial boundary conditions. Three further points need to be made. One, the development so far has assumed that the initial values for the succeeding time steps are simply those of the end values of the preceding time step. In fact, however, at each vehicle stop the velocity errors, u and v , are set to zero, and the integration continues with $u = v = 0$ as a starting condition. These boundary conditions can be incorporated by simply replacing the $\hat{\phi}$ operator by

$$\bar{\psi}_{1k} = \hat{\phi}_{1k}(\delta_{1k} - M_{1k})$$

where

$$M_{1k} = \begin{matrix} 0 & l \neq k; \quad 1, \quad k \neq u, \quad v \\ 1 & l = k = u, v \end{matrix}$$

So finally we have

$$\underline{\mu}^{n+1} = \prod_{i=0}^n \bar{\psi}_{1u}^i + \sum_{j=0}^{n-1} \prod_{i=j+1}^n \bar{\psi}_{1\hat{\lambda}}^i \hat{\psi}_{1j} + \hat{\lambda}^n \underline{\psi}^n \quad (36)$$

Two, the $\underline{\psi}^n$ vectors are, in general, a function of time -- at least parametrically through the motion of the vehicle. In order to achieve best accuracy, we should time center the values taken to represent $\underline{\psi}$. Thus, for the gyro drift rates $\alpha^{n-1} = \alpha(t = (t_{n+1} + t_n)/2)$ gives best accuracy. The vertical deflections do not couple into the evolution of the system while the vehicle is stopped, so that the optimum here is $\xi^{n-1} = \xi(t = t_{n+1} - \tau_{n-1}/2)$. Since the vehicle motion is probably near to being symmetric about this time, we may write

$\xi^{n-1} = \xi((\bar{X}^n + \bar{X}^{n-1})/2, [Y^n + Y^{n-1}]/2)$ as a more convenient form of good accuracy. Thus, we use

$$\underline{y}^{n-1} = \begin{bmatrix} \xi((\bar{X}^n + \bar{X}^{n-1})/2, [Y^n + Y^{n-1}]/2) \\ \eta([X^n + X^{n-1}]/2, [Y^n + Y^{n-1}]/2) \\ 0 \\ \alpha([t_{n+1} + t_n]/2) \\ \beta([t_{n+1} + t_n]/2) \\ \gamma([t_{n+1} + t_n]/2) \end{bmatrix} \quad (37)$$

as the most accurate representation.

Last, equation (36) is underdetermined. We want an overdetermined system, so as to be able to use a least squares solution for the vertical deflections and gyro biases. Thus, we substitute for ξ^n and η^n by the form

$$\xi^n = \sum_{k=1}^N a_{nk} \bar{\xi}_k + \sum_{k=1}^N b_{nk} \bar{\eta}_k$$

and

$$\eta^n = \sum_{k=1}^N c_{nk} \bar{\xi}_k + \sum_{k=1}^N d_{nk} \bar{\eta}_k$$

The terms with $\bar{\eta}$ in the determination of ξ , and vice versa, enter because the deflections are strongly cross correlated. The exposition is simplified if we define

$$r_{2i-1} \cong \xi^i$$

$$r_{2i+2} \cong \eta^i$$

then
$$r_i^{(e)} = \sum_{k=1}^{2N} a_{ik} r_k$$

where the (e) superscript denotes an estimate. We would like to minimize the variance of the estimation error, or

$$\begin{aligned} F \equiv (r_i - r_i^{(e)})^2 &= r_i^2 - 2r_i r_i^{(e)} + r_i^{(e)2} \\ &= r_i^2 - 2r_i \sum_{k=1}^{2N} a_{ik} r_k + \sum_{k=1}^{2N} \sum_{l=1}^{2N} a_{ik} a_{il} r_k r_l \end{aligned}$$

To minimize F, we find

$$\frac{\delta F}{\delta a_{ik}} = 0 = -2r_i r_k + 2 \sum_{l=1}^{2N} a_{il} r_k r_l \quad (38)$$

Taking expectation values of both terms in (38) gives

$$\langle r_i r_k \rangle = \sum_{l=1}^{2N} a_{il} \langle r_k r_l \rangle$$

The covariances are matrices and we can solve for a, and have

$$a_{ik} = \langle r_i r_l \rangle \langle r_l r_k \rangle^{-1}$$

where $\langle r_l r_k \rangle^{-1}$ is the inverse matrix of $\langle r_l r_k \rangle$. To specify

$$\langle r_{21-1} r_{21-1} \rangle = \langle \xi^i \xi^i \rangle$$

$$\langle r_{21-1} r_{21-2} \rangle = \langle \xi^i \eta^i \rangle = \langle \eta^i \xi^i \rangle \quad (39)$$

$$\langle r_{21-2} r_{21-2} \rangle = \langle \eta^i \eta^i \rangle$$

The needed covariances are usually estimated using the assumption of an isotropic, homogeneous covariance. That is the power spectrum of gravity anomaly fluctuations is a function of neither position nor direction. Then the anomaly covariance is a function only of the distance between two points. The covariance of the vertical deflections can be derived given the anomaly covariance (Shaw et al. 1969).

A number of models have been used for the anomaly covariance.

If $\phi_{gg}(r)$ is the anomaly covariance, then

$$\phi_{\xi\xi} = \sigma_\delta^2 [\phi_{gg}(r)/\sigma_g^2 + (\sin^2\theta - \cos^2\theta) f_c(r)]$$

$$\phi_{\eta\eta} = \sigma_\delta^2 [\phi_{gg}(r)/\sigma_g^2 + (\cos^2\theta - \sin^2\theta) f_c(r)]$$

$$\phi_{\xi\eta} = \phi_{\eta\xi} = -2\sigma_\delta^2 \sin\theta \cos\theta f_c(r)$$

where σ_g^2 is the anomaly variance, $\sigma_\delta = \sigma_g/g_0 \sqrt{2}$, and $r = (\underline{X}^2 + \underline{Y}^2)/D$ with D the correlation length. The angle θ is $\tan^{-1}(Y/X)$. The function $f_c(r)$ can be derived once ϕ_{gg} is known. A number of models for the anomaly have been used. For example,

exponential, Bessel, (Shaw et al, 1969) and second order Markovian (Kasper, 1971) models give

exponential

$$\phi_{gg} = \sigma_g^2 \exp(-r)$$

$$f_c(r) = \frac{2}{r^2} - (1 + 2/r + 2/r^2) e^{-r}$$

Bessel

$$\phi_{gg} = \sigma_g^2 r K_1(r)$$

$$f_c(r) = 4/r^2 - 2K_0(r) - (4/r + r) K_1(r)$$

2nd Order Markovian

$$\phi_{gg} = \sigma_g^2 e^{-r} [1 + r]$$

$$f_c = 6/r^2 - e^{-r} [r + 3 + 6/r + 6/r^2]$$

thus, $\langle \xi^1 \eta^1 \rangle = -2\sigma_g^2 \sin \theta \cos \theta f_c(r_{11})$

where $\theta = \tan^{-1} [(Y^1 - Y^1) / (X^1 - X^1)]$

$$r_{11} = \frac{1}{D} [(X^1 - X^1)^2 + (Y^1 - Y^1)^2]^{1/2}$$

and the other elements can be filled similarly.

Rewriting (36) to concentrate on the u^n and v^n variables, we have

$$\begin{aligned} \begin{matrix} u^n \\ v^n \end{matrix} &= \begin{bmatrix} n-1 & & & \\ \Pi & & & \\ i=0 & & & \end{bmatrix} \begin{matrix} \psi^i \\ \psi^i \\ \psi^i \\ \psi^i \end{matrix} \begin{matrix} u^0 \\ v^0 \\ u^0 \\ v^0 \end{matrix} + \begin{bmatrix} n-2 & n-1 & & \\ \Sigma & \Pi & & \\ j=0 & i=j+1 & & \end{bmatrix} \begin{matrix} \psi^i \\ \psi^i \\ \psi^i \\ \psi^i \end{matrix} \begin{matrix} u^j \\ v^j \\ u^j \\ v^j \end{matrix} \\ &+ \hat{\Lambda}^{n-1} \begin{matrix} u^{n-1} \\ v^{n-1} \\ u^{n-1} \\ v^{n-1} \end{matrix} \end{aligned} \quad (40)$$

where the $[]_{ik}$ notation means the i, k 'th element of the matrix enclosed in the bracket. It is also convenient to rewrite \underline{y} as a sum of vectors, i.e.

$$\underline{y}^n = \underline{a} + \delta \underline{a}^n + \underline{b}^n$$

where

$$\underline{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}, \quad \delta \underline{a}^n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta \alpha^n \\ \delta \beta^n \\ \delta \gamma^n \end{bmatrix} \quad \text{and} \quad \underline{b}^n = \begin{bmatrix} g \xi^n \\ -g \eta^n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The \underline{a} vector is the mean gyro drift rates over the mission, and the $\delta \underline{a}$ vector is the deviations of the drifts from the average. Since we assume

$$\xi^n = r_{2n-1} = \sum_{k=1}^{2N} a_{2n-1,k} \tilde{r}_k$$

$$\eta^n = r_{2n-2} = \sum_{k=1}^{2N} a_{2n-2,k} r_k$$

$$b^n = \sum_{k=1}^{2N} \begin{pmatrix} \langle r_{2n-1} \tilde{r}_1 \rangle \langle \tilde{r}_1 \tilde{r}_k \rangle \\ \langle r_{2n-2} \tilde{r}_1 \rangle \langle \tilde{r}_1 \tilde{r}_k \rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{r}_k$$

and (40) becomes

$$\begin{pmatrix} u^n \\ v^n \end{pmatrix} = \left\{ \left[\sum_{i=0}^{n-1} \psi^i \right]_{u,v,k} u \right. \quad (41)$$

$$+ \left[\sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} \psi^i \hat{\Lambda}^j + \hat{\Lambda}^{n-1} \right]_{u,v,k} a_k$$

$$+ \sum_{j=0}^{n-2} \left[\sum_{i=j+1}^{n-1} \psi^i \hat{\Lambda}^j \right]_{u,v,k} \delta a_k^j + \hat{\Lambda}^{n-1} \delta a_k^{n-1}$$

$$+ \sum_{m=1}^{2N} \left[\sum_{j=0}^{n-2} \left(\sum_{i=j+1}^{n-1} \psi^i \hat{\Lambda}^j \right)_{u,v,k} \begin{pmatrix} \langle r_{2j-1} \tilde{r}_1 \rangle \langle r_1 \tilde{r}_m \rangle^{-1} \\ \langle r_{2j-2} \tilde{r}_1 \rangle \langle r_1 \tilde{r}_m \rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$+ \hat{\Lambda}^{u-1} \begin{pmatrix} \langle r_{2n-3} \tilde{r}_1 \rangle \langle \tilde{r}_1 \tilde{r}_m \rangle^{-1} \\ \langle r_{2n-4} \tilde{r}_1 \rangle \langle \tilde{r}_1 \tilde{r}_m \rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{r}_m \Big) = F_{uv}^n$$

which is in a form suitable for least squares analysis. The terms involving δa do not enter into the solution, except to provide an estimate of the weighting of each term of the optimal solution. However, for the current report we shall use even weights and thus drop those terms from the equation. We have then

$$F_{uv}^n = \frac{u^n}{v^n} - A_{uv}^n \mu_k^0 - B_{uv,k}^n a_k - C_{uv,m}^n \bar{r}_m \quad (42)$$

where A, B, and C are defined by (32). We want to minimize

$$\sum_{n=1}^M \sum_{u,v} F_{uv}^n{}^2 = G \quad \text{which implies}$$

$$\frac{\delta G}{\delta a_k} = -2 \sum_{n=1}^M \sum_{u,v} F_{uv}^n B_{uv,k}^n = 0 \quad (43)$$

$$\frac{\delta G}{\delta \bar{r}_m} = -2 \sum_{n=1}^M \sum_{u,v} F_{uv}^n C_{uv,m}^n \bar{r}_m = 0$$

which gives the following set of simultaneous equations

$$a_1 \sum_{n=1}^M \sum_{u,v} B_{uv,1}^n B_{uv,k}^n + \bar{r}_m \sum_{n=1}^M \sum_{u,v} C_{uv,m}^n B_{uv,k}^n$$

$$= \sum_{n=1}^M \sum_{u,v} \left(\frac{u^n}{v^n} - A_{uv,1}^n \mu_1^0 \right) B_{uv,k}^n \quad \text{permuted over } k \quad (44)$$

and

$$a_1 \sum_{n=1}^M \sum_{u,v} B_{u,1}^n C_{v,p}^n + F_m \sum_{n=1}^M \sum_{u,v} C_{u,m}^n C_{v,p}^n \quad \text{permuted over } p$$

$$= \sum_{n=1}^M \sum_{u,v} \left(\xi_{v,1}^n - A_{u,1}^n \xi_{u,1}^0 \right) C_{v,p}^n \quad \begin{matrix} p=1,2 \\ 2N,2N-1 \end{matrix}$$

which then give the desired gyro rates and deflections of the vertical.

The permutation over p is not carried out for $p=1, 2$ or $p=2N, 2N-1$.

These are the values of η and ξ at the start and finish of the mission and are known beforehand.

III. Machine Coded Algorithm

This section presents a FORTRAN source listing of a coded version of the algorithm developed in section II. The program has been debugged to the extent possible without actual test data. Notes on the programming are contained within the comment cards in the listing.

Computer printouts not necessary for
understanding of the report.

PROGRAM FOR THE OPTIMIZED, POST-MISSION DETERMINATION OF
THE DEFLECTION OF THE VERTICAL USING RGSS DATA
PROGRAM PRODUCED BY PHOENIX CORPORATION

CUMMUN CPHI,SPHI,TPHI
DIMENSION K(50)
DIMENSION TGO(51),TSTOP(51),U(51),V(51),XX(51),YY(51),XHAT(51),
2 YHAT(51),X(50),Y(50),START(6),STEM(6)
DIMENSION PSI(50,6,6),XL(50,6,6),DUMMY(6,6),DUM2(6,6),DUM3(6,6),
2DUM4(6,6)
DIMENSION COEF(100,100),DEFL(100,100),COVAR(100,100),
1 SNMAT(100,100),F(100),VAR(100),SUMAT(100)
EQUIVALENCE (SUMAT(100,1),SUMAT(1))
DATA G/9,80665E2/

INPUT DATA

N1 = # OF POINTS AT WHICH A POSITION IS SPECIFIED
THE FIRST AND LAST POINTS BEING THE START AND STOP
OF THE VEHICLE - THE REMAINDER THE POINTS AT WHICH THE
DEFLECTION IS TO BE DETERMINED.
MSTOP = NUMBER OF VEHICLE STOPS IN A GIVEN MISSION
TGO(I) = TIME SPENT TRAVELLING ON I-TH LEG
TSTOP(I) = TIME SPENT STOPPED ON I-TH LEG
U(I) = X OR NRTH VELOCITY ERROR AT END OF I-TH LEG
V(I) = EAST OR Y VELOCITY ERROR AT END OF I-TH LEG
XX(I) = NORTH POSITION AT END OF I-TH LEG
YY(I) = EAST POSITION AT END OF I-TH LEG
XHAT, YHAT = NORTH AND EAST POSITIONS AT WHICH THE
DEFLECTION IS TO BE DETERMINED. ' EXCEPTIONS ARE
THE FIRST AND LAST VALUES OF XHAT, YHAT WHICH ARE
THE STARTING PLACE AND FINAL STOPPING PLACE OF THE
MISSION
TLAT = TERRESTRIAL LATITUDE OF THE MISSION
X10, ETA0, XIFIN, ETAFIN = THE VALUES OF THE NORTH ANDEAST
DEFLECTIONS OF THE VERTICAL AT THE START AND STOP OF
THE MISSION RESPECTIVELY
START(I) = INITIAL CONDITION VECTOR, CONTAINS IN SEQUENCE
G*X10, G*ETA0, 0., 0., ETA0, X10

1 READ(5,1) MSTOP,N1
1 FORMAT(2I5)
MS = MSTOP + 1
2 READ(5,2)(K(I),TGO(I),TSTOP(I),U(I),V(I),XX(I),YY(I),I=1,MS)
2 FORMAT(15,6E10,4)
101 WRITE(6,101)(1,TGO(I),TSTOP(I),U(I),V(I),XX(I),YY(I),I=1,MS)
101 FORMAT(1X,15,6(3X,E10,4))
C
102 READ(5,2)(K(I),XHAT(I),YHAT(I),I=1,N1)
102 WRITE(6,102)(K(I),XHAT(I),YHAT(I),I=1,N1)
102 FORMAT(1H1,(15,2E14,4))
C
3 READ(5,3)TLAT,X00,ETA0,XIFIN,ETAFIN
3 READ(5,3)(START(I),I=1,6)
3 FORMAT(6E12,4)

```

WRITE(6,103)MSTOP,N1,(START(I),I=1,6),TLAT,XI0,ETA0,XIFIN,ETAFIN
103 FORMAT(1H0,2I8/2(1X
      SPHI = SIN(TLAT)
      CPHI = COS(TLAT)
      TPHI = SPHI/CPHI

```

INITIALIZE THE TIME ADVANCEMENT OPERATORS

```
CALL ADVANS
```

FIND THE CENTERED VEHICLE POSITION FOR EACH TIME STEP

```

DO 200 I = 1,MSTOP
X(I) = (XX(I+1) + XX(I))/2.
200 Y(I) = (YY(I+1) + YY(I))/2.

```

DEFINE THE TIME SHIFT OPERATORS FOR EACH TIME STEP
 THE PSI OPERATOR GIVES THE CONTRIBUTION COMING FROM
 THE VALUES OF THE VARIABLES THEMSELVES AT THE
 BEGINNING OF THE TIME STEP
 THE XL OPERATOR GIVES THE CONTRIBUTION FROM THE
 DRIVING TERMS

```

DO 300 I = 1,MSTOP
CALL TIME(TGUR(I),TSTOP(I),PSI,XL,N,X,Y)
300 CONTINUE

```

DETERMINATION OF THE COEFFICIENT MATRIX OF THE SET OF LINEAR EQUATIONS CONTAINING THE UNKNOWN QUANTITIES.

COEF(*,2*N1) CONTAINS THE CONSTANT TERM
 COEF(*,2*N1-1) I # 3,2,1 CONTAINS THE TERMS INVOLVING
 COEF(*,J) J # 1 - 2*N1 - 4 CONTAINS THE TERMS INVOLVING
 ALPHA,BETA,GAMMA, RESP. THE GYRO DRIFT RATES ABOUT
 VERTICAL,NORTH, EAST AXES, RESP.
 THE XI : ETA VALUES AT THE CHOSEN POINTS

INITIALIZATION OF THE CONSTANT PART OF THE COEF MATRIX
 THE PSI OPERATOR IS SUCCESSIVELY OPERATED ON THE
 INITIAL CONDITION VECTOR, START, TO PRODUCE THE
 EFFECT OF THE INITIAL CONDITIONS AT THE START OF THE
 MISSION ON THE U AND V ERROR VELOCITIES. THE DIFFERENCE
 BETWEEN THESE QUANTITIES FORMS PART OF THE TERM INVOLV-
 BETWEEN THESE QUANTITIES FORMS PART OF THE CONSTANT
 TERM, A CONTRIBUTION DUE TO THE DEFLECTION OF THE VERTICA
 TERM, A CONTRIBUTION DUE TO THE DEFLECTION OF THE
 OF THE VERTICAL IS SUBTRACTED LATER.

```

DO 1000 I = 1,MSTOP
CALL EQUIV(I,PSI,DUMMY)
DO 1010 KK = 1,6
STEM(KK)=0.
DO 1010 L = 1,6
1010 STEM(KK) = STEM(KK) + DUMMY(KK,L) * START(L)
DO 1020 L = 1,6
1020 START(L) = STEM(L)
COEF(2*I-1,N2) = U(I) - START(1)

```

1000 COEF(2*I,N2) = V(I) - START(2)
CONTINUE

DETERMINATION OF THE TERMS IN THE COEF MATRIX WHICH DEPEND
ON THE DEFLECTIONS OF THE VERTICAL

FIRST STEP: DEFINE A MATRIX = DEFL = WHICH GIVES THE VARA
FIRST STEP: DEFINE A MATRIX = DEFL = WHICH GIVES THE DEPENDENCE
OF THE VELOCITY ERRORS ON THE VALUES OF THE
DEFLECTIONS AT EACH OF THE MIDPOINTS OF EACH TRAVEL LR
DEFLECTIONS AT EACH OF THE MIDPOINTS OF EACH TRAVEL LEG

DO 1100 I = 1,MSTOP
CALL EQUIV(I,XL,DUMMY)
DEFL(2*I-1,2*I-1) = DUMMY(1,1)
DEFL(2*I-1,2*I) = DUMMY(1,2)
DEFL(2*I,2*I-1) = DUMMY(2,1)
DEFL(2*I,2*I) = DUMMY(2,2)
IF (I,EW,MSTOP) GO TO 1101
KK = I + 1
DO 1110 N = KK,MSTOP
CALL EQUIV(N,PSI,DUM2)
CALL XMAT(DUM2,DUMMY,DUM3)
CALL EQUAL(DUM3,DUMMY)
DEFL(2*N-1,2*I-1) = DUMMY(1,1)
DEFL(2*N-1,2*I) = DUMMY(1,2)
DEFL(2*N,2*I-1) = DUMMY(1,1)
DEFL(2*N,2*I) = DUMMY(2,2)

1110 CONTINUE
1100 CONTINUE

SECOND STEP: DEFINE THE MATRIX = COVAR = WHICH, WHEN
MULTIPLIED BY DEFL GIVES THE DEPENDENCE OF THE COEF
MATRIX ON THE DEFLECTIONS AT THE DESIRED POINTS

CALL COLLOC(MSTOP,N1,COVAR,X,Y,XHAT,YHAT)
N2P = N2 - 4

THIRD STEP: DO THE MULTIPLICATION

DO 1150 I = 1,M2
DO 1150 J = 1,N2P
COEF(I,J) = 0.
DO 1150 L = 1,M2
1150 COEF(I,J) = COEF(I,J) + DEFL(I,L) * COVAR(L,J+2)
DO 1160 I = 1,M2

UPDATE THE CONSTANT PART OF THE COEF MATRIX TO ACCOUNT FOR THE
KNOWN VALUES OF THE DEFLECTION AT THE START AND STOP
OF THE MISSION

DO 1155 L = 1,M2
COEF(I,N2) = COEF(I,N2) - G * XI0 * DEFL(I,L) * COVAR(L,1)
- G * ETA0 * DEFL(I,L) * COVAR(L,2)
- G * XIFIN * DEFL(I,L) * COVAR(L,N2-1)
- G * ETAFIN * DEFL(I,L) * COVAR(L,N2)

4W2

1155 CONTINUE
1160 CONTINUE

FILLING IN THOSE PARTS OF THE COEF MATRIX THAT DEPEND ON THE
GYRO DRIFT RATES

DO 1210 N = 1,3
DO 1210 I = 1,2
1210 COEF(I,N2-4+N) = XL(I,I,3+N)
CALL EQUIV(1,XL,DUMMY)
C
DO 1200 I = 2,MSTOP
CALL EQUIV(I,PSI,DUM2)
CALL EQUIV(1,XL,DUM3)
CALL XMAI(DUM2,DUMMY,DUM4)
CALL ADD(DUM4,DUM3,DUMMY)
DO 1220 N = 1,3
1220 COEF(2+I-1,N2-4+N) = DUMMY(1,3+N)
1200 COEF(2*I,N2-4+N) = DUMMY(2,3+N)
CONTINUE
C
N2M = N2 - 1

DEFINING THE MATRIX = SQMAT = WHICH SOLVED GIVES THE DESIRED
LEAST SQUARES SOLUTION FOR THE DEFLECTIONS OF THE
VERTICAL AND THE GYRO DRIFT RATES.

DO 1300 I = 1,N2M
DO 1300 J = 1,N2
SQMAT(I,J) = 0.
DO 1300 N = 1,M2
1300 SQMAT(I,J) = SQMAT(I,J) + CUEN(K,I) * COEF(N,J)

AFTER SUBROUTINE SOLVE, SQMAT(*,2*N) CONTAINS THE SOLUTION
VECTOR, THE LAST THREE ARE THE GYRO RATES AND THE
REST ARE THE DEFLECTIONS OF THE VERTICAL.
THE LAST COLUMN IS EQUIVALENCED TO A VECTOR SUMAT

CALL SOLVE(SQMAT)
DO 1400 I = 1,M2
F(I) = 0.

DETERMINATION OF THE ACTUAL VARIANCE OF THE SOLUTION

DO 1400 J = 1,N2M
1400 F(I) = F(I) + COEF(I,J) * SUMAT(J)
DO 1450 I = 1,M2
VAR(I) = (F(I) - COEF(I,N2))**2
1450 SUMSQ = SUMSQ + VAR(I)
SIGMA = SQRT(SUMSQ/FLUAT(M2-1))
NN = N2 - 4
DO 1460 I = 1,NN
SUMAT(I) = SUMAT(I)/G

C
C

OUTPUT THE FINAL RESULTS

```
WRITE(6,110) SOMAT(NN+1),SOMAT(NN+2),SOMAT(NN+3)
110 FORMAT(1H1,' FINAL RESULTS '//GYRO DRIFT RATES, ALPHA',E12.4,5X
2,'BETA',E12.4,5X,'GAMMA',E12.4//DEFLECTIONS OF VERT//' XI'
3,'17X','ETA',17X'NORTH POS',11X,'EAST POS'//)
WRITE(6,111) (SOMAT(I),SOMAT(I+1),XHAT(I),YHAT(I),I=1,N2,2)
WRITE(6,112) SUMSU,SIGMA,(I,VAR(I),I = 1,M2)
112 FORMAT(1H1,' VARIANCE OF SOLUTION',E12.4,5X,' SIGMA',E12.4/
,' INDIVIDUAL SQ ERRORS'/(1X,15,5X,E12.4))
STOP
END
```

SUBROUTINE COLLOC(M,N1,COVAR,X,Y,XHAT,YHAT)

THIS SUBROUTINE PRODUCES A MATRIX = COVAR = THAT PRODUCES
VALUES FOR THE DEFLECTION OF THE VERTICAL AT POINTS, I,
FROM THE VALUES OF THE DEFLECTIONS AT OTHER POINTS, J.
THIS IS DONE BY STATISTICAL COLLOCATION,
FOR A DERIVATION OF THE METHOD SEE THE PHOENIX CORP. REPORT

DIMENSION CV1(50,50),CVINV(50,50),CV2(50,50),COVAR(50,50),
2 X(50),Y(50),XHAT(50),YHAT(50)
SIGG2 = SIGMA(1.)
SIGD2 = SIGD(1.)

DETERMINE THE COVARIANCES BETWEEN THE DEFLECTIONS AT THE
BASIS POINTS. ODD INICES DENOTE XI VALUES, EVEN
INDICES ETA VALUES. THE COVARIANCES ARE DERIVED
UNDER THE ASSUMPTION OF ISOTROPIC, HOMOGENEOUS
COVARIANCE OF THE GRAVITY ANOMALY.

DO 500 I = 1,N1
DO 500 J = 1,N1
R = SQRT((XHAT(I) - XHAT(J))**2 + (YHAT(I) - YHAT(J))**2)
STH = (YHAT(I) - YHAT(J))/R
CTH = (XHAT(I) - XHAT(J))/R
CV1(2*I-1,2*J-1) = SIGD2 * (PHIGG(R)/SIGG2 + (STH**2-CTH**2)*FC(R))
CV1(2*I-1,2*J) = SIGD2*(PHIGG(R)/SIGG2+(CTH**2- STH**2)*FC(R))
CV1(2*I-1,2*J) = -2. * SIGD2 * STH * CTH * FC(R)
CV1(2*I,2*J-1) = CV1(2*I-1,2*J)
500 CONTINUE

INVERSION OF THE CV1 MATRIX IS FOUND IN CVINV

CALL MATINV(CV1,CVINV)
DO 600 K = 1,M

DETERMINATION OF THE COVARIANCES BETWEEN THE BASIS SET AND
THE SET OF POINTS DETERMINED BY THE MISSION LEGS.

DO 600 I = 1,N1
R = SQRT((X(K) - XHAT(I))**2 + (Y(K)-YHAT(I))**2)
STH = (Y(K) - YHAT(I))/R
CTH = (X(K) - XHAT(I))/R
CV2(2*K-1,2*I-1) = SIGD2 * (PHIGG(R)/SIGG2 + (STH**2- CTH**2)*FC(R))
CV2(2*K-1,I*I) = -2. *SIGD2*STH*CTH*FC(R)
CV2(2*K,I*I-1) = CV2(2*K-1,2*I)
600 CONTINUE
N2 = 2*N1
M2 = 2*M

PRODUCTION OF THE COVAR MATRIX BY MULTIPLICATION OF CV2 BY
CVINV

DO 700 L = 1,M2
DO 700 K = 1,N2
COVAR(L,K) = 0.
DO 700 I = 1,N2

```

700 COVAR(L,K) = COVAR(L,K) + CV2(L,I)*CVINV(I,K)
RETURN
END
FUNCTION SIGMA(X)

```

```

THIS FUNCTION AND ITS ENTRIES GIVE THE NECESSARY VALUES FOR
THE COMPUTATION OF THE DEFLECTION COVARIANCES.
CURRENTLY, THE FUNCTION ASSUMES A SECOND ORDER MARKOVIAN
STRUCTURE FOR THE ANOMALY COVARIANCE. VAR IS THE
VARIANCE OF THE ANOMALY AND D IS THE CORRELATION
LENGTH

```

```

DATA VAR,D,G,ROOT2/3.36E-2,4E6,9.80665E2,1.41421/
SIGMA = VAR
RETURN
ENTRY SIGD(X)
SIGD = VAR/G/ROOT2
RETURN
ENTRY PHIGG(R)
Q = R/D
PHIGG = VAR*EXP(-Q)*(1.+Q)
RETURN
ENTRY FC(R)
Q = R/D
FC = 6./Q**2 * EXP(-Q) * (Q+3. + 6./Q**2)
RETURN
END
SUBROUTINE ADVANS

```

```

THIS SUBROUTINE PROVIDES THE VALUES OF THE NECESSARY TIME
SHIFT MATRICES.
THIS ENTRY INITIALIZES THE VALUES OF THE NECESSARY
EIGENVECTOR MATRICES

```

```

COMMON CPHI,SPHI,TPHI
COMPLEX R(6),P(2),C(6),C2(6),D(2),D2(2),S(6,6),SINV(6,6),T(3,3),
2 TINV(3,3),PH(6,6),PHI(6,6),VL(6,6),VLI(6,6),PU(3,3),VP(3,3)
2 DIMENSION RPHI(6,6),RVLI(6,6),RPU(3,3),RVP(3,3),RPHO(6,6),
2 DUMMY(6,6),DUM2(6,6),PSI(50,6,6),XL(50,6,6)
DATA G,REARTH,OMEGA/9.80665E2,6.37103E8,7.29212E-5/
OMEGAS = SQRT(G/REARTH)
DO 200 I = 1,6
DO 200 J = 1,6
200 RPHU(I,J) = 0.
DO 210 I = 1,3
210 RPHU(I,I) = 1.

```

```

FIRST FOR THE TIME WHEN THE VEHICLE IS IN MOTION

```

```

R CONTAINS THE ROOTS OF THE SECULAR EQUATION

```

```

R(1) = (0.,1.) * OMEGAS
R(2) = -R(1)
A1 = OMEGA**2 + OMEGAS**2
B1 = SQRT(OMEGA**4 + OMEGAS**4 + 2.*OMEGA**2*OMEGAS**2*(1.+2.*CPHI2))
R(3) = (0.,1)*SQRT(0.5 * A1 + 0.5 * B1)

```

```

R(4) = -R(3)
R(5) = (1.,0.)*SQRT(0.5*b1-0.5*A1)
R(6) = -R(5)

```

```

      FILL IN THE VALUES FOR THE TRANSFORMATION MATRIX, S, AND
      ITS INVERSE, SINV

```

```

DO 500 I = 1,6
A = (R(1)**2 + OMEGAS**2)
B = 2./R(1)*A**2*(R(1)**4 + (OMEGA*OMEGAS*CPHI)**2)
  2 + 4. * R(1)**3 * (OMEGA*OMEGAS*SPHI)**2
S(1,1) = A/B/REARTH*((OMEGA*CPHI)**2 + R(1)**2)
S(1,2) = 2.*R(1)**3*OMEGA*SPHI/B/REARTH
S(1,3) = (OMEGA*CPHI)**2 * A * R(1)/B/REARTH
S(1,4) = -OMEGA * CPHI * R(1)**2*A/B
S(1,5) = OMEGA*SPHI*R(1)**2*(R(1)**2-OMEGAS**2)/B
S(1,6) = A/B * R(1)**3
SINV(1,1) = G*A/R(1)
SINV(2,1) = G*OMEGA*SPHI
SINV(3,1) = G*A/R(1)**2
SINV(4,1) = -(A**2 + (R(1)*OMEGA*SPHI)**2)/(R(1)*OMEGA*CPHI)
SINV(5,1) = -R(1) * OMEGA*SPHI
500 SINV(6,1) = A

```

```

      NOW DO THE SAME FOR WHEN THE VEHICLE IS STOPPED

```

```

P(1) = (0.,1.)* OMEGA
P(2) = -P(1)
A1 = 1./SQRT(2.)
T(1,1) = -CPHI * A1
T(1,2) = SPHI * A1
T(1,3) = (0.,1.)*A1
T(2,1) = T(1,1)
T(2,2) = T(1,2)
T(2,3) = -T(1,3)
T(3,1) = SPHI
T(3,2) = CPHI
T(3,3) = 0.
TINV(1,1) = -CPHI * A1
TINV(1,2) = T(1,1)
TINV(1,3) = SPHI
TINV(2,1) = SPHI * A1
TINV(2,2) = TINV(2,1)
TINV(2,3) = CPHI
TINV(3,1) = (0.,1.)* A1
TINV(3,2) = -TINV(3,1)
TINV(3,3) = 0
ENTRY TIME(T1,T2,PSI,XL,N,XPOS,YPOS)

```

```

      THIS ENTRY ACTUALLY CALCULATES THE TIME SHIFT MATRICES
      T1 IS THE TIME THE VEHICLE IS MOVING
      T2 IS THE TIME STOPPED
      THE DERIVATION OF THE VARIOUS OPERATIONS PERFORMED
      HERE IS FOUND IN THE PHOENIX CORP. REPORT

```

```

T = T1 * 0.5
DO 600 K = 1,6

```

```

600 C(L) = CEXP(R(K)*T)
    C2(K) = C(L) * C(K)
    DO 700 I = 1,6
    DO 700 J = 1,6
    PH(I,J) = (0.,0.)
    PHI(I,J) = (0.,0.)
    VL(I,J) = (0.,0.)
    VLI(I,J) = (0.,0.)
    DO 700 K = 1,6
    PH(I,J) = PH(I,J) + C2(K)*SINV(I,K)*S(K,J)
    PHI(I,J) = PHI(I,J) + R(K) * C(K)*SINV(I,K)*S(K,J)

```

```

700 VLI(I,J) = VLI(I,J) + (C2(K)-(1.,0.))/R(K)*SINV(I,K)+S(K,J)
    VL(I,J) = VL(I,J) + ((K)*SINV(I,K)*S(K,J))
    DO 800 I = 1,6
    PHI(1,I) = PHI(1,I) + PH(4,I)*(YPOS(N+1)-YPOS(N))
    PHI(2,I) = PHI(2,I) - PH(4,I)*(XPOS(N+1)-XPOS(N))
    VLI(1,I) = VLI(1,I) + VL(4,I) * (YPOS(N+1) - YPOS(N))
    VLI(2,I) = VLI(2,I) - VL(4,I) * (XPOS(N+1) - XPOS(N))
    DO 810 I = 1,6
    DO 810 J = 1,6
    RPHI(I,J) = REAL(PHI(I,J))
    RVLI(I,J) = REAL(VLI(I,J))
    D(1) = CEXP(P(1) * T2)
    D(2) = CEXP(P(2) * T2)
    DO 900 I = 1,3
    DO 900 J = 1,3
    PO(I,J) = D(1) * TINV(I,1)*T(1,J) + D(2) * TINV(I,2)*T(2,J)
    + TINV(I,3) * T(3,J)
    VP(I,J) = (D(1)-1.)/P(1)*TINV(I,1)*T(1,J)
    + (D(2)-1.)/P(2) * TINV(I,2) * T(2,J) + T2 * TINV(I,3) * T(3,J)
    DO 910 I = 1,3
    DO 910 J = 1,3
    RPO(I,J) = REAL(PO(I,J))
    RVP(I,J) = REAL(VP(I,J))
    RPHI(3+I,3) = -OMEGA*CPHI/REARTH*RVPI(I,1)
    DO 920 I = 1,3
    DO 920 J = 1,3
    RVPO(3+I,3+J) = RVP(I,J)
    RPHI(3+I,3+J) = RPHI(I,J)
    CALL XMAT(RPHI,RVPO,DUMMY)
    CALL EWUI(N,PSI,DUMMY)
    DO 950 I = 1,6
    DO 950 J = 1,2
    PSI(N,I,J) = 0.
    CALL XMAT(RPHI,RVPO,DUMMY)
    CALL ADD(RVLI,DUMMY,DUM2)
    CALL EWUI(N,XL,DUM2)
    RETURN
    END

```

CCCCCCCC

HANDY MATRIX OPERATIONS NOT FOUND IN FORTRAN

EQUIV MAKES A SMALL SEC
EQUIV SETS A SMALL 6X6 MATRIX EQUAL TO A SECTION A LARGER
ONE, EQUI IS THE INVERSE
ADD ADDS TWO MATRICES. XMAT MULTIPLIES TWO MATRICES AND
EQUAL SETS ONE MATRIX EQUAL TO ANOTHER

```
200 SUBROUTINE EQUIV(N,A,B)
    DIMENSION A(50,6,6),B(6,6)
    DO 200 I = 1,6
    DO 200 J = 1,6
    B(I,J) = A(N,1,J)
    RETURN
    END
    SUBROUTINE XMAT(A,B,C)
    DIMENSION A(6,6),B(6,6),C(6,6)
    DO 200 I = 1,6
    DO 200 J = 1,6
    C(I,J) = 0.
    DO 200 K = 1,6
    C(I,J) = C(I,J) + A(I,K) * B(K,J)
    RETURN
    END
```

```
200 SUBROUTINE EQUAL(A,B)
    DIMENSION A(6,6),B(6,6)
    DO 200 I = 1,6
    DO 200 J = 1,6
    B(I,J) = A(I,J)
    RETURN
    END
```

```
200 SUBROUTINE ADD(A,B,C)
    DIMENSION A(6,6),B(6,6),C(6,6)
    DO 200 I = 1,6
    DO 200 J = 1,6
    C(I,J) = A(I,J) + B(I,J)
    RETURN
    END
```

```
200 SUBROUTINE EQUI(N,A,B)
    DIMENSION A(50,6,6),B(6,6)
    DO 200 I = 1,6
    DO 200 J = 1,6
    A(N,1,J) = B(I,J)
    RETURN
    END
```

References:

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Shaw, L., Paul, I. and Henrikson, P. 1969, Journal of Geophysical Research, 74, 4259.

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Block 20. Continued.

mission -- as well as the data on the IPS velocity errors at each stop.

The purpose of this report is to develop the equations for the position, velocity, and tilt angle errors into a useable algorithm for the optimal estimation of the deflections of the vertical. As a result, three major analytic tasks are presented. This mathematical development enables the production of a machine algorithm for use with actual data. The final section of the report contains a coded version of the algorithm with explanatory comments.

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