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Nonlinear Optimal Trajectories Using Successive Linearization

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The Aerospace Corporation
El Segundo, Calif. 90245

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A method of designing trajectories for maneuvering reentry vehicles is developed in this paper. Optimal control theory using linear perturbation guidance and a quadratic penalty function is employed to design trajectories for these vehicles. The vehicle equations of motion are represented by a five-dimension state vector. The problem is formulated by linearizing about a reference trajectory. A linear servomechanism problem, rather than a regulator problem, is solved, leading to a Riccati matrix differential equation

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NONLINEAR OPTIMAL TRAJECTORIES USING SUCCESSIVE LINEARIZATION*

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Abstract

A method of designing trajectories for maneuvering reentry vehicles is developed in this paper. Optimal control theory using linear perturbation guidance and a quadratic penalty function is employed to design trajectories for these vehicles. The vehicle equations of motion are represented by a five-dimension state vector. The problem is formulated by linearizing about a reference trajectory. A linear servomechanism problem, rather than a regulator problem, is solved, leading to a Riccati matrix differential equation and an auxiliary vector differential equation which are solved backwards in time. Successive linearization consists of iteratively generating new trajectories by linearizing about a reference trajectory created from a prior iteration. The cost function is adjusted between iterations to shape the trajectory. Qualitative guidelines for selecting these cost functions are given. Test cases are shown, with emphasis on designing a trajectory to intercept a target point.

Introduction

The objective of this paper is to investigate the application of optimal control theory to design trajectories for maneuvering reentry vehicles. The trajectory must satisfy specified conditions, and possibly be optimal in some sense. A general formulation of the problem leads to the Euler-Lagrange equations, which are two-point boundary value problems, or the Hamilton-Jacoby-Bellman partial differential equation.

Much of the recently published research related to maneuvering reentry vehicles is concerned with determining an optimal control to enable the vehicle to follow a prescribed trajectory.[1-3] The methods used for solving these trajectory-following problems are variations of the perturbation guidance schemes discussed in Bryson and Ho and in various other texts.[4-5] The basic perturbation guidance scheme is concerned with time-varying linear systems with quadratic performance criteria and small deviations from a nominal solution.

The specified conditions for an optimal trajectory are given in terms of terminal conditions. For accuracy trajectories, the impact point and final flight path angle are specified. Efficient flight path shaping leads to steeper terminal flight path angles, which reduce navigation errors. Range extension and evasion trajectories can also be specified by the terminal conditions. In each case, the initial conditions are fixed. The one variable that needs attention throughout the entire flight is the control vector, since its magnitude is required to not exceed specified values and it plays an important role in determining total flight time and final velocity.

Specifying only terminal conditions allowed the use of an extended linear perturbation approach with quadratic performance criteria. The desired trajectory is significantly different from an initially available nominal or reference trajectory. Therefore, an iterative approach, or a sequence of trajectories, can be developed, each trajectory being a reference nominal for the subsequent iteration.

The iterative method of successively generating and linearizing about nonlinear trajectories as described in this paper permits penalty factors of the quadratic cost function to be adjusted between iterations for the purpose of shaping the final trajectory to meet the required goals and to ensure convergence if necessary. The method of successive linearization is similar to the method of quasilinearization, or generalized Newton-Raphson approach.[5-6] There is an advantage for the method described in this paper, however: direct access of intermediate computations is available, enabling the iterative adjustments of the problem parameters.

The second section of this paper briefly summarizes the applicable linear perturbation theory, including the servomechanism technique. The nonlinear and linearized equations of motion are given, and the section concludes with a more detailed description of the successive linearization method. The third section contains a discussion of the evaluation of the weighting matrices. The last section contains descriptions of test cases to further illustrate the method and to demonstrate its usefulness.

Linear Perturbation Theory

Linear Servomechanism Technique

The nonlinear differential equations describing the vehicle motion are given by

$$\dot{x} = f(x, u) \tag{1}$$

with x the state vector, u the control vector, and f a vector valued function of the state and control. Linearizing about a nominal state and control function, $x_o(t)$ and $u_o(t)$, and defining

$$\left.\begin{aligned} \delta x &= x - x_o \\ \delta u &= u - u_o \end{aligned}\right\} \tag{2}$$

gives

$$\delta\dot{x} = F\,\delta x + G\,\delta u \tag{3}$$

F and G are time-varying Jacobian matrices whose elements f_{ij} and g_{ij} are given by

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**Member of the Technical Staff, Guidance and Control Division, Member AIAA.

$$f_{ij} = \frac{\partial f_i(x, u)}{\partial x_j}$$

$$g_{ij} = \frac{\partial f_i(x, u)}{\partial u_j} \qquad (4)$$

where f_i, x_j, and u_j are the components of their respective vectors, and where f_{ij} and g_{ij} are evaluated along a nominal or reference trajectory.

The objective of the optimization process is to generate a trajectory which has specified desirable properties. For the reentry vehicle trajectory problems discussed in this paper, the terminal conditions are specified. Designating this desired terminal state $\eta(t_f)$, we formulate the problem by minimizing the quadratic cost function

$$J = \frac{1}{2}\left\{\left\|\delta x(t_f) - \left(\eta(t_f) - x_o(t_f)\right)\right\|_S^2 + \int_{t_o}^{t_f} \|\delta x(t)\|_Q^2 + \|\delta u(t)\|_R^2 \, dt\right\} \qquad (5)$$

The first term of Eq. (5) represents the penalty for a deviation from the desired terminal state at the final time. The elements of the S matrix, therefore, would be much larger than the corresponding elements of the Q matrix. The first term inside of the integral sign represents a penalty for deviations of the perturbed trajectory from the nominal trajectory. Since the desired trajectory is not necessarily close to the nominal, Q would normally be small. However, this term needs some weighting to keep the perturbed trajectory close to the nominal to allow the linear perturbation approximations to be valid. The second term inside of the integral sign represents the penalty for control variations about the nominal, and needs to be weighted to ensure that the control does not exceed specified bounds.

A solution to the minimization problem stated in Eq. (5) is given in Sage,[5] Page 96, or Bryson and Ho,[4] Chapter 5. The solution is

$$\delta u = - R^{-1} G'(P\,\delta x - \xi) \qquad (6)$$

with the matrix P(t) and vector $\xi(t)$ given by

$$\dot{P} = - PF - F'P + PGR^{-1} G' P - Q$$
$$P(t_f) = S \qquad (7)$$

$$\dot{\xi} = (- F' + PGR^{-1} G')\,\xi$$
$$\xi(t_f) = S\left(\eta(t_f) - x_o(t_f)\right) \qquad (8)$$

where prime (′) denotes transpose.

Equation (7) is the same Riccati equation obtained for the linear regulator problem, and Eq. (8) is an auxiliary equation needed for the servomechanism problem.

The algorithm for solving the linearized servomechanism problem is as follows:

1. Generate a nominal trajectory $x_o(t)$ with a nominal control $u_o(t)$, and compute and store the F(t) and G(t) matrices.

2. Initialize P and ξ at the terminal time, according to Eqs. (7) and (8), and integrate backwards from t_f to t_o. Compute and store the gain matrix

$$K = R^{-1} G'P \qquad (9)$$

and control vector

$$k = R^{-1} G'\xi \qquad (10)$$

3. Initialize $x(t_o)$ and integrate Eq. (1) forward in time with

$$u = - K\,\delta x + k + u_o \qquad (11)$$

with $\delta x = x - x_o$ to obtain x(t), a nonlinear solution, and an optimal linearized control u(t).

System Description, Linearized Equations of Motion

The coordinate system used to describe the equations of motion was centered on the surface of a flat, nonrotating earth, with x pointing downrange, y crossrange, and z positive upward along the local vertical and passing through the vehicle at time equals zero. The point mass representation of the vehicle is shown in Figure 1. The state equations were

$$\dot{x} = s \cos\gamma \cos\psi \qquad (12)$$

$$\dot{y} = s \cos\gamma \sin\psi \qquad (13)$$

$$\dot{z} = - s \sin\gamma \qquad (14)$$

$$\dot{\psi} = q\frac{A}{m}\frac{C_L}{s}\frac{u_1}{\cos\gamma} \qquad (15)$$

$$\dot{\gamma} = -q\frac{A}{m}\frac{C_L}{s}u_2 \qquad (16)$$

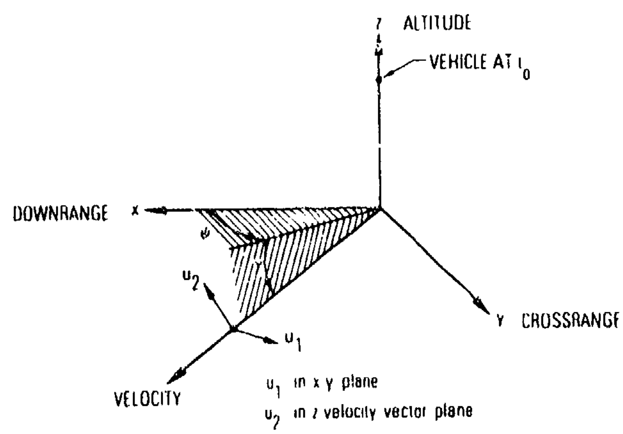


Figure 1. Coordinate definition.

$$\dot{s} = -q\frac{A}{m}C_D(u_1, u_2) \quad (17)$$

with

x, y, z	= the position coordinates of the vehicle
ψ, γ	= two angles defining the velocity vector
s	= vehicle speed
u_1, u_2	= control components
q	= dynamic pressure = $(1/2)\rho s^2$
A/m	= reference area to mass ratio
C_L	= slope of the lift coefficient
$C_D(u_1, u_2)$	= nonlinear drag coefficient
ρ	= air density = $\rho_o \exp(-z/h_s)$
ρ_o, h_s	= constants defining air density approximation

The control components u_1 and u_2 are the components of angle of attack. The direction of the force associated with angle of attack is shown in Figure 1. This model is applicable to either a cruciform configuration vehicle or a bank-to-turn vehicle with negligibly small autopilot time constants.

There is no direct control of the speed. The state model X was therefore represented by the five-state vector

$$X = \begin{bmatrix} x \\ y \\ z \\ \psi \\ \gamma \end{bmatrix}$$

The speed equation (17) was used for the solution of the nonlinear state equations (12) through (16), but was not considered an element of the state vector for the purpose of solving the linearized equations.

The F and G matrices, given in Eqs. (3) and (4) for the linearized model, were dimensioned, respectively, (5×5) and (5×2), with all of the elements equal to zero except those listed.

$$f_{14} = -s \cos\gamma \sin\gamma$$

$$f_{15} = -s \cos\psi \sin\gamma$$

$$f_{24} = s \cos\gamma \cos\psi$$

$$f_{25} = -s \sin\psi \sin\gamma$$

$$f_{35} = -s \cos\gamma$$

$$f_{43} = -C_1 s u_1 \exp(-z/h_s)\frac{1}{h_s \cos\gamma} \quad (18)$$

$$f_{45} = C_1 s u_1 \exp(-z/h_s)\frac{\sin\gamma}{\cos^2\gamma}$$

$$f_{53} = C_1 s u_1 \exp(-z/h_s)\frac{1}{h_s}$$

$$g_{41} = C_1 s \exp(-z/h_s)\frac{1}{\cos\gamma}$$

$$g_{52} = -C_1 s \exp(-z/h_s) \quad (18) \text{ cont}$$

with the constant $C_1 = (1/2)\, C_L (A/m)\rho_o$.

The linearized perturbation differential equation is given by Eq. (3) with

$$F = \begin{bmatrix} 0 & 0 & 0 & f_{14} & f_{15} \\ 0 & 0 & 0 & f_{24} & f_{25} \\ 0 & 0 & 0 & 0 & f_{35} \\ 0 & 0 & f_{43} & 0 & f_{45} \\ 0 & 0 & f_{53} & 0 & 0 \end{bmatrix} \quad (19)$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ g_{41} & 0 \\ 0 & g_{52} \end{bmatrix} \quad (20)$$

The gain matrix K, Eq. (9), and control vector k, Eq. (10), are, respectively, a (5×2) matrix and a two-dimension vector.

Successive Linearization Technique

The successive linearization method iteratively applies the algorithm described in the preceding section, with the nominal trajectory and control for each iteration being the solution of the prior iteration.

The aerodynamically controlled reentry vehicle problems discussed in this paper were started with the nominal control $u_o(t) = 0$. After an initial reference trajectory is generated with this control, a perturbed trajectory is generated using an optimal controller derived from the linearized equations and the initial cost function penalty matrices. When this first pass is completed, we have a perturbed reference trajectory $x_o(t)$ and a nominal $u_o(t) \neq 0$. A control, $u(t)$, associated with a trajectory $X(t)$ that has the required specified properties can be obtained by a judicious selection of the weighting matrices Q, R, and S.

The process is illustrated in Figure 2. The technique for the selection of the weighting matrices is described in the next section.

The gain matrix K and the auxiliary control vector k for each iteration are related to the selected weighting matrices Q, R, and S; Eqs. (7) through (10). A qualitative relationship between the weighting matrices and the defined performance

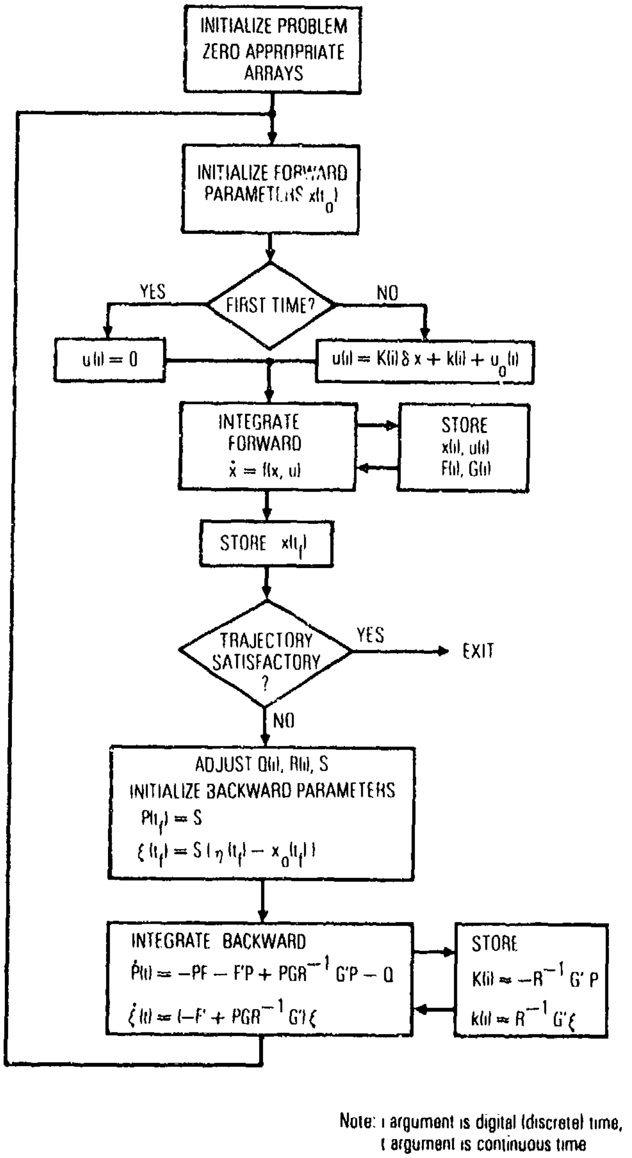


Figure 2. Successive linearization technique.

criterion can be developed. The weighting matrices can then be changed between iterations, based on the measured conditions, to achieve the desired performance. Specific methods for shaping the trajectories of the reentry vehicle are shown in the sections on "Evaluation of the Weighting Matrices" and the "Test Cases."

Evaluation of Weighting Matrices Q, R, S

The performance of the vehicle, as characterized by the shape of the final trajectory, is determined by the values of the Q, R, and S matrices. No general analytic method is available for computing these matrices for the time varying system described. Sworder and Wells[3] demonstrate an analytic approach which can be used when making certain simplying assumptions, and de Virgilio, Wells, and Schiring[1] suggest a procedure for determining these matrices. The commonly used rule-of-thumb method, suggested by Bryson and Ho,[4] is to proportion the weights according to the reciprocal of the squares of the maximum acceptable deviation from the nominal.

$$S^{-1} = n \times \text{maximum acceptable value of } x(t_f)\, x'(t_f)$$

$$Q^{-1} = n \times (t_f - t_o) \times \text{maximum acceptable value of } x(t)\, x'(t)$$

$$R^{-1} = r \times (t_f - t_o) \times \text{maximum acceptable value of } u(t)\, u'(t).$$

with n the state vector dimension, r the control vector dimension, and $(t_f - t_o)$ the limits of integration of the cost function, Eq. (5).

The trajectories developed using the method described in this paper were not necessarily concerned with minimizing a deviation from a reference. The relationship between the solution and the weighting matrices is roughly governed by the inverse square law, and this information was used to aid in choosing the value for the weighting matrices.

The qualitative effect of the weighting matrices on the shape of the final trajectory can easily be determined by referring to the cost function, Eq. (5): large Q tending to hold the perturbed trajectory close to the reference; large R tending to reduce the amount of control; and large S tending to reduce the error between the perturbed final state and the target point. Other trajectory properties have been specified, which also must be controlled with Q, R, S selection: the total time of flight and the final velocity. These relationships are shown in Table 1, which gives qualitative guidelines for adjusting the weighting matrices from one iteration to the next.

Because the desired trajectory is generally not close to the reference trajectory, Q should be as small as practical. Q must be at least positive semi-definite. The Q matrix functions as a stabilizing controller, and generally needs small positive numbers on the diagonal to ensure a stable solution. The desired trajectory must come close to the target point, suggesting large positive numbers for the S matrix.

Experience has shown the chief influence on the shape of the trajectory is the relative weighting of the elements of the S matrix. Essentially, the perturbed trajectory will terminate much closer to the target point than the nonmaneuvering impact point on the first iteration. The relative weights on the terminal position components and flight path angle, although having second order effects on the final state, appear to strongly influence the trajectory shape. These component weightings can be adequately accomplished by manipulating the diagonal components, with the off-diagonal elements set equal to zero.

The control vector u is subject to a direct and an indirect hard constraint. The direct constraint is u_{max} u where

$$|u| \le u_{max}\, u \tag{21}$$

with $|u| = (u_1^2 + u_2^2)^{1/2}$.

Table 1. Qualitative relationships for adjusting weighting matrices.

Objective	Matrix Adjustment Q	R	S
Follow Reference	Increase	Increase	Decrease
Intercept Target	Decrease	Decrease	Increase
Reduce Control	Decrease	Increase	Decrease
Decrease Time of Flight	Increase if perturbed trajectory is longer than nonmaneuvering trajectory; decrease otherwise.	Increase if perturbed trajectory is longer than nonmaneuvering trajectory; decrease otherwise.	Increase altitude component; decrease horizontal components. Component refers to diagonal element of matrix.
Increase Terminal Speed	Increase if perturbed trajectory terminal speed is lower than nonmaneuvering trajectory; decrease otherwise.	Increase if perturbed trajectory terminal speed is lower than nonmaneuvering trajectory; decrease otherwise.	Increase altitude component; decrease horizontal components. Component refers to diagonal element of matrix.

The control must also limit the normal acceleration to an acceptable value, a_{max}. The acceleration is given by

$$a_g = \left(\frac{q\, C_L\, u}{W/Ag}\right)\left(\frac{1}{g}\right) \tag{22}$$

with

a_g = acceleration in g's

q = dynamic pressure

C_L = slope of the lift coefficient

W/Ag = mass to reference area ratio

g = gravity

The indirect constraint is

$$|u| \leq u_{max\ a}$$

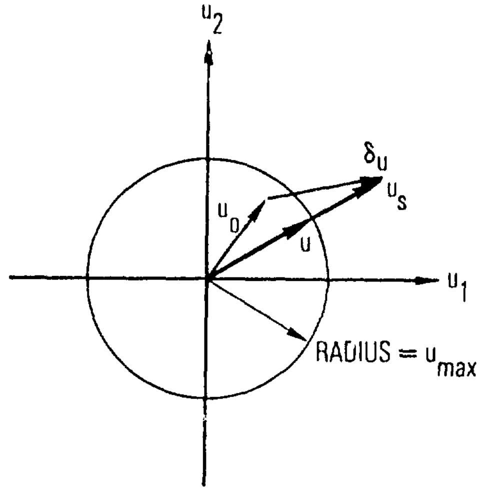
with

$$u_{max\ a} = \left(\frac{W}{A}\right)\frac{1}{q}\frac{1}{C_L}\, a_{max} \tag{23}$$

Then,

$$u_{max} \leq u_{max\ u},\ u_{max\ a} \tag{24}$$

The total control (nominal plus perturbed) is tested at each point along the trajectory. If it exceeds u_{max}, it is limited as shown in Figure 3, and the R matrix is modified for the next iteration according to



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u_s = \begin{bmatrix} u_{s_1} \\ u_{s_2} \end{bmatrix}$$

IF $|u_s| > u_{max}$

$$u_i = \frac{u_{max}}{|u_s|}\, u_{s_i}$$

Figure 3. Scaling the control when exceeding the constraint.

$$r_{ii\ new} = \left(\frac{u_s}{u_{max}}\right)^2 r_{ii} \tag{25}$$

with

r_{ii} = i^{th} diagonal element of R matrix

$r_{ii\ new}$ = i^{th} diagonal element of R matrix for the next iteration

u_s = $u_o + \delta u$

As shown in Figure 3, u_s is the total control from the linear formulas when $u_o + \delta u$ exceeds u_{max}.

Test Cases

The procedures described in this paper were used to generate trajectories designed to achieve various objectives. Two sample cases are described in this section. The descriptions are for the purpose of illustrating the technique and demonstrating the types of results that can be obtained. Numerical results are given in terms of normalized data: linear distances are referenced to h, the initial altitude, and angles are referenced to $\alpha = u_{max\ u}$, the maximum angle of attack.*

Specified Target and Terminal Flight Path Angle – No Acceleration Constraint

A basic trajectory is one that has a fixed initial state, impacts at a specified location, approaches

*Vehicle parameters and aerodynamic data for the evaluation of the trajectory generation scheme were similar to the examples used in an unclassified report with unlimited distribution.[1]

the target along a specified diveline (terminal flight path angle), and minimizes control effort.[2]

The minimum control effort aids in achieving the additional objectives of increased terminal velocity and minimum flight time (see Table 1). Conditions for this case and computed results are listed in Table 2. Figures 4 through 14 are computer-generated plots of trajectory data.

The weighting on position deviations along the trajectory was zero. Weighting on the angles along the trajectory corresponded roughly to deviations of 0.25; weighting on the control (angle of attack) corresponded roughly to 0.25, with a hard constraint of 1.0. Terminal weightings corresponded roughly to position errors of 0.003, a very large allowable azimuth deviation and a flight path angle error of 0.25.

Figures 4, 5, and 6 show the trajectories. The first perturbed trajectory (Iteration 1) came fairly close to the final trajectory. The terminal flight path angle was significantly different than the target value of 2.617. Subsequent iterations improved the trajectory as indicated in Table 2. After the third iteration, the differences between the actual and targeted terminal conditions were reasonably small: downrange error = 0.00156, crossrange error = 0, flight path angle error = 0.006.

The flight time of 6.125 sec and speed of 0.2819/sec at the last iteration, compared to the nonmaneuvering values of 3.998 sec and 0.3925/sec, demonstrates the trade-off between the maneuvering and nonmaneuvering cases.

Table 2. Specified target — no acceleration constraint

Test Conditions

Initial Conditions:

$x_o = 0,\ y_o = 0,\ z_o = z_o/h = 1.0$

$\dot{s}_o/h = 0.593/\text{sec},\ \psi_o = 0,\ \gamma_o = 1.309$

Weighting Matrices:

$Q_{44} = Q_{55} = 20/(t_f - t_o),\ R_{11}^{-1} = R_{22}^{-1} = 0.01\ (t_f - t_o)$

$S_{11} = S_{22} = S_{33} = 2 \times 10^{-5},\ S_{44} = 10^{-3},\ S_{55} = 20$

All Other Elements = 0

Control Constraint:

$u_{max\ u} = 1.0 \qquad u_{max\ a} = \text{none}$

Terminal Conditions	Iteration No. 0	1	2	3	Target Condition
x/h	1.73207	2.50873	2.49833	2.49843	2.50000
y/h	0	0.33220	0.33327	0.33333	0.33333
γ/α	1.309	2.425	2.603	2.623	2.617
Time, sec	3.998	6.119	6.125	6.125	---
Speed/h, 1/sec	0.3925	0.2924	0.2878	0.2819	---

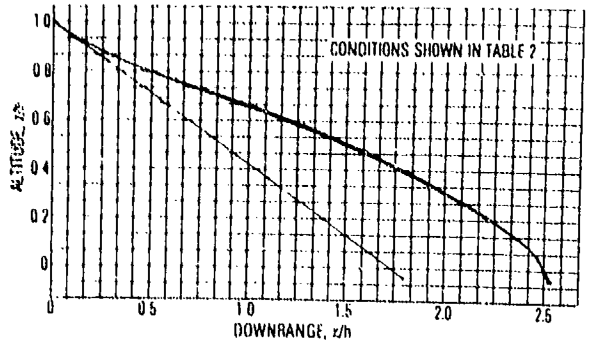


Figure 4. Trajectory in the x-z plane.

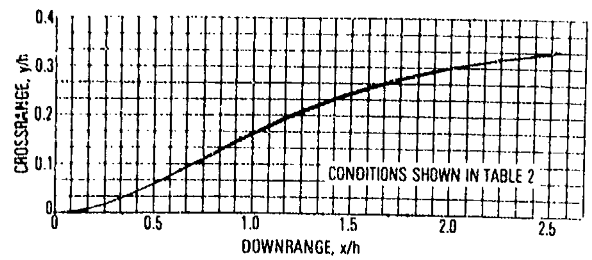


Figure 5. Trajectory in the x-y plane.

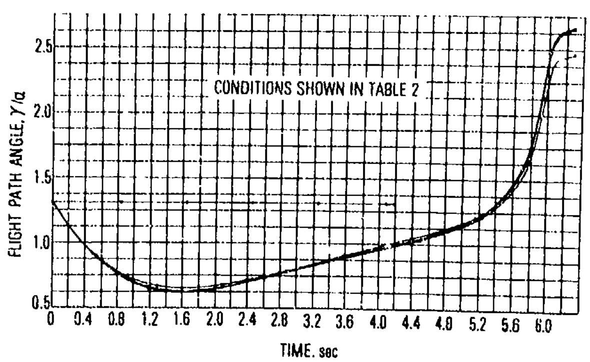


Figure 6. Flight path angle.

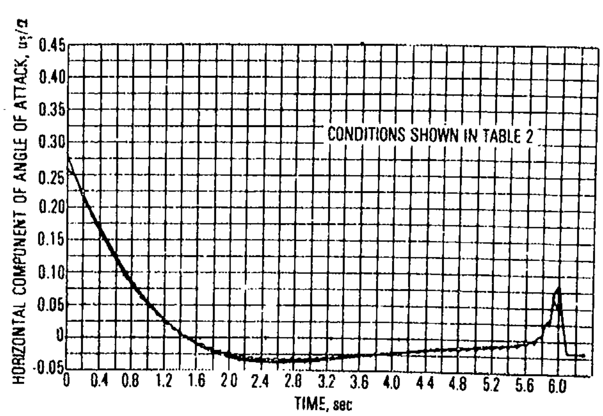


Figure 7. Horizontal control.

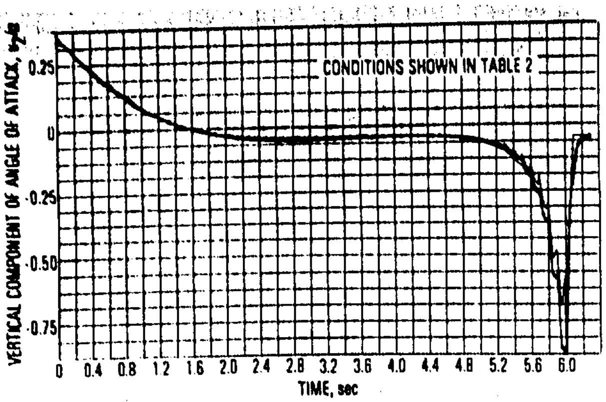


Figure 8. Vertical control.

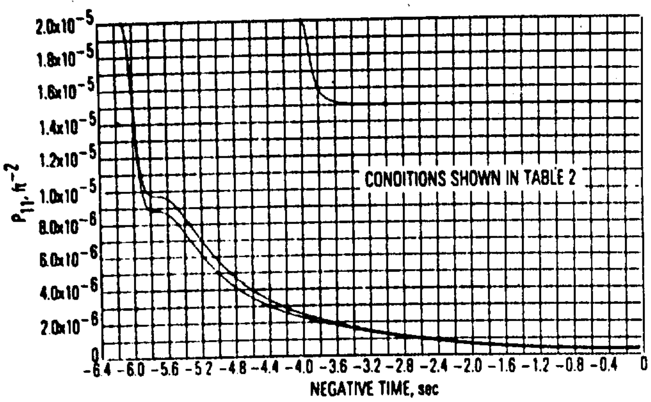


Figure 9. An element of the P matrix.

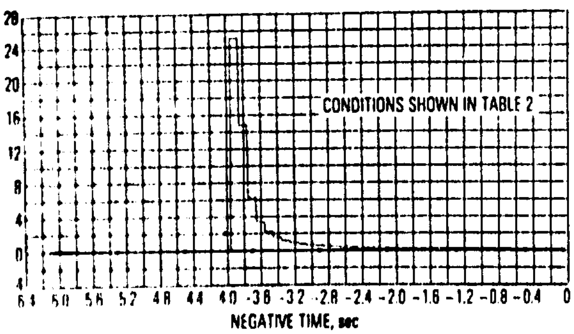


Figure 10. Horizontal component of auxiliary control.

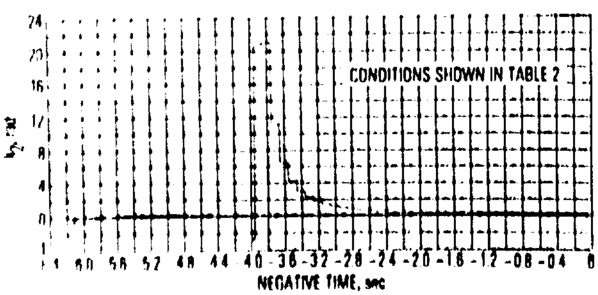


Figure 11. Vertical component of auxiliary control.

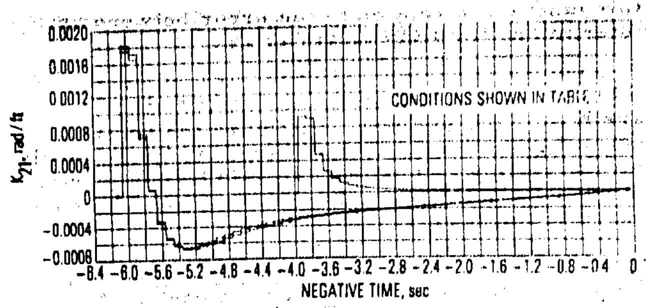


Figure 12. Gain element K_{21}.

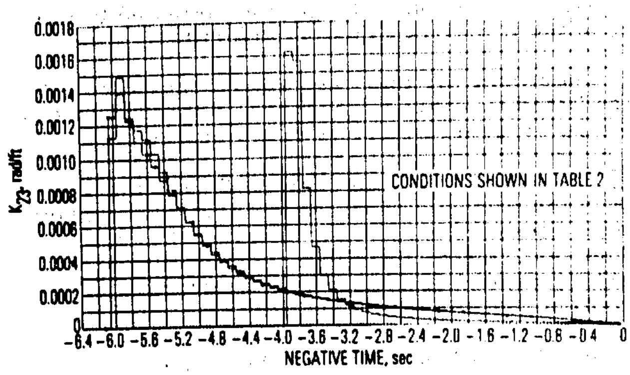


Figure 13. Gain element K_{23}.

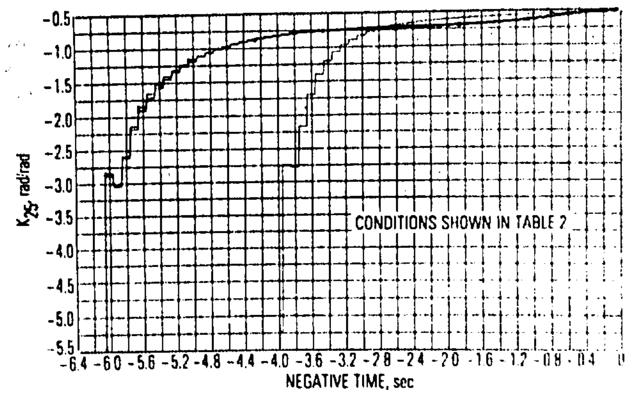


Figure 14. Gain element K_{25}.

Improvements in flight time and final speed could have been made at the expense of larger terminal errors in position or angle.

Figures 7 and 8 show the control functions. The first iteration appears rather ragged due to the discretization method (sampling of 0.1-sec intervals). The control u_1 is the angle of attack in the horizontal plane, and u_2 the angle of attack in the pitch plane. At no time was the 1.0 constraint reached.

Figure 9 shows a typical element P_{11} of the P matrix in the solution of the Riccati equation. The solution goes from left to right, with the time axes labeled with a minus sign. The P matrix is significantly different between the first iteration and subsequent iterations. Figures 10 and 11 show the solution to the auxiliary control vector k. The first solution is very large compared to the subsequent solutions, as might be expected, since this function is related to the error between an interim and the solution trajectory.

Figures 12 through 14 show three elements of the (2 × 5) gain matrix K. The three elements are K_{21}, K_{23}, and K_{25}, which are the vertical control gains for x, z, and Y errors, respectively. An interesting fact is that the gain K_{21} is mostly of opposite polarity for Iterations 2 and 3 as compared to Iteration 1.

Specified Target and Terminal Flight Path Angle — Angle of Attack and Acceleration Constraint

A more realistic test case is one that is limited both in maximum angle of attack and normal acceleration. An in-plane target was selected to examine the R matrix adjusting algorithm.

Conditions for this case and computed results are listed in Table 3. Figures 15 through 18 are computer-generated plots of trajectory data.

The initial weighting matrices were the same as in the previous case, except that out-of-plane parameters were ignored. Figures 15 and 16 show the trajectories. The first perturbed trajectory came fairly close to the terminal target position, but the terminal flight path angle error was large as seen in Figure 16 and Table 3. An identical case had previously been tried with no acceleration constraint (not reported herein), and the terminal flight path angle error was considerably smaller on the first iteration. Subsequent iterations improve the trajectory as indicated in Table 3. After the fourth iteration, the differences between the actual and targeted terminal conditions were small: downrange error = 0.00073, flight path

Table 3. Specified target — acceleration constraint

Test Conditions

Initial Conditions:

$x_o = 0$, $y_o = 0$, $z_o/h = 1.0$

$s_o/h = 0.593/\text{sec}$, $\Psi_o = 0$, $Y_o = 1.309$

Weighting Matrices:

$Q_{44} = Q_{55} = 20/(t_f - t_o)$, $R_{11}^{-1} = R_{22}^{-1} = 0.01\ (t_f - t_o)$

$S_{11} = S_{22} = S_{33} = 2 \times 10^{-5}$, $S_{44} = S_{55} = 20$

All Other Elements = 0

Control Constraint:

$u_{max\ u} = 1.0$ $\quad u_{max\ a}/\alpha = 0.16$ at t = 0

Terminal Conditions		Iteration No. 0	1	2	3	4	Target Condition
x/h	ft	1.73207	2.50377	2.50530	2.50133	2.50073	2.5000
y/h	ft	0	0	0	0	0	0
Y/α	rad	1.309	1.713	1.854	1.921	1.939	1.950
Time, sec		3.998	5.856	5.871	5.874	5.876	-
Speed/h, 1/sec		0.3925	0.33307	0.3313	0.3310	0.3310	-

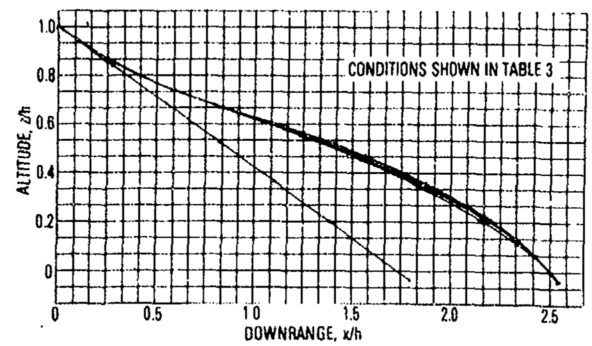


Figure 15. Trajectory in the x-z plane, constraint problem.

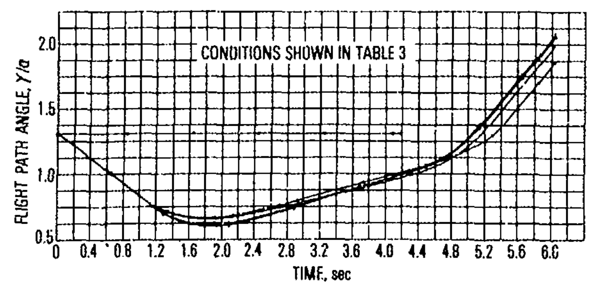


Figure 16. Flight path angle, constraint problem.

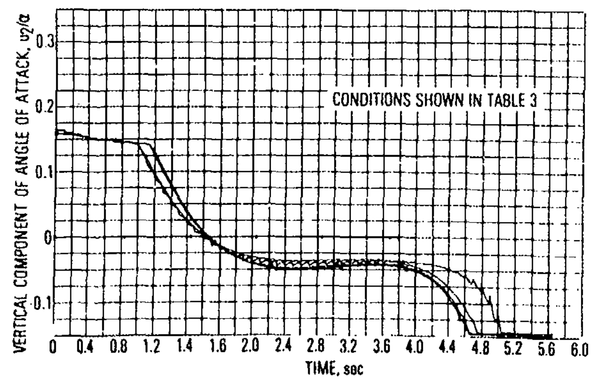


Figure 17. Vertical control, constraint problem.

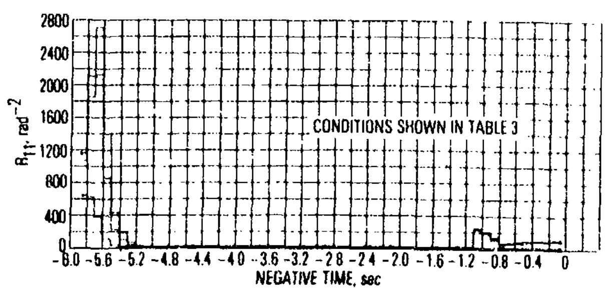


Figure 18. An element of the R matrix, $R_{11} = R_{22}$.

angle error = 0.011. Referring to Table 3, we see the final flight path angle approaching the targeted value with very small changes of flight time or final speed showing the effectiveness of the R weighting matrix.

Figure 17 shows the control function. Because of 0.16 constraint, the magnitude of u_2 is limited to about 0.16 at the start and again toward the end of the flight. As R is adjusted between iterations, the boundary riding portion of the trajectory is extended, from about 1.0 to 1.2 sec at the start and from 5.4 to 4.8 sec toward the end. We can clearly see the control going from a proportional controller to a bang-bang type, which is known to be optimal for certain types of problems. Figure 18 shows one element of the R matrix for this case. Initially R is constant, about 25 (R_1^{-1}} = 0.01 (4)). After the first iteration, we see R growing at the extremities of the trajectory, where the proportional controller is requesting a control magnitude greater than the prescribed limit.

Conclusions

This paper demonstrates that reentry trajectories can be designed using linear optimal control theory and a method of successive linearization. A unique feature of successive linearization is the ability to adjust the weighting matrices between iterations to shape the trajectory to meet specified requirements. An algorithm for computing values of the R weighting matrix gives an improvement over the trial-and-error methods that have previously been used.

A more general aspect of the paper is the development of a method of applying optimal control theory to nonlinear problems. We have not proved that the solutions using successive linearizations will converge. When they do converge, we have not proved that the solution is optimal in any sense. We have shown, however, for the cases considered, that by proper choices of the penalty function weighting matrices a solution can be made to converge to a trajectory having desirable specified terminal properties.

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