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lambda lambda sub 1 lambda sub N an element of the set R to the nth power sub 1 sub N the sum from j = 1 to j = N of lambda sub j A sub j

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If $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^n$, $B, A_1,\dots,A_N$ are bounded linear operators from a Banach space X to a Banach space Z, the concept of a simple eigenvalue for the operator $B - \Sigma_{j=1}^N \lambda_j A_j$ is defined. It is then shown that bifurcation always occurs at simple eigenvalues and the results are applied to a second order ordinary differential equation with boundary conditions at three distinct points.

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BIFURCATION FROM SIMPLE EIGENVALUES FOR SEVERAL PARAMETER FAMILIES*

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BIFURCATION FROM SIMPLE EIGENVALUES FOR SEVERAL PARAMETER FAMILIES

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Jack K. Hale

Abstract: If $\lambda = (\lambda_1,\ldots,\lambda_N) \in \mathbb{R}^n$, $B, A_1,\ldots,A_N$ are bounded linear operators from a Banach space X to a Banach space Z, the concept of a simple eigenvalue for the operator $B - \sum_{j=1}^{N} \lambda_j A_j$ is defined. It is then shown that bifurcation always occurs at simple eigenvalues and the results are applied to a second order ordinary differential equation with boundary conditions at three distinct points.

1. Introduction.

Suppose X, Z are Banach spaces, $\mathbb{R}$ is the real line, $B: X \to Z$, $A: X \to Z$ are bounded linear operators, $\lambda \in \mathbb{R}$. An element $\lambda_0 \in \mathbb{R}$ is said to be a simple eigenvalue of the pair of operators (B,A) if $\dim \mathfrak{N}(B-\lambda_0 A) = 1 = \operatorname{codim} \mathscr{R}(B-\lambda_0 A)$, $Ax_0 \notin \mathscr{R}(B)$ where $x_0 \in \mathfrak{N}(B-\lambda_0 A)$, $x_0 \neq 0$ and $\mathfrak{N}, \mathscr{R}$ denote respectively the null space and range of operators (see Crandall and Rabinowitz [3]). If $A = I$, the identity operator, and $X = Z$, this is the usual definition of simple eigenvalue. Although elementary, it is a fundamental result in bifurcation theory that λ_0 is always a bifurcation point for a smooth family of functions

$$M: \mathbb{R} \times X \to Z, \quad M(\lambda,0) = 0, \quad \lambda \in \mathbb{R}$$

provided that λ_0 is a simple eigenvalue of the pair of operators (B,A) where B,A are defined by

$$M(x,\lambda_0+\mu) = [B - (\lambda_0+\mu)A]x + O(|\mu|^2|x|+|x|^2)$$

as $|\mu|$, $|x| \to 0$.

In the applications, there are many cases where the function M depends on several eigenvalue parameters. These additional parameters often have the effect of increasing the dimension of the null space of the operator corresponding to the linear approximation and lead to secondary bifurcations (see, for example, Bauer, Keller, Reiss [2], Keener [5], List [6]). There are, however, other applications where the additional parameters are needed in order to obtain a feasible eigenvalue problem. For example, for a second order ordinary differential equation with boundary conditions specified at three distinct points, one cannot expect to have a complete system of eigenvalues if only one parameter is used (see, for example, Atkinson [1]). The literature on such equations is extensive (see, for example, Källstrom and Sleeman [4] for references).

It is the purpose of this paper to begin a study of bifurcation for problems of the latter type. More specifically, suppose $B,A_1,\ldots,A_N$ are bounded linear operators taking X into Z. An N-tuple $\lambda_0 = (\lambda_1^0,\ldots,\lambda_N^0)$ of real numbers is said to be a <u>simple eigenvalue of the operators</u> $(B,A_1,\ldots,A_N)$ if $L(\lambda) \overset{\text{def}}{=} B - \sum_{j=1}^{N} \lambda_j A_j$ satisfies

(i) $\dim \mathfrak{N}(L(\lambda_0)) = 1$;

(ii) $L(\lambda_0)$ is Fredholm of index $1 - N$;

(iii) $[A_1\, \mathfrak{N}(L(\lambda_0)),\ldots,A_N\, \mathfrak{N}(L(\lambda_0))] \oplus \mathscr{R}(L(\lambda_0)) = Z$

where [ ] denotes the span. For the case $N = 1$, this definition coincides with the previous definition.

We show below that a simple eigenvalue of $(B, A_1, \ldots, A_N)$ is always a bifurcation point for a smooth family of functions

$$M: \mathbb{R}^N \times X \to Z, \quad M(\lambda, 0) = 0, \quad \lambda \in \mathbb{R}^N$$

provided

$$M(\lambda, x) = L(\lambda_0 + \mu)x + O(|\mu|^2 |x| + |x|^2)$$

as $|\mu|, |x| \to 0$, $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$.

If $\nu \in \mathbb{R}$ is another parameter, $p \in Z$ is given, we describe the bifurcation diagram for the functions

$$M(\lambda, x) + \nu p$$

under some generic conditions on the function (λ_0, uy_0) where $u \in \mathbb{R}$ and $[y_0] = \mathfrak{N}(L(\lambda_0))$. In particular, we obtain the familiar cusp for several parameter families.

Finally, a specific example previously discussed by Thomas and Zachmann [8] of a second order ordinary differential equation with boundary conditions at three distinct points is considered.

2. Bifurcation theory. Suppose

$$M: \mathbb{R}^N \times X \to Z$$
$$M(\lambda, 0) = 0, \quad \lambda \in R^n$$
$$M(\lambda, x) = L(\lambda)x + N(\lambda, x)$$
$$L(\lambda) = B - \sum_{j=1}^{N} \lambda_j A_j$$
$$N(\lambda, x) = O(|\lambda - \lambda_0|^2 |x| + |x|^2) \quad \text{as} \quad |\lambda - \lambda_0|, |x| \to 0,$$

where $\lambda_0 \in \mathbb{R}^N$ is fixed and M has continuous derivatives up through order two.

<u>Theorem 2.1</u>. If λ_0 is a simple eigenvalue of $(B,A_1,\ldots,A_N)$ then λ_0 is a bifurcation point for $M(\lambda,x)$. More specifically, there is a neighborhood $V \subset \mathbb{R}^N \times X$ of $(0,0)$ such that the solutions $(\lambda,x) \in V$ of

$$M(\lambda,x) = 0 \tag{2.1}$$

are given by

$$x = y_0 u + z^*(y_0 u, \lambda^*(u))$$

where $\lambda^*(u)$ is a continuously differentiable function of $u \in \mathbb{R}$ with $\lambda^*(0) = \lambda_0$, $[y_0] = \mathfrak{N}(L(\lambda_0))$, $z^*\colon \mathfrak{N}(L(\lambda_0)) \times \mathbb{R}^N \to X$ is a continuously differentiable function satisfying $z^*(y,\lambda_0) = O(|y|^2)$ as $|y| \to 0$.

<u>Proof</u>: Let $X = X_0 \oplus X_1$, $Z = Z_0 \oplus Z_1$,

$$X_0 = \mathfrak{N}(L(\lambda_0)) = [y_0],$$

$$Z_1 = \mathscr{R}(L(\lambda_0)).$$

Since λ_0 is assumed to be a simple eigenvalue of $(B,A_1,\ldots,A_N)$, we may choose

$$Z_0 = [A_1 y_0,\ldots,A_N y_0].$$

Let $U: X \to X_0$, $I - U: X \to X_1$, $E: Z \to Z_1$, $I - E: Z \to Z_0$ be projections defined by the above decomposition of X,Z.

We may now apply the method of Liapunov-Schmidt to Equation (2.1). If $x = y + z$, $y \in X_0$, $z_0 \in X_1$, then the equation

$$EM(y+z,\lambda_0+\mu) = 0$$

has a unique solution $z^*(y,\mu) \in X_1$ for y,μ in a neighborhood of $(0,0) \in X_0 \times \mathbb{R}^N$ and $z^*(y,0) = O(|y|^2)$ as $|y| \to 0$. This solution has continuous derivatives. Thus, every solution of Equation (2.1) near $(0,0)$ must be obtained as

$$x = y + z^*(y,\mu)$$

where (y,μ) satisfy the bifurcation equations

$$(I-E)M(y+z^*(y,\mu),\lambda_0+\mu) = 0. \tag{2.2}$$

If $y = uy_0$, $u \in \mathbb{R}$, then Equation (2.2) can be written in terms of the basis vectors $A_j y_0$ with components $F_j(u,\mu)$ as

$$\begin{aligned}\sum_{j=1}^{N} F_j A_j y_0 &= (I-E)L(\lambda_0+\mu)(uy_0+z^*(uy_0,\mu)) \\ &\quad + (I-E)N(\lambda_0+\mu,uy_0+z^*(uy_0,\mu)) \\ &= u\sum_{j=1}^{N} \lambda_j A_j y_0 + O(|\mu|^2|u| + |u|^2).\end{aligned}$$

Since the vectors $\{A_j y_0\}$ are linearly independent, it follows that

$$F_j(u,\mu) = \mu_j u + O(|\mu|^2|u| + |u|^2), \qquad j = 1,2,\ldots,N. \tag{2.3}$$

Solving Equation (2.2) is equivalent to solving the equations $F_j(u,\mu) = 0$, $j = 1,2,\ldots,N$. Since each F_j vanishes for $u = 0$, we may define $G_j(u,\mu) = F_j(u,\mu)/u$ and know that all solutions except $u = 0$ are obtained from the equations

$$G_j(u,\mu) = 0, \quad j = 1,2,\ldots,N.$$

Since $G_j(0,\mu) = \mu_j$, $j = 1,2,\ldots,N$, we may apply the implicit function theorem to complete the proof.

Theorem 2.1 corresponds to the situation where $x = 0$ is a solution of Equation (2.1) for all $\lambda \in \mathbb{R}^N$. What happens if there are additional parameters in the problem and zero is not a solution of the equation for the parameters not zero? For example, consider the equation

$$M(\lambda,x) + \nu p = 0 \tag{2.4}$$

where M is the same function as before, $\nu \in \mathbb{R}$ and p is a given element of Z. Also, suppose λ_0 is a simple eigenvalue of $(B,A_1,\ldots,A_N)$ and y_0 is a basis for the null space of $L(\lambda_0)$. By imposing some additional conditions on the nonlinear terms of $M(0,uy_0)$ one can discuss completely the solutions of Equation (2.4) near $\lambda = 0$, $\nu = 0$, $x = 0$. We do not consider the most general situation but merely show how one can obtain the analogue of the familiar cusp.

Using the same notation as before, we know that $[A_1y_0,\ldots,A_Ny_0]$ is a complementary subspace of $\mathscr{R}(L(\lambda_0))$. With E the previous projection onto $\mathscr{R}(L(\lambda_0))$, let

$$I - E = Q_1 + \cdots + Q_N$$

where Q_j is a projection onto A_jy_0 and $Q_jQ_k = 0,\ j \neq k$. Let

$$\alpha_1 A_1 y_0 = \frac{\partial^3}{\partial u^3} Q_1 M(0,uy_0)$$

$$\beta_1 A_1 y_0 = Q_1 p,$$

Suppose

$$\alpha_1 \neq 0,\ \beta_1 \neq 0,\ \frac{\partial^2}{\partial u^2} Q_1 M(0,uy_0)\Big|_{u=0} = 0. \tag{2.5}$$

If the bifurcation functions $F_j(\mu,\nu,u)$, $j = 1,2,\ldots,N$, are defined as in the proof of Theorem 2.1, then

$$\frac{\partial^2 F}{\partial\mu\partial u}\Big|_{(0,0,0)} = I \tag{2.6}$$

$$F_1(\mu,\nu,u) = \alpha_1 u^3 + \mu_1 u + \nu\beta_1 + O(|u|^4+|\mu|^2|u|+|\mu||u|^2+|\nu|^2+|\nu\mu|+|\nu u|)$$

as $\mu,\nu,u \to 0$.

If

$$F_j(0,\nu,0) = \nu f_j(\nu),\quad j = 1,2,\ldots,N \tag{2.7}$$

and

$$\tilde{F}_1 = F_1, \quad \tilde{\mu}_1 = \mu_1$$

$$\tilde{F}_j = (-f_j F_1 + f_1 F_j)/u \tag{2.8}$$

$$\tilde{\mu}_j = \tilde{F}_j(\mu,\nu,0), \quad j = 2,3,\ldots,N,$$

then the bifurcation equations $F_j(\mu,\nu,u) = 0$, $j = 1,2,\ldots,N$, are equivalent to the equations

$$\tilde{F}_j(\tilde{\mu},\nu,u) = 0 \tag{2.9}$$

The functions $\tilde{F}_j$ satisfy the properties

$$\tilde{F}_j(0,0,0) = 0, \quad j = 2,3,\ldots,N$$

$$\left.\frac{\partial(\tilde{F}_2,\ldots,\tilde{F}_N)}{\partial(\tilde{\mu}_2,\ldots,\tilde{\mu}_N)}\right|_{(0,0,0)} = I.$$

Therefore, the Implicit Function Theorem implies the equations

$$\tilde{F}_j = 0, \quad j = 2,3,\ldots,N,$$

have a unique solution $\tilde{\mu}_j^*(\mu_1,\nu,u)$, $j = 2,3,\ldots,N$, in a neighborhood of zero with $\tilde{\mu}_j^*(0,0,0) = 0$. Therefore, the bifurcation equations are equivalent to the single equation

$$\tilde{F}_1(\mu_1,\tilde{\mu}_2^*,\ldots,\tilde{\mu}_N^*,\nu,u) = 0. \tag{2.10}$$

The latter equation in μ_1,ν,u has the form

$$\alpha_1 u^3 + \mu_1 u + \beta_1 \nu + \text{h.o.t.} = 0 \qquad (2.11)$$

where h.o.t. denotes higher order terms which are $O(u^4+\mu_1^2|u|+|\mu_1||u|^2+|\nu|^2+|\nu\mu_1|+|\nu u|)$ as $\mu_1,\nu,u \to 0$. The bifurcation curve for Equation (2.11) in the parameter space μ_1,ν is obtained from the multiple solutions of (2.11); that is, μ_1,ν are obtained parametrically in terms of u as the solution of Equation (2.11) and the equation

$$3\alpha_1 u^2 + \mu_1 + \text{h.o.t.} = 0$$

where h.o.t. denotes $O(|u|^3+\mu_1^2+|\mu_1 u|+|\nu|)$. This implies $\mu_1 = -3\alpha_1 u^2 + O(|u|^3)$, $\nu = 2\alpha_1 u^3/\beta_1 + O(|u|^4)$ as $|u| \to 0$. This is the familiar cusp and is shown in Figure 1 for $\alpha_1 < 0$, $\beta_1 > 0$.

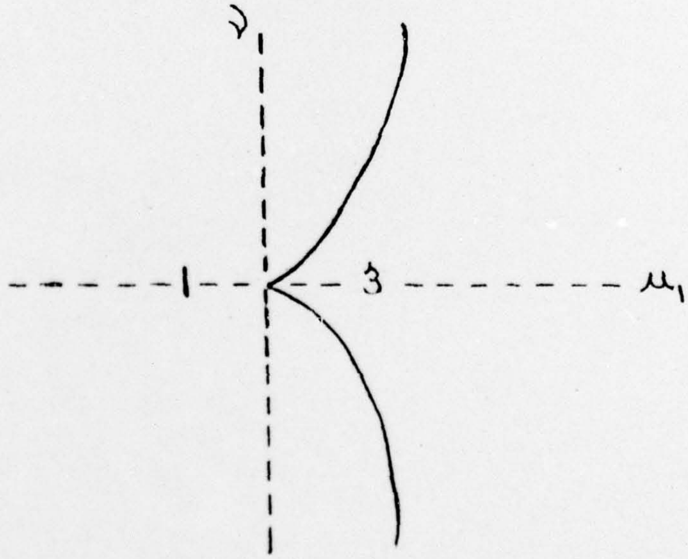

Figure 1

There is one solution of Equation (2.4) to the left of the cusp and three solutions to the right.

3. An application. Suppose $\alpha < \beta < \gamma$ are real numbers, $p > 0$ is continuously differentiable on $[\alpha,\gamma]$, q,a,b are continuous on $[\alpha,\gamma]$, λ_1,λ_2 are real parameters and define

$$L_0 y = -(py')' + qy, \qquad (' = \frac{d}{dx})$$

$$L(\lambda)y = L_0 y + \lambda_1 ay + \lambda_2 by.$$

Also, suppose $f(x,y,y')$ is continuous in x, continuously differentiable in y,y', $f(x,0,0) = 0$, $\partial f(x,0,0)/\partial(y,y') = 0$. For given numbers $\alpha_1,\beta_1,\gamma_1 \in [0,\pi)$, consider the boundary value problem

$$M(\lambda,y) \stackrel{\text{def}}{=} L(\lambda)y + f(\cdot,y,y') = 0 \quad \text{on} \quad (\alpha,\gamma) \tag{3.1}$$

$$y(\alpha)\cos\alpha_1 - y'(\alpha)\sin\alpha_1 = 0 \tag{3.2}$$

$$y(\beta)\cos\beta_1 - y'(\beta)\sin\beta_1 = 0 \tag{3.3}$$

$$y(\gamma)\cos\gamma_1 - y'(\gamma)\sin\gamma_1 = 0. \tag{3.4}$$

Under appropriate conditions on a,b, it has been shown by Sleeman [7] that there exists an eigenvalue λ_0 of $L(\lambda)$ with $\dim \mathfrak{N}(L(\lambda_0)) = 1$. For example, if a is positive on (α,γ) and b has a positive maximum on (α,β) and a negative minimum on (β,γ), then there are an infinite set of such eigenvalues and these are characterized by the eigenvector y_0 having m zeros on (α,β) and n zeros on (β,γ), all zeros being simple. In the following, we assume, that the functions a,b satisfy conditions which ensure that the eigenfunction y_0 has simple zeros on $[\alpha,\gamma]$.

Let $X = \{y \in W^{2,2}(\alpha,\gamma)$: y satisfies the boundary conditions (3.2), (3.3), (3.4)$\}$ and let $Z = L^2(\alpha,\gamma)$.

Lemma 3.1. $\mathscr{R}(L(\lambda_0)) = \{f \in Z: \int_\alpha^\beta fy_0 = 0,\ \int_\beta^\gamma fy_0 = 0\}$ where $[y_0] = \mathfrak{N}(L(\lambda_0))$; that is, $L(\lambda_0)$ is Fredholm of index -1.

Proof: If we write the original three point boundary value problem on $[\alpha,\gamma]$ as two separate boundary value problems,

$$L(\lambda_0)y = f \quad \text{on} \quad (\alpha,\beta) \tag{3.5}$$

plus boundary conditions (3.2), (3.3)

$$L(\lambda_0)y = f \quad \text{on} \quad (\beta,\gamma) \tag{3.6}$$

plus boundary conditions (3.3), (3.4)

for $f \in Z$, then it is well-known that these two problems have a solution if and only if

$$f \in Z, \quad \int_\alpha^\beta fy_0 = 0, \quad \int_\beta^\gamma fy_0 = 0. \tag{3.7}$$

Therefore, if $f \in \mathscr{R}(L(\lambda_0))$, it is necessary that f satisfy (3.7). To show that it is sufficient, suppose f satisfies (3.7) and let y_1, y_2 be solutions of problems (3.6), (3.7) respectively given by

$$y_1 = \delta_1 y_0 + K_1 f$$

$$y_2 = \delta_2 y_0 + K_2 f$$

where $K_1 f$, $K_2 f$ are any given solutions of Problems (3.6), (3.7) respectively and δ_1, δ_2 are arbitrary constants.

We must show there exist δ_1, δ_2 such that the function y on (α,γ) given by $y = y_1$ on (α,β), $y = y_2$ on (β,γ) satisfies the three point boundary value problem on $[\alpha,\gamma]$. We need only show that y, y' are continuous at β.

If $\beta_1 = 0$, then $y_1(\beta) = y_2(\beta) = 0$. Therefore, $y(x)$ is continuous at $x = \beta$. The function $y'(x)$ is continuous at $x = \beta$ if and only if $y_1'(\beta) = y_2'(\beta)$; that is,

$$(\delta_1 - \delta_2) y_0'(\beta) = (K_2 f)'(\beta) - (K_1 f)'(\beta).$$

Since $y_0'(\beta) \neq 0$, we may solve for $\delta_1 - \delta_2$.

If $\beta_1 \neq 0$, then the conditions for $y(x), y'(x)$ to be continuous at $x = \beta$ are equivalent to (we have used (3.3))

$$\begin{aligned} (\delta_1 - \delta_2) y_0(\beta) &= (K_2 f)(\beta) - (K_1 f)(\beta) \\ (\cos\beta_1)(\delta_1 - \delta_2) y_0(\beta) &= (\cos\beta_1)[(K_2 f)(\beta) - (K_1 f)(\beta)]. \end{aligned} \tag{3.8}$$

If $\beta_1 = \pi/2$, then $y'(\beta) = 0$. Since $y_0(x)$ has simple zeros, it follows that $y_0(\beta) \neq 0$ and we may solve for $\delta_1 - \delta_2$ to make equations (3.8) satisfied. If $\beta_1 \neq 0, \pi/2$, then both equations are equivalent in (3.8). Also, $y_0(\beta) \neq 0$ and we can solve for $\delta_1 - \delta_2$. This completes the proof of the lemma.

Lemma 3.2. If

$$\det\begin{bmatrix} \int_\alpha^\beta ay_0^2 & \int_\alpha^\beta by_0^2 \\ \int_\beta^\gamma ay_0^2 & \int_\beta^\gamma by_0^2 \end{bmatrix} \neq 0 \tag{3.9}$$

then λ_0 is a simple eigenvalue of $L(\lambda)$.

Proof: From Lemma 3.1, it remains only to show that ay_0, by_0 are linearly independent and do not belong to $\mathfrak{R}(L(\lambda_0))$. However, Relation (3.9) clearly implies all of these properties and the lemma is proved.

Lemma 3.2 and Theorem 2.1 imply there is a bifurcation at $\lambda = \lambda_0$ for the original boundary value problem (3.1)-(3.4). This result is the same as the one of Thomas and Zachmann [8].

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