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ECOM-76-1337-F

MULTIMODE FILTERS

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<p>This report describes work on Multi Mode Filters performed from 1 June 1976 to 28 February 1977. The objective of this program is to develop design criteria and establish tradeoffs for the transduction of acoustic bulk waves in crystal stacks, for use in filtering applications. A primary goal of the program is to explore methods of obtaining miniature low-loss acoustic filters using interactions among the three modes in each crystal layer, which depend on the thickness coordinate.</p>		

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20. ABSTRACT (continued)

This report is concerned with Phase I of this overall Multi Mode Stacked Crystal Filter Program: The detailed requirements of this phase were to consider:

a. Number of crystal plates in the stack. Emphasis is placed on two- and three-plate stacks, exclusive of bonding or coupling layers, relative thicknesses of the plates, and bonds or coupling layers.

b. Materials comprising each crystal plate, bond, or coupling layer. Emphasis is placed on quartz and highly piezoelectric materials, such as lithium niobate. The material parameters used in all of the stacked filter programs are arbitrary; however, most results are illustrated using the material parameters of AT-cut quartz or are illustrated using the material parameters of AT-cut quartz or multiples of them. Results for berlinite are included (Section 2).

c. Number of thickness modes coupled piezoelectrically, in a given plate, to the electrode system. Number of thickness modes, coupled mechanically at the interfaces, for achieving the filter response. Emphasis is placed on one- and two-shear or quasi-shear modes.

d. Crystallographic orientation of each plate. Emphasis is placed on rotated Y-cuts of quartz and lithium niobate and relative rotation, about the common thickness axis, of the various plates with respect to each other.

e. Bonding materials for attaching resonator plates together. Techniques for accomplishing bonding so that welded interface boundary conditions are approached as closely as possible, and effects of finite thickness of bonds and bond viscosity are considered.

Bond effects for the single mode case are illustrated (Section 2) for bonds of various thicknesses and size relative to the crystal plate sizes. The effects of bond viscosity (Q) are also illustrated.

f. Various electroding arrangements, and interconnections between layers. Time did not permit an evaluation of this aspect of the stacked filter. When dealing with two plates, only a 180 degree phase reversal between the input and output electrodes of the stack is possible. An augmentation of the actual plate coordinate equivalent circuit (Section 3) is discussed in Section 5. This is appropriate to this problem and allows for arbitrary electrical interconnections between plate electrodes of the plates in a stack of more than two elements.

While additional design criteria would be desirable, most of the specific requirements of Phase I of this Multi Mode stacked crystal filter program have been accomplished.

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TABLE OF CONTENTS

<u>Section</u>	<u>Title</u>	<u>Page</u>
1.	INTRODUCTION	1-1
2.	SINGLE MODE BOND STUDIES	2-1
	A. TWO CRYSTAL SINGLE MODE FILTERS	2-1
	B. BOND PARAMETERS	2-4
	(1) Bond Material and Thickness	2-13
	(2) Bond Area to Transducer Area Ratio	2-13
	C. TRANSDUCER MATERIAL	2-13
	D. THREE-CRYSTAL QUARTZ FILTER	2-18
3.	THEORETICAL INVESTIGATIONS OF MULTIMODE STACKED FILTERS	3-1
	A. DEVELOPMENT OF THE SECULAR EQUATION AND PROGRAM CROT	3-2
	B. SOLUTION OF THE SECULAR EQUATION AND PRO- GRAM SYMEIG	3-18
	C. TRANSFORMATION TO NORMAL COORDINATES, NOR- MAL MODE IMPEDANCE MATRIX AND PROGRAM VCOUP	3-36
	D. TRANSFORMATION TO ACTUAL PLATE COORDINATES FROM THE NORMAL COORDINATE SYSTEM	3-72
	E. MULTIMODE STACKED FILTERS AND MODE PRO- GRAMS	3-84
	F. COMPUTER GRAPHS OF TYPICAL MODE PROGRAM RESULTS	3-113
4.	DEVICE FABRICATION AND TEST	4-1
	A. RESONATOR MATERIALS	4-1
	B. BONDING TECHNIQUES	4-2
	C. ELECTRODE CONSIDERATIONS	4-6
	D. EXPERIMENTAL RESULTS	4-6
	E. PROGRAM SIGNIFICANCE	4-12
5.	SUGGESTIONS FOR FURTHER WORK	5-1

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1. INTRODUCTION

This report describes work on Multi Mode Filters performed from June 1, 1976 to February 28, 1977 under Contract No. DAAB07-76-C-1337.

The objective of this program is to develop design criteria and establish tradeoffs for the transduction of acoustic bulk waves in crystal stacks, for use in filtering applications. A primary goal of the program is to explore methods of obtaining miniature low-loss acoustic filters using interactions among the three modes in each crystal layer, which depend on the thickness coordinate.

This report is concerned with Phase I of this overall Multi Mode Stacked Crystal Filter Program: the detailed requirements of this phase were to consider:

1. Number of crystal plates in the stack. Emphasis is placed on two and three-plate stacks, exclusive of bonding or coupling layers, relative thicknesses of the plates, and bonds or coupling layers.

The MØDE programs (Section 3) handle only two plates in a stack with intimate contact between the plates. These plates are allowed independent thicknesses and material properties. The single mode programs (Section 2) allow for two and three plates in the stacked crystal filter, with bonds between the layers. These programs assume that the plates are made of identical material but are allowed to be of different thicknesses. The bonds in these single mode programs are all assumed to be identical in a single stack.

Possible procedures for including arbitrary bonds in the MØDE programs and for including additional plates in the stacks of these programs are also described (Section 5).

2. Materials comprising each crystal plate, bond, or coupling layer. Emphasis is placed on quartz and highly piezoelectric materials, such as lithium niobate. The material parameters used in all of the stacked filter programs are arbitrary; however, most results are illustrated using the material parameters of AT-cut quartz or multiples of them. Results for berlinite are included (Section 2).
3. Number of thickness modes coupled piezoelectrically, in a given plate, to the electrode system. Number of thickness modes, coupled mechanically at the interfaces, for achieving the filter response. Emphasis is placed on one- and two-shear or quasi-shear modes. The various MØDE programs (Section 3) allow for from one to three piezoelectrically coupled modes in each plate. This number can vary between plates in a stack. These programs also allow for from one to three mechanically coupled modes at the boundaries of the plates.

4. Crystallographic orientation of each plate. Emphasis is placed on rotated Y-cuts of quartz and lithium niobate and relative rotation, about the common thickness axis, of the various plates with respect to each other.

The programs CRØT, SYMEIG and VCØUP (Section 3) allow for calculation of the required input parameters for the MØDE programs using any arbitrary orientation of the plates with respect to the standard X, Y, Z axes. Operation of these programs is illustrated for AT-cut, which is a rotated Y-cut of quartz. The MØDE2 and MØDE3 programs allow for an arbitrary angle of rotation about the common thickness axis between the plates in the stack.

5. Bonding Materials for attaching resonator plates together. Techniques for accomplishing bonding so that welded interface boundary conditions are approached as closely as possible, and effects of finite thickness of bonds and bond viscosity are considered.

Bond effects for the single mode case are illustrated (Section 2) for bonds of various thicknesses and size relative to the crystal plate sizes. The effects of bond viscosity (Q) are also illustrated.

Techniques for including bonds in the MØDE programs are also discussed (Section 5).

Information relative to bonding materials and techniques for accomplishing bonding is described (Section 4).

6. Various electroding arrangements, and interconnections between layers. Time did not permit an evaluation of this aspect of the stacked filter. When dealing with two plates only a 180° phase reversal between the input and output electrodes of the stack is possible. An augmentation of the actual plate coordinate equivalent circuit (Section 3) is discussed in Section 5. This is appropriate to this problem and allows for arbitrary electrical interconnections between plate electrodes of the plates in a stack of more than two elements.

While additional design criteria would be desirable it appears that most of the specific requirements of Phase I of this Multi Mode stacked crystal filter program have been accomplished.

2. SINGLE MODE BOND STUDIES

In this section Mason's equivalent circuit for piezoelectric transducers with constant flux density (in-line field model) is used to investigate the effects of bond parameters on multimode stacked filters. This model is shown in Figure 2.1. This same circuit can also be used to represent the bond between elements in the stack by eliminating the electromechanical transformer and associated electrical input network.

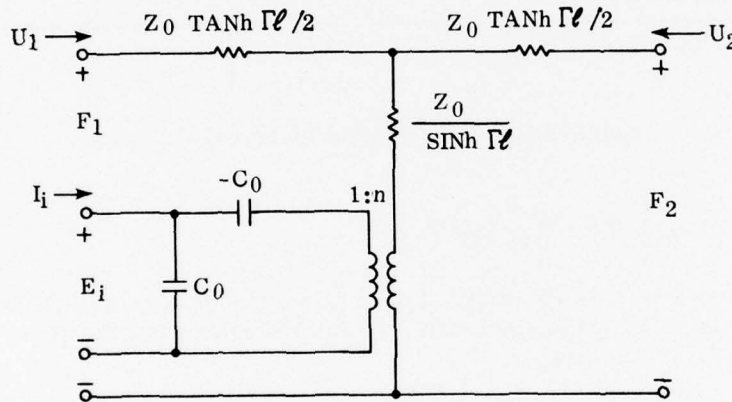


Figure 2.1. Mason's Equivalent Circuit of Single Mode Transducer with Constant Flux Density D (In-Line Field Model).

A. TWO-CRYSTAL SINGLE MODE FILTERS

Mason's equivalent circuit for piezoelectric transducers can be used to devise an equivalent electrical circuit that represents bonded crystals operating in a single shear mode. An equivalent circuit for two bonded crystals of the same dimensions is given in Figure 2.2. The impedances are found from Mason's equivalent circuit to be

$$Z_1 = Z_{OT} / \sinh (\Gamma_T \ell_T) \quad 2.1$$

$$Z_2 = Z_{OT} / \tanh (\Gamma_T \ell_T / 2) \quad 2.2$$

$$Z_3 = Z_{OB} / \tanh (\Gamma_B \ell_B / 2) \quad 2.3$$

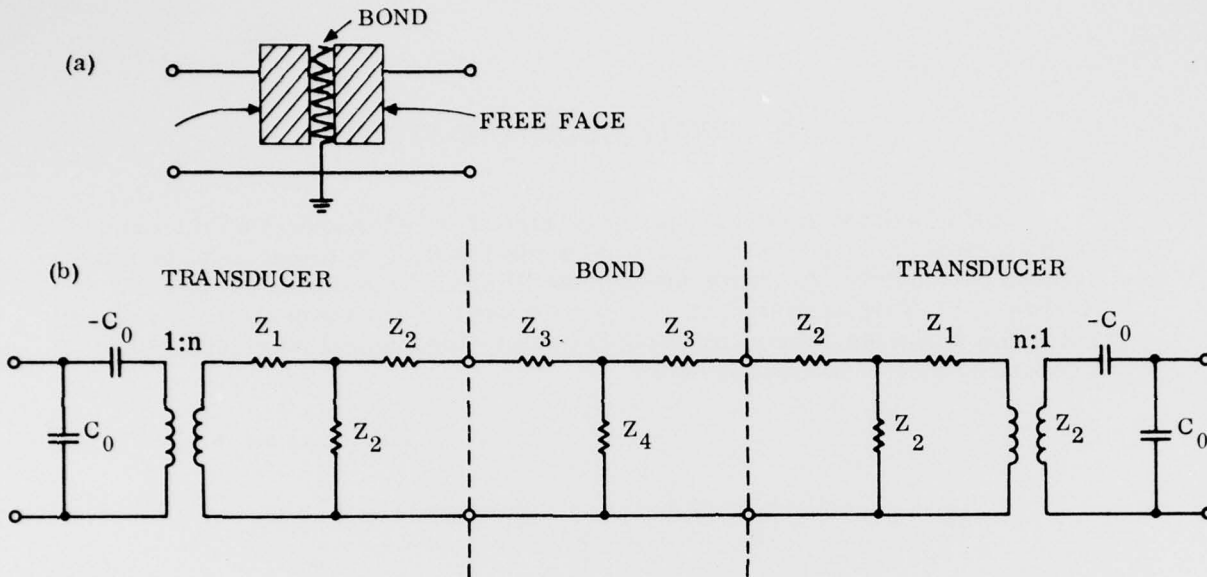


Figure 2.2. Two Bonded Identical Crystals.

$$Z_4 = Z_{OB} \sinh(\Gamma_B \ell_B) \quad 2.4$$

where $Z_{OB} = Z_{OT} \sinh(\Gamma_T \ell_T)$ and Z_{OT} , Γ_T and Z_{OB} , Γ_B , ℓ_B are the characteristic impedance, complex propagation constant, and thickness for the transducer and bond, respectively.

C_0 and n are clamped capacitance and electromechanical transformer turns ratio for the transducers.

The complex propagation constant, Γ , is given by

$$\Gamma = \frac{\omega}{c} \left(1 + \frac{j}{2Q} \right), \quad 2.5$$

where ω is the angular frequency,
 c is the velocity of propagation, and
 Q is the mechanical Q of the material.

The capacitance C_0 and electromechanical transformer turns ratio n are given by

$$C_0 = \epsilon^S A / \ell = \epsilon^T (1 - k^2) A / \ell \quad 2.6$$

$$n = k \sqrt{2f_0 C_0 Z_{OT}}, \quad 2.7$$

where ϵ^S is the dielectric constant for the transducer crystal
 k is the coefficient of electromechanical coupling

A is the transducer active area
 ℓ is the transducer thickness
 f_0 is the fundamental resonant frequency, and
 Z_{OT} is the transducer impedance

The transducer crystal thickness, ℓ , is chosen to be one-half wavelength at the frequency of mechanical resonance and electrical antiresonance, f_0

$$\ell = c/2f_0 \quad 2.8$$

The impedance, Z_0 , of an arbitrary transducer or bond material is given by

$$Z_0 = A\rho c, \quad 2.9$$

where A is the cross-sectional area,
 ρ is the material density and
 c is the velocity of propagation in the material.

Note that from the Mason's equivalent circuit of Figure 2.1, the free faces of the crystals are assumed to have zero force. Hence, the terminals representing the free faces are connected to ground in the equivalent circuit. It is convenient to reduce the equivalent circuit of Figure 2.1 to the equivalent form shown in Figure 2.3 to avoid dealing with the simultaneous occurrence of poles in the branch impedance functions at resonance. The modified impedances are:

$$Z_1' = 2Z_{OT}/\tanh(\Gamma_T \ell_T/2) \quad 2.10$$

$$Z_2' = 2Z_{OT} \tanh(\Gamma_T \ell_T/2) \quad 2.11$$

A FORTRAN computer program, MMPL0T, was written for two bonded crystal of arbitrary dimensions, and material types using the equivalent circuit

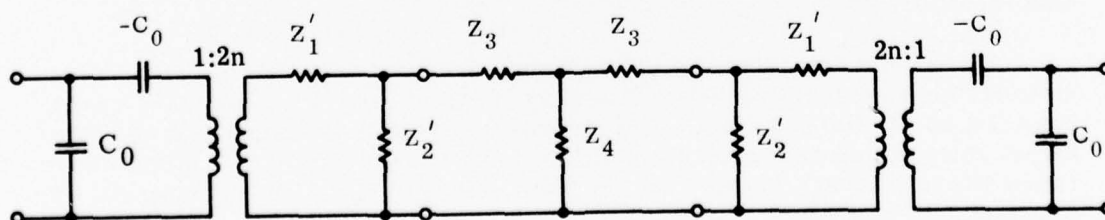


Figure 2.3. Equivalent Circuit of Two Bonded Identical Crystals.

form of Figure 2.3. A listing of MMPLLOT is shown in Table 2.1 along with an equivalent circuit indicating the nomenclature employed.

A multimode filter, consisting of two epoxy bonded identical AT-cut quartz crystals operating in the pure thickness shear modes, was analyzed.

For AT cut quartz the material constants are

$$\rho = 2.65 \times 10^3 \text{ kg/m}^3$$

$$c = 3.32 \times 10^3 \text{ m/s}$$

$$Q > 10000$$

$$\epsilon = 4.58$$

$$k = 0.088$$

For a general epoxy bonding material the constants are:

$$\rho = 1.7 \times 10^3 \text{ kg/m}^3$$

$$c = 2.848 \times 10^3 \text{ m/s}$$

$$Q \approx 10$$

The crystals are one-half wavelength thick at 10 MHz. Electrical mismatch loss (power input to the crystal stack compared to the maximum available power) and Power out/Power in versus frequency are plotted in Figures 2.4 and 2.5, respectively, over the range of 1-50 MHz assuming a .01 mil thick epoxy bond and 50 ohm generator and load impedances. The mismatch curve of Figure 2.4 approaches the type of response expected for two crystals in intimate contact with zero bond thickness. The resonances at 5, 10 and 15 MHz correspond to the 1st, 2nd and 3rd harmonics, where the two-crystal thickness corresponds to one-half wavelength. A null occurs at 20 MHz, where both crystals are one wavelength thick.

An expanded view of the mismatch response at 5 MHz is given in Figure 2.6. The percent bandwidth is very small at .043% when using 50 ohm terminations. The bandwidth is increased to .34% by matching the filter with 1600 ohm terminations.

Since the mechanical Q's of most bond materials are virtually unknown, mismatch loss versus frequency was plotted in Figure 2.7 for assumed epoxy bond Q's of 10, 100 and 1000. The higher Q decreases the bandwidth and gives higher mismatch loss. Inaccurate values of Q will not have much effect in the range 100 to 1000 and higher, but accurate values of Q for low Q materials are important.

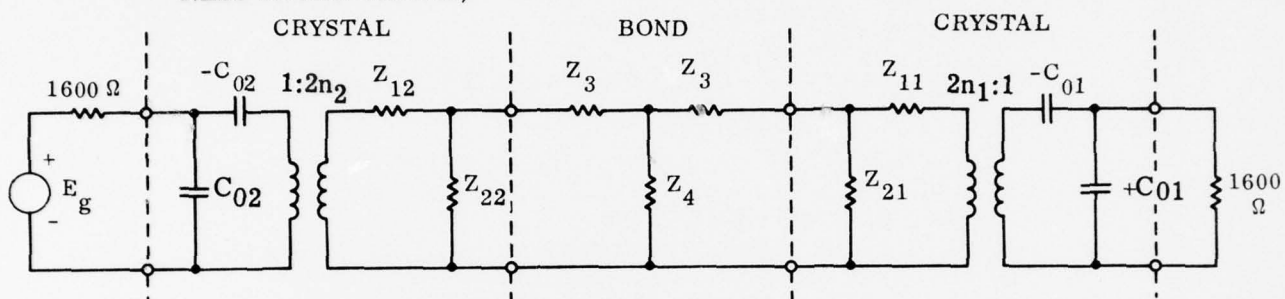
B. BOND PARAMETERS

The bond material, bond thickness, and bond area relative to the transducer active area are the important parameters for controlling the 3 dB bandwidth, insertion loss*, and passband ripple of the bonded crystal filter.

* Insertion loss includes mismatch loss plus the losses in the bond and crystals.

TABLE 2.1

A. EQUIVALENT CIRCUIT FOR TWO BONDED QUARTZ CRYSTALS OF ARBITRARY DIMENSIONS AND BOND MATERIAL TYPE (SINGLE THICKNESS SHEAR MODES)



$$Z_{11} = 2Z_{OT} / \tanh(\Gamma_T \ell_{T1} / 2)$$

$$Z_{21} = 2Z_{OT} \tanh(\Gamma_T \ell_{T1} / 2)$$

$$Z_{12} = 2Z_{OT} / \tanh(\Gamma_T \ell_{T2} / 2)$$

$$Z_{22} = 2Z_{OT} \tanh(\Gamma_T \ell_{T2} / 2)$$

$$Z_3 = Z_{OB} \tanh(\Gamma_B \ell_B / 2)$$

$$Z_4 = Z_{OB} / \sinh(\Gamma_B \ell_B)$$

The 1600-ohm load and generator impedances were chosen to maximize the bandwidth when using the AT-cut quartz transducers.

TABLE 2.1 (CONT'D)

B. COMPUTER PROGRAM, MMPL0T

```

10$:IDENT:865400-276-1265,ALK EP-3 135
20$:LIMITS:10,,,5000
30$:OPTION:F0RTRAN
40$:F0RTY:NLNO,NFORM
50$:LIMITS:03,26K
60$:REMOTE:$$,E1
70$:REMOTE:P*,E1
80     PARAMETER NPTS=1000,NPLTS=1
90     COMPLEX E,AMP,E0,EG
100    COMPLEX Z11,Z21,Z12,Z22,Z3,Z4,ZC1,ZC2,ZI
110    COMPLEX GT1,GT2,GB,GB2
120    COMPLEX SINH,TANHX
130    DIMENSION KI(NPLTS),PMIS(1000),PILOS(1000)
140    DIMENSION IX(10),IY(10),ID(10),BUF(1000)
150    CHARACTER DA*(NPLTS)/"*/
160    DATA IY(1),IY(2)/19,19HINSERTION LOSS (DB)/
170    DATA IX(1),IX(2)/15,15HFREQUENCY (KHZ)/
180    DATA ID(1),ID(2)/23,23HAL KACHELMYER EP-3 135/
190    PI=3.14159
200    YINCH=8.
210    XINCH=10.
230    F0=1.E7
231    IND=1
232    I1=100
233    I2=700
234    N1=4
235    N2=4
236 502  IF(IND.EQ.1)GO TO 501
238    I1=400
239    I2=400
240    N1=1
242    I2=10
243 501  CONTINUE
244    IPLOT=1
245    D=10.E-3
250    AREA=PI*D*D/4.
260C    TRANSDUCER CONSTANTS
270    XK=.088
280    CT=3.32E3
290    DT=2.65E3
300    ZOT=AREA*CT*DT
310    OT=1000.
320C    BOND CONSTANTS
330    CB=1.2E3
340    DB=19.33E3
345    DO 5 IRATIO=I1,I2,100
345    RATIO=FLOAT(IRATIO)
347    IF(IRATIO.EQ.700)RATIO=50.
350    ZOB=AREA*CB*DB/RATIO
360    OB=100.
370C    TRANSDUCER DIMENSIONS
380    TL1=CT/(2.*F0)
390    TL2=TL1

```

TABLE 2.1 (CONT'D)

```

400C      BOND THICKNESS
410      DO 5 N=N1,N2
420      BL=FLOAT(N)*2.54E-6
430C      CAPACITANCE CO
440      EP=4.58*8.85E-12
450      CO1=EP*(1.-XK*XK)*AREA/TL1
460      CO2=EP*(1.-XK*XK)*AREA/TL2
470C      COUPLING COEFFICIENT PHI
480      PH1=XK*SQRT(2.*FO*CO1*ZOT)
490      PH2=XK*SQRT(2.*FO*CO2*ZOT)
500      WRITE(6,11)CO1,CO2,PH1,PH2
510 11    FORMAT(3X,"CO1=",E12.4," CO2=",E12.4," PH1=",F9.5,
515      &" PH2=",F9.5)
520      PH1=2.*PH1
530      PH2=2.*PH2
540      IFS=99500
550      IFE=102000
560      INC=5
565      K=0
570C      LOOP THRU FREQUENCY
580      DO 1 I=IFS,IFE,INC
590      K=K+1
600      FREQ=FLOAT(I)*1.E2
610C      PROPAGATION CONSTANTS
620      BT=2.*PI*FREQ/CT
630      BB=2.*PI*FREQ/CB
640      AT=BT/(2.*QT)
650      AB=BB/(2.*QB)
660      GT1=TL1*CMPLX(AT,BT)/2.
670      GT2=TL2*CMPLX(AT,BT)/2.
680      GB=BL*CMPLX(AB,BB)
690      GB2=GB/2.
700C      COMPLEX IMPEDANCES
710      Z11=2.*ZOT/TANH(GT1)
720      Z21=2.*ZOT*TANH(GT1)
730      Z12=2.*ZOT/TANH(GT2)
740      Z22=2.*ZOT*TANH(GT2)
750      Z3=ZOB*TANH(GB)
760      Z4=ZOB/SINH(GB)
770      ZC1=CMPLX(0.,-1./(2.*PI*FREQ*CO1))
780      ZC2=CMPLX(0.,-1./(2.*PI*FREQ*CO2))
790C      CIRCUIT NETWORK EQUATIONS
800      SCLF=6.25E-4
810      EO=CMPLX(1.,0.)
820      AMP=EO/ZC1+SCLF*CMPLX(1.,0.)
830      E=(EO-AMP*ZC1)*PH1
840      AMP=AMP/PH1
850      H=E*AMP*Z11
860      AMP=AMP+E/Z21
870      H=E*AMP*Z3
880      AMP=AMP+E/Z4
890      E=E*AMP*Z3
900      AMP=AMP+E/Z22
910      H=(E+AMP*Z12)/PH2
920      AMP=AMP*PH2

```

TABLE 2.1 (CONT'D)

```

930      E=E-AMP*ZC2
940      AMP=AMP+E/ZC2
950      EIM=20.*ALOG10(CABS(E))
960      ARGEI=ATAN(AIMAG(E)/REAL(E))
970      ZI=E/AMP
980      EG=AMP*1600.+E
990C     POWER AND VOLTAGE RATIOS
1000     P50=((CABS(EG)/2.)**2)/1600.
1010     P=REAL(E*CONJG(AMP))
1020     PMIS(K)=10.*ALOG10(P/P50)
1030     P0=SCLF
1040     PILOS(K)=10.*ALOG10(P0/P50)
1090 1    CONTINUE
1100     WRITE(6,10)(PMIS(J),J=1,K)
1105     WRITE(6,10)(PILOS(J),J=1,K)
1110     FE=FLOAT(IFE)/10.
1120     DEL=FLOAT(INC)
1130     FS=FLOAT(IFS)/10.
1140     YMAX=0.
1150     YMIN=-24.
1160     DO 100 J=1,K
1185     IF(PILOS(J).LT.YMIN)PILOS(J)=YMIN
1190 100   CONTINUE
1195     IF(IPLOT.NE.1)GO TO 503
1200     CALL CPLOT(0)(PILOS,K,YMIN,YMAX,YINCH,FS,FE,XINCH,IY,IX,ID,BUF)
1202     GO TO 504
1205 503   CALL REPLOT(PILOS)
1206 504   IPLOT=IPLOT+1
1210 5     CONTINUE
1215     IND=IND+1
1216     IF(IND.EQ.2)GO TO 502
1220     CALL PLOT(0.,0.,999)
1230     STOP
1240 10    FORMAT(V)
1250     END
1260     COMPLEX FUNCTION SINH(Z)
1270     COMPLEX Z
1280     SINH=(CEXP(Z)-CEXP(-Z))/2.
1290     RETURN
1300     END
1310     COMPLEX FUNCTION TANHX(Y)
1320     COMPLEX Y
1330     TANHX=(CEXP(Y)-CEXP(-Y))
1340     TANHX=TANHX/(CEXP(Y)+CEXP(-Y))
1350     RETURN
1360     END
1380$*LIBRARY:L1
1390$*EXECUTE
1400$*REMOTE:$$,E1
1410$*REMOTE:P#,E1
1415$*LIMITS:10,,,5000
1420$*PRMFL:L1,R,S,ADEUSERS/ADELIB
1430$*TAPE:17,X17DD,,E7098,,CALCOMP-1265
1440$*ENDJOB

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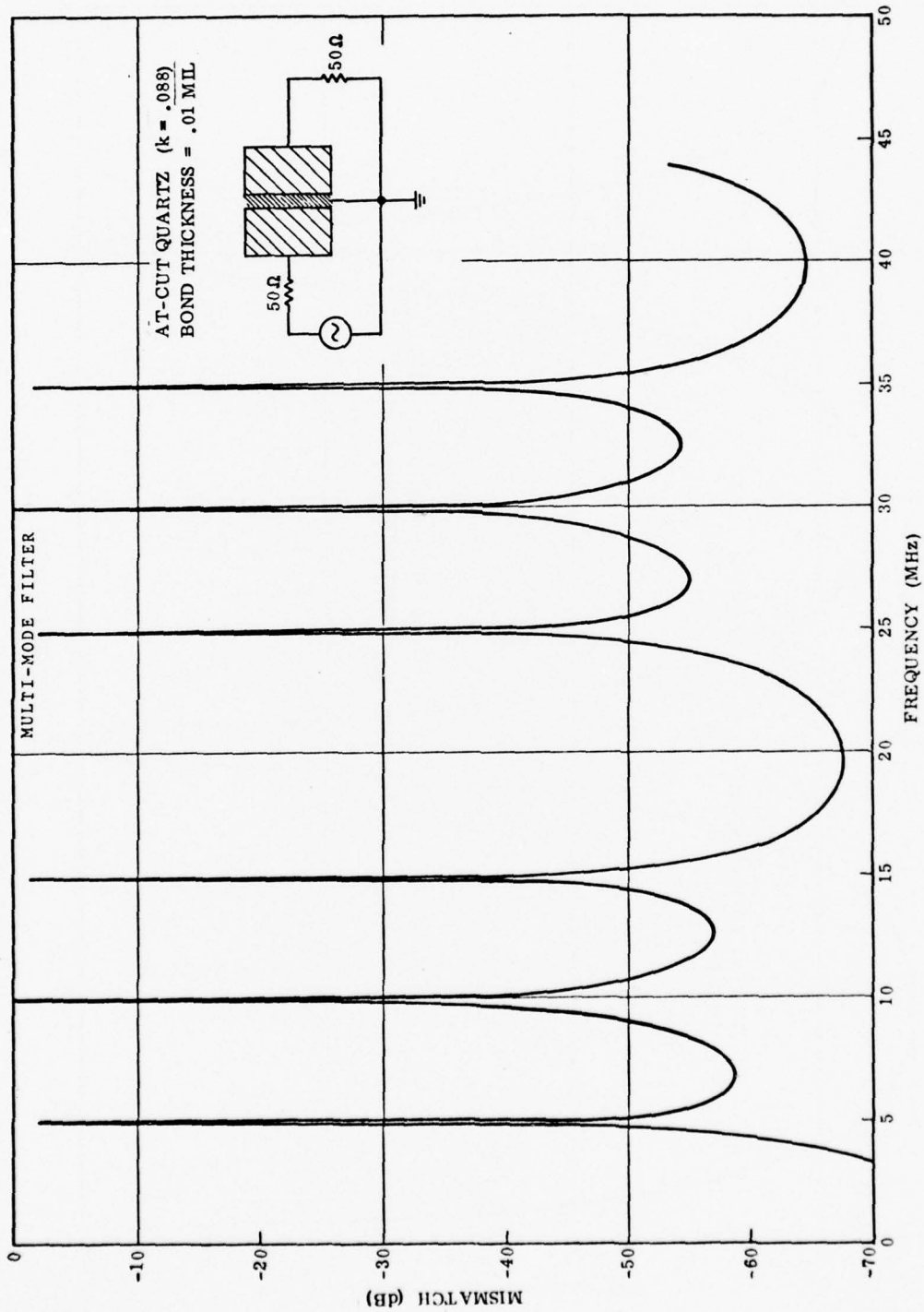


Figure 2.4. Mismatch Versus Frequency.

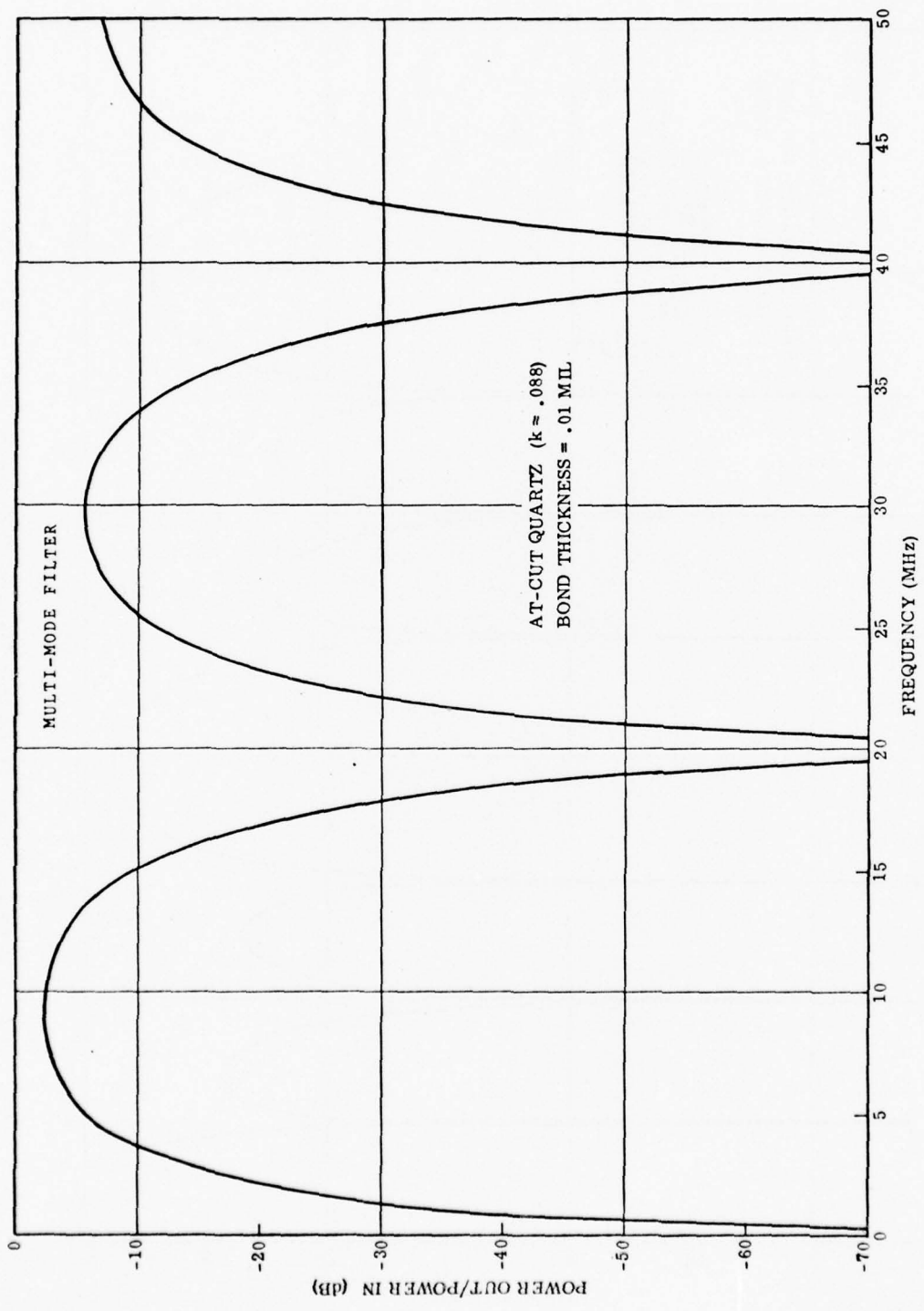


Figure 2.5. Power Out/Power In Versus Frequency.

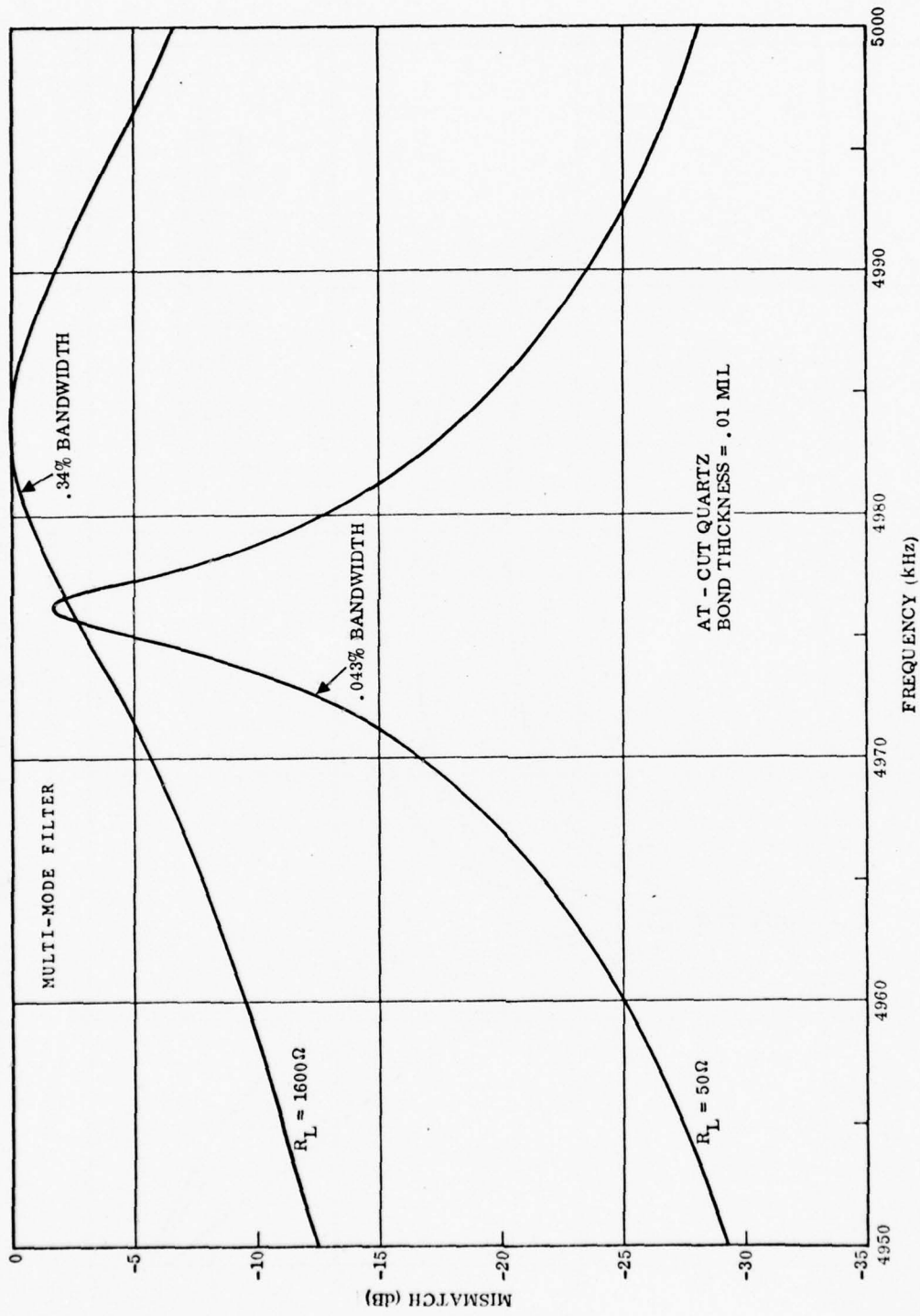


Figure 2.6. Mismatch Versus Frequency.

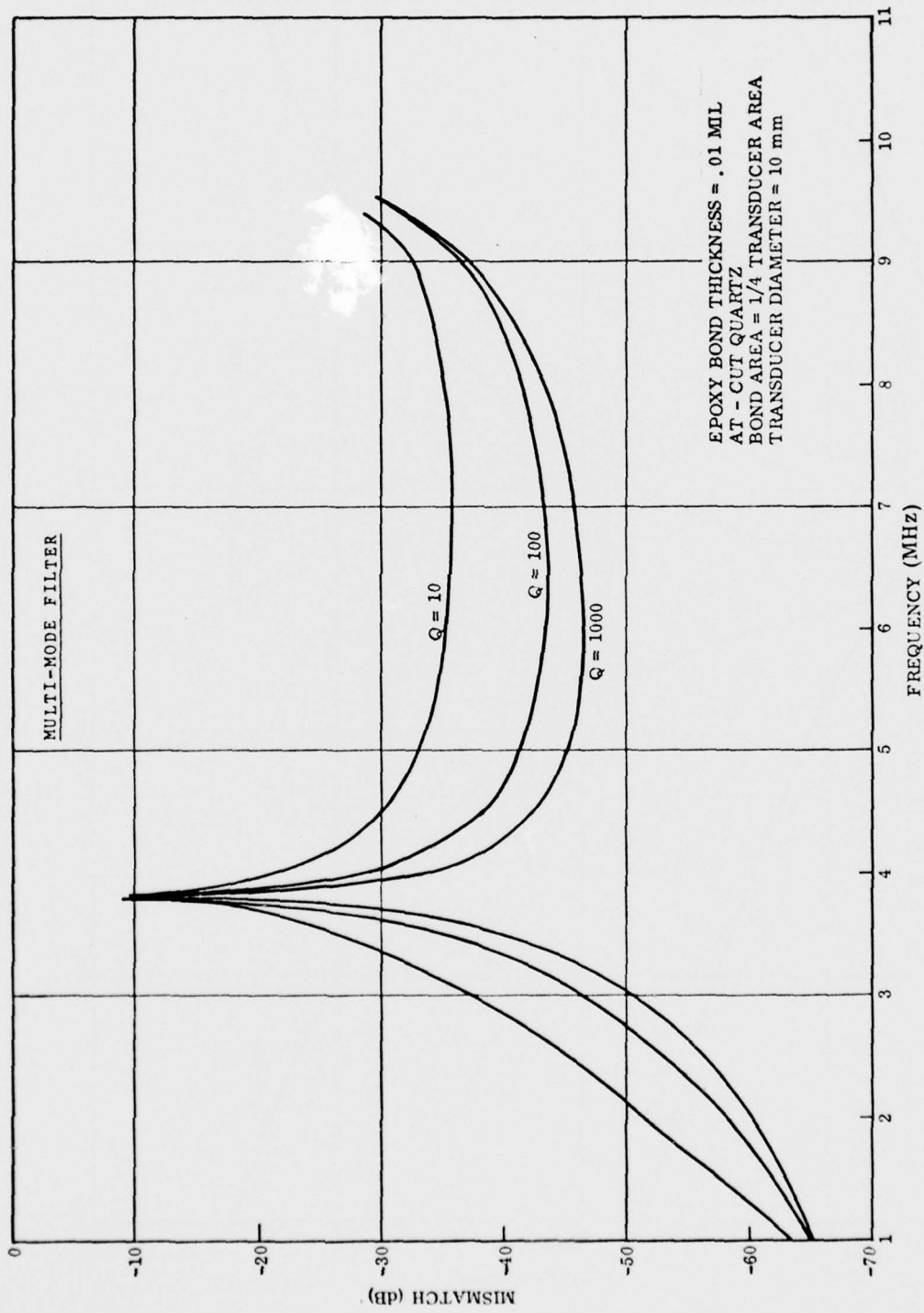


Figure 2.7. Effect of Bond Q On Mismatch Versus Frequency.

For a gold bond the appropriate material constants are:

$$\rho = 19.333 \times 10^3 \text{ kg/m}^3$$

$$c = 1.2 \times 10^3 \text{ m/s}$$

$$Q \geq 100$$

(1) Bond Material and Thickness

Insertion loss versus frequency over the passband at 10 MHz for the two crystal AT-cut quartz filter is plotted in Figures 2.8 and 2.9 using epoxy and gold bonds, respectively, for a series of bond thicknesses from .1 mil to 1 mil in steps of .1 mil. The bond area to active transducer area ratio was held constant at 1/400. The generator and load impedances are 1600 ohms in order to give a maximum bandwidth. Allowing 3 dB passband ripple, the epoxy bond gives a .65% bandwidth while the gold bond gives a .75% bandwidth. This corresponds to a 15% increase in bandwidth due to the better impedance match of the gold bond.

The family of curves clearly indicate the effect of bond thickness on the frequency response. As the bond thickness decreases, the crystals become more coupled, thus separating the peaks and giving a larger passband ripple.

(2) Bond Area to Transducer Area Ratio

Insertion loss versus frequency over the passband at 10 MHz for the two crystal AT-cut quartz filter is plotted in Figures 2.10 and 2.11 for a series of bond area to transducer active area ratios from 1/50 to 1/600. The bond thickness was held constant at .4 mil. The curves indicate that the area ratio plays the same role as that of the bond thickness. As the bond area to transducer area ratio becomes larger, the transducer crystals become more coupled with the same effect of separating the peaks with larger passband ripple.

The bond tends to act as a capacitive coupling ($c = \epsilon A/d$), where A corresponds to the area ratio and d corresponds to the bond thickness. Area ratio and bond thickness can be traded off to yield the desired passband response.

C. TRANSDUCER MATERIAL

AT-cut quartz was chosen for the stacked crystal (multi-mode) filter because of its pure mode characteristics. However, its low coupling coefficient, $k = .088$, limits the amount of bandwidth that can be obtainable.

A new material, berlinite, has the same pure thickness shear mode characteristics of AT-cut quartz but with a higher coupling coefficient of $k = .143$. The velocity of propagation, c , and density, ρ , for berlinite are:

$$c = 2.87 \times 10^3 \text{ m/s}$$

$$\rho = 2.62 \times 10^3 \text{ kg/m}^3$$

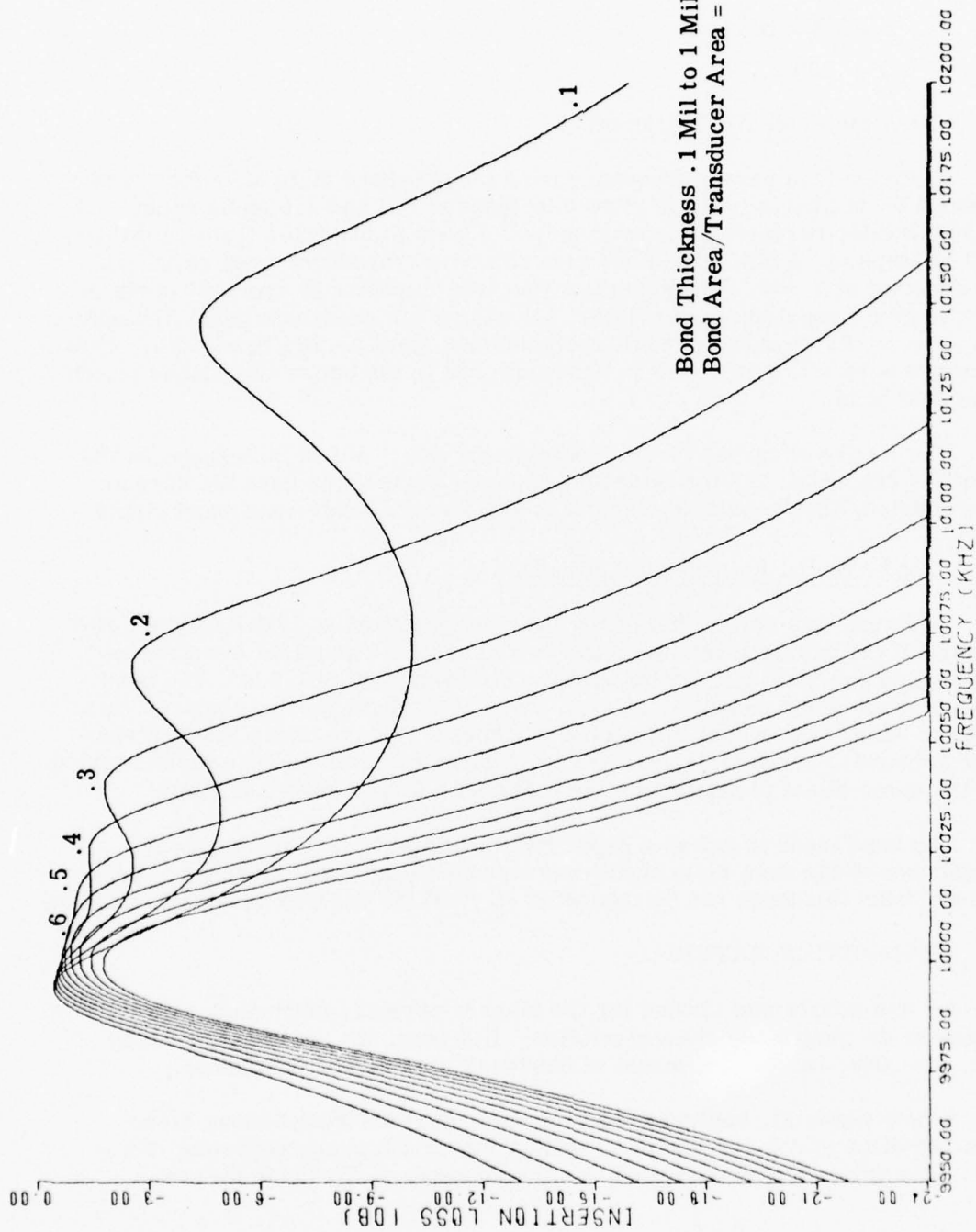


Figure 2.8. Insertion Loss Versus Frequency for Epoxy Bonded Two Crystal AT-Cut Quartz Filter.

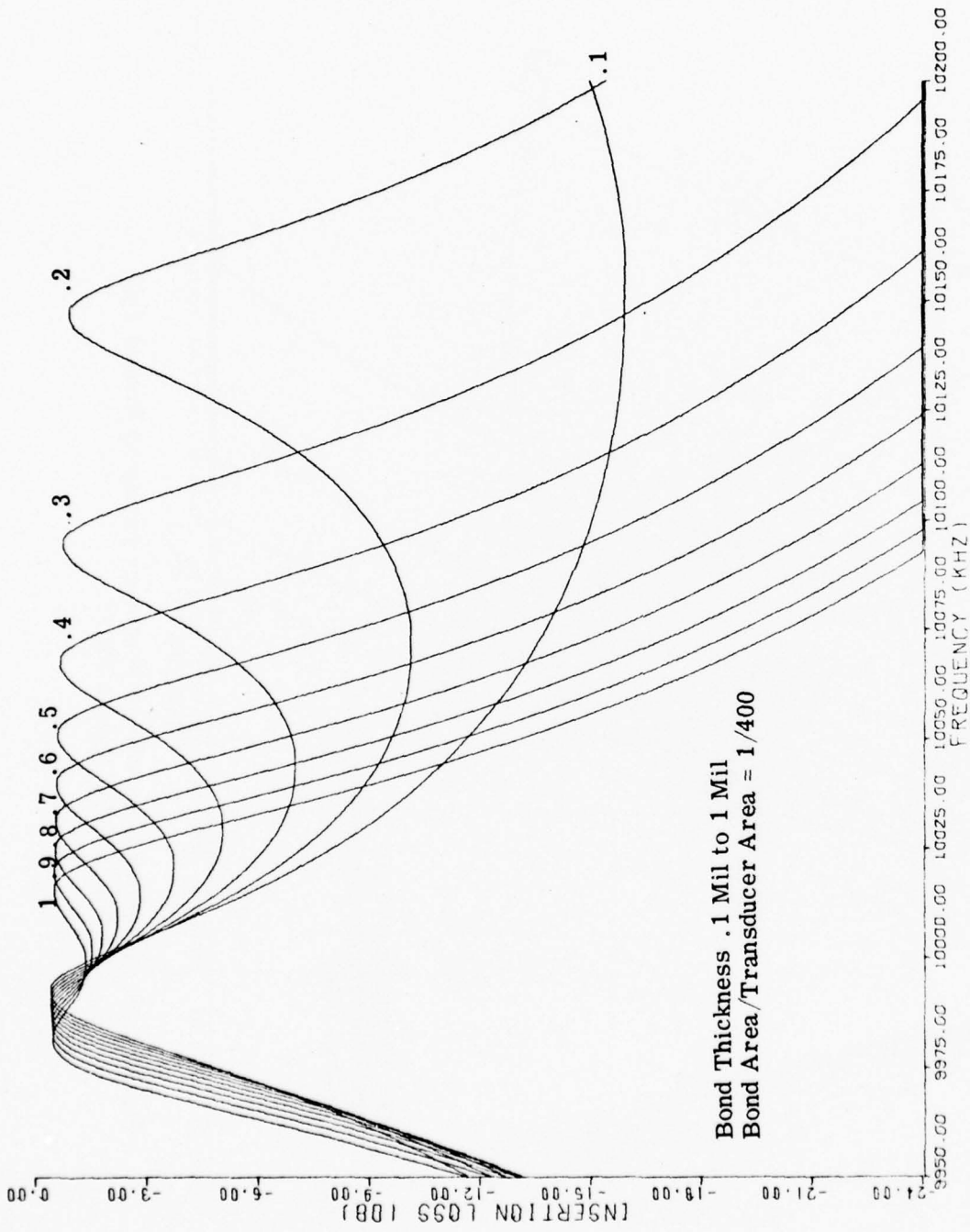


Figure 2.9. Insertion Loss Versus Frequency for Gold Bonded Two Crystal AT-Cut Quartz Filter.

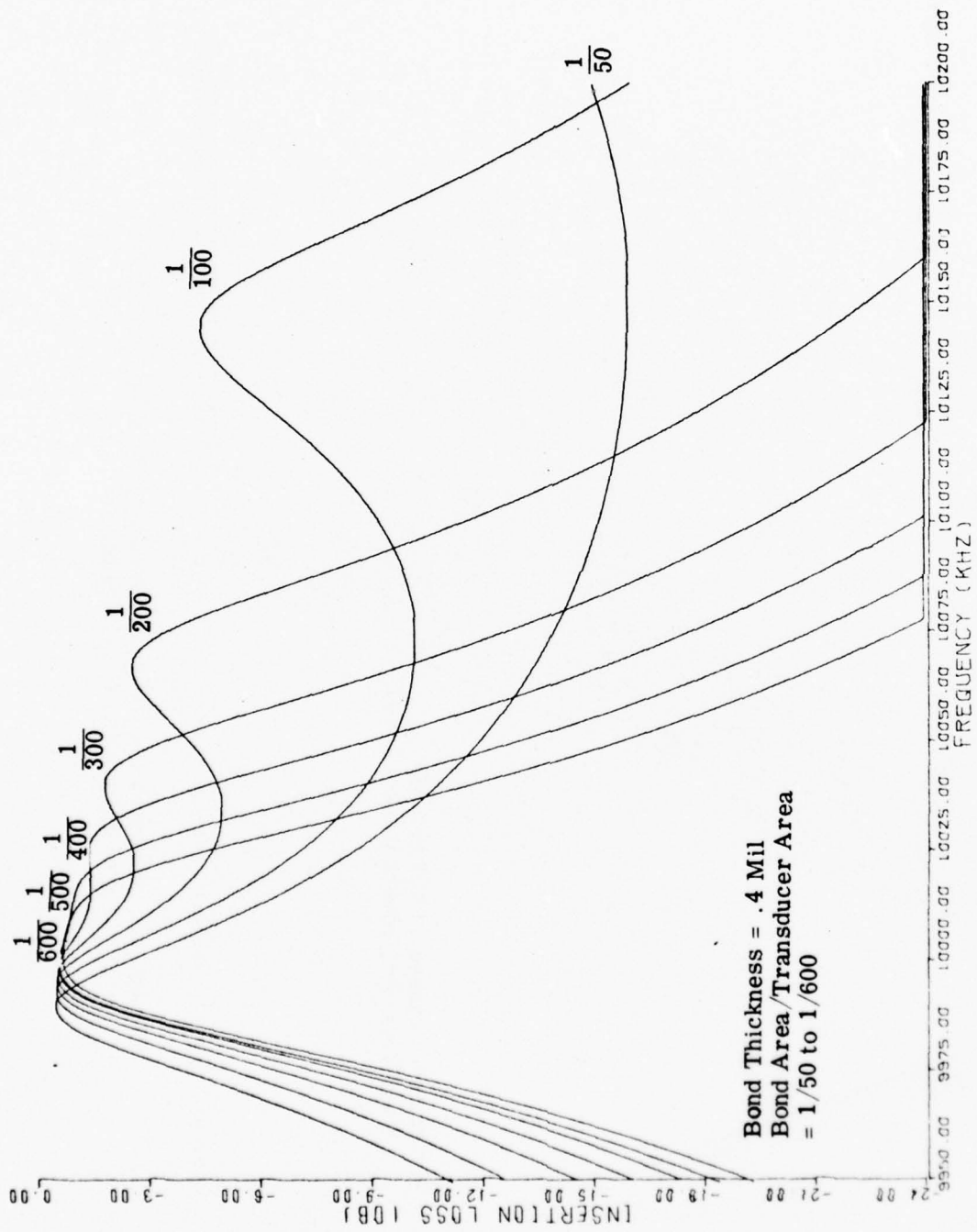


Figure 2.10. Insertion Loss Versus Frequency for Epoxy Bonded Two Crystal AT-Cut Quartz Filter.

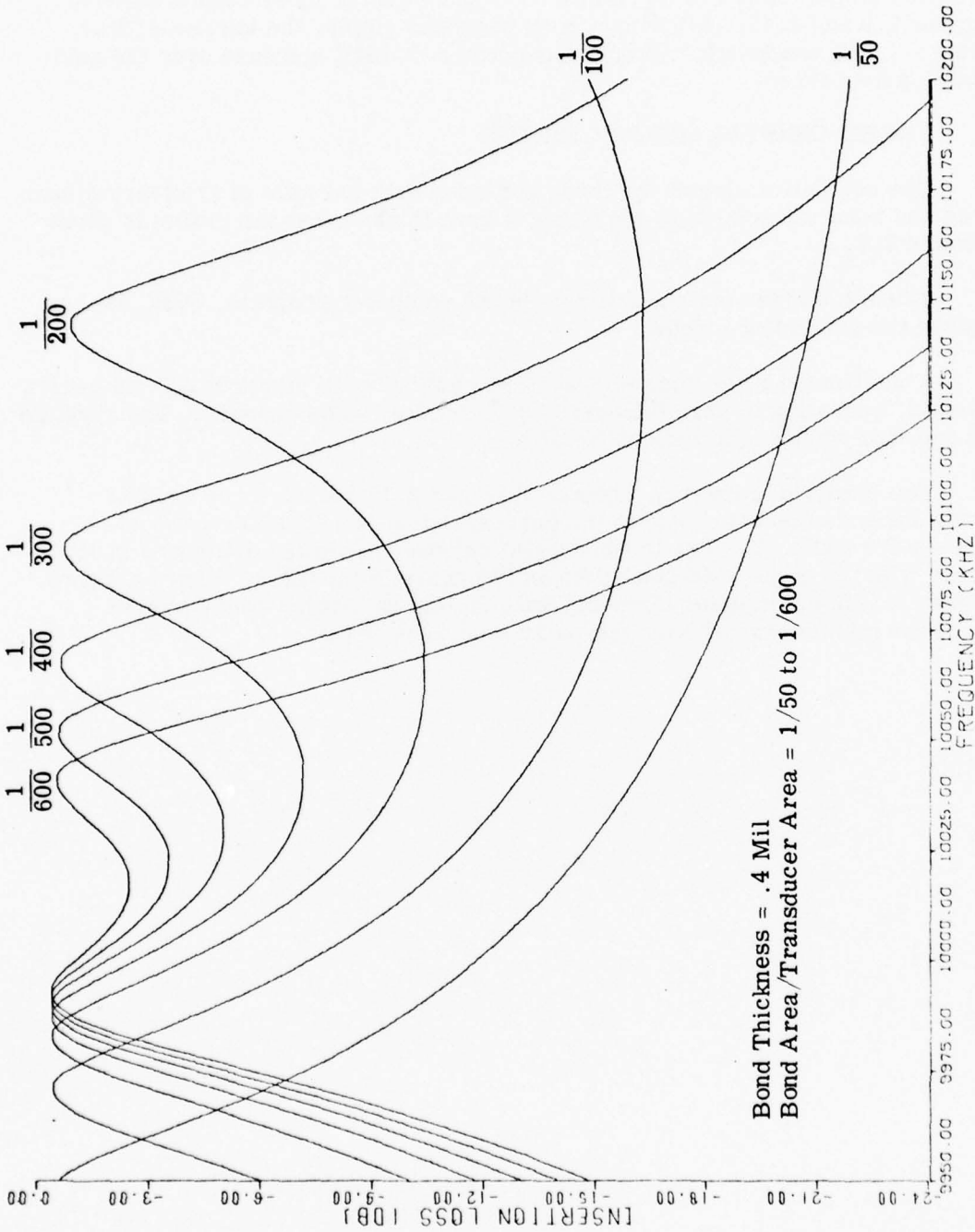


Figure 2.11. Insertion Loss Versus Frequency for Gold Bonded Two Crystal AT-Cut Quartz Filter.

The family of curves in Figures 2.12 and 2.13 for the gold-bonded two crystal berlinite filter can be compared to their quartz filter counterparts in Figures 2.9 and 2.11. Allowing a 3 dB passband ripple, the berlinite filter yields a 1.98% bandwidth. This corresponds to a 164% increase over the gold-bonded quartz filter.

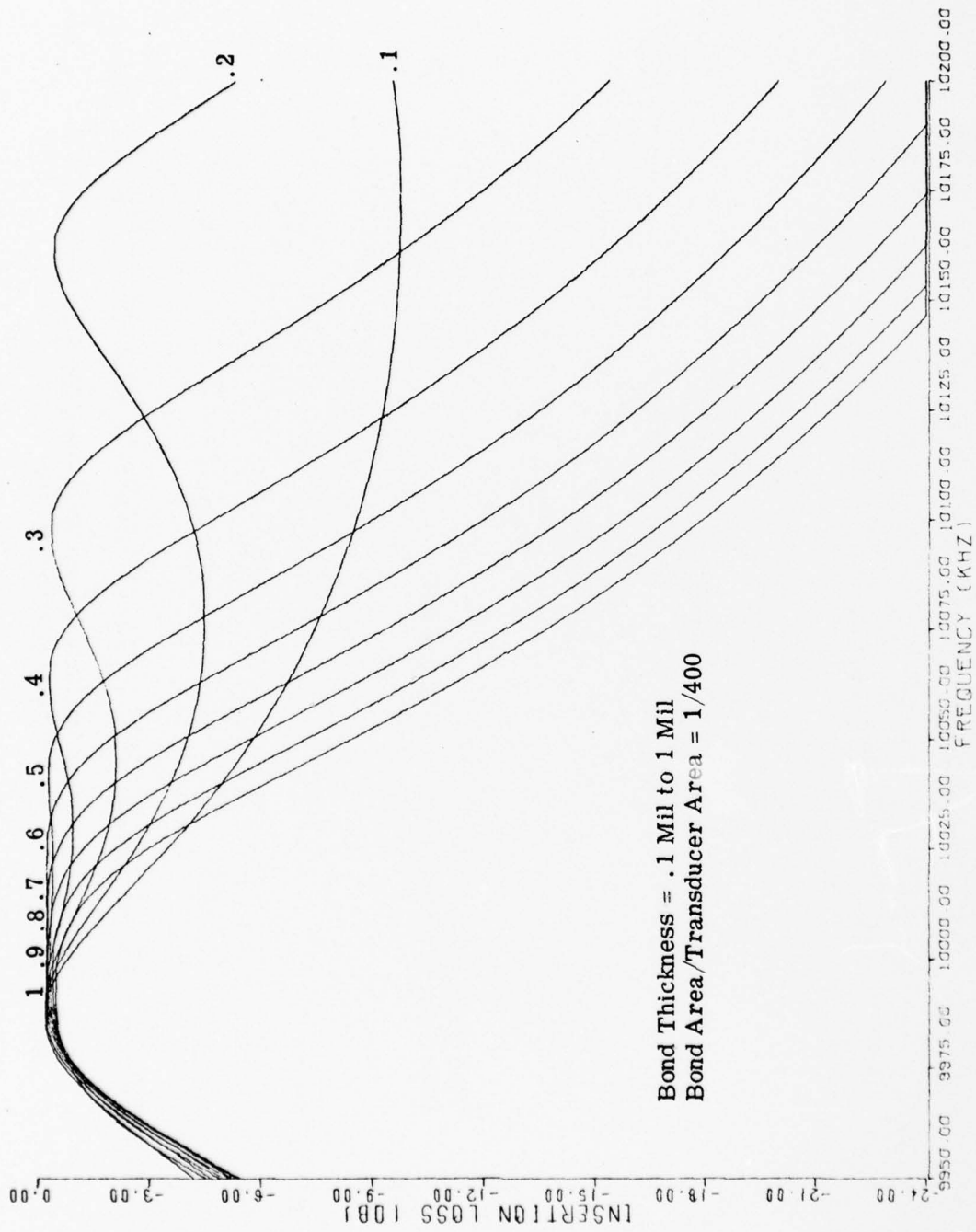
D. THREE-CRYSTAL QUARTZ FILTER

The equivalent circuit for three bonded quartz crystals of arbitrary dimensions and bond material type operating in pure thickness shear modes is given in Table 2.2.

Table 2.2 also contains a listing of the computer program, 3CB, used to analyze the equivalent circuit.

A multimode filter consisting of three gold-bonded identical AT-cut quartz crystals, operating in pure thickness shear modes, was proposed. The crystals are one-half wavelength thick at 10 MHz.

The family of curves in Figures 2.14 and 2.15 for the three crystal quartz filter can be compared to the corresponding two crystal curves of Figures 2.9 and 2.11. The three-crystal curves are shifted downward in frequency with the same 3 dB bandwidth as the two-crystal filter. There appears to be no advantage in using three crystals in this particular configuration where the middle crystal only serves as a coupling layer.



Bond Thickness = .1 Mil to 1 Mil
 Bond Area/Transducer Area = 1/400

Figure 2.12. Insertion Loss Versus Frequency for Gold Bonded Two Crystal Berlinite Filter.

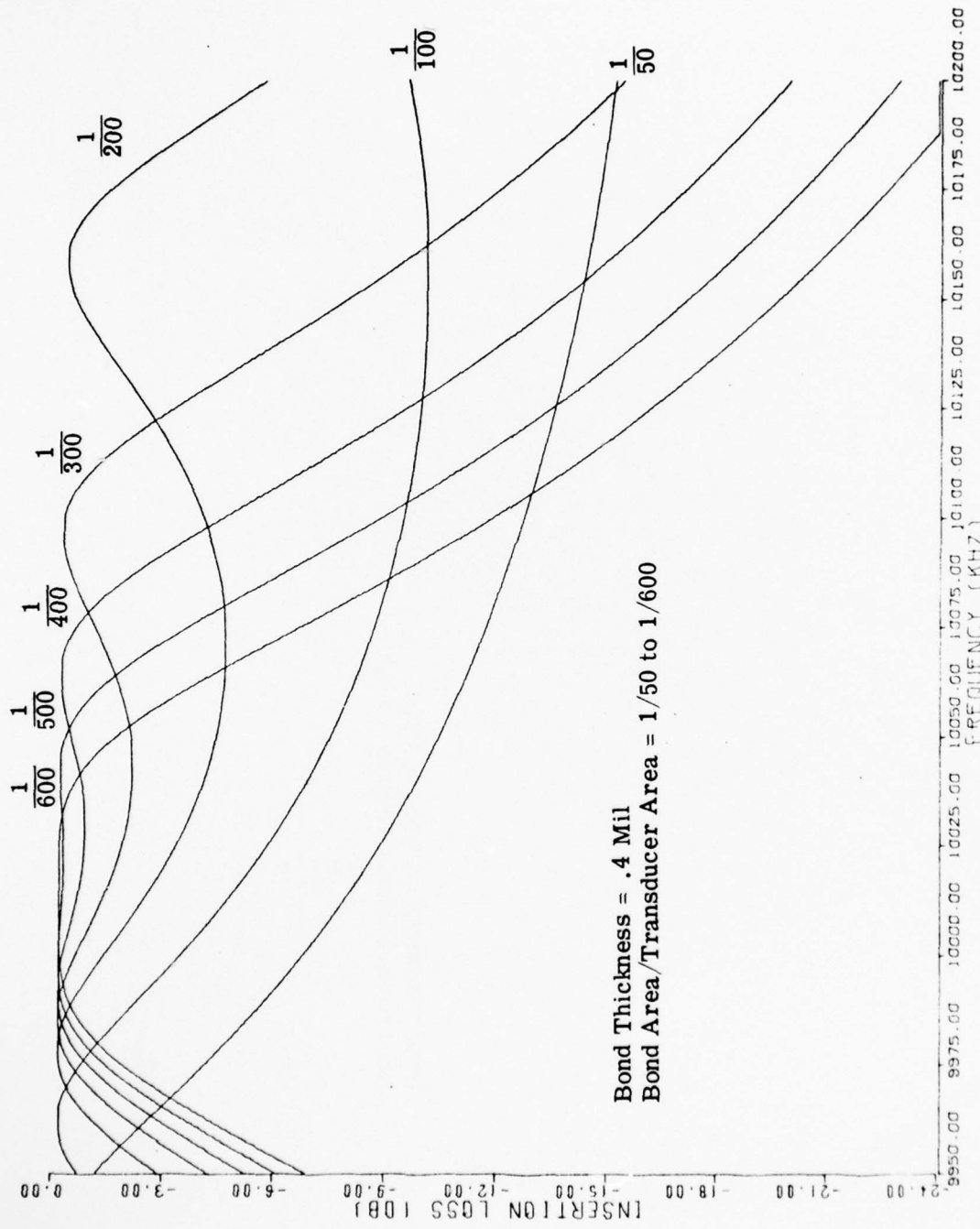
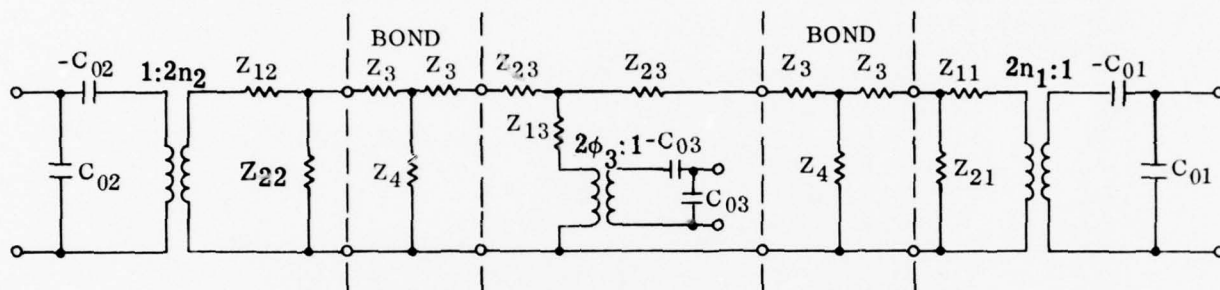


Figure 2.13. Insertion Loss Versus Frequency for Gold Bonded Two Crystal Berlinitite Filter.

TABLE 2.2

A. EQUIVALENT CIRCUIT FOR THREE BONDED QUARTZ CRYSTALS OF ARBITRARY DIMENSIONS AND BOND MATERIAL TYPE (SINGLE THICKNESS SHEAR MODES)



$$Z_{11} = 2Z_{OT} / \tanh(\Gamma_T \ell_{T1} / 2)$$

$$Z_{21} = 2Z_{OT} \tanh(\Gamma_T \ell_{T1} / 2)$$

$$Z_{12} = 2Z_{OT} / \tanh(\Gamma_T \ell_{T2} / 2)$$

$$Z_{22} = 2Z_{OT} \tanh(\Gamma_T \ell_{T2} / 2)$$

$$Z_{13} = Z_{OT} / \sinh(\Gamma_T \ell_{T3})$$

$$Z_{23} = Z_{OT} \tanh(\Gamma_T \ell_{T3} / 2)$$

$$Z_3 = Z_{OB} \tanh(\Gamma_B \ell_B / 2)$$

$$Z_4 = Z_{OB} / \sinh(\Gamma_B \ell_B)$$

TABLE 2.2 (CONT'D)

B. COMPUTER PROGRAM, 3CB

```

10$: IDENT: 888400-276-1265, ALK EP-3 135
20$: LIMITS: 10, , , 5000
30$: OPTION: FORTRAN
40$: FORTY: NLNO, NFORM
50$: LIMITS: 03, 26K
60$: REMOTE: $$, E1
70$: REMOTE: P*, E1
80     PARAMETER NPPTS=1000, NPLTS=1
90     COMPLEX E, AMP, E0, EG
100    COMPLEX Z11, Z21, Z12, Z22, Z3, Z4, ZC1, ZC2, ZI
105    COMPLEX Z13, Z23
110    COMPLEX GT1, GT2, GB, GB2
120    COMPLEX SINH, TANHX
130    DIMENSION K1(NPLTS), PMIS(1000), PILOS(1000)
140    DIMENSION IX(10), IY(10), ID(10), BUF(1000)
150    CHARACTER DA*(NPLTS)/* */
160    DATA IY(1), IY(2)/19, 19HINSERTION LOSS (DB)/
170    DATA IX(1), IX(2)/15, 15HFREQUENCY (KHZ)/
180    DATA ID(1), ID(2)/23, 23HAL KACHELMYER EP-3 135/
190    PI=3.14159
200    YINCH=8.
210    XINCH=10.
220    ISHORT=1
230    FO=1.E7
231    IND=1
232    I1=100
233    I2=700
234    N1=4
235    N2=4
236 502  IF(IND.EQ.1)GO TO 501
238    I1=400
239    I2=400
240    N1=1
242    N2=10
243 501  CONTINUE
244    IPLOT=1
245    D=10.E-3
250    AREA=PI*D*D/4.
260C    TRANSDUCER CONSTANTS
270    XK=.088
280    CT=3.32E3
290    DT=2.65E3
300    ZOT=AREA*CT*DT
310    QT=10000.
320C    BOND CONSTANTS
330    CB=1.2E3
340    DB=19.333E3
345    DO 5 IRATIO=11, 12, 100
346    RATIO=FLOAT(IRATIO)
347    IF(IRATIO.EQ.700)RATIO=50.
350    ZOB=AREA*CB*DB/RATIO
360    QB=100.

```

TABLE 2.2 (CONT'D)

```

370C      TRANSDUCER DIMENSIONS
380      TL1=CT/(2.*F0)
390      TL2=TL1
395      TL3=TL1
400C      BOND THICKNESS
410      DO 5 N=N1,N2
420      BL=FLOAT(N)*2.54E-6
430C      CAPACITANCE CO
440      EP=4.58*8.85E-12
450      CO1=EP*(1.-XK*XK)*AREA/TL1
460      CO2=EP*(1.-XK*XK)*AREA/TL2
465      CO3=EP*(1.-XK*XK)*AREA/TL3
470C      COUPLING COEFFICIENT PHI
480      PH1=XK*SQRT(2.*F0*CO1*ZOT)
490      PH2=XK*SQRT(2.*F0*CO2*ZOT)
495      PH3=XK*SQRT(2.*F0*CO3*ZOT)
500      WRITE(6,11)CO1,CO2,PH1,PH2
510 11    FORMAT(3X,"CO1=",E12.4," CO2=",E12.4," PH1=",F9.5,
515      &" PH2=",F9.5)
520      PH1=2.*PH1
530      PH2=2.*PH2
540      IFS=99500
550      IFE=102000
560      INC=5
565      K=0
570C      LOOP THRU FREQUENCY
580      DO 1 I=IFS,IFE,INC
590      K=K+1
600      FREQ=FLOAT(I)*1.E2
610C      PROPAGATION CONSTANTS
620      BT=2.*PI*FREQ/CT
630      BB=2.*PI*FREQ/CB
640      AT=BT/(2.*QT)
650      AB=BB/(2.*QB)
660      GT1=TL1*CMPLX(AT,BT)/2.
670      GT2=TL2*CMPLX(AT,BT)/2.
675      GT3=TL3*CMPLX(AT,BT)/2.
680      GB=BL*CMPLX(AB,BB)
690      GB2=GB/2.
700C      COMPLEX IMPEDANCES
710      Z11=2.*ZOT/1ANH(GT1)
720      Z21=2.*ZOT*1ANH(GT1)
730      Z12=2.*ZOT/1ANH(GT2)
740      Z22=2.*ZOT*1ANH(GT2)
744      Z13=ZOT/SINH(2.*GT3)
745      IF(ISHORT.EQ.1)Z13=Z13+CMPLX(0.,PH3*PH3/(2.*PI*FREQ*CO3))
746      Z23=ZOT*1ANH(GT3)
750      Z3=ZOB*1ANH(GB2)
760      Z4=ZOB/SINH(GB)
770      ZC1=CMPLX(0.,-1./(2.*PI*FREQ*CO1))
780      ZC2=CMPLX(0.,-1./(2.*PI*FREQ*CO2))

```

TABLE 2.2 (CONT'D)

```

790C      CIRCUIT NETWORK EQUATIONS
800      SCLF=6.25E-4
810      EO=CMPLX(1.,0.)
820      AMP=EO/ZC1+SCLF*CMPLX(1.,0.)
830      E=(EO-AMP*ZC1)*PH1
840      AMP=AMP/PH1
850      E=E+AMP*Z11
860      AMP=AMP+E/Z21
870      E=E+AMP*Z3
880      AMP=AMP+E/Z4
882      E=E+AMP*(Z3+Z23)
884      AMP=AMP+E/Z13
886      E=E+AMP*(Z3+Z23)
888      AMP=AMP+E/Z4
890      E=E+AMP*Z3
900      AMP=AMP+E/Z22
910      E=(E+AMP*Z12)/PH2
920      AMP=AMP*PH2
930      E=E-AMP*ZC2
940      AMP=AMP+E/ZC2
950      EIM=20.*ALOG10(CABS(E))
960      ARGEL=ATAN(AIMAG(E)/REAL(E))
970      ZI=E/AMP
980      EG=AMP*1600.+E
990C      POWER AND VOLTAGE RATIOS
1000     P50=((CABS(EG)/2.)**2)/1600.
1010     P=REAL(E*CONJG(AMP))
1020     PMIS(K)=10.*ALOG10(P/P50)
1030     P0=SCLF
1040     PILOS(K)=10.*ALOG10(P0/P50)
1090 1    CONTINUE
1100     WRITE(6,10)(PMIS(J),J=1,K)
1105     WRITE(6,10)(PILOS(J),J=1,K)
1110     FE=FLOAT(IFE)/10.
1120     DEL=FLOAT(INC)
1130     FS=FLOAT(IFS)/10.
1140     YMAX=0.
1150     YMIN=-24.
1160     DO 100 J=1,K
1185     IF(PILOS(J).LT.YMIN)PILOS(J)=YMIN
1190 100  CONTINUE
1195     IF(IPL0T.NE.1)GO TO 503
1200     CALL CPLOT 0(PILOS,K,YMIN,YMAX,YINCH,FS,FE,XINCH,IY,IX,ID,BUF)
1202     GO TO 504
1205 503  CALL REPLOT(PILOS)
1206 504  IPL0T=IPL0T+1
1210 5    CONTINUE
1215     IND=IND+1
1216     IF(IND.EQ.2)GO TO 502
1220     CALL PLOT(0.,0.,999)
1230     STOP
1240 10   FORMAT(V)
1250     END

```

TABLE 2.2 (CONT'D)

```
1260      COMPLEX FUNCTION SINH(Z)
1270      COMPLEX Z
1280      SINH=(CEXP(Z)-CEXP(-Z))/2.
1290      RETURN
1300      END
1310      COMPLEX FUNCTION TANHX(Y)
1320      COMPLEX Y
1330      TANHX=CEXP(Y)-CEXP(-Y)
1340      TANHX=1TANHX/(CEXP(Y)+CEXP(-Y))
1350      RETURN
1360      END
1380$:LIBRARY:L1
1390$:EXECUTE
1400$:REMOTE:$$,E1
1410$:REMOTE:P*,E1
1415$:LIMITS:10,,,5000
1420$:PRMFL:L1,R,S,ADeUSERS/ADeLIB
1430$:TAPE:17,X17DD,,E7197,,CALCOMP-1265
1440$:ENDJOB
```

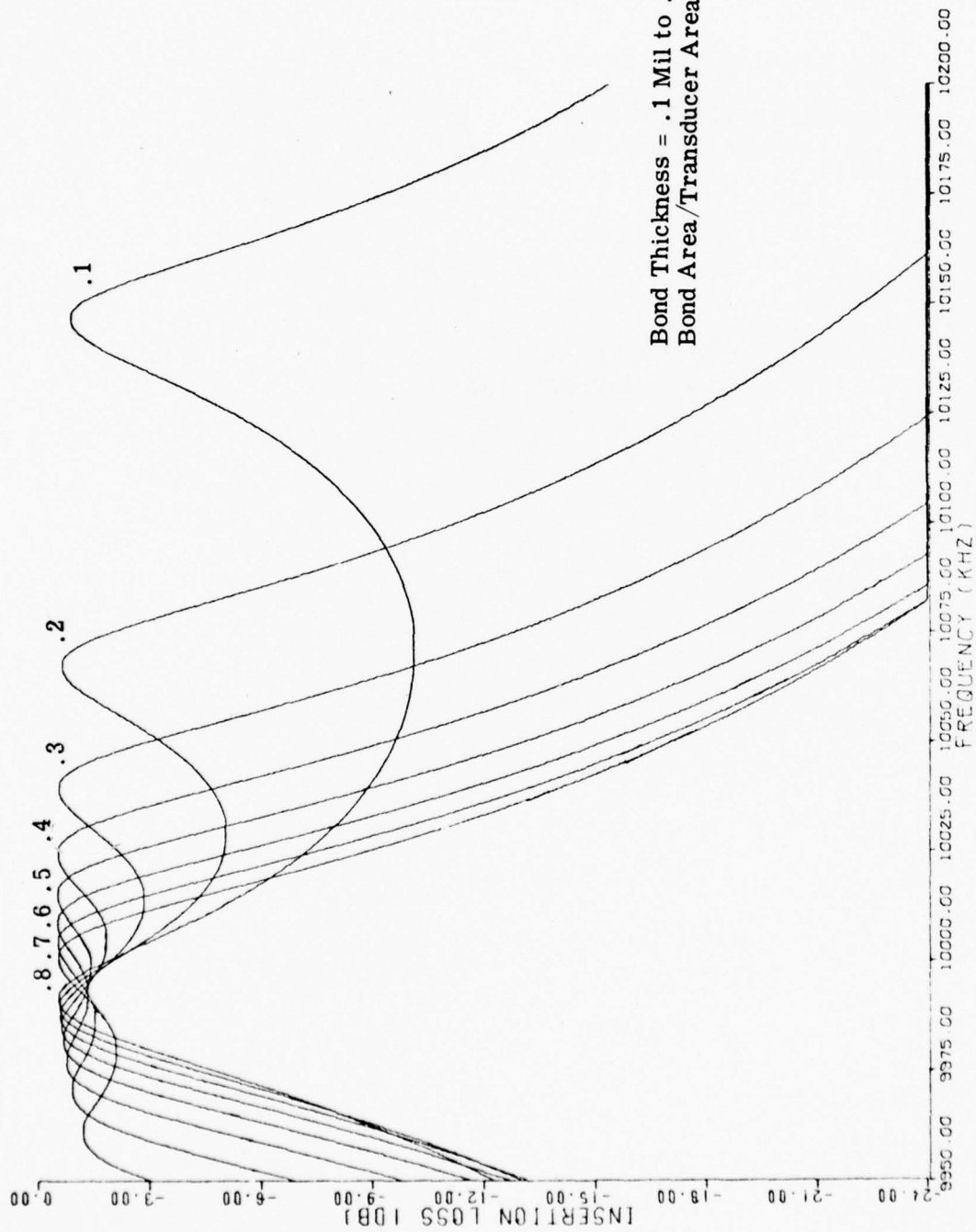


Figure 2.14. Insertion Loss Versus Frequency for Gold Bonded Three Crystal AT-Cut Quartz Filter.

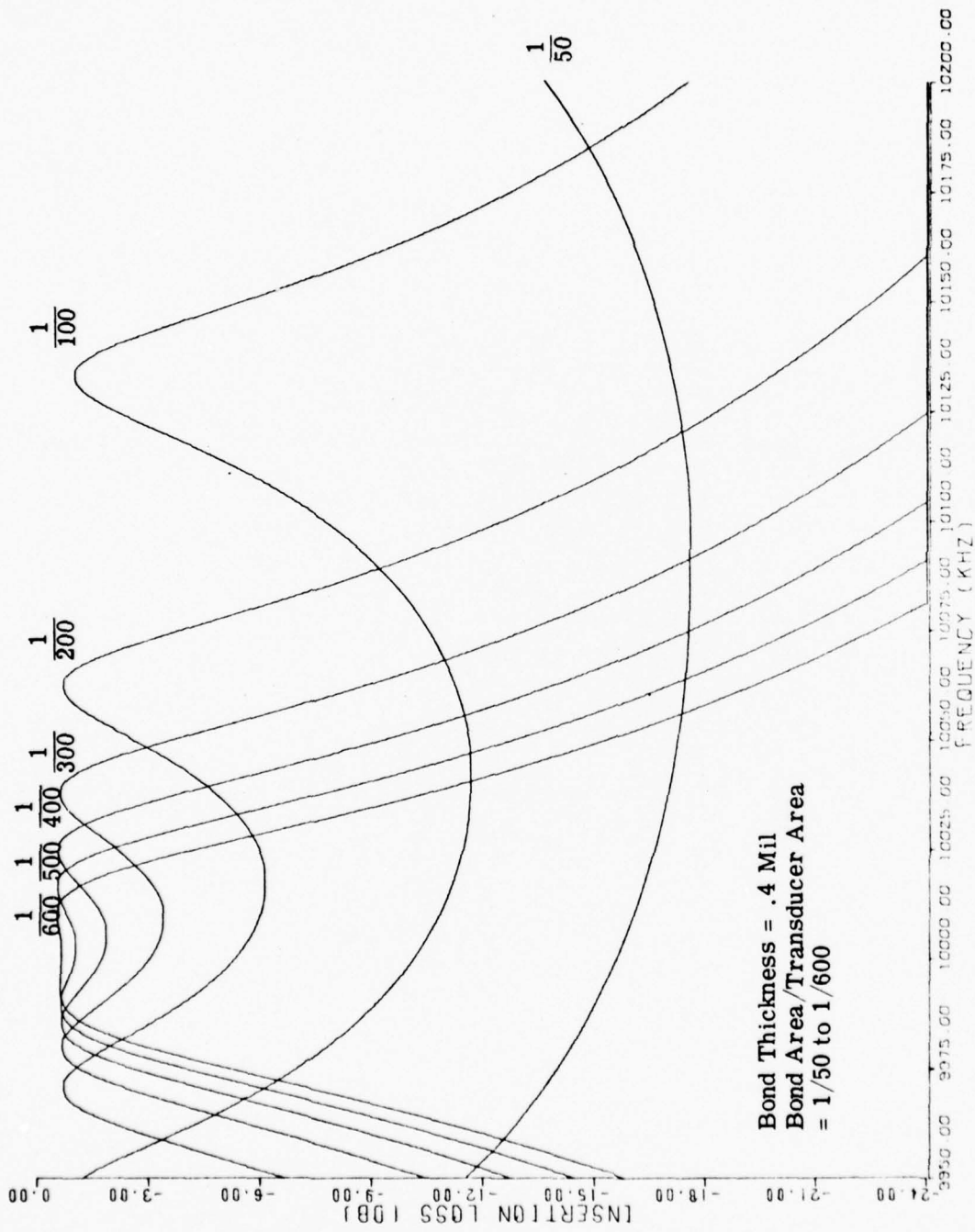


Figure 2.15. Insertion Loss Versus Frequency for Gold Bonded Three Crystal AT-Cut Quartz Filter.

3. THEORETICAL INVESTIGATIONS OF MULTIMODE STACKED FILTERS

The multimode stacked filter investigations in this section will make use of the transmission line model of a piezoelectric plate as developed by Dr. Ballato [1] instead of the Mason Equivalent circuit used previously.

A thorough derivation of this model is given in reference [1] ; however a brief discussion of its derivation will be included here as an aid to understanding the programs and results to be presented.

An attempt will also be made to present the ideas on a fairly elementary level so that the programs can be interpreted by those who are unfamiliar with crystal physics and tensor notation.

This derivation will be split into sections; the computer programs developed during the course of this contract will be presented at those points in the derivation where they are used.

In this presentation the following symbols and conventions will be used:

T_{ij} = stress (force per unit area)

(T_{ij} = component of force in the x_i direction transmitted across the face of the plate perpendicular to the x_j direction)

u_j = mechanical displacement

S_{ij} = strain

D_i = electric displacement

E_i = electric field

ρ = mass density

ϕ = electric potential

c_{ijkl}^E = elastic stiffness at constant electric field

e_{kij} = piezoelectric stress constant

ϵ_{ik}^S = dielectric permittivity at constant strain

Einstein summation convention (when a letter subscript occurs twice in the same term, summation with respect to that subscript is carried out)

$$a_i = \sum_{j=1}^3 T_{ij} q_j = T_{ij} q_j$$

An index (subscript) preceded by a comma denotes differentiation with respect to the space coordinate represented by that index

$$u_{i,j} = \partial u_i / \partial x_j$$

A dot above a variable denotes differentiation with respect to time

$$\dot{u}_j = \frac{\partial u_j}{\partial t}, \quad \ddot{u}_j = \frac{\partial^2 u_j}{\partial t^2}$$

All Latin indexes have the range 1, 2 and 3.

A. DEVELOPMENT OF THE SECULAR EQUATION AND PROGRAM CROT

The piezoelectric plate under consideration is assumed to be represented in an orthogonal right hand coordinate system x_i with one of the coordinates in the plate thickness direction.

The pertinent set of equations governing the behavior of this plate are: [2]

the stress equations of motion derived from Newton's second law

$$T_{ij,i} = \rho \ddot{u}_j, \quad (1)$$

the equations defining mechanical strain,

$$S_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (2)$$

the charge equation of electrostatics (the divergence relation from Maxwell's equations in the absence of free charge),

$$D_{i,i} = 0, \quad (3)$$

the electric field-electric potential relation for a quasi-static case,

$$E_k = -\phi_{,k} \quad (4)$$

and the linear piezoelectric constitutive relations characterizing the medium,

$$T_{ij} = c_{ijkl}^E S_{kl} - e_{kij} E_k \quad (5)$$

and

$$D_i = e_{ikl} S_{kl} + \epsilon_{ik}^S E_k \quad (6)$$

Equation (5) is Hooke's law, extended to take into account piezoelectricity; Eq. (6) is the constitutive relation for a dielectric medium extended to include piezoelectricity.

Substituting Eqs. (2) and (4) into Eqs. (5) and (6) yields

$$T_{ij} = c_{ijlk}^E u_{l,k} + e_{kij} \phi_{,k} \quad (7)$$

$$D_i = e_{ilk} u_{l,k} - \epsilon_{ik}^S \phi_{,k} \quad (8)$$

where use has been made of the following symmetry relations among the material constants [3], [4]:

$$\begin{aligned} c_{ijkl} &= c_{ijlk} = c_{jikl} = c_{klij} \\ e_{ijk} &= e_{ikj} \\ \epsilon_{ij} &= \epsilon_{ji} \end{aligned} \quad (9)$$

Equations (7) and (8) can be substituted in Eqs. (1) and (3) to yield

$$T_{ij,i} = c_{ijkl}^E u_{l,ki} + e_{kij} \phi_{,ki} = \rho \ddot{u}_j \quad (10)$$

$$D_{i,i} = e_{ilk} u_{l,ki} - \epsilon_{ik}^S \phi_{,ki} = 0 \quad (11)$$

Assuming a wave propagating in the i^{th} direction (the plate thickness direction) with electric potential applied in the same direction, and that there is no dependence on the other two directions, $q_{,j} = q_{,k} = 0$. Equation (11) can be solved for ϕ ;

$$\phi_{,ii} = \frac{e_{ili}}{\epsilon_{ii}^S} u_{l,ii} \quad (12)$$

This equation can, in turn, be substituted into Eq. (10) to yield

$$\left\{ c_{ijli}^E + e_{ijj} \left(\frac{e_{ili}}{\epsilon_{ii}^S} \right) \right\} u_{l,ii} = \rho \ddot{u}_j \quad (13)$$

It is convenient to write this equation as

$$\bar{c}_{ijki}^E u_{k,ii} = \rho \ddot{u}_j, \quad (14)$$

where

$$\bar{c}_{ijki}^E = c_{ijli}^E + e_{ij} \left(\frac{e_{ili}}{S} \right) \frac{1}{\epsilon_{ii}} \quad (15)$$

are the "piezoelectrically stiffened" elastic stiffnesses at constant normal electric displacement and constant tangential electric field.

If u_j takes the standard time factor form, $e^{j\omega t}$, and is only a function of the coordinate, x_i , in the thickness direction, then

$$u_j = u_j(x_i; t) = u_j(x_i) e^{j\omega t}, \quad (16)$$

and Eq. (14) becomes

$$\bar{c}_{ijki}^E u_{k(x_i),ii} = -\rho \omega^2 u_j(x_i) \quad (17)$$

or

$$\bar{c}_{ijki}^E u_{k(x_i),ii} + \rho \omega^2 u_j(x_i) = 0.$$

If it is further assumed that $u_k(x_i)$ represents plane waves traveling in the x_i direction with wave number, η , so that

$$u_j(x_i) = u_j(\hat{x}_j) e^{\pm j\eta x_i}, \quad (18)$$

where \hat{x}_j indicates only that the direction of u_j is along the x_j direction, the equations in (17) are

$$\bar{c}_{ijki}^E (-\eta^2 u_k(\hat{x}_k)) + \rho \omega^2 u_j(\hat{x}_j) = 0. \quad (19)$$

It is convenient at this point to define

$$c = \frac{\rho \omega^2}{\eta^2}; \quad (20)$$

then Eq. (19) becomes Eq. (21) where the direction indicator, \hat{x}_k , has been dropped since it is now known that u_k lies in the x_k direction and that i must take on the value corresponding to the propagation direction x_i ,

$$\bar{c}_{ijki}^E u_k - c u_j = 0. \quad (21)$$

This expression can be written in a clearer form by introducing the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} . \quad (22)$$

Then Eq. (21) becomes

$$(\bar{c}_{ijki}^E - c\delta_{jk}) u_k = 0 . \quad (23)$$

Cramer's rule, applied to this set of equations for the u_k displacement proportionality constants, shows that non-trivial solutions can exist only if

$$\left| \bar{c}_{ijki}^E - c\delta_{jk} \right| = 0 . \quad (24)$$

Equation (24) is commonly referred to as the characteristic or secular equation for the allowed modes of vibration.

The whole purpose of this exercise is to show that the equations and results to be developed later are not dependent on any particular choice of the coordinate lying in the thickness direction of the piezoelectric plate. The form of the equations remain the same; only the constants change in value.

From now on only plates with x_3 lying in the thickness direction will be considered. For this case the equations in Eq. (23) become:

$$\begin{aligned} (\bar{c}_{3113}^E - c) u_1 + \bar{c}_{3123}^E u_2 + \bar{c}_{3133}^E u_3 &= 0 \\ \bar{c}_{3213}^E u_1 + (\bar{c}_{3223}^E - c) u_2 + \bar{c}_{3233}^E u_3 &= 0 \\ \bar{c}_{3313}^E u_1 + \bar{c}_{3323}^E u_2 + (\bar{c}_{3333}^E - c) u_3 &= 0 , \end{aligned} \quad (25)$$

which can be simplified by using the symmetry relations in Eq. (9).

In order for non-trivial solutions of these equations to exist, it is necessary that the determinant of the coefficients of the u 's be equal to zero, as shown in Eq. (24). Equation (24), or its explicit counterpart from Eq. (25), is a cubic equation in c , of the form,

$$c^3 + qc^2 + rc + s = 0 . \quad (26)$$

The first task in the quest for a multi-mode plate model is to solve Eq. (24). However, in deriving Eq. (24), a tacit assumption has been made. The derivation assumes that piezoelectric material constants are known values in the coordinate frame assigned to the plate. Unfortunately, this is not usually

the case. When measurements are made on piezoelectric crystals the tensor components of the material constants are specified in terms of a rectangular set of coordinate axes, which coincide with the crystallographic axes of the material, insofar as this is geometrically possible.

In crystallography, they talk about the a , b , c axes of the crystal; these axes are parallel to the edges of the unit cell, which is the parallelepiped out of which the crystal can be constructed. These a , b , c axes may or may not be mutually perpendicular. The IEEE^[5] has established a standard for associating a rectangular set of X , Y , Z axes with the crystallographic a , b , c , axes, for each of the seven systems into which crystals are commonly classified, depending on their degree of symmetry. These seven systems are, in turn, divided into point groups or classes according to their symmetry, with respect to a point. There are thirty-two of these classes but only twenty of them exhibit piezoelectric properties. It is this system and class notation that determines which of the possible material constant components are non-zero.

The orientation of a crystal plate in terms of the rectangular X , Y , Z axes is specified in the IEEE standards by means of a rotational symbol of the form $(A\ B\ a\ b\ c)\ \Phi/\Theta/\Psi$, where

A is the initial direction of the plate thickness before rotation,

B is the direction of the plate length before rotation,

$a = t, l, \text{ or } w$ depending on which direction is the axis of first rotation,

$b = t, l, \text{ or } w$ depending on which edge is used for the second rotation, and

$c = t, l, w$ according to the edge used for the third rotation.

Φ, Θ, Ψ indicate the corresponding rotation angles with sign where a positive angle is a rotation counterclockwise, looking toward the origin from the positive end of the axis of rotation. This positive end of the axis of rotation is the end that initially pointed in the positive directions of X, Y, Z . In this rotational symbol only as many specifications are used as are needed to completely specify the plate orientation. Unfortunately, people have a tendency (in the literature) to use names for various crystal cuts such as AC-cut quartz or AT-cut quartz, or they refer to them by names such as $35\text{-}1/4^\circ$ rotated Y-cut, without supplying the rotational symbol. In this example AT-cut quartz and a $35\text{-}1/4^\circ$ rotated Y-cut of quartz are the same, with a rotational symbol $(YX1)\ 35\text{-}1/4^\circ$.

There are also left and right-hand crystals, which, in the case of quartz, leads to different numerical values for some of the material coefficients.

It is assumed now that we know the orientation of the plate under consideration in Eq. (25), relative to the standard X, Y, Z reference frame for the plate material; hence we are ready to obtain the required material coefficients. Now, another confusion factor presents itself. The usual way of listing material

coefficients is in a two-subscript (or less) notation rather than the tensor notation used to develop Eq. (25). The conversion to this two-subscript, or engineer notation, from the four subscript tensor notation is shown in Table 3-1[3]. Furthermore, the tensor symmetry relations shown in Eq. (9) reduce the 81 possible combinations for the elastic stiffness constants to a maximum of 21, the 27 possible combinations for the piezoelectric stress constants to 18, and the nine possible dielectric permittivities to a maximum of six. Tables 3-2 and 3-3, respectively, show the tensor coefficients and their engineering counterparts for the stiffness and piezoelectric stress constants. The tensor and engineer forms of the dielectric permittivity are identical; the only reduction taking place is that $\epsilon_{ij}^S = \epsilon_{ji}^S$.

TABLE 3-1. CONVERSION FROM TENSOR TO ENGINEERING NOTATION

$$C_{ijkl} = C_{pq}$$

$$e_{ikl} = e_{ip}$$

$$T_{ij} = T_p ; S_{ij} = S_p \text{ when } i = j, p = 1, 2, 3$$

$$2S_{ij} = S_p \text{ when } i \neq j, p = 4, 5, 6$$

$$i, j, k, \ell = 1, 2, 3$$

$$p, q = 1, 2, 3, 4, 5, 6$$

ij or kℓ	p or q
11	1
22	2
33	3
23 or 32	4
31 or 13	5
12 or 21	6

This means that the matrices of the material coefficients of interest in this program will take the form shown in Figure 3-1a for the most general piezoelectric material. In this case there are 45 different non-zero coefficients. Figure 3-1b shows the same matrices for quartz, which was used in the theoretical investigations of this contract. In this case only 14 non-zero different coefficients are present and some of these differ only in sign.

Figure 3-2 shows a general form of the coordinate rotation problem about a common origin. We assume the material coefficients are known in the old x_1, x_2, x_3 (XYZ) system, and that the plate under consideration in Eq. (25) is oriented with the plate thickness direction in the x'_3 direction of the new coordinate system, x'_1, x'_2, x'_3 .

TABLE 3-2. RELATIONSHIP OF TENSOR STIFFNESS COEFFICIENTS TO
 ENGINEER COEFFICIENTS $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{kl ij}$
 $C_{pq} = C_{qp}$

C_{1111}	C_{1112}	C_{1113}	C_{1121}	C_{1122}	C_{1123}	C_{1131}	C_{1132}	C_{1133}
C_{11}	C_{16}	C_{15}	C_{16}	C_{12}	C_{14}	C_{15}	C_{14}	C_{13}
C_{1211}	C_{1212}	C_{1213}	C_{1221}	C_{1222}	C_{1223}	C_{1231}	C_{1232}	C_{1233}
C_{16}	C_{66}	C_{56}	C_{66}	C_{26}	C_{46}	C_{56}	C_{46}	C_{36}
C_{1311}	C_{1312}	C_{1313}	C_{1321}	C_{1322}	C_{1323}	C_{1331}	C_{1332}	C_{1333}
C_{15}	C_{56}	C_{55}	C_{56}	C_{25}	C_{45}	C_{55}	C_{45}	C_{35}
C_{2111}	C_{2112}	C_{2113}	C_{2121}	C_{2122}	C_{2123}	C_{2131}	C_{2132}	C_{2133}
C_{16}	C_{66}	C_{56}	C_{66}	C_{26}	C_{46}	C_{56}	C_{46}	C_{36}
C_{2211}	C_{2212}	C_{2213}	C_{2221}	C_{2222}	C_{2223}	C_{2231}	C_{2232}	C_{2233}
C_{12}	C_{26}	C_{25}	C_{26}	C_{22}	C_{24}	C_{25}	C_{24}	C_{23}
C_{2311}	C_{2312}	C_{2313}	C_{2321}	C_{2322}	C_{2323}	C_{2331}	C_{2332}	C_{2333}
C_{14}	C_{46}	C_{45}	C_{46}	C_{24}	C_{44}	C_{45}	C_{44}	C_{34}
C_{3111}	C_{3112}	C_{3113}	C_{3121}	C_{3122}	C_{3123}	C_{3131}	C_{3132}	C_{3133}
C_{15}	C_{56}	C_{55}	C_{56}	C_{25}	C_{45}	C_{55}	C_{45}	C_{35}
C_{3211}	C_{3212}	C_{3213}	C_{3221}	C_{3222}	C_{3223}	C_{3231}	C_{3232}	C_{3233}
C_{14}	C_{46}	C_{45}	C_{46}	C_{24}	C_{44}	C_{45}	C_{44}	C_{34}
C_{3311}	C_{3312}	C_{3313}	C_{3321}	C_{3322}	C_{3323}	C_{3331}	C_{3332}	C_{3333}
C_{13}	C_{36}	C_{35}	C_{36}	C_{23}	C_{34}	C_{35}	C_{34}	C_{33}

TABLE 3-3. RELATIONSHIP OF TENSOR PIEZOELECTRIC STRESS COEFFICIENTS TO ENGINEERING COEFFICIENTS $e_{ijk} = e_{ikj}$

e_{111}	e_{112}	e_{113}	e_{121}	e_{122}	e_{123}	e_{131}	e_{132}	e_{133}
e_{11}	e_{16}	e_{15}	e_{16}	e_{12}	e_{14}	e_{15}	e_{14}	e_{13}
e_{211}	e_{212}	e_{213}	e_{221}	e_{222}	e_{223}	e_{231}	e_{232}	e_{233}
e_{21}	e_{26}	e_{25}	e_{26}	e_{22}	e_{24}	e_{25}	e_{24}	e_{23}
e_{311}	e_{312}	e_{313}	e_{321}	e_{322}	e_{323}	e_{331}	e_{332}	e_{333}
e_{31}	e_{36}	e_{35}	e_{36}	e_{32}	e_{34}	e_{35}	e_{34}	e_{33}

By considering the scalar or dot product of the unit vectors i, j, k , in the old coordinate system with the unit vectors i', j', k' in the new coordinate system the following relationship can be determined.

The coordinates of any point, P, with respect to the new axes are given, in terms of the coordinates of P in the old axes, by the following relations:

$$\begin{aligned}
 x'_1 &= \cos \alpha_1 x_1 + \cos \beta_1 x_2 + \cos \gamma_1 x_3 \\
 x'_2 &= \cos \alpha_2 x_1 + \cos \beta_2 x_2 + \cos \gamma_2 x_3 \\
 x'_3 &= \cos \alpha_3 x_1 + \cos \beta_3 x_2 + \cos \gamma_3 x_3
 \end{aligned}
 \tag{27}$$

It is more convenient to represent these direction cosines by the symbols l, m , and n where,

$$\begin{aligned}
 l_p &= \cos \alpha_p \\
 m_p &= \cos \beta_p \\
 n_p &= \cos \gamma_p
 \end{aligned}
 \tag{28}$$

A. Most General Crystal (Triclinic) Class 1

$$c_{pq}^E = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}$$

$$e_{ip} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ -e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix}$$

$$\epsilon_{ij}^S = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

B. Quartz Trigonal Class 32

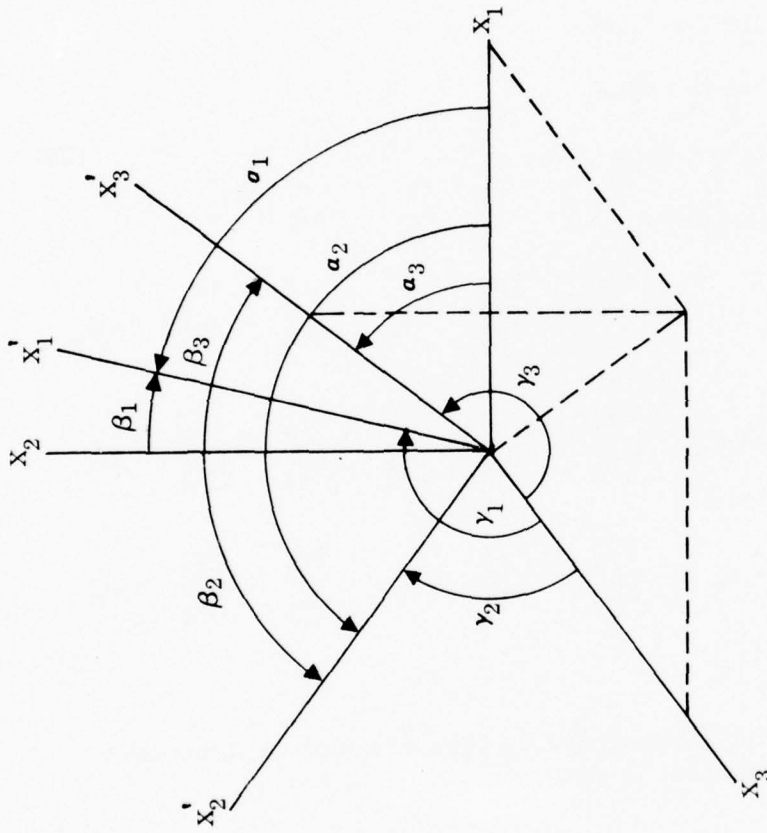
$$c_{pq}^E = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{bmatrix}$$

$$c_{66} = \frac{1}{2} (c_{11} - c_{12})$$

$$e_{ip} = \begin{bmatrix} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon_{ij}^S = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

Figure 3-1. Matrices for Coefficients of Piezoelectric Materials



$\alpha_1, \beta_1, \gamma_1$ = ANGLES BETWEEN NEW X'_1 AXIS AND OLD X_1, X_2, X_3 AXES RESPECTIVELY

$\alpha_2, \beta_2, \gamma_2$ = ANGLES BETWEEN NEW X'_2 AXIS AND OLD X_1, X_2, X_3 AXES RESPECTIVELY

$\alpha_3, \beta_3, \gamma_3$ = ANGLES BETWEEN NEW X'_3 AXIS AND OLD X_1, X_2, X_3 AXES RESPECTIVELY

Figure 3-2. Rotation of Rectangular Axes About a Common Origin

Then

$$\begin{aligned}x'_1 &= l_1 x_1 + m_1 x_2 + n_1 x_3 \\x'_2 &= l_2 x_1 + m_2 x_2 + n_2 x_3 \\x'_3 &= l_3 x_1 + m_3 x_2 + n_3 x_3 \quad ,\end{aligned}\tag{29}$$

while the coordinates of the point, P, with respect to the old axes are given in terms of the coordinates of P in the new axes by

$$\begin{aligned}x_1 &= l_1 x'_1 + l_2 x'_2 + l_3 x'_3 \\x_2 &= m_1 x'_1 + m_2 x'_2 + m_3 x'_3 \\x_3 &= n_1 x'_1 + n_2 x'_2 + n_3 x'_3 \quad .\end{aligned}\tag{30}$$

From these equations we obtain:

$$\begin{aligned}\frac{\partial x'_1}{\partial x_1} &= \frac{\partial x_1}{\partial x'_1} = l_1 & \frac{\partial x'_2}{\partial x_1} &= \frac{\partial x_1}{\partial x'_2} = l_2 & \frac{\partial x'_3}{\partial x_1} &= \frac{\partial x_1}{\partial x'_3} = l_3 \\ \frac{\partial x'_1}{\partial x_2} &= \frac{\partial x_2}{\partial x'_1} = m_1 & \frac{\partial x'_2}{\partial x_2} &= \frac{\partial x_2}{\partial x'_2} = m_2 & \frac{\partial x'_3}{\partial x_2} &= \frac{\partial x_2}{\partial x'_3} = m_3 \\ \frac{\partial x'_1}{\partial x_3} &= \frac{\partial x_3}{\partial x'_1} = n_1 & \frac{\partial x'_2}{\partial x_3} &= \frac{\partial x_3}{\partial x'_2} = n_2 & \frac{\partial x'_3}{\partial x_3} &= \frac{\partial x_3}{\partial x'_3} = n_3\end{aligned}\tag{31}$$

The general tensor transformation for a tensor of rank, n, from the x coordinates to the x' coordinates, is given by Mason [6], as,

$$x'_{k1} \cdots x'_{kn} = \frac{\partial x'_{k1}}{\partial x_{j1}} \frac{\partial x'_{k2}}{\partial x_{j2}} \cdots \frac{\partial x'_{kn}}{\partial x_{jn}} x_{j1} \cdots x_{jn}\tag{32}$$

In order to determine the material coefficients of concern in this program it is only necessary to deal with coefficients of the fourth rank and below.

If c'_{ijkl} , e'_{ijk} and ϵ'_{ij} represent the desired coefficients in the new reference frame (x'_1, x'_2, x'_3) , and c_{mnop} , e_{lmn} , ϵ_{kl} represent the known coefficients in the old reference frame (x_1, x_2, x_3) , Eq. (32) takes the form shown for each of the coefficients:

$$\begin{aligned}
c'_{ijkl} &= \frac{\partial x'_i}{\partial x_m} \frac{\partial x'_j}{\partial x_n} \frac{\partial x'_k}{\partial x_o} \frac{\partial x'_l}{\partial x_p} c_{mnop} \\
e'_{ijk} &= \frac{\partial x'_i}{\partial x_l} \frac{\partial x'_j}{\partial x_m} \frac{\partial x'_k}{\partial x_n} e_{lmn} \\
\epsilon'_{ij} &= \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} \epsilon_{kl}
\end{aligned} \tag{33}$$

It is more convenient to use the equivalent expression shown in Eq. (31) and to place these direction cosines in a matrix array A,

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} . \tag{34}$$

In this terminology Eqs. (29), (30), and (33) become, in tensor notation,

$$\begin{aligned}
x'_i &= a_{ij} x_j \\
x_i &= a_{ji} x'_j \\
c'_{ijkl} &= a_{im} a_{jn} a_{ko} a_{lp} c_{mnop} \\
e'_{ijk} &= a_{il} a_{jm} a_{kn} e_{lmn} \\
\epsilon'_{ij} &= a_{ik} a_{jl} \epsilon_{kl}
\end{aligned} \tag{35}$$

The program for carrying out this transformation of the material coefficients, and for determining the piezoelectrically stiffened coefficients, \bar{c}_{ijki} , in the secular equation (Eq. 24) is listed in Table 3-4. This program was written for operation on the Honeywell Information Systems series 6000/600 computer, as installed at the General Electric ESD facility in Syracuse, in the Y FOR mode of time-sharing operation. However, most of the statements are standard Fortran IV and, hence, should be adaptable to any similar computing system.

An attempt was made to make the program self-explanatory but the following observations will help.

Line 10 is a first line run command and is not required. Lines 150 through 262 are the elastic stiffness constants, piezoelectric stress constants, and dielectric permittivities in the primary (unrotated or XYZ) reference frame. For the program illustrated they are Bechmann's [7] values for left-hand quartz. The order of input is that for the triclinic crystal shown in Figure 3-1. The

TABLE 3-4. LISTING OF PROGRAM CR0T FOR THE TRANSFORMATION OF PIEZOELECTRIC COEFFICIENTS AND GENERATION OF THE ELEMENTS IN THE SECULAR EQUATION

```

10** RUNH-04=(CORE=22)
100C PROGRAM TO TRANSFORM THE COEFFICIENTS OF THE PIEZOELECTRIC
101C CONSTITUTIVE EQUATIONS FROM ONE SET OF COORDINATES
102C TO A ROTATED SET OF COORDINATES
110 INTEGER 0,P
120 DIMENSION G(21),GG(6,6),GGGG(3,3,3,3),IA(3,3)
130 DIMENSION C(21),CC(6,6),CCCC(3,3,3,3),A(3,3)
132 DIMENSION PS(18),PSI(3,3,3)
134 DIMENSION E(18),EEE(3,3,3)
136 DIMENSION DP(6),DPI(3,3)
138 DIMENSION EPS(6),EPSI(3,3)
139 DIMENSION CR(3,3)
140 DATA IA/1,6,5,6,2,4,5,4,3/
150C BECHMANN'S VALUES FOR LEFT HAND QUARTZ
159C THE ELASTIC STIFFNESS CONSTANTS
160 G(1)=86.74E9
162 G(2)=6.99E9
164 G(3)=11.91E9
166 G(4)=-17.91E9
168 G(5)=0.
170 G(6)=0.
172 G(7)=36.74E9
174 G(8)=11.91E9
176 G(9)=17.91E9
178 G(10)=0.
180 G(11)=0.
182 G(12)=107.2E9
184 G(13)=0.
186 G(14)=0.
188 G(15)=0.
190 G(16)=57.94E9
192 G(17)=0.
194 G(18)=0.
196 G(19)=57.94E9
198 G(20)=-17.91E9
200 G(21)=39.88E9
210C THE PIEZOELECTRIC STRESS CONSTANTS
212 PS(1)=0.171
214 PS(2)=-0.171
216 PS(3)=0.
218 PS(4)=-0.0406
220 PS(5)=0.
222 PS(6)=0.
224 PS(7)=0.
226 PS(8)=0.
228 PS(9)=0.
230 PS(10)=0.
232 PS(11)=0.0406
234 PS(12)=-0.171
236 PS(13)=0.
238 PS(14)=0.
240 PS(15)=0.
242 PS(16)=0.
244 PS(17)=0.
246 PS(18)=0.

```

TABLE 3-4 (Cont'd)

```

250C THE DIELECTRIC PERMITTIVITIES
252 DP(1)=39.21E-12
254 DP(2)=0.
256 DP(3)=0.
258 DP(4)=39.21E-12
260 DP(5)=0.
262 DP(6)=41.03E-12
300C X3 IS IN THE THICKNESS DIRECTION
310C THIS IS AT CUT QUARTZ (YXL)THETA
320 THEIA=35.25
360 PI=3.1415926
369C THESE ARE DIRECTION COSINES BETWEEN NEW AND OLD AXES
370 RL1=1.
380 RM1=0.
390 RV1=0.
400 RL2=0.
410 RM2=SIN(THETA*PI/180.)
420 RN2=-COS(THETA*PI/180.)
430 RL3=0.
440 RM3=-RN2
450 RN3=RM2
460 A(1,1)=RL1
470 A(1,2)=RM1
480 A(1,3)=RN1
490 A(2,1)=RL2
500 A(2,2)=RM2
510 A(2,3)=RV2
520 A(3,1)=RL3
530 A(3,2)=RM3
540 A(3,3)=RN3
600C CONVERT THE 21 ELASTIC CONSTANTS G(I) INTO ENGINEERING
601C NOTATION GG(M,N)
610 N=0
620 DO 10 I=1,6
630 DO 10 J=1,6
640 N=N+1
650 GG(I,J)=G(N)
660 GG(J,I)=GG(I,J)
670 10 CONTINUE
680C CONVERT FROM ENGINEERING NOTATION GG(M,N) INTO
681C TENSOR NOTATION GGGG(I,J,K,L)
690 DO 20 I=1,3
700 DO 20 J=1,3
710 DO 20 K=1,3
720 DO 20 L=1,3
730 M=IAC(I,J)
740 N=IAC(K,L)
750 GGGG(I,J,K,L)=GG(M,N)
760 20 CONTINUE
770C COMPUTE THE COMPONENTS OF THE ROTATED MATRIX
780 DO 40 I=1,3
790 DO 40 J=1,3
800 DO 40 K=1,3
810 DO 40 L=1,3
820 CCCC(I,J,K,L)=0.
830 DO 30 M=1,3
840 DO 30 N=1,3
850 DO 30 O=1,3

```

TABLE 3-4 (Cont'd)

```

860 D0 30 P=1,3
870 CCCC(I,J,K,L)=A(I,M)*A(J,N)*A(K,O)*A(L,P)*GGGG(M,N,O,P)
871& +CCCC(I,J,K,L)
880 30 CONTINUE
890 M=IA(I,J)
900 N=IA(K,L)
910 CC(M,N)=CCCC(I,J,K,L)
920 40 CONTINUE
970 PRINT 900
980 PRINT 910
990C CONVERT BACK TO OBTAIN THE 21 ROTATED ELASTIC CONSTANTS C(I)
1000 N=0
1010 D0 50 I=1,6
1020 D0 50 J=1,6
1030 N=N+1
1040 C(N)=CC(I,J)
1050 PRINT 920, N, G(N), C(N)
1060 50 CONTINUE
1100C CONVERT THE 18 PIEZOELECTRIC STRESS CONSTANTS PS(M)
1101C INTO TENSOR NOTATION PST(I,J,K)
1110 D0 60 I=1,3
1120 D0 60 J=1,3
1130 D0 60 K=1,3
1140 M=IA(J,K)+(I-1)*6
1150 PST(I,J,K)=PS(M)
1160 60 CONTINUE
1170C COMPUTE THE COMPONENTS OF THE ROTATED MATRIX
1180 D0 80 I=1,3
1190 D0 80 J=1,3
1200 D0 80 K=1,3
1210 EEE(I,J,K)=0.
1220 D0 70 L=1,3
1230 D0 70 M=1,3
1240 D0 70 N=1,3
1250 EEE(I,J,K)=A(I,L)*A(J,M)*A(K,N)*PST(L,M,N)+EEE(I,J,K)
1260 70 CONTINUE
1270C CONVERT BACK TO OBTAIN THE 18 ROTATED
1271C PIEZOELECTRIC STRESS CONSTANTS E(I)
1280 M=IA(J,K)+(I-1)*6
1290 E(M)=EEE(I,J,K)
1300 80 CONTINUE
1310 PRINT 930
1320 PRINT 910
1330 D0 90 N=1,18
1340 PRINT 920, N, PS(N), E(N)
1350 90 CONTINUE
1400C CONVERT THE 6 DIELECTRIC PERMITTIVITIES DP(N)
1401C INTO TENSOR NOTATION DPT(I,J)
1410 N=0
1420 D0 100 I=1,3
1430 D0 100 J=1,3
1440 N=N+1
1450 DPT(I,J)=DP(N)
1460 DPT(J,I)=DPT(I,J)
1470 100 CONTINUE
1480C COMPUTE THE COMPONENTS OF THE ROTATED MATRIX
1490 D0 120 I=1,3
1500 D0 120 J=1,3
1510 EPST(I,J)=0.

```

TABLE 3-4 (Cont'd)

```

1520 D0 110 K=1,3
1530 D0 110 L=1,3
1540 EPST(I,J)=A(I,K)*A(J,L)*DPT(K,L)+EPST(I,J)
1550 110 CONTINUE
1560 120 CONTINUE
1570 PRINT 940
1580 PRINT 910
1590C CONVERT BACK TO OBTAIN THE 6 ROTATED DIELECTRIC
1591C PERMITTIVITIES
1600 N=0
1610 D0 130 I=1,3
1620 D0 130 J=1,3
1630 N=N+1
1640 EPS(N)=EPST(I,J)
1650 PRINT 920, N, DPC(N), EPS(N)
1660 130 CONTINUE
1700 900 FORMAT(14X, 'THE ELASTIC STIFFNESS CONSTANTS ARE', //)
1710 910 FORMAT(11X, 'N', 4X, 'BEFORE ROTATION', 7X, 'AFTER ROTATION')
1720 920 FORMAT(11X, 12, 3X, E15.8, 6X, E15.8)
1730 930 FORMAT(/, 12X, 'THE PIEZOELECTRIC STRESS CONSTANTS ARE', //)
1740 940 FORMAT(/, 15X, 'THE DIELECTRIC PERMITTIVITIES ARE', //)
1810C THIS PART USES THE ROTATED VALUES TO CALCULATE THE
1811C STIFFENED MATRIX ELEMENTS IN THE SECULAR EQUATION
1812C OF THE FORM CB(J,K)-DELTA(J,K)*C=0
1815C IN THE FOLLOWING I REPRESENTS THE COORDINATE IN
1816C THE THICKNESS DIRECTION. I.E. FOR X3 I=3
1820 I=3
1830 D0 140 J=1,3
1840 D0 140 K=1,3
1850 CB(J,K)=0.
1860 CBC(J,K)=CCCC(I,J,K,I)+EEE(I,I,J)*EEE(I,I,K)/EPST(I,I)
1870 140 CONTINUE
1880 PRINT 950,I
1890 PRINT 960
1900 PRINT 970, ((CB(K,J), J=1,3), K=1,3)
1910 950 FORMAT(/, 'FOR X', I1, ' IN THE THICKNESS DIRECTION')
1920 960 FORMAT(/, 10X, 'THE ELEMENTS IN THE SECULAR EQUATION ARE')
1930 970 FORMAT(/, 3X, E15.8, 6X, E15.8, 6X, E15.8)
1950C THESE CB(I,J) VALUES IN TURN CAN BE USED TO CALCULATE
1951C THE COEFFICIENTS OF THE CUBIC EQUATION IN C
1955C IF C**3+Q*C**2+R*C+S=0 THEN IN TERMS OF CB(I,J)
1960 Q=-(CB(1,1)+CB(2,2)+CB(3,3))
1970 R=CB(2,2)*(CB(3,3)+CB(1,1))+CB(1,1)*CB(3,3)
1980& -(CB(2,3)*CB(3,2)+CB(1,2)*CB(2,1)+CB(3,1)*CB(1,3))
1990 S=CB(2,1)*(CB(1,2)*CB(3,3)-CB(1,3)*CB(3,2))
2000& +CB(2,2)*(CB(1,3)*CB(3,1)-CB(1,1)*CB(3,3))
2010& +CB(2,3)*(CB(3,2)*CB(1,1)-CB(1,2)*CB(3,1))
2020 PRINT 980
2030 PRINT 990,Q
2040 PRINT 991,R
2050 PRINT 992,S
2060 980 FORMAT(5X, 'IN C**3 +Q*C**2 + R*C + S =0')
2070 990 FORMAT(8X, 'Q=', E15.8)
2080 991 FORMAT(8X, 'R=', E15.8)
2090 992 FORMAT(8X, 'S=', E15.8)
3000C FOR USE IN CALCULATING THE COUPLING
3001C COEFFICIENTS AND CAPACITANCE WE NEED
3002C THE FOLLOWING VALUES

```

TABLE 3-4 (Cont'd)

```

3010 PRINT 995
3020 PRINT 996, (EEE(1,1,J),J=1,3)
3030 PRINT 997
3040 PRINT 998,EPST(1,1)
3050 995 FORMAT(/,2X,'THE APPROPRIATE PIEZOELECTRIC ',
3051& 'STRESS CONSTANTS ARE')
3060 996 FORMAT(/,10X,'E(1) = ',E15.8,/,10X,
3061& 'E(2) = ',E15.8,/,10X,'E(3) = ',E15.8,/)
3070 997 FORMAT(/,2X,'THE APPROPRIATE DIELECTRIC ',
3071& 'PERMITTIVITY IS')
3080 998 FORMAT(/,10X,'EP = ',E15.8)
3090 STOP
4000 END

```

matrices are inputted by rows, and only elements that have not been previously defined are entered. This input data arrangement is shown in Table 3-5. Lines 300 through 430 specify the plate orientation in terms of the direction cosines in Eqs. (28), (29), and (34).

The transformation matrix, [8] lines 730 and 740, loaded as shown in line 140, follows Table 3-2. The input at line 1820 designates the coordinate that lies in the plate thickness direction; the appropriate coefficients in the secular equation are calculated as well as the appropriate piezoelectric stress constants and dielectric permittivity for this orientation of the plate.

A printout of the results of this program, using AT-cut quartz as the plate material with x_3 in the thickness direction, is shown in Table 3-6.

To switch material it is necessary to change statements 150 to 262. To change orientations it is necessary to change line 300 to 450, and line 1820. Table 3-7 shows these changes and a printout for AT-cut quartz with x_2 in the thickness direction. These particular values check with those given by Tiersten^[3].

B. SOLUTION OF THE SECULAR EQUATION AND PROGRAM SYMEIG

As a result of the output of the CROT program we have numerical values for the \bar{c}_{ijki}^E coefficients in Eqs. (23) and (24), in general, or in Eq. (25) for the plate thickness direction, restricted to lie in the x_3 direction. We also know the values of the coefficients of c in Eq. (26). Since the array of \bar{c}_{ijki}^E coefficients is a symmetrical matrix array with real coefficients, the characteristic numbers or eigenvalues of the secular equation will be real. It also turns out that they will be positive since this is a positive definite matrix.

Thus, the solution of Eqs. (24) or (26) will yield three real, positive roots, $c^{(i)}$, from which three real wave numbers, $\eta^{(i)}$, can be obtained from Eq. (20) for a specified value of ω . If these values of $c^{(i)}$ are substituted into Eqs. (23) or (25), a set of ratios among the $u_k^{(i)}$ amplitudes can be determined for each value of $c^{(i)}$. These ratios can then be used to construct a new ref-

TABLE 3-5. INPUT DATA ARRANGEMENT

Elastic Stiffness Coefficients

Input Symbol (N)	Output Symbol (N)	Engineering Equivalents	Tensor Equivalents
G(1)	C(1)	C_{11}	C_{1111}
G(2)	C(2)	$C_{12} = C_{21}$	$C_{1122} = C_{2211}$
G(3)	C(3)	$C_{13} = C_{31}$	$C_{1133} = C_{3311}$
G(4)	C(4)	$C_{14} = C_{41}$	$C_{1123} = C_{1132} = C_{2311} = C_{3211}$
G(5)	C(5)	$C_{15} = C_{51}$	$C_{1113} = C_{1131} = C_{1311} = C_{3111}$
G(6)	C(6)	$C_{16} = C_{61}$	$C_{1112} = C_{1121} = C_{1211} = C_{2111}$
G(7)	C(7)	C_{22}	C_{2222}
G(8)	C(8)	$C_{23} = C_{32}$	$C_{2233} = C_{3322}$
G(9)	C(9)	$C_{24} = C_{42}$	$C_{2223} = C_{2232} = C_{2322} = C_{3222}$
G(10)	C(10)	$C_{25} = C_{52}$	$C_{2213} = C_{2231} = C_{1322} = C_{3122}$
G(11)	C(11)	$C_{26} = C_{62}$	$C_{2212} = C_{2221} = C_{1222} = C_{2122}$
G(12)	C(12)	C_{33}	C_{3333}
G(13)	C(13)	$C_{34} = C_{43}$	$C_{3323} = C_{3332} = C_{2333} = C_{3233}$
G(14)	C(14)	$C_{35} = C_{53}$	$C_{3313} = C_{3331} = C_{1333} = C_{3133}$
G(15)	C(15)	$C_{36} = C_{63}$	$C_{3312} = C_{3321} = C_{1233} = C_{2133}$
G(16)	C(16)	C_{44}	$C_{2323} = C_{2332} = C_{3223} = C_{3232}$
G(17)	C(17)	$C_{45} = C_{54}$	$C_{2313} = C_{2331} = C_{1323} = C_{3123}$ $= C_{3213} = C_{3231} = C_{1332} = C_{3132}$
G(18)	C(18)	$C_{46} = C_{64}$	$C_{2312} = C_{2321} = C_{1223} = C_{1232}$ $= C_{2123} = C_{2132} = C_{3212} = C_{3221}$
G(19)	C(19)	C_{55}	$C_{1331} = C_{1313} = C_{3113} = C_{3131}$
G(20)	C(20)	$C_{56} = C_{65}$	$C_{1312} = C_{1321} = C_{1213} = C_{1231}$ $= C_{2113} = C_{2131} = C_{3112} = C_{3121}$
G(21)	C(21)	C_{66}	$C_{1212} = C_{1221} = C_{2112} = C_{2121}$

TABLE 3-5 (Cont'd)

Piezoelectric Stress Constants

Input Symbol (N)	Output Symbol (N)	Engineering Equivalents	Tensor Equivalents
PS(1)	E(1)	e_{11}	e_{111}
PS(2)	E(2)	e_{12}	e_{122}
PS(3)	E(3)	e_{13}	e_{133}
PS(4)	E(4)	e_{14}	$e_{123} = e_{132}$
PS(5)	E(5)	e_{15}	$e_{113} = e_{131}$
PS(6)	E(6)	e_{16}	$e_{112} = e_{121}$
PS(7)	E(7)	e_{21}	e_{211}
PS(8)	E(8)	e_{22}	e_{222}
PS(9)	E(9)	e_{23}	e_{233}
PS(10)	E(10)	e_{24}	$e_{223} = e_{232}$
PS(11)	E(11)	e_{25}	$e_{213} = e_{231}$
PS(12)	E(12)	e_{26}	$e_{212} = e_{221}$
PS(13)	E(13)	e_{31}	e_{311}
PS(14)	E(14)	e_{32}	e_{322}
PS(15)	E(15)	e_{33}	e_{333}
PS(16)	E(16)	e_{34}	$e_{323} = e_{332}$
PS(17)	E(17)	e_{35}	$e_{313} = e_{331}$
PS(18)	E(18)	e_{36}	$e_{312} = e_{321}$

TABLE 3-5 (Cont'd)

Dielectric Permittivities at Constant Strain

Input Symbol (N)	Output Symbol (N)	Engineering Equivalentents	Tensor Equivalentents
DP(1)	EPS(1)	ϵ_{11}^S	ϵ_{11}^S
DP(2)	EPS(2)	$\epsilon_{12}^S = \epsilon_{21}^S$	$\epsilon_{12}^S = \epsilon_{21}^S$
DP(3)	EPS(3)	$\epsilon_{13}^S = \epsilon_{31}^S$	$\epsilon_{13}^S = \epsilon_{31}^S$
DP(4)	EPS(4)	ϵ_{22}^S	ϵ_{22}^S
DP(5)	EPS(5)	$\epsilon_{23}^S = \epsilon_{32}^S$	$\epsilon_{23}^S = \epsilon_{32}^S$
DP(6)	EPS(6)	ϵ_{33}^S	ϵ_{33}^S

TABLE 3-6. RESULTS OF CROT FOR AT CUT QUARTZ WITH X₃ IN THE THICKNESS DIRECTION

2511 02/23/77 12.259

THE ELASTIC STIFFNESS CONSTANTS ARE

i	BEFORE ROTATION	AFTER ROTATION
1	0.35740000E+11	0.35740000E+11
2	0.38900000E+10	0.27153374E+11
3	0.11210000E+11	-0.12530741E+12
4	-0.17940000E+11	0.36595338E+12
5	0.	0.
6	0.	0.
7	0.35740000E+11	0.19243551E+12
8	0.11210000E+11	-0.74142732E+12
9	0.17210000E+11	-0.25212727E+12
10	0.	0.
11	0.	0.
12	0.10720000E+12	0.12275638E+12
13	0.	-0.57004231E+10
14	0.	0.
15	0.	0.
16	0.37840000E+11	0.38811527E+11
17	0.	0.
18	0.	0.
19	-0.37950000E+11	0.29012315E+11
20	-0.17210000E+11	-0.25335712E+12
21	0.38900000E+10	0.63366930E+11

THE PIEZOELECTRIC STRESS CONSTANTS ARE

i	BEFORE ROTATION	AFTER ROTATION
1	0.17100000E+00	0.17100000E+00
2	-0.17100000E+00	-0.13503257E-01
3	0.	-0.15231178E+00
4	-0.10500000E-01	-0.57344247E-01
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	0.	0.
10	0.	0.
11	0.13500000E-01	-0.57072126E-01
12	-0.17100000E+00	-0.75025134E-01
13	0.	0.
14	0.	0.
15	0.	0.
16	0.	0.
17	0.	-0.94904857E-01
18	0.	-0.10757213E+00

TABLE 3-6 (Cont'd)

THE DIELECTRIC PERMITTIVITIES ARE

n	BEFORE ROTATION	AFTER ROTATION
1	0.39210000E-10	0.39210000E-10
2	0.	0.
3	0.	0.
4	0.39210000E-10	0.40423765E-10
5	0.	-0.85730335E-12
6	0.41030000E-10	0.39816236E-10

FOR X3 IN THE THICKNESS DIRECTION

THE ELEMENTS IN THE SECULAR EQUATION ARE

0.29239228E 11	0.	0.
0.	0.38611527E 11	-0.57094245E 10
0.	-0.57094232E 10	0.12976634E 12

IN $Cx^3 + 0x^2 + Px + S = 0$
 $P = -0.19751709E 12$
 $S = 0.99012206E 22$
 $S = -0.11555234E 33$

THE APPROPRIATE PIEZOELECTRIC STRESS CONSTANTS ARE

$e(1) = -0.9499467E-01$
$e(2) = 0.$
$e(3) = 0.$

THE APPROPRIATE DIELECTRIC PERMITTIVITY IS

$\epsilon = 0.39215236E-10$

TABLE 3-7. AT-CUT QUARTZ WITH X_2 IN THE THICKNESS DIRECTION

a) Material Constants

150C BECHMANN'S VALUES FOR LEFT HAND QUARTZ

159C THE ELASTIC STIFFNESS CONSTANTS

160 G(1)=36.74E9
 162 G(2)=6.99E9
 164 G(3)=11.91E9
 166 G(4)=-17.91E9
 168 G(5)=0.
 170 G(6)=0.
 172 G(7)=36.74E9
 174 G(8)=11.91E9
 176 G(9)=17.91E9
 178 G(10)=0.
 180 G(11)=0.
 182 G(12)=107.2E9
 184 G(13)=0.
 186 G(14)=0.
 188 G(15)=0.
 190 G(16)=57.94E9
 192 G(17)=0.
 194 G(18)=0.
 196 G(19)=57.94E9
 198 G(20)=-17.91E9
 200 G(21)=39.83E9

210C THE PIEZOELECTRIC STRESS CONSTANTS

212 P_S(1)=0.171
 214 P_S(2)=-0.171
 216 P_S(3)=0.
 218 P_S(4)=-0.0406
 220 P_S(5)=0.
 222 P_S(6)=0.
 224 P_S(7)=0.
 226 P_S(8)=0.
 228 P_S(9)=0.
 230 P_S(10)=0.
 232 P_S(11)=0.0406
 234 P_S(12)=-0.171
 236 P_S(13)=0.
 238 P_S(14)=0.
 240 P_S(15)=0.
 242 P_S(16)=0.
 244 P_S(17)=0.
 246 P_S(18)=0.

250C THE DIELECTRIC PERMITTIVITIES

252 D_P(1)=39.21E-12
 254 D_P(2)=0.
 256 D_P(3)=0.
 258 D_P(4)=39.21E-12
 260 D_P(5)=0.
 262 D_P(6)=41.03E-12

TABLE 3-7. (Cont'd)

b) Plate Orientation

300C X2 IS IN THE THICKNESS DIRECTION
 310C THIS IS AT CUI QUARTZ (YXL)THETA
 320 THETA=35.25
 360 PI=3.1415926
 369C THESE ARE DIRECTION COSINES BETWEEN NEW AND OLD AXES
 370 RL1=1.
 380 RM1=0.
 390 RN1=0.
 400 RL2=0.
 410 RM2=COS(THETA*PI/180.)
 420 RN2=SIN(THETA*PI/180.)
 430 RL3=0.
 440 RM3=-RN2
 450 RN3=RM2

1810C THIS PART USES THE ROTATED VALUES TO CALCULATE THE
 1811C STIFFENED MATRIX ELEMENTS IN THE SECULAR EQUATION
 1812C OF THE FORM $CBC(J,K)-DELTA(CJ,K)*C=0$
 1813C IN THE FOLLOWING I REPRESENTS THE COORDINATE IN
 1816C THE THICKNESS DIRECTION. I.E. FOR X3 I=3
 1820 I=2

c) Results

25121 05/14/77 15.063

THE ELASTIC STIFFNESS CONSTANTS ARE

	BEFORE ROTATION	AFTER ROTATION
1	1.05740000E 11	0.85740000E 11
2	1.69900000E 10	-0.82538741E 10
3	1.11910000E 11	0.27153374E 11
4	-0.11791000E 11	-0.38295352E 10
5	0.	0.
6	0.	0.
7	1.05740000E 11	0.12976534E 12
8	1.11910000E 11	-0.74104737E 10
9	1.17910000E 11	0.57034240E 10
10	0.	0.
11	0.	0.
12	1.10720000E 12	0.10263751E 12
13	0.	0.99212510E 10
14	0.	0.
15	0.	0.
16	1.67940000E 11	0.33611527E 11
17	0.	0.
18	0.	0.
19	0.57940000E 11	0.63606906E 11
20	-0.11791000E 11	0.25335716E 10
21	0.39850000E 11	0.29013015E 11

TABLE 3-7. (Cont'd)

THE PIEZOELECTRIC STRESS CONSTANTS ARE

N	BEFORE ROTATION	AFTER ROTATION
1	0.17100000E 00	0.17100000E 00
2	-0.17100000E 00	-0.15231173E 00
3	0.	-0.13688267E-01
4	-0.40660000E-01	0.67043287E-01
5	0.	0.
6	0.	0.
7	0.	0.
8	0.	0.
9	0.	0.
10	0.	0.
11	0.40660000E-01	0.10767213E 00
12	-0.17100000E 00	-0.94904867E-01
13	0.	0.
14	0.	0.
15	0.	0.
16	0.	0.
17	0.	-0.76095134E-01
18	0.	0.67072126E-01

THE DIELECTRIC PERMITTIVITIES ARE

N	BEFORE ROTATION	AFTER ROTATION
1	0.39210000E-10	0.39210000E-10
2	0.	0.
3	0.	0.
4	0.39210000E-10	0.39816236E-10
5	0.	0.85780385E-12
6	0.41030000E-10	0.40423765E-10

FOR X2 IN THE THICKNESS DIRECTION

THE ELEMENTS IN THE SECULAR EQUATION ARE

$$\begin{array}{ccc}
 0.29239228E 11 & 0. & 0. \\
 0. & 0.12976534E 12 & 0.57004240E 10 \\
 0. & 0.57004246E 10 & 0.38611527E 11
 \end{array}$$

IN $C^{**3} + Q^{**}C^{**2} + R^{**}C + S = 0$
 $Q = -0.12761709E 12$
 $R = 0.99012206E 22$
 $S = -0.14555234E 33$

THE APPROPRIATE PIEZOELECTRIC STRESS CONSTANTS ARE

$$\begin{array}{l}
 E(1) = -0.94904867E-01 \\
 E(2) = 0. \\
 E(3) = 0.
 \end{array}$$

THE APPROPRIATE DIELECTRIC PERMITTIVITY IS

$$EP = 0.39816236E-10$$

erence system in which the modes of vibration are displayed mutually independent of each other*. This is the so-called normal coordinate reference system in which the allowed modes of operation are termed the normal modes. The directions of these normal coordinates are given by the characteristic vendors or eigenvectors associated with each $c^{(i)}$. This set of ratios among the $u_k^{(i)}$ amplitudes will be designated as the $\beta_k^{(i)}$ components of the i^{th} eigenvector associated with each $c^{(i)}$. For example $c^{(1)}$ is associated with the values $\beta_1^{(1)}$, $\beta_2^{(1)}$, $\beta_3^{(1)}$, which each $c^{(2)}$ is associated with the values $\beta_1^{(2)}$, $\beta_2^{(2)}$, $\beta_3^{(2)}$.

Furthermore, if the components for each (i) are normalized so that

$$\beta_k \beta_k = 1 \text{ or } \beta_1^{(i)} \beta_1^{(i)} + \beta_2^{(i)} \beta_2^{(i)} + \beta_3^{(i)} \beta_3^{(i)} = 1, \quad (36)$$

then these $\beta_k^{(i)}$ are the direction cosines of the particle displacement for each of the three modes, (i), in relation to the plate coordinates. The vector set formed in this fashion is also orthonormal and has the properties,

$$\beta_j^{(i)} \beta_k^{(i)} = \beta_m^{(j)} \beta_m^{(k)} = \delta_{jk}. \quad (37)$$

The general procedure, therefore, is to solve Eq. (26) for its roots or determine them from Eq. (24) and substitute into Eq. (23) for each root, to determine the u_k values appropriate to this root. Then normalize these to unity and construct the eigenvector for this root; then repeat the procedure for the next root, etc. If, working with Eq. (26), this cubic equation can be reduced to the form

$$d^3 + a d + b = 0,$$

where $c = d - (q/3)$ (38)

$$a = \frac{1}{3} (3r - q^2)$$

$$b = \frac{1}{27} (2q^3 - 9qr + 27s),$$

then, letting $d = m \cos \Theta$, where $m = 2 \sqrt{\frac{-a}{3}}$, $\cos 3 \Theta = \frac{-4b}{m^3}$,

gives the roots [14] (39)

$$d_1 = m \cos \Theta_1, \quad d_2 = m \cos \left(\Theta_1 + \frac{2\pi}{3} \right), \quad d_3 = m \cos \left(\Theta_1 + \frac{4\pi}{3} \right).$$

Here Θ_1 is the smallest value that satisfies $\cos 3 \Theta = -\frac{4b}{m^3}$.

* In the general case the number of roots of c is $c^{(k)}$ and the normal coordinate reference system has k dimensions instead of three, as in our special case.

Instead of following this procedure and writing a program for it, the Honeywell Information Series 600/6000 Time-Sharing Systems has a program available that is in the ESD computers time-sharing library. This program solves the equation and computes the eigenvectors as well as giving the eigenvalues. This "canned" routine was used to solve Eq. (25) and to obtain the eigenvectors satisfying Eq. (40).

$$(\bar{c}_{3jk} E - c^{(i)} \delta_{jk}) \beta_k^{(i)} = 0 \quad \begin{array}{l} \text{(no sum over} \\ \text{i is taken here)} \end{array} \quad (40)$$

The instructions for this "canned program" called SYMEIG, together with a listing of it, are shown in Table 3-8. This program is written in Time-Sharing FORTRAN, but the conversion to a more conventional form is relatively obvious.

The output of this program is shown in Table 3-9 when the required results from CROT in Table 3-6 were supplied. Both the Yes and No responses to the instruction question are shown.

Figure 3-3 uses these results for AT-cut quartz, with x_3 in the thickness direction, to clarify the previous discussion. It shows the construction of the eigenvectors for this simple 3-mode case, and the relationship of the 3-dimensional normal mode axes to the plate axes. For waves propagating along the x_3 plate axis we have one pure shear mode (particle displacement at right angles to the propagation direction), $\beta^{(1)}$, one quasi-shear mode, $\beta^{(2)}$, and one quasi-longitudinal mode, $\beta^{(3)}$. For waves propagating along the normal mode axis, 3, we have two pure shear modes, $\beta^{(1)}$ and $\beta^{(2)}$, and one pure longitudinal mode (particle displacement in the direction of propagation direction), $\beta^{(3)}$.

Another program for the purpose of debugging, was written at this time. Its purpose was an essential test of the results obtained up to this point.

The nine separate equations in Eq. (40) (three for each value of $c^{(i)}$) can be written as follows (where $CB_{jk} = \bar{c}_{3jk}$ in matrix form):

$$\begin{bmatrix} CB_{11} & CB_{12} & CB_{13} \\ CB_{21} & CB_{22} & CB_{23} \\ CB_{31} & CB_{32} & CB_{33} \end{bmatrix} \begin{bmatrix} \beta_1^{(1)} & \beta_1^{(2)} & \beta_1^{(3)} \\ \beta_2^{(1)} & \beta_2^{(2)} & \beta_2^{(3)} \\ \beta_3^{(1)} & \beta_3^{(2)} & \beta_3^{(3)} \end{bmatrix} = \begin{bmatrix} \beta_1^{(1)} & \beta_1^{(2)} & \beta_1^{(3)} \\ \beta_2^{(1)} & \beta_2^{(2)} & \beta_2^{(3)} \\ \beta_3^{(1)} & \beta_3^{(2)} & \beta_3^{(3)} \end{bmatrix} \begin{bmatrix} c^{(1)} & 0 & 0 \\ 0 & c^{(2)} & 0 \\ 0 & 0 & c^{(3)} \end{bmatrix} \quad (41)$$

This matrix can be pre-multiplied on each side by β inverse = $[\beta]^{-1}$ to yield

$$[\beta]^{-1} \begin{bmatrix} CB_{11} & CB_{12} & CB_{13} \\ CB_{21} & CB_{22} & CB_{23} \\ CB_{31} & CB_{32} & CB_{33} \end{bmatrix} \begin{bmatrix} \beta_1^{(1)} & \beta_1^{(2)} & \beta_1^{(3)} \\ \beta_2^{(1)} & \beta_2^{(2)} & \beta_2^{(3)} \\ \beta_3^{(1)} & \beta_3^{(2)} & \beta_3^{(3)} \end{bmatrix} = \begin{bmatrix} c^{(1)} & 0 & 0 \\ 0 & c^{(2)} & 0 \\ 0 & 0 & c^{(3)} \end{bmatrix}, \quad (42)$$

TABLE 3-8. INSTRUCTIONS FOR SYMEIG AND THE PROGRAM LISTING OF SYMEIG FOR CALCULATING THE EIGENVALUES AND EIGENVECTORS

a) Instructions

This FORTRAN program calculates the eigenvectors and eigenvalues of a real, symmetrical matrix.

METHOD

The method used is Jacobi's method, which was adapted for computer use by von Neumann. The method consists of applying to the matrix a system of plane rotations given by orthogonal matrices designed to reduce the off-diagonal elements to zero. The eigenvalues are then the diagonal elements of the original matrix and, if the eigenvectors were desired, they are developed as the columns of the product of the orthogonal matrices.

INSTRUCTIONS

Enter data as requested. For further instructions run the program.

SAMPLE SOLUTION

SYMEIG

*RUN

```

DO YOU WANT INSTRUCTIONS?
= YES
INSTRUCTIONS FOR SYMEIG
THIS PROGRAM CALCULATES THE EIGENVECTORS AND EIGENVALUES OF A
REAL, SYMMETRICAL MATRIX. THE METHOD USED IS JACOBI'S ITERATION
METHOD. THE METHOD CONSISTS OF APPLYING TO THE MATRIX A SYSTEM
OF PLANE ROTATIONS GIVEN BY ORTHOGONAL MATRICES MADE TO REDUCE
THE OFF-DIAGONAL ELEMENTS TO ZERO. THIS PROGRAM USES FOUR
ARGUMENTS. A IS THE NAME OF A TWO-DIMENSIONAL ARRAY CONTAINING
THE REAL, SYMMETRIC MATRIX IN ITS FIRST N ROWS AND COLUMNS
R IS THE NAME OF THE TWO-DIMENSIONAL ARRAY WHICH WILL CONTAIN
THE EIGENVECTORS IN ITS FIRST N COLUMNS.
N IS AN INTEGER VARIABLE OR CONSTANT GIVING THE ORDER
OF THE MATRIX.
MV IS AN INTEGER VARIABLE OR CONSTANT WHICH MUST BE 0 OR 1.
IF IT IS 0 BOTH EIGENVECTORS AND EIGENVALUES ARE FORMED.
IF IT IS ONE ONLY THE EIGENVALUES ARE FOUND.
ENTER THE ORDER OF MATRIX AND THE MATRIX SEPARATED BY COMMAS.
= 3
= 1,1,5
= 1,1,25
= .5,25,2
THE MATRIX IS
  1.0000000E+00  1.0000000E+00  5.0000000E-01
  1.0000000E+00  1.0000000E+00  2.5000000E-01
  5.0000000E-01  2.5000000E-01  2.0000000E+00
EACH EIGENVALUE FOLLOWED BY CORRESPONDING EIGENVECTOR
-1.6647290E-02
-7.2120713E-01  6.8634924E-01  9.3727956E-02
 1.4801212E+00
 4.4428103E-01  5.6210938E-01  -6.9760113E-01
 2.5365255E+00
 5.3148334E-01  4.6147330E-01  7.1032929E-01

```

TABLE 3-8. (Cont'd)

b) Program Listing

```

1* GR-500 LINE TIME STARTED LIBRARY PROGRAMS
2* SEE GR-1624 FOR EXTERNAL DOCUMENTATION
3*
4*
5* REVISION NOVEMBER 12, 1969
6*
7* SYMBIS
8*
9*   DIMENSION A(30,30),B(30,30)
10*   ASCII YES,NO,ANS;YES="YES";NO="NO"
11*   PRINT 99;PRINT:"SYMBIS";PRINT 99;99 FOR(1(/////))
12*   PRINT:"DO YOU WANT INSTRUCTIONS?"
13*   I=1
14*   IF(ANS.EQ.YES)GO TO 11
15*   IF(ANS.EQ.NO)GO TO 3
16*   PRINT:"ANSWER ONLY WITH YES OR NO."
17*   GO TO 1
18*
19* 11 PRINT:"INSTRUCTIONS FOR SYMBIS"
20*   PRINT:"THIS PROGRAM CALCULATES THE EIGENVECTORS AND EIGENVALUES OF AN"
21*   PRINT:"REAL, SYMMETRICAL MATRIX. THE METHOD USED IS JACOBI'S ITERATION"
22*   PRINT:"METHOD. THE METHOD CONSISTS OF APPLYING TO THE MATRIX A SYSTEM"
23*   PRINT:"OF PLANE ROTATIONS GIVEN BY ORTHOGONAL MATRICES MADE TO REDUCE"
24*   PRINT:"THE OFF-DIAGONAL ELEMENTS TO ZERO. THIS PROGRAM USES FOLLO"
25*   PRINT:"WING ARGUMENTS. A IS THE NAME OF A TWO-DIMENSIONAL ARRAY CONTAINING"
26*   PRINT:"THE REAL, SYMMETRIC MATRIX IN ITS FIRST N ROWS AND COLUMNS"
27*   PRINT:"A IS THE NAME OF THE TWO-DIMENSIONAL ARRAY WHICH WILL CONTAIN"
28*   PRINT:"THE EIGENVECTORS IN ITS FIRST N COLUMNS."
29*   PRINT:"N IS AN INTEGER VARIABLE OR CONSTANT GIVING THE ORDER"
30*   PRINT:"OF THE MATRIX."
31*   PRINT:"M IS AN INTEGER VARIABLE OR CONSTANT WHICH MUST BE 0 OR 1."
32*   PRINT:"IF IT IS 0 BOTH EIGENVECTORS AND EIGENVALUES ARE FORMED."
33*   PRINT:"IF IT IS ONE ONLY THE EIGENVALUES ARE FOUND."
34*   PRINT:"ENTER THE ORDER OF MATRIX AND THE MATRIX SEPARATED BY COMMAS."
35*   READ 1,((A(I,J),I=1,N),J=1,N)
36*   PRINT:"THE MATRIX IS"
37*   DO 10 I=1,N
38*   DO 10 J=1,N
39*   PRINT:(A(I,J),J=1,N)
40*   PRINT:""
41*   CALL MATRIG(A,P,N,W)
42*   PRINT:"EACH EIGENVALUE IS FOLLOWED BY CORRESPONDING EIGENVECTOR"
43*   DO 20 I=1,4
44*   PRINT:""
45*   PRINT:A(I,1)
46*   DO 20 PRINT:(C(J,I),J=1,N)
47*   CONTINUE
48*   STOP*END
49*
50*   SUBROUTINE MATRIG(A,P,N,W)
51*   DIMENSION A(30,30),B(30,30)
52*   IF(W-1)GOTO,25,10
53*   DO 20 J=1,N
54*   DO 20 I=1,N
55*   B(I,J)=0.0
56*   B(I,I)=1.0
57*   WRITE(20) B
58*   WRITE(20) A
59*   DO 30 I=1,N
60*   DO 30 J=1,N

```

TABLE 3-8. (Cont'd)

```

210      DO 35 J=1,N
220      IF(I-J) 30,35,30
230 30      ANORM=ANORM+A(I,J)*A(I,J)
240 35      CONTINUE
250      IF(ANORM) 105,105,40
250 40      ANORM=SQRT(ANORM*2.)
270      ANRMX=ANORM*1.0E-6/FLOAT(N)
230C      INITIALIZE INDICATORS AND COMPUTE THRESHOLD
290      IND=0
300      THR=ANORM
310 45      THR=THR/N
315      N1=N-1
320 50      DO 149 L=1,N1
325          LI=L+1
330          DO 149 M=LI,N
340C              COMPUTE SINE AND COSINE
350 62      IF(ABS(A(L,M))-THR) 149,65,65
350 65      IND=1
370          X=0.5*(A(L,L)-A(M,M))
380 68      Y=A(L,M)
390          Z=SQRT(Y*Y+X*X)
400          Y=-A(L,M)/Z
410          IF(X) 70,75,75
420 70      Y=-Y
430 75      SINX=Y/SQRT(2.0*(1.0+(SQRT(1.0+Y*Y))))
440          SINX2=SINX*SINX
450 78      COSX=SQRT(1.0-SINX2)
450          COSX2=COSX*COSX
470          SINCS=SINX*COSX
430C      ROTATE L AND M COLUMNS
490          DO 90 I=1,N
500          IF(I-L) 80,90,80
510 80      IF(I-M) 85,90,85
520 85      X=A(I,L)*COSX-A(I,M)*SINX
530          A(I,M)=A(I,L)*SINX+A(I,M)*COSX
540          A(I,L)=X
550 90      CONTINUE
560          X=2.0*A(L,M)*SINCS
570          Y=A(L,L)*COSX2+A(M,M)*SINX2-X
580          X=A(L,L)*SINX2+A(M,M)*COSX2+X
590          A(L,M)=(A(L,L)-A(M,M))*SINCS+A(L,M)*(COSX2-SINX2)
600          A(L,L)=Y
610          A(M,M)=X
620          DO 100 I=1,N
630          A(L,I)=A(I,L)
640          A(M,I)=A(I,M)
650          IF(MV-1) 95,100,95
660 95      X=R(I,L)*COSX-R(I,M)*SINX
670          R(I,M)=R(I,L)*SINX+R(I,M)*COSX
680          R(I,L)=X
690 100     CONTINUE
700 149     CONTINUE
710 150     IF(IND=1) 160,155,160
720 155     IND=0

```

TABLE 3-8. (Cont'd)

```

730      GO TO 51
740 165  IF(1/R-ANR(X))165,165,45
750 165  DO 165 I=1,N
760      DO 165 J=1,N
770      IF(A(I,I)-A(J,J))170,165,165
780 170  X=A(I,I)
790      A(I,I)=A(J,J)
800      A(J,J)=X
810      IF(W-1)175,170,175
820 175  DO 180 K=1,N
830      X=A(K,I)
840      A(K,I)=A(K,J)
850 180  A(K,J)=X
860 185  CONTINUE
870 190  RETURN
99999  STOP
99999  END

```

TABLE 3-9. OUTPUT OF SYMEIG FOR AT CUT QUARTZ WITH X₃ IN THICKNESS DIRECTION

SYMEIG

DO YOU WANT INSTRUCTIONS?

=YES

INSTRUCTIONS FOR SYMEIG

THIS PROGRAM CALCULATES THE EIGENVECTORS AND EIGENVALUES OF A REAL, SYMMETRICAL MATRIX. THE METHOD USED IS JACOBI'S ITERATION METHOD. THE METHOD CONSISTS OF APPLYING TO THE MATRIX A SYSTEM OF PLANE ROTATIONS GIVEN BY ORTHOGONAL MATRICES MADE TO REDUCE THE OFF-DIAGONAL ELEMENTS TO ZERO. THIS PROGRAM USES FOUR ARGUMENTS. A IS THE NAME OF A TWO-DIMENSIONAL ARRAY CONTAINING THE REAL, SYMMETRIC MATRIX IN ITS FIRST N ROWS AND COLUMNS. B IS THE NAME OF THE TWO-DIMENSIONAL ARRAY WHICH WILL CONTAIN THE EIGENVECTORS IN ITS FIRST N COLUMNS. N IS AN INTEGER VARIABLE OR CONSTANT GIVING THE ORDER OF THE MATRIX.

MV IS AN INTEGER VARIABLE OR CONSTANT WHICH MUST BE 0 OR 1.

IF IT IS 0 BOTH EIGENVECTORS AND EIGENVALUES ARE FORMED.

IF IT IS ONE ONLY THE EIGENVALUES ARE FOUND.

ENTER THE ORDER OF MATRIX AND THE MATRIX SEPARATED BY COMMAS.

=3,0.29239223E11,0.,0.

=0.,0.33611527E11,-0.57004232E10

=0.,-0.57004232E10,0.12976534E12

THE MATRIX IS

0.29239223E 11	0.	0.
0.	0.33611527E 11	-0.57004245E 10
0.	-0.57004232E 10	0.12976534E 12

EACH EIGENVALUE FOLLOWED BY CORRESPONDING EIGENVECTOR

0.29239223E 11		
0.10000000E 01	0.	0.

0.38256431E 11		
0.	0.99806542E 00	0.62172465E-01

0.13012144E 12		
0.	-0.62172465E-01	0.99806542E 00

TABLE 3-9 (Cont'd)

37 EIG

DO YOU WANT INSTRUCTIONS?

= 0

ENTER THE ORDER OF MATRIX AND THE MATRIX SEPARATED BY COMMAS.

= 3, 0.29239228E11, 0., 0.

= 0., 0.38611527E11, -0.57004232E10

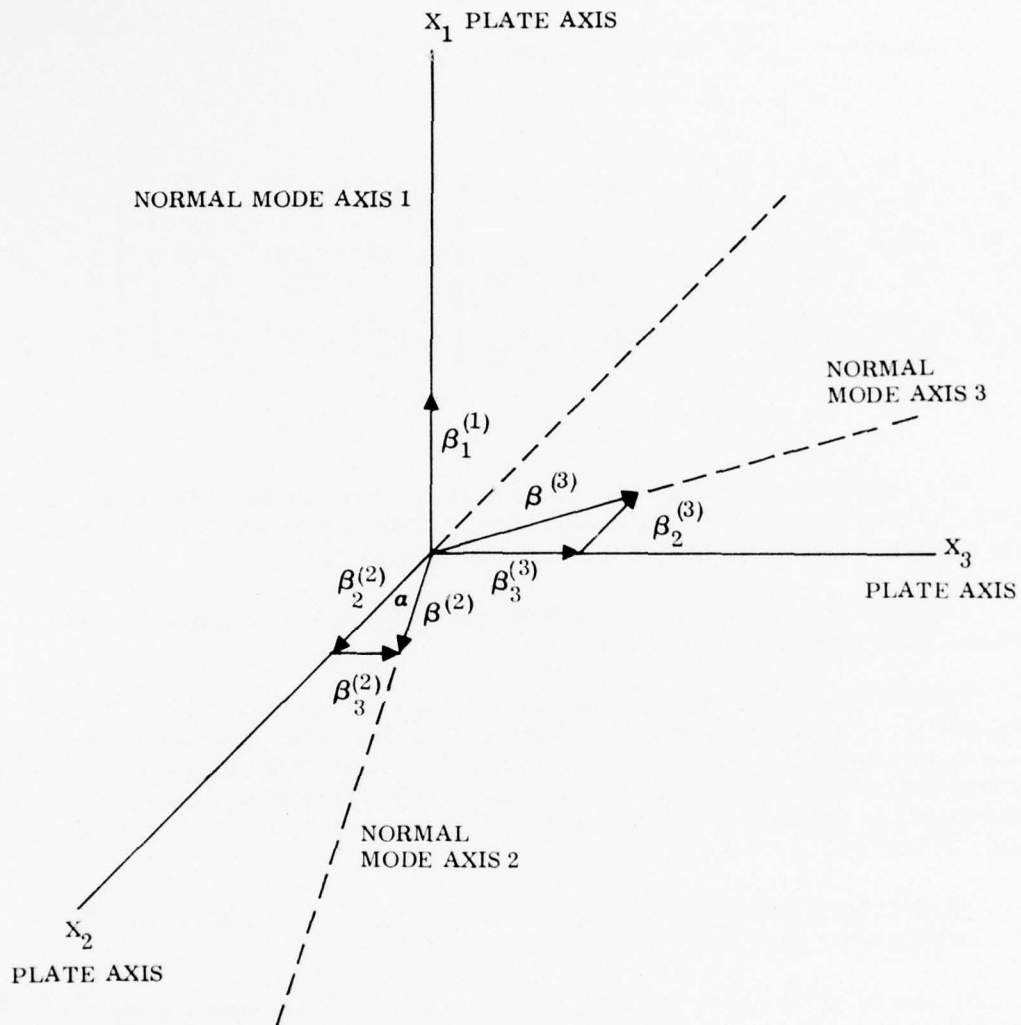
= 0., -0.57004245E10, 0.12975534E12

THE MATRIX IS

0.29239228E 11	0.	0.
0.	0.38611527E 11	-0.57004245E 10
0.	-0.57004232E 10	0.12975534E 12

EACH EIGENVALUE FOLLOWED BY CORRESPONDING EIGENVECTOR

0.29239228E 11		
0.10000000E 01	0.	0.
0.38256431E 11		
0.	0.99806542E 00	0.62172465E-01
0.13012144E 12		
0.	-0.62172455E-01	0.99806542E 00



ANGLE BETWEEN NORMAL MODE AXIS 2 AND X_2 IS α

ANGLE BETWEEN NORMAL MODE AXIS 2 AND X_3 IS $90 - \alpha$

$$\cos \alpha = \frac{\beta_2^{(2)}}{\beta^{(2)}} = \beta_2^{(2)} \quad \beta^{(2)} = 1 - \beta^{(1)} - \beta^{(3)}$$

$$\cos (90 - \alpha) = \sin \alpha = \frac{\beta_3^{(2)}}{\beta^{(2)}} = \beta_3^{(2)}$$

Figure 3-3. Relation of Normal Coordinate Axes to Plate Axes for AT-Cut Quartz with X_3 in the Thickness Direction

Since $[\beta]^{-1} [\beta] = 1$.

However if $[\beta]$ is an orthogonal matrix^[9]

β transpose = $[\beta]^T$ should be equal to $[\beta]^{-1}$;

hence,

$$\begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} \end{bmatrix} \begin{bmatrix} CB_{11} & CB_{12} & CB_{13} \\ CB_{21} & CB_{22} & CB_{23} \\ CB_{31} & CB_{32} & CB_{33} \end{bmatrix} \begin{bmatrix} \beta_1^{(1)} & \beta_1^{(2)} & \beta_1^{(3)} \\ \beta_2^{(1)} & \beta_2^{(2)} & \beta_2^{(3)} \\ \beta_3^{(1)} & \beta_3^{(2)} & \beta_3^{(3)} \end{bmatrix} = \begin{bmatrix} c^{(1)} & 0 & 0 \\ 0 & c^{(2)} & 0 \\ 0 & 0 & c^{(3)} \end{bmatrix} \quad (43)$$

The program Norm listed in Table 3-10 carries out Eq. (43) using, as inputs, the secular matrix coefficients, CB_{jk} , from CROT and the $\beta_k^{(i)}$ values from SYMEIG. It should return the correct values of $c^{(i)}$.

The direction of the plate thickness can be made to correspond to that used in CROT by changing line 120.

Table 3-11 shows the output of this program for the example in Tables 3-6 and 3-9. The output should be a diagonal matrix whose elements correspond to the eigenvalues in Table 3-9. Notice that the output is not diagonal. However a closer examination shows that the off-diagonal terms are errors in the last-listed digit, or beyond, of the diagonal terms and, hence, represent computer round-off errors.

C. TRANSFORMATION TO NORMAL COORDINATES, NORMAL MODE IMPEDANCE MATRIX AND PROGRAM VCOUP

At this point we carry out a transformation to normal coordinates^{[15],[16]} by defining the following quantities.

$$T_{3j}^o = \beta_i^{(j)} T_{3i} \quad (44)$$

$$u_j^o = \beta_i^{(j)} u_i \quad (45)$$

$$e_{33j}^o = \beta_i^{(j)} e_{33i} \quad (46)$$

TABLE 3-10. LISTING OF PROGRAM NORM FOR DIAGONALIZATION OF THE SECULAR MATRIX COEFFICIENTS

```

100C THIS PROGRAM CONVERTS THE COEFFICIENTS IN THE SECULAR
101C EQUATION TO DIAGONAL FORM
110 DIMENSION CB(3,3),B(3,3),CN(3,3)
120 I=3
130C THIS MATRIX STORES THE ELEMENTS OF THE SECULAR EQUATION
140 CB(1,1)=0.29239228E11
150 CB(1,2)=0.
160 CB(1,3)=0.
170 CB(2,1)=0.
180 CB(2,2)=0.38611527E11
190 CB(2,3)=-0.57004245E10
200 CB(3,1)=0.
210 CB(3,2)=-0.57004232E10
220 CB(3,3)=0.12976634E12
250C THIS MATRIX STORES THE EIGENVECTORS OF CB BY COLUMNS
251C THE COMPONENTS OF THE EIGENVECTORS ARE THE
252C DIRECTION COSINES BETWEEN THE NORMAL MODE
253C AXES AND THE PLATE AXES(B(1,1)=COSINE OF
254C ANGLE BETWEEN N1 AND X1,B(2,1) BETWEEN N1 AND X2,ETC)
260 B(1,1)=0.1E1
270 B(2,1)=0.
280 B(3,1)=0.
290 B(1,2)=0.
300 B(2,2)=0.99806542
310 B(3,2)=0.62172451E-1
320 B(1,3)=0.
330 B(2,3)=-0.62172451E-1
340 B(3,3)=0.99806542
370C THIS MATRIX SHOULD CONTAIN THE EIGENVALUES ON
371C THE MAIN DIAGONAL. IT REPRESENTS B1*CB*B
372C BT IS THE TRANSPOSE OF B
380 D0 10 L=1,3
390 D0 10 M=1,3
400 D0 10 J=1,3
410 D0 10 K=1,3
420 CN(L,M)=B(J,L)*B(K,M)*CB(J,K)+CN(L,M)
430 10 CONTINUE
440 PRINT 900,I
450 PRINT 910
460 PRINT 920,((CN(I,J),J=1,3),I=1,3)
500 STOP
510 900 FORMAT(/,'FOR X',I1,' IN THE THICKNESS DIRECTION')
520 910 FORMAT(/,5X,'THE ELEMENTS IN THE NORMAL COORDINATE',
521& ' EQUATION ARE')
530 920 FORMAT(/,3X,E15.8,6X,E15.8,6X,E15.8)
540 END

```

TABLE 3-11. OUTPUT OF NØRM FOR AT CUT QUARTZ WITH X₃ IN THE THICKNESS DIRECTION

331 31 02/20/77 15.592

FOR X₃ IN THE THICKNESS DIRECTION

THE ELEMENTS IN THE LOCAL COORDINATE POSITION ARE

0.22139228E+11	0.	0.
0.	0.31256437E+11	-0.12404836E+11
0.	0.31516140E+02	0.13012144E+12

Furthermore, since the $\beta_i^{(j)}$ are orthonormal, the inverse transformations are

$$T_{3j} = \beta_j^{(i)} T_{3i}^o \quad (47)$$

$$u_j = \beta_j^{(i)} u_i^o \quad (48)$$

$$e_{33j} = \beta_j^{(i)} e_{33i}^o \quad (49)$$

Equation (44) for example gives:

$$T_{31}^o = \beta_1^{(1)} T_{31} + \beta_2^{(1)} T_{32} + \beta_3^{(1)} T_{33} \quad (50)$$

In terms of these new quantities Eq. (1) for i in the x_3 direction becomes

$$T_{3j,3}^o = -\rho \omega^2 u_j^o \quad (51)$$

Returning to Eq. (12) for x_3 in the thickness direction (i.e. $i = 3$) we have,

$$\phi_{,33} = \left(\frac{e_{3k3}}{\epsilon_{33}^s} \right) u_{k,33}, \quad (52)$$

which can be integrated to give,

$$\phi = \left(\frac{e_{3k3}}{\epsilon_{33}^s} \right) u_k + a_3 x_3 + b_3 \quad (53)$$

This in turn can be substituted into Eq. (7) for $i = 3$, $k = 3$ to yield,

$$T_{3j} = c_{3jk}^E u_{k,3} + e_{33j} \left(\frac{e_{3k3}}{\epsilon_{33}} \right) u_{k,3} + e_{33j} a_3 . \quad (54)$$

Multiplying Eq. (54) by $\beta_j^{(i)}$ and using Eq. (15) leads to,

$$T_{3i}^0 = \beta_j^{(i)} T_{3j} = \bar{c}_{3jk}^E \beta_j^{(i)} u_{k,3} + \beta_j^{(i)} e_{33j} a_3 ; \quad (55)$$

but from Eq. (40) or Eq. (25),

$$\bar{c}_{3jk}^E \beta_j^{(i)} = c_{kj}^{(i)} \delta_{kj} \beta_j^{(i)} , \quad (\text{no sum over } i) \quad (56)$$

so that

$$T_{3i}^0 = c_{kj}^{(i)} \delta_{kj} \beta_j^{(i)} u_{k,3} + e_{33i}^0 a_3 \quad (\text{no sum over } i) \quad (57)$$

$$\text{or } T_{3i}^0 = c_{ij}^{(i)} \beta_j^{(i)} u_{j,3} + e_{33i}^0 a_3 \quad (\text{no sum over } i) ,$$

from which we obtain

$$T_{3i}^0 = c_{ij}^{(i)} u_{i,3}^0 + e_{33i}^0 a_3 \quad (\text{no sum over } i) . \quad (58)$$

This equation can be substituted into Eq. (51) to give

$$c_{ij}^{(i)} u_{i,33}^0 + \rho \omega^2 u_i^0 = 0 \quad (\text{no sum over } i)$$

or

$$\begin{aligned} c^{(1)} u_{1,33}^0 + \rho \omega^2 u_1^0 &= 0 \\ c^{(2)} u_{2,33}^0 + \rho \omega^2 u_2^0 &= 0 \\ c^{(3)} u_{3,33}^0 + \rho \omega^2 u_3^0 &= 0 . \end{aligned} \quad (59)$$

This clearly shows that the normal coordinate motions, u_i^0 , are uncoupled.

If use is made of the symmetry relations of Eq. (9), it is possible to write

$$\begin{aligned}
 e_{3k3} u_k &= (\beta_k^{(i)} e_{3i3}^o) (\beta_k^{(m)} u_m^o) \\
 &= \beta_k^{(i)} \beta_k^{(m)} e_{3i3}^o u_m^o \\
 &= \delta_{im} e_{3i3}^o u_m^o = e_{3i3}^o u_i^o,
 \end{aligned} \tag{60}$$

where the orthonormal properties of the $\beta_k^{(i)}$ in Eq. (37) are used.

This shows that

$$e_{3k3} u_k = e_{3k3}^o u_k^o \tag{61}$$

is invariant under the transformation.

Equation (53) for the potential then becomes

$$\phi = (e_{3k3}^o / \epsilon_{33}^s) u_k^o + a_3 x_3 + b_3, \tag{62}$$

which shows that the x_3 -directed component of an electric field, arising from particle displacement due to passage of a wave, reverses direction when the wave direction reverses.

It is now desirable to show that Eqs. (59) can be represented by transmission line equations. The standard lossless transmission line equations [17]

$$\begin{aligned}
 \frac{\partial V}{\partial z}(z; t) &= -L \frac{\partial I}{\partial t}(z; t) \\
 \frac{\partial I}{\partial z}(z; t) &= -C \frac{\partial V}{\partial t}(z; t),
 \end{aligned} \tag{63}$$

where L and C represent the inductance and capacitance per unit length, can be put in the desired form by the following procedure.

Let $V(z; t)$ and $I(z; t)$ take the same form as $u_1(x_1; t)$ in Eqs. (16) and (18), so that

$$\begin{aligned}
 V(z; t) &= V e^{+j\eta z} e^{+j\omega t} = V e^{j\omega(t - \frac{\eta}{\omega} z)} = V e^{j\omega(t - \frac{z}{v})} \\
 I(z; t) &= I e^{+j\eta z} e^{+j\omega t} = I e^{j\omega(t - \frac{\eta}{\omega} z)} = I e^{j\omega(t - \frac{z}{v})}
 \end{aligned} \tag{64}$$

represent sinusoidal time functions propagating in the z direction with a velocity, v .

Eqs. (64) can be substituted into (63) to yield

$$\begin{aligned} -j\eta V &= -j\omega LI \\ -j\eta I &= -j\omega CV \text{ or } I = \frac{\omega C}{\eta} V . \end{aligned} \quad (65)$$

If the second equation of Eq. (65) is substituted into the first,

$$-j\eta V = -j\omega \frac{2LCV}{\eta} \text{ or } \left(1 - \frac{\omega^2 LC}{\eta^2}\right) V = 0 . \quad (66)$$

Equation (66) can be satisfied for arbitrary values of V only if

$$\eta = \omega \sqrt{LC} . \quad (67)$$

The following identification between the velocity of propagation, v , and the wave number, η , must be made from the last form of Eq. (64):

$$v = \frac{\omega}{\eta} \text{ or } \eta = \frac{\omega}{v} \quad v = \frac{1}{\sqrt{LC}} . \quad (68)$$

If a transmission line characteristic impedance and admittance are defined as

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{Y_0} , \quad (69)$$

then, for an arbitrary wavenumber $\eta^{(i)}$, Eqs. (63) can be put in the form,

$$V_{,3}^{(i)} = -j\omega \sqrt{\frac{L^{(i)}}{C^{(i)}}} \sqrt{L^{(i)} C^{(i)}} I^{(i)} = -j\eta^{(i)} Z_0^{(i)} I^{(i)} \quad (70)$$

$$I_{,3}^{(i)} = -j\omega \sqrt{\frac{C^{(i)}}{L^{(i)}}} \sqrt{L^{(i)} C^{(i)}} V^{(i)} = -j\eta^{(i)} Y_0^{(i)} V^{(i)} ,$$

where it is assumed that the wave is propagating in the x_3 direction.

If the second of Eqs. (70) is differentiated with respect to x_3 and substituted into the first equation, $V^{(i)}$ can be eliminated, and vice versa, to yield

$$I_{,33}^{(i)} + \left(\eta^{(i)}\right)^2 I^{(i)} = 0 \quad (71)$$

$$V_{,33}^{(i)} + \left(\eta^{(i)}\right)^2 V^{(i)} = 0 .$$

Both of these equations are identical to Eq. (59) when Eq. (20) is used, so it should be possible, at least, to determine that there is a correspondence between the normal mode displacements and the transmission line variables.

Also, note that by writing Eqs. (16) and (18) in the same form as the last of Eq. (64), the acoustic wave velocity for each $\eta^{(i)}$ is given by

$$v^{(i)} = \frac{\omega}{\eta^{(i)}} = \sqrt{c^{(i)}/\rho} \quad (72)$$

Since we are free to make corresponding wave numbers of the transmission line and elastic waves equal, Eq. (59) amounts to using three transmission lines (one for each i), and supporting a wave propagating with the velocity, $v^{(i)}$ from Eq. (72), if the other variables in the acoustic wave equations can be made consistent with Eq. (70).

Equations (51) and (58) represent a set relating T_{3i}^0 and u_i^0 , which are very similar to the first-order differential equations of Eq. (70). However, the $e_{33i}^0 a_3$ term of Eq. (58) hinders this analogy. This term is piezoelectric in nature and a spatial constant; hence if T_{3i}^0 is separated into two parts by defining

$$T_{3i}^0 = \tilde{T}_{3i}^0 + \bar{T}_{3i}^0 \quad \text{with} \quad \bar{T}_{3i}^0 = e_{33i}^0 a_3, \quad (73)$$

then, because \bar{T}_{3i}^0 is a constant,

$$\bar{T}_{3i,3}^0 = 0, \quad (74)$$

so that Eqs. (51) and (58) become

$$\begin{aligned} \tilde{T}_{3j,3}^0 &= -\rho\omega^2 u_i^0 \\ c^{(i)} u_{i,3}^0 &= \tilde{T}_{3i}^0, \end{aligned} \quad (75)$$

which now has the structure of Eq. (70).

To complete the analogy between the acoustic waves and the transmission line it is necessary to pair corresponding quantities. For reasons, given in Ref. [1], the following choices are made:

$$\begin{aligned} V^{(i)} &= A \tilde{T}_{3i}^0 \\ I^{(i)} &= -\dot{u}_i^0 = -j\omega u_i^0 \\ Z_0^{(i)} &= A\rho v^{(i)} \end{aligned}$$

$$\begin{aligned}
 c^{(i)} &= \rho (v^{(i)})^2 \\
 Z_0^{(i)} &= 1/Y_0^{(i)} \\
 \eta^{(i)} &= \omega/v^{(i)}
 \end{aligned}
 \tag{76}$$

$I^{(i)}$ is the negative of the particle velocity in the normal coordinate system. The negative sign is a result of the convention for positive power flow in acoustics. If a positive tensile force is applied to an elastic body a positive outward particle velocity is produced but, under these conditions, power is said to produce a power flow into the body. Hence, the outward velocity associated with an inward power flow produces the negative sign.

The factor, A , in Eq. (76) is a portion of the area perpendicular to the wave propagation, i.e. normal to x_3 .

Still following Dr. Ballato's presentation, we now turn to an examination of the thickness modes of an electroded, piezoelectric crystal plate with traction-free surfaces, driven by an electric field in the thickness direction, (since this is a clever way to incorporate the electrical circuit into the transmission line model). Figure 3-4 is a sketch of the situation to be analyzed. The plate is laterally

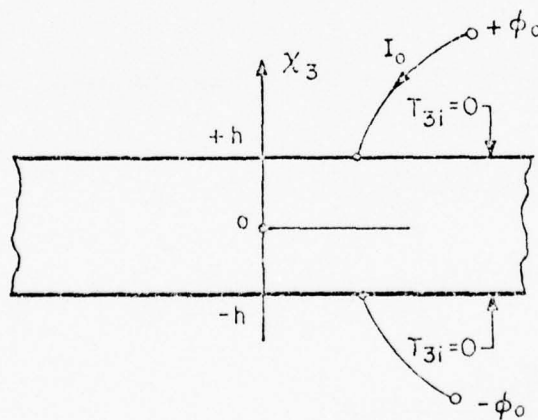


Figure 3-4. Unbounded, Traction-Free, Piezoelectric Plate. Thickness Excitation of Thickness Modes

unbounded, of thickness $2h$, with upper and lower surfaces at $x_3 = +h$ and $-h$, respectively. These surfaces are also assumed to be maintained at potentials $+\phi_0$ and $-\phi_0$, respectively. Lateral coordinates are of no interest and the $e^{j\omega t}$ time factor is ignored.

At the plate boundaries the conditions to be satisfied are:

$$\begin{aligned}
 T_{3j} &= 0 \text{ at } x_3 = \pm h \\
 \phi &= \pm \phi_0 \text{ at } x_3 = \pm h
 \end{aligned}
 \tag{77}$$

If Cramer's rule is applied to Eqs. (47) it is seen that, if the untransformed stress, T_{3j} , vanishes at the surfaces, the transformed stress, T_{3j}^0 , must also vanish since, to satisfy the orthogonality condition that the transpose of the β matrix is equal to its inverse, the determinant of the β 's must be non-zero. Hence

$$T_{3i}^0 = 0 \text{ at } x_3 = \pm h . \quad (78)$$

We seek a solution of Eq. (59) that satisfies Eqs. (77) and (78) when inserted into Eqs. (58) and (62). From now on until further notice the Einstein convention is dropped and no sum is to be taken unless it is specifically indicated. To this end we select

$$u_i^0 = U_i \sin \eta^{(i)} x_3 , \quad (79)$$

so that

$$u_{i,3}^0 = U_i \eta^{(i)} \cos \eta^{(i)} x_3 \text{ and } u_{i,33}^0 = -U_i (\eta^{(i)})^2 \sin \eta^{(i)} x_3 ,$$

which satisfies Eq. (59), provided

$$(\eta^{(i)})^2 = \frac{\rho \omega^2}{c^{(i)}} , \quad (80)$$

which it does, from Eq. (20). If Eq. (79) is now placed into Eq. (58) and use is made of Eq. (78),

$$\begin{aligned} T_{3i}^0 &= c^{(i)} u_{i,3}^0 + e_{33i}^0 a_3 \\ &= c^{(i)} \eta^{(i)} U_i \cos \eta^{(i)} x_3 + e_{33i}^0 a_3 = 0 \text{ at } x_3 = \pm h . \end{aligned} \quad (81)$$

Hence,

$$U_i = \frac{-e_{33i}^0 a_3}{c^{(i)} \eta^{(i)} \cos \eta^{(i)} h} . \quad (82)$$

Using Eq. (62) along with Eq. (79) yields:

$$\begin{aligned} \phi &= \left(\frac{e_{3k3}^0}{\epsilon_{33}^s} \right) u_k^0 + a_3 x_3 + b_3 \quad (\text{sum over } k) \\ &= \frac{e_{313}^0}{\epsilon_{33}^s} U_1 \sin \eta^{(1)} x_3 + \frac{e_{323}^0}{\epsilon_{33}^s} U_2 \sin \eta^{(2)} x_3 \end{aligned} \quad (83)$$

$$+ \frac{e_{333}^0}{\epsilon_{33}^s} U_3 \sin \eta^{(3)} x_3 + a_3 x_3 .$$

Substituting from Eq. (82) yields:

$$\begin{aligned} \phi = & \frac{-e_{313}^0 e_{331}^0 a_3 \sin \eta^{(1)} x_3}{\epsilon_{33}^s c^{(1)} \eta^{(1)} \cos \eta^{(1)} h} - \frac{e_{323}^0 e_{332}^0 a_3 \sin \eta^{(2)} x_3}{\epsilon_{33}^s c^{(2)} \eta^{(2)} \cos \eta^{(2)} h} \\ & - \frac{e_{333}^0 e_{333}^0 a_3 \sin \eta^{(3)} x_3}{\epsilon_{33}^s c^{(3)} \eta^{(3)} \cos \eta^{(3)} h} + a_3 x_3 + b_3 \end{aligned} \quad (84)$$

Letting $\phi = \pm \phi_0$ at $x_3 = \pm h$ leads to

$$\phi_0 + (-\phi_0) = 0 = 2b_3 .$$

Thus

$$b_3 = 0 \quad (85)$$

and

$$\phi_0 = a_3 h \left(1 - \frac{1}{h} \sum_{i=1}^3 \left(\frac{e_{3i3}^0 e_{33i}^0}{\epsilon_{33}^s c^{(i)}} \right) \frac{\tan \eta^{(i)} h}{\eta^{(i)}} \right) . \quad (86)$$

This leads to:

$$a_3 = \frac{\phi_0/h}{\left\{ 1 - \sum_{i=1}^3 \frac{e_{3i3}^0 e_{33i}^0}{\epsilon_{33}^s c^{(i)}} \frac{\tan \eta^{(i)} h}{\eta^{(i)}} \right\}}$$

Now we define:

$$(k^{(i)})^2 = \frac{e_{3i3}^0 e_{33i}^0}{\epsilon_{33}^s c^{(i)}} , \quad (88)$$

where no sum over i is taken, so that

$$(k^{(1)})^2 = \frac{e_{313}^o e_{331}^o}{\epsilon_{33}^s c^{(1)}} = \frac{(e_{331}^o)^2}{\epsilon_{33}^s c^{(1)}} , \quad (k^{(2)})^2 = \frac{(e_{332}^o)^2}{\epsilon_{33}^s c^{(2)}} , \quad (k^{(3)})^2 = \frac{(e_{333}^o)^2}{\epsilon_{33}^s c^{(3)}} ,$$

when use is made of the symmetry relations for e_{ijk}^o . Here $k^{(i)}$ is the piezoelectric coupling coefficient for the i^{th} mode in the Thickness Excitation of Thickness Mode (TETM) case.

Putting Eq. (82) into Eq. (58) and Eq. (79) along with Eq. (87) leads to

$$T_{3i}^o = e_{333i}^o a_3 \left(1 - \frac{\cos \eta^{(i)} x_3}{\cos \eta^{(i)} h} \right) \quad (89)$$

$$u_i^o = \frac{-e_{33i}^o \sin \eta^{(i)} x_3}{hc^{(i)} \eta^{(i)} \cos \eta^{(i)} h} \left\{ \frac{\phi}{1 - \sum_{j=1}^3 (k^{(j)})^2 \frac{\tan \eta^{(j)} h}{\eta^{(j)} h}} \right\} \quad (90)$$

Substituting Eq. (53) into Eq. (8), specialized to the 3-direction by letting $i = k = 3$, leads to:

$$D_3 = -\epsilon_{33}^s a_3 = \frac{-\epsilon_{33}^s \phi_0 / h}{\left\{ 1 - \sum_{j=1}^3 (k^{(j)})^2 \frac{\tan \eta^{(j)} h}{\eta^{(j)} h} \right\}} \quad (91)$$

Now, consider a portion of the plate having a lateral area, A , as introduced in connection with Eq. (76). The current I_0 , intercepted by this area is equal to

$$I_0 = -A \dot{D}_3 = -j \omega A D_3 . \quad (92)$$

The minus sign is here because, at the positive (upper) electrode, the surface normal points in the direction of minus x_3 within the crystal.

Considering this plate to be an electrical network, one sees an admittance

$$Y_{in} (\text{TETM}) = I/E = I_0 / 2\phi_0 . \quad (93)$$

Defining a capacitance C_0 by

$$C_0 = A \epsilon_{33}^s / 2h , \quad (94)$$

Equation (93) becomes

$$Y_{in}^{(TETM)} = \frac{j\omega C_0}{\left\{ 1 - \sum_{p=1}^3 (k^{(p)})^2 \frac{\tan \eta^{(p)} h}{\eta^{(p)} h} \right\}}, \quad (95)$$

by substituting Eqs. (86), (88), (91), (92) and (94).

We now assume $Y_{in}^{(TETM)}$ can be represented by the configuration shown in Figure 3-5. Then

$$\begin{aligned} Y_{in}^{(TETM)} &= j\omega C_0 + \frac{(-j\omega C_0)Y_{(TL)}}{Y_{(TL)} - j\omega C_0} \\ &= \frac{j\omega C_0}{\left(1 - \frac{Y_{(TL)}}{j\omega C_0} \right)} \end{aligned} \quad (96)$$

Equations (95) and (96) are identical if

$$Y_{(TL)} = j\omega C_0 \left(\sum_{p=1}^3 (k^{(p)})^2 \frac{\tan \eta^{(p)} h}{\eta^{(p)} h} \right) \quad \left(\begin{array}{l} \text{only} \\ \text{indicated sum} \\ \text{over } p \end{array} \right) \quad (97)$$

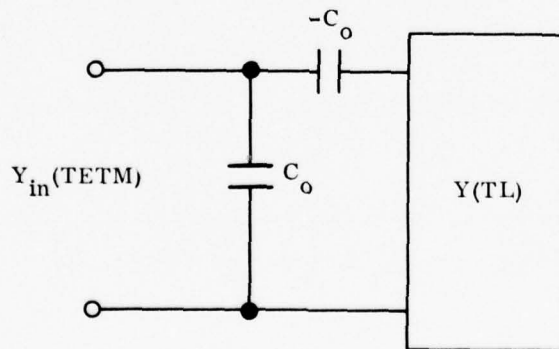


Figure 3-5. Assumed Input Admittance Network

This can be interpreted as the sum of three admittances in parallel, with each admittance containing one tangent function; i.e.,

$$Y_{(TL)} = \sum_{i=1}^3 Y_{TL}^{(i)}, \quad (98)$$

where

$$Y_{TL}^{(i)} = j\omega C_0 \left(\frac{e_{3i3}^o e_{33i}^o}{\epsilon_{33}^s c^{(i)}} \right) \frac{\tan \eta^{(i)} h}{\eta^{(i)} h}.$$

Using the variables defined in Eq. (76) this can be written as

$$\begin{aligned} Y_{TL}^{(i)} &= \frac{jA}{2h^2} \frac{e_{3i3}^o e_{33i}^o}{\rho v^{(i)}} \tan \eta^{(i)} h = \frac{jA}{2h^2} e_{3i3}^o \frac{e_{33i}^o \tan \eta^{(i)} h}{Z_0} \\ &= \frac{A^2}{4h^2} e_{3i3}^o e_{33i}^o \frac{2 \tan \eta^{(i)} h}{-jZ_0^{(i)}} = \frac{A^2}{4h^2} e_{3i3}^o e_{33i}^o \frac{1}{-j \frac{Z_0}{2} \cot \eta^{(i)} h}. \end{aligned} \quad (99)$$

The input impedance of an open-circuited lossless transmission line of length, h , characteristic impedance, Z_0 , and propagation constant, η , is

$$Z_{0c} = -jZ_0 \cot \eta h; \quad (100)$$

and the impedance of two of these lines in parallel is $-j \frac{Z_0}{2} \cot \eta h$.

Hence, we are tempted to represent $Y_{TL}^{(i)}$ as shown in Figure 3-6. For this configuration

$$Y_{TL}^{(i)} = \frac{1}{Z_{TL}^{(i)}} = \frac{1}{\frac{1}{(n_i)^2} \left(-j \frac{Z_0}{2} \cot \eta^{(i)} h \right)} \quad (101)$$

Equation (101) will agree with Eq. (99) if

$$n_i = \frac{A \sqrt{e_{3i3}^o e_{33i}^o}}{2h} = \frac{A e_{33i}^o}{2h}. \quad (102)$$

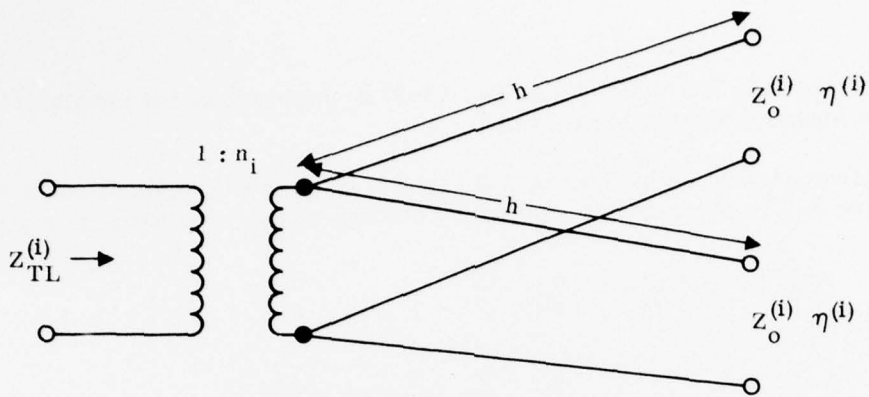


Figure 3-6. Assumed Configuration for $Y_{TL}^{(i)}$

Figure 3-6 can be redrawn as shown in Figure 3-7, since, in this circuit, no current will flow at the center of the transmission line so the wires in Figure 3-7 can be cut.

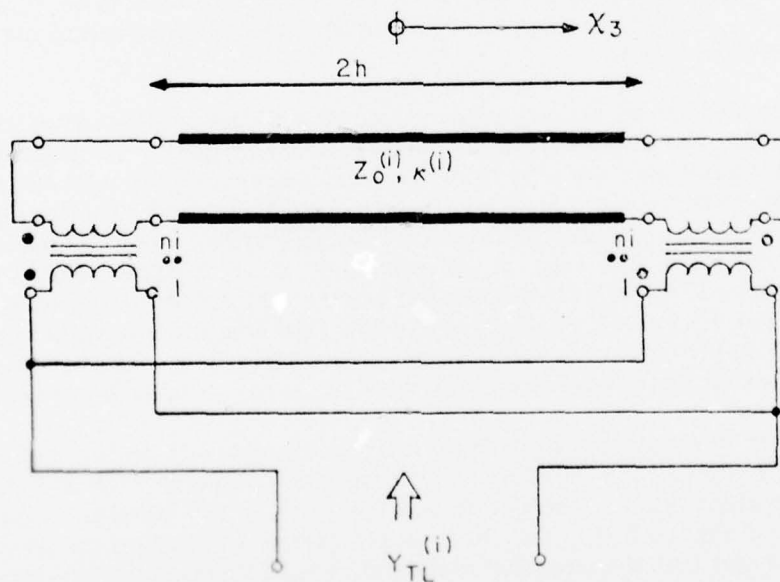


Figure 3-7. Traction-Free Plate. Representation of Single Thickness Mode, Electrical Input Circuit Omitted

In Figures 3-6 and 3-7 and Eq. (102) n_i represents the piezoelectric turns ratio in Mason's Equivalent Circuit.

In connection with Figures 3-6 and 3-7 it should be pointed out that, if $-C_0$ in Figure 3-5 is ignored, the secondary voltage of the transformer is given by:

$$E_S^{(i)} = n_i 2\phi_0 = A e_{33i}^0 \frac{\phi_0}{h}$$

$$= A e_{33i}^0 a_3 \left(1 - \sum_{j=1}^3 k^{(j)} \frac{\tan \eta^{(j)} h}{\eta^{(j)} h} \right) \quad (103)$$

Hence, $E_S^{(i)}$ is proportional to $A e_{33i}^0 a_3$ but, by Eq. (73), $E_S^{(i)}$ is proportional to $A \tilde{T}_{3i}^0$. Thus, while the voltage variables associated with the waves on the transmission lines are $A \tilde{T}_{3i}^0$, the electromechanical transformer's secondary voltages can be identified with $A \tilde{T}_{3i}^0$, so that the circuit accounts for the total stress, $A T_{3i}^0$.

Dr. Ballato goes into several ramifications on drawing the network as shown in Figure 3-7; these will not be dealt with here. All that we emphasize here is that, if the transformers are placed at the ends, as shown, then the polarity dots must be as indicated, in order for this to be a valid representation of Figure 3-6. Figure 3-8 shows the complete 3-mode equivalent circuit for this traction-free case.

Figure 3-9 shows the resulting network with the traction-free condition removed so that the short circuits of Figure 3-8 are gone. Because it pertains to normal coordinates, the port variables are superscripted with the degree sign. The port variables are numbered so that the left side (bottom or $-h$ side of the crystal plate) of the transmission line supporting mode (i) leads to port $(i)^0$, while the right side (top or $+h$ side of the crystal plate) leads to port $(i+3)^0$. Ports 1^0 to 6^0 are mechanical ports and port 7^0 is an electrical port. We also define V_π^0 and I_π^0 ($\pi = 1, 2, \dots, 7$) as the voltages and currents appropriate to port π , with the sign conventions shown in Figure 3-9. At present, no attempt is made to match these port variables with the stresses and displacements.

In order to obtain the impedance matrix appropriate to this network it is convenient to replace the distributed transmission lines in Figure 3-9 with their lumped equivalent form. The equivalent tee form of the lossless transmission line, shown in Figure 3-10, may be ascertained by evaluating its open circuit, short circuit and transfer impedances, and comparing them to the transmission line equations. Using this equivalence Figure 3-9 can be redrawn as shown in Figure 3-11.

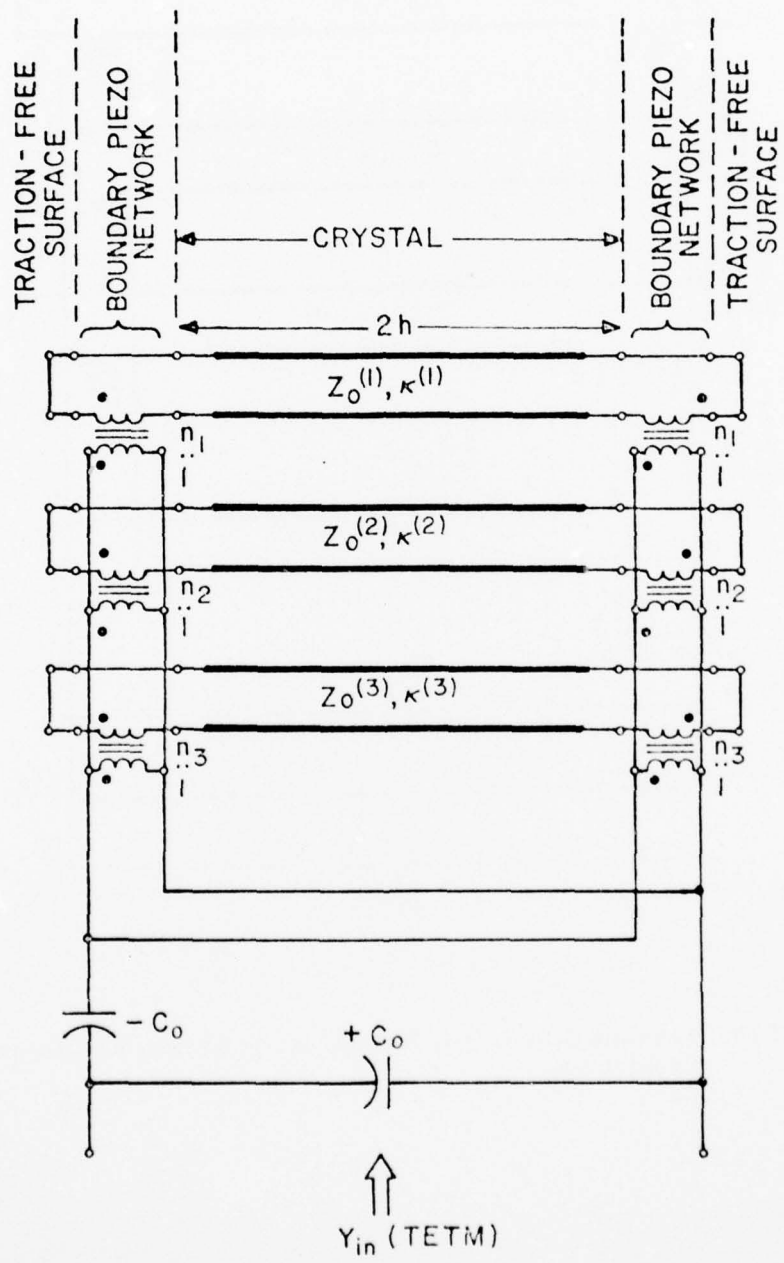


Figure 3-8. Equivalent Network Analog Representation of Traction-Free Plate, TETM

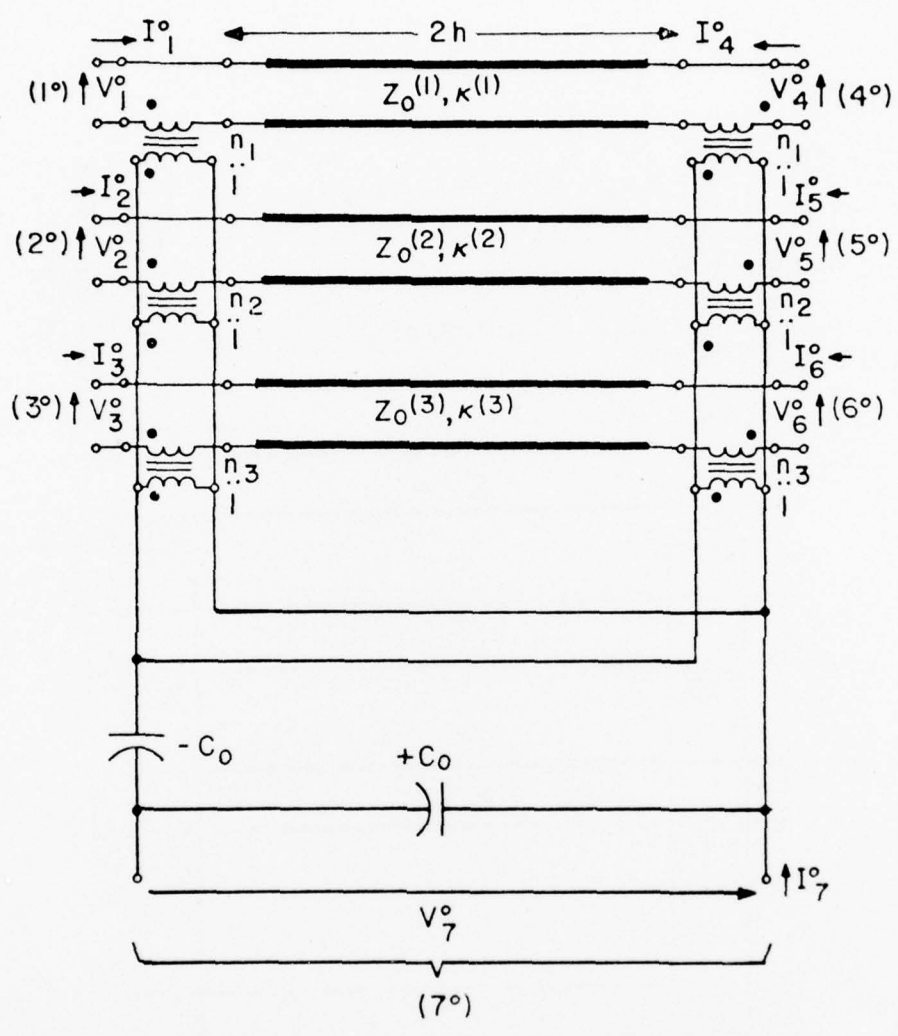
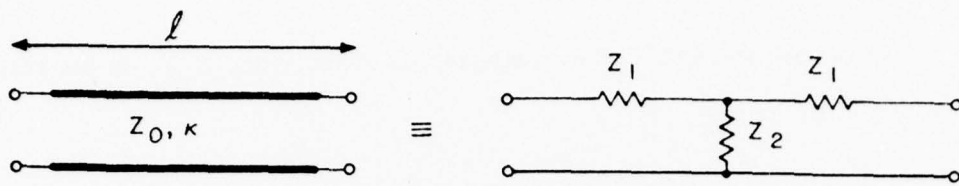


Figure 3-9. Possible Seven-Port Normal Mode Equivalent Circuit of a Crystal Plate



$$Z_1 = \frac{Z_0}{j \sin \theta} (\cos \theta - 1) = j Z_0 \tan(\theta/2)$$

$$Z_2 = \frac{Z_0}{j \sin \theta} ; \theta = \kappa l$$

Figure 3-10. Lumped, Tee, Form of a Transmission-Line Section

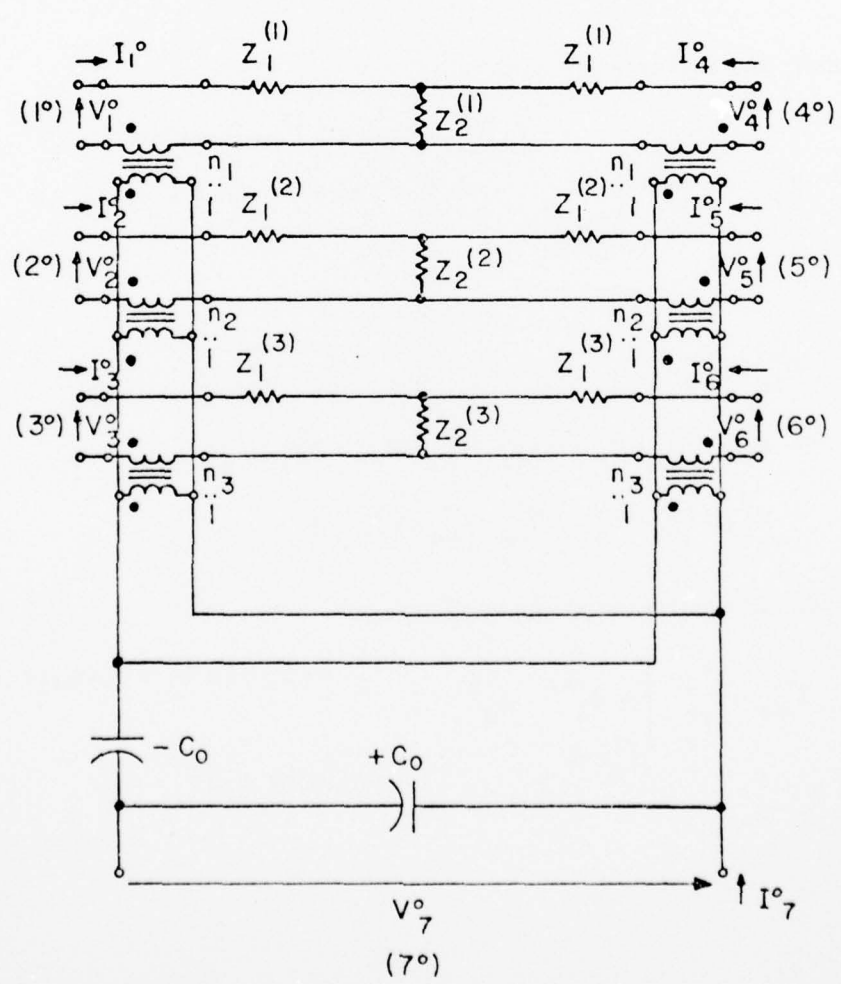


Figure 3-11. Lumped Circuit for Evaluating TETM Electro-Mechanical Impedance Matrix

It is now necessary to evaluate terms of the type, $Z_{\pi\xi}^0$, in the relation

$$V_{\pi}^0 = Z_{\pi\xi}^0 I_{\xi}^0, \quad (104)$$

where the Greek indices range from 1 to 7. Because this is a lumped, linear, passive, bilateral network, and because of the sign convention for currents and voltage shown in Figure 3-11, the overall matrix will be symmetrical about its principal diagonal. Furthermore, all mechanical port-driving point impedances will be identical in form, differing only in the mode index number. The same will be true for all mechanical transfer impedances connecting ports on the same transmission line. Likewise all electromechanical transfer impedances will differ only in mode index number.

When the electrical port (7^0) is open-circuited so that $I_7^0 = 0$, the two capacitances, $+C_0$ and $-C_0$, add together to produce shorts at the piezoelectric transformers. This completely decouples the transmission lines from each other.

Using this last effect means that:

$$Z_{\pi\xi}^0 = \left. \frac{V_{\pi}^0}{I_{\xi}^0} \right|_{I_{\beta}^0=0} \quad \begin{array}{l} I_{\beta}^0 \text{ means all } I_{\beta}^0 = 0 \text{ except } \beta = \xi \\ \xi = 1 \text{ to } 6 \text{ except } \xi \neq \pi \neq \pi \pm 3 \end{array} \quad (105)$$

Hence

$$\begin{aligned} Z_{12}^0 &= Z_{13}^0 = Z_{15}^0 = Z_{16}^0 = Z_{21}^0 = Z_{23}^0 = Z_{24}^0 = Z_{26}^0 = Z_{31}^0 = Z_{32}^0 \\ &= Z_{34}^0 = Z_{35}^0 = Z_{42}^0 = Z_{43}^0 = Z_{45}^0 = Z_{46}^0 = Z_{51}^0 = Z_{53}^0 \\ &= Z_{54}^0 = Z_{56}^0 = Z_{61}^0 = Z_{62}^0 = Z_{64}^0 = Z_{65}^0 = 0 \end{aligned}$$

The mechanical driving point impedances are

$$Z_{\pi\pi}^0 = \left. \frac{V_{\pi}^0}{I_{\pi}^0} \right|_{I_{\beta}^0=0} = Z_1^{(k)} + Z_2^{(k)} \quad \begin{array}{l} I_{\beta}^0 \text{ means all } I_{\beta}^0 = 0 \text{ except } \beta = \pi \\ k = \pi \text{ or } \pi - 3 \end{array}$$

(106)

$$\begin{aligned}
Z_{11}^0 = Z_{44}^0 &= \frac{Z_o^{(1)}}{j \tan \theta_1} \\
Z_{22}^0 = Z_{55}^0 &= \frac{Z_o^{(2)}}{j \tan \theta_2} \\
Z_{33}^0 = Z_{66}^0 &= \frac{Z_o^{(3)}}{j \tan \theta_3} ,
\end{aligned}
\tag{106}$$

(continued)

where $\theta_i = 2h\eta^{(i)}$.

The mechanical transfer impedances between ports on the same line are:

$$Z_{\pi; \pi+3}^0 = \left. \frac{V_{\pi}^0}{I_{\pi+3}^0} \right|_{I_{\beta}^0 = 0} = Z_2^{(k)}$$

I_{β}^0 means all $I_{\beta}^0 = 0$ except $\beta = \pi + 3$
 $\pi = 1$ to 3
 $k = \pi$

$$Z_{\pi; \pi+3}^0 = Z_{\pi+3; \pi}^0$$

$$\begin{aligned}
Z_{14}^0 = Z_{41}^0 &= \frac{Z_o^{(1)}}{j \sin \theta_1} \\
Z_{25}^0 = Z_{52}^0 &= \frac{Z_o^{(2)}}{j \sin \theta_2} \\
Z_{36}^0 = Z_{63}^0 &= \frac{Z_o^{(3)}}{j \sin \theta_3}
\end{aligned}
\tag{107}$$

This accounts for all the impedances except those related to the electrical port.

With all the mechanical ports open-circuited all the electrical mechanical transformers are open-circuited, so that no current can flow through $-C_0$. Hence the electrical port driving point impedance is

$$Z_{77}^0 = \left. \frac{V_7^0}{I_7^0} \right|_{I_{\beta=0, \beta \neq 7}^0} = \frac{1}{j\omega C_0}
\tag{108}$$

The electromechanical transfer impedances are

$$Z_{7;\pi}^0 = \frac{V_7^0}{I_\pi^0} \left| I_\beta^0 = 0, \beta \neq \pi \right. \quad \text{and} \quad Z_{\pi;7}^0 = \frac{V_\pi^0}{I_7^0} \left| I_\beta^0 = 0, \beta \neq 7 \right. \quad (109)$$

With all the ports open-circuited except mechanical port π , the only current flowing is that at port π . This current must flow through the electromechanical transformer closest to port π with turns ratio 1 to n_π or 1 to $n_{\pi-3}$. This produces a primary current of $n_\pi I_\pi$ or $n_{\pi-3} I_\pi$ flowing out of the polarity dot terminal, since $I_7^0 = 0$. This current must flow through C_0 and produce a voltage

$\frac{n_\pi I_\pi}{j\omega C_0}$ or $\frac{n_{\pi-3} I_\pi}{j\omega C_0}$. A similar argument can be employed to determine $Z_{\pi;7}^0$, since I_7^0 must flow through C_0 to produce a primary voltage, $-\frac{I_7^0}{j\omega C_0}$, or a secondary voltage, $-\frac{n_\pi I_7^0}{j\omega C_0}$ or $-\frac{n_{\pi-3} I_7^0}{j\omega C_0}$, in relation to the polarity dots of the transformers. This voltage appears at port π as a voltage, $V_\pi = \frac{n_\pi I_7^0}{j\omega C_0}$ or $\frac{n_{\pi-3} I_7^0}{j\omega C_0}$.

$$\begin{aligned} Z_{17}^0 &= Z_{47}^0 = Z_{71}^0 = Z_{74}^0 = \frac{n_1}{j\omega C_0} \\ Z_{27}^0 &= Z_{57}^0 = Z_{72}^0 = Z_{75}^0 = \frac{n_2}{j\omega C_0} \\ Z_{37}^0 &= Z_{67}^0 = Z_{73}^0 = Z_{76}^0 = \frac{n_3}{j\omega C_0} \end{aligned} \quad (110)$$

Now that all the terms in this matrix have been evaluated, it is necessary to show it represents the normal coordinate matrix for a piezoelectric plate.

Following the earlier analogy to a transmission line in Eq. (76), let

$$\begin{aligned} V_\pi^0 &= AT_{3i}^0 \quad (-h) \quad (\pi = i = 1, 2, 3) \\ V_\pi^0 &= AT_{3i}^0 \quad (+h) \quad (\pi = i + 3 = 4, 5, 6) \\ V_\pi^0 &= V_7^0 \quad (\pi = 7) \end{aligned} \quad (111)$$

where T_{3i}^0 ($\pm h$) refers to the values of T_{3i}^0 at the upper or top of the plate and the lower (bottom) surfaces of the plate, respectively. The choice of V 's here pertains to the total stress, T_{3i}^0 , and not simply the wavy portion, \tilde{T}_{3i}^0 , as in Equation (76).

The currents are taken as

$$\begin{aligned}
 I_{\xi}^0 &= -\dot{u}_i^0(-h) = -j\omega u_i^0(-h) \quad (\xi = i = 1, 2, 3) \\
 I_{\xi}^0 &= +u_i^0(+h) = +j\omega u_i^0(+h) \quad (\xi = i + 3 = 4, 5, 6) \\
 I_{\xi}^0 &= I_7^0 \quad (\xi = 7) ,
 \end{aligned}
 \tag{112}$$

where $u_i^0(\pm h)$ again refers to values of u_i^0 at $x_3 = \pm h$. The reason for the sign reversal in the equations for I_{ξ}^0 , above, is that it is desirable to define the port currents as being directed into the positive voltage terminal of a port, while the transmission line equations (which Eq. (76) represents) do not allow the sense of the current to change with respect to the voltage, as the wave progresses down the line.

With these choices of variables the impedance matrix elements depend on quotients of stress components and components of displacement in the normal-coordinate system, and are given by,

$$Z_{\pi \xi}^0 = V_{\pi}^0 / I_{\xi}^0 ,
 \tag{113}$$

with all currents equal to zero except for I_{ξ}^0 .

With these definitions, when all the currents except I_7 are equal to zero we are forcing all the displacement components, at the top and bottom faces of the plate, in the normal coordinate framework, to be zero. This corresponds to completely clamping the plate at the top and bottom surfaces so that the plate cannot move. Therefore we do not expect Z_{77}^0 to be the reciprocal of $Y_{in}^{(TETM)}$ in Eq. (95), since it was derived on the traction-free basis, where the stresses vanished in the normal coordinate system and, hence, for that case the plate was completely free to move.

Letting u_i^0 ($i = 1, 2, 3$) to be identically zero everywhere, satisfies Eq. (59) and reduces Eq. (62) to

$$\phi = a_3 x_3 + b_3 .
 \tag{114}$$

The boundary condition is that $\phi = \pm \phi_0$ at $x_3 = \pm h$ can only be satisfied if

$$b_3 = 0
 \tag{115}$$

$$a_3 = \phi_0 / h .$$

Specializing Eq. (8) to the 3-direction by letting $i = k = 3$ and then, substituting Eq. (53), yields

$$D_3 = -\epsilon_{33}^S a_3 \quad (116)$$

For this case, using Eq. (92),

$$\begin{aligned} I_7^0 &= -j\omega A D_3 = \\ &= j\omega A \epsilon_{33}^S a_3 = j\omega A \frac{\epsilon_{33}^S}{h} \phi_0 \quad (117) \end{aligned}$$

and

$$V_7^0 = 2\phi_0 \quad (113)$$

So Eq. (113) becomes

$$Z_{77}^0 = \frac{2\phi_0}{\frac{j\omega A \epsilon_{33}^S \phi_0}{h}} = \frac{1}{j\omega C_0} \quad (118)$$

where Eq. (94) was employed.

The remaining impedances require one of the u_i^0 to be finite and the other two to be zero. This obviously satisfies Eq. (59) for the two $u_i^0 = 0$ components and makes four of the mechanical currents equal to zero. The fifth component of mechanical current can be made to vanish if the non-zero u_i^0 is chosen to be zero at the appropriate surface ($\pm h$) and non-zero at the other ($\mp h$). This can be accomplished by choosing the non-zero u_i^0 as

$$u_i^0 = U_i \sin \eta^{(i)} (h \pm x_3) \quad (119)$$

However, we also require, for these impedances, that $I_7^0 = 0$. Using Eqs. (92) and (116) this means that

$$I_7^0 = -j\omega A D_3 = j\omega A \epsilon_{33}^S a_3 = 0 \quad (120)$$

or $a_3 = 0$.

Substituting this requirement into Eq. (58) leads to

$$T_{3i}^0 = c^{(i)} u_{i,3}^0, \quad (121)$$

for the case under consideration.

Hence, the normal coordinate stresses corresponding to the zero components of normal coordinate displacements, u_i^0 , of mode index, i , are zero and so are V_π^0 , corresponding to these mode indexes from Eq. (111), and the corresponding $Z_{\pi\xi}^0$ of Eq. (113).

For example, take

$$u_2^0 = u_3^0 = 0 \quad (122)$$

$$u_1^0 = U_1 \sin \eta^{(1)} (h-x_3),$$

so that Eq. (112) gives $I_2^0 = I_3^0 = I_5^0 = I_6^0 = 0$, and let $I_7^0 = 0$. Eq. (121) requires that

$$T_{32}^0 = T_{33}^0 = 0, \quad (123)$$

so that, from Eq. (111),

$$V_2^0 = V_3^0 = V_5^0 = V_6^0 = 0. \quad (124)$$

The choice for u_1^0 in Eq. (122) leads to

$$I_1^0 = -j\omega u_1^0(-h) = -j\omega U_1 \sin(2h\eta^{(1)}) = -j\omega U_1 \sin \Theta_1$$

$$I_4^0 = j\omega u_1^0(+h) = 0 \quad (125)$$

$$T_{31}^0 = c^{(1)} u_{1,3}^0 = -c^{(1)} U_1 \eta^{(1)} \cos \eta^{(1)} (h-x_3).$$

Substituting this value of T_{31}^0 into Eq. (111) gives

$$V_1^0 = A T_{31}^0(-h) = -c^{(1)} \eta^{(1)} A U_1 \cos(\eta^{(1)} 2h) = -\eta^{(1)} c^{(1)} A U_1 \cos \Theta_1$$

$$V_4^0 = A T_{31}^0(+h) = -c^{(1)} \eta^{(1)} U_1 A. \quad (126)$$

Then, from Eqs. (113) and (124),

$$Z_{21}^0 = Z_{31}^0 = Z_{51}^0 = Z_{61}^0 = 0 . \quad (127)$$

If, in place of u_1^0 in Eq. (122), we had chosen u_2^0 to be finite and let $u_1^0 = 0$, then $V_1^0 = V_3^0 = V_4^0 = V_6^0 = 0$ in Eq. (124), while $I_2^0 \neq 0$ and $I_5^0 = 0$ in Eq. (125), so that

$$Z_{12}^0 = Z_{32}^0 = Z_{42}^0 = Z_{62}^0 = 0 . \quad (128)$$

Repeating the argument, with u_3^0 finite, yields

$$Z_{13}^0 = Z_{23}^0 = Z_{43}^0 = Z_{53}^0 = 0 . \quad (129)$$

Returning, now, to the finite impedances resulting from the choice in Eq. (122), the values in Eqs. (125) and (126) lead to

$$Z_{11}^0 = \frac{V_1^0}{I_1^0} = \frac{\eta^{(1)} c^{(1)} A \cos \Theta_1}{j \omega \sin \Theta_1} = \frac{Z_o^{(1)}}{j \tan \Theta_1} \quad (130)$$

$$Z_{41}^0 = \frac{V_4^0}{I_1^0} = \frac{c^{(1)} \eta^{(1)} A}{j \omega \sin \Theta_1} = \frac{Z_o^{(1)}}{j \sin \Theta_1} \quad (131)$$

where definitions in Eq. (76) have been employed and $\Theta_i = 2h\eta^{(i)}$.

If the choice in (122) had been

$$u_1^0 = U_1 \sin \eta^{(1)} (h + x_3) \quad (132)$$

the result would have been

$$\begin{aligned} I_1^0 &= -j \omega u_1^0 (-h) = 0 \\ I_4^0 &= j \omega u_1^0 (+h) = j \omega U_1 \sin (\eta^{(1)} 2h) = j \omega U_1 \sin \Theta_1 \\ T_{31}^0 &= c^{(1)} U_1 \eta^{(1)} \cos ((h + x_3) \eta^{(1)}) . \end{aligned} \quad (133)$$

This leads to

$$\begin{aligned}
 V_1^0 &= A T_{31}^0 (-h) = c^{(1)} \eta^{(1)} U_1 A \\
 V_4^0 &= A T_{31}^0 (+h) = c^{(1)} \eta^{(1)} A U_1 \cos (\eta^{(1)} 2h) .
 \end{aligned} \tag{134}$$

Substituting these values into Eq. (113) leads to

$$\begin{aligned}
 Z_{44}^0 &= \frac{V_4^0}{I_4^0} = \frac{Z_o^{(1)}}{j \tan \Theta_1} = Z_{11}^0 \\
 Z_{14}^0 &= \frac{V_1^0}{I_4^0} = \frac{Z_o^{(1)}}{j \sin \Theta_1} = Z_{41}^0 ,
 \end{aligned} \tag{135}$$

while the requirement that $u_2^0 = u_3^0 = 0$ still requires $V_2^0 = V_3^0 = V_5^0 = V_6^0 = 0$. Hence,

$$Z_{24}^0 = Z_{34}^0 = Z_{54}^0 = Z_{64}^0 = 0 . \tag{136}$$

It is now obvious that repeating this procedure, using u_2^0 and u_3^0 in place of u_1^0 in Eqs. (122) and (132), and setting $u_1^0 = 0$, will yield:

$$\begin{aligned}
 Z_{22}^0 &= Z_{55}^0 = \frac{Z_o^{(2)}}{j \tan \theta_2} \\
 Z_{25}^0 &= Z_{52}^0 = \frac{Z_o^{(2)}}{j \sin \theta_2} \\
 Z_{33}^0 &= Z_{66}^0 = \frac{Z_o^{(3)}}{j \tan \theta_3} \\
 Z_{36}^0 &= Z_{63}^0 = \frac{Z_o^{(3)}}{j \sin \theta_3} \\
 Z_{15}^0 &= Z_{35}^0 = Z_{45}^0 = Z_{65}^0 = Z_{16}^0 = Z_{26}^0 = Z_{46}^0 = Z_{56}^0 = 0 .
 \end{aligned} \tag{137}$$

Thus, all the elements except the electromechanical transfer impedances have been calculated.

These can be obtained by, again, requiring $I_7^0 = 0$. Using Eqs. (92) and (116) leads to

$$I_7^0 = -j\omega A D_3 = j\omega A \epsilon_{33}^S a_3 = 0 \quad (138)$$

so that

$$a_3 = 0 \quad (139)$$

If this value is substituted into Eq. (62) it becomes

$$\phi = \left(\frac{e_{3k3}^0}{s \epsilon_{33}} \right) u_k^0 + b_3, \quad (140)$$

where the sum over k is to be taken.

With the sign convention adopted in Figure 9,

$$V_7^0 = [\phi_{(x_3=+h)} - \phi_{(x_3=-h)}] \quad (141)$$

If the choice for mechanical displacements in Eq. (122) is made Eqs. (140) and (141) become

$$\phi = \frac{e_{313}^0}{s \epsilon_{33}} u_1^0 + b_1 \quad (142)$$

$$V_7^0 = \left[\frac{e_{313}^0}{s \epsilon_{33}} u_1^0(+h) - \frac{e_{313}^0}{s \epsilon_{33}} u_1^0(-h) \right],$$

which, for this particular choice for the form of u_1^0 , leads to

$$V_7^0 = \frac{-e_{313}^0}{s \epsilon_{33}} U_1 \sin(\eta^{(1)} 2h) \quad (143)$$

When Eqs. (143) and (125) are substituted into Eq. (113),

$$Z_{71}^0 = \frac{V_7^0}{I_1^0} = \frac{e_{313}^0}{j\omega \epsilon_{33}^S} = \frac{n_1}{j\omega C_0} \quad (144)$$

where Eqs. (94) and (102) have been used.

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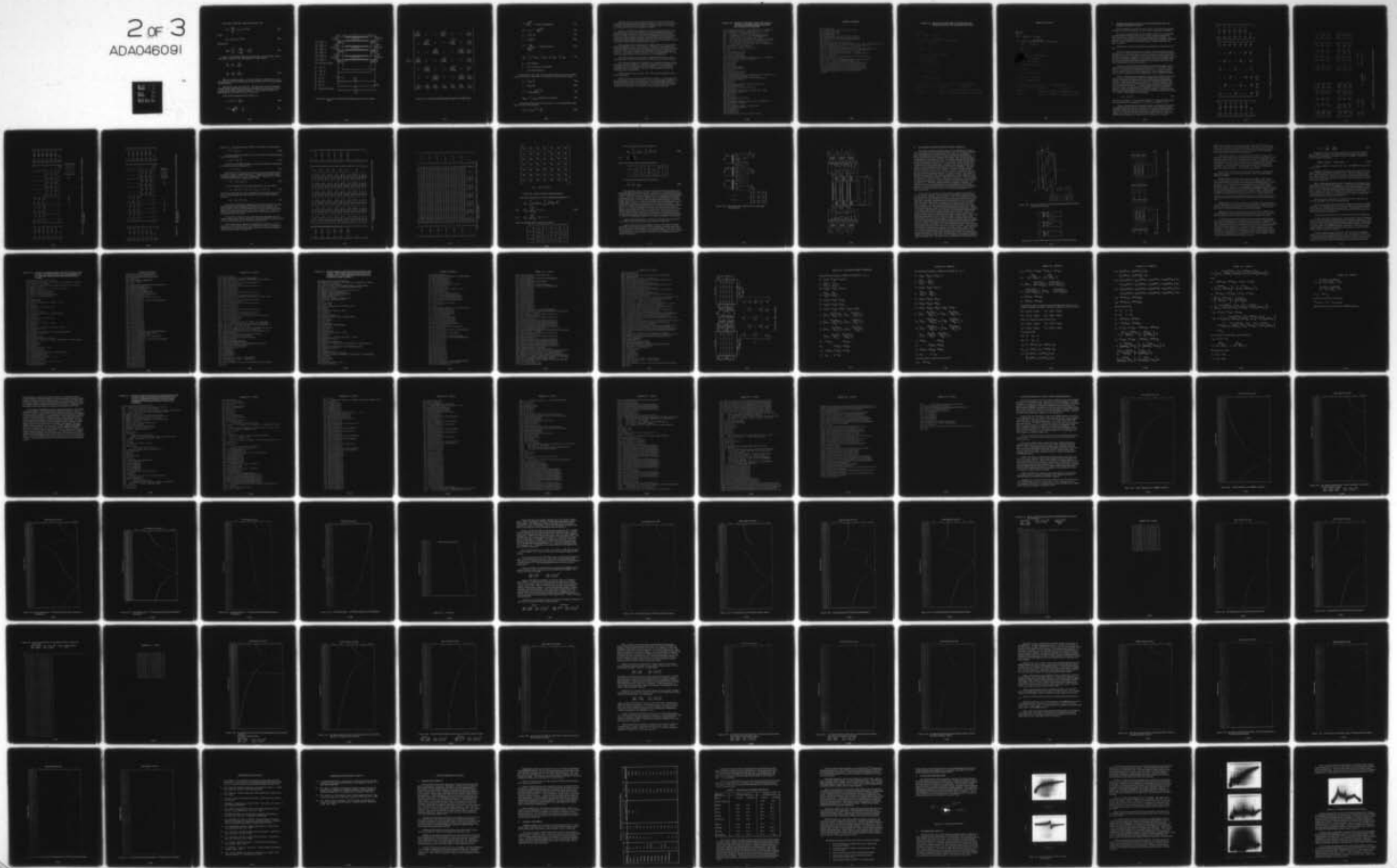
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If the choice of u_1^0 in Eq. (132) had been made, then

$$V_7^0 = \frac{e_{313}^0}{\epsilon_{33}^s} U_1 \sin(\eta^{(1)} 2h), \quad (145)$$

while

$$I_4^0 = j\omega U_1 \sin(\eta^{(1)} 2h). \quad (146)$$

This leads to

$$Z_{74}^0 = \frac{V_7^0}{I_4^0} = \frac{e_{313}^0}{j\omega \epsilon_{33}^s} = \frac{n_1}{j\omega C_0}. \quad (147)$$

Again it is obvious that repeating this procedure, using u_2^0 and u_3^0 in place of u_1^0 in Eqs. (122) and (132), and setting $u_1^0 = 0$, will result in

$$Z_{72}^0 = Z_{75}^0 = \frac{n_2}{j\omega C_0}$$

$$Z_{73}^0 = Z_{76}^0 = \frac{n_3}{j\omega C_0} \quad (148)$$

While it should be possible, in a similar fashion, to show that $Z_{\xi 7} = Z_{7\xi}$, the requirement that this matrix must be symmetrical about the principal diagonal will be used here.

The goal has finally been achieved. The equivalent circuit of a thickness-excited thickness mode plate in the normal coordinate reference frame is shown in Figure 3-12 (which is identical to Figure 9). The Normal Coordinate Impedance Matrix of this plate is given in Figure 3-13.

In this matrix the parameters of interest are:

$$\theta_i = 2h \eta^{(1)} = \frac{2h\omega}{v^{(i)}} \quad (149)$$

$$\eta^{(i)} = \omega \sqrt{\frac{\rho}{c^{(i)}}} = \frac{\omega}{v^{(i)}} \quad (20)$$

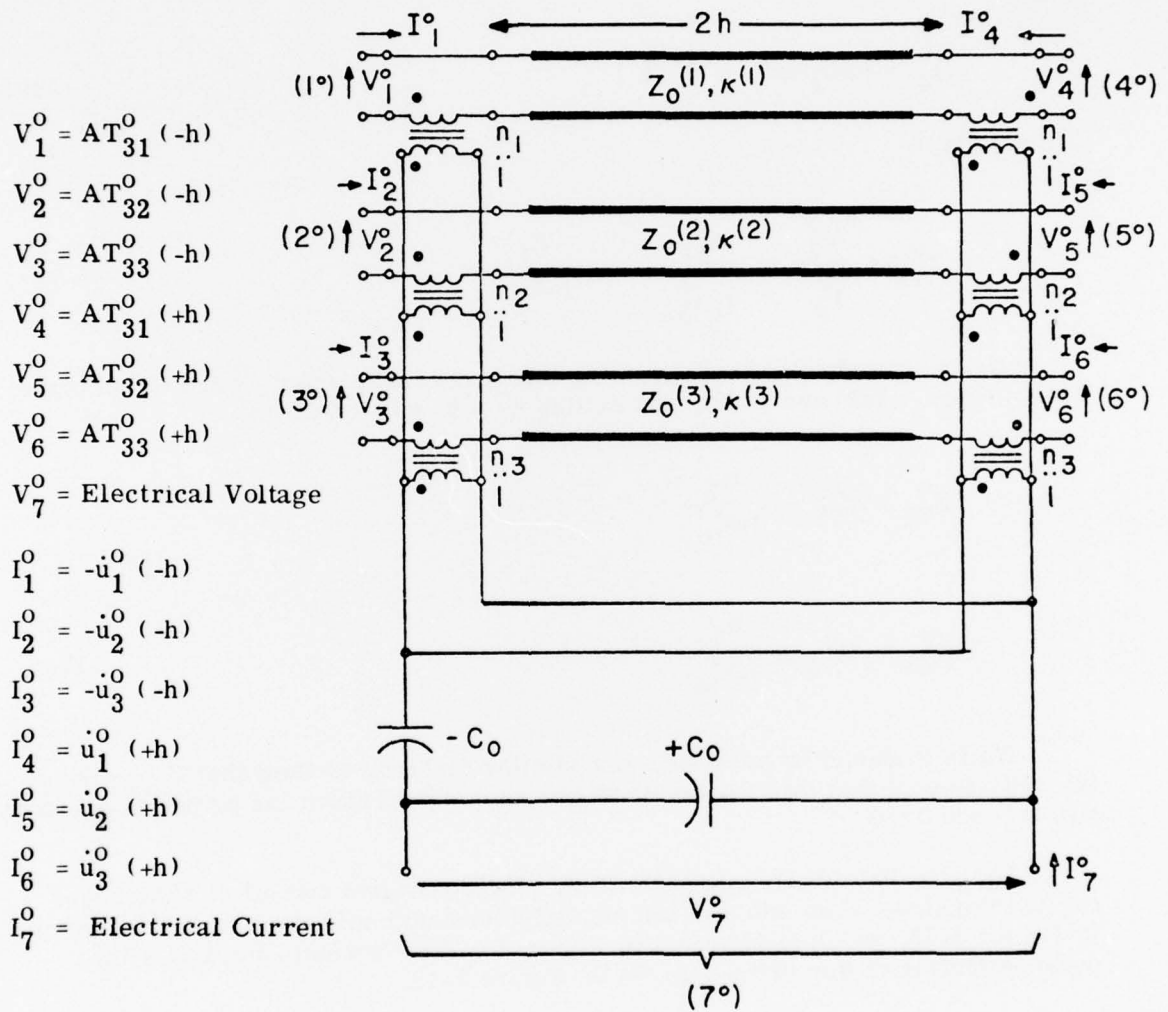


Figure 3-12. Seven-Port, Normal-Mode, Equivalent Circuit, for a TETM Plate

$$[Z^o] = \begin{bmatrix}
 \frac{Z_0^{(1)}}{j \tan \theta_1} & 0 & 0 & \frac{Z_0^{(1)}}{j \sin \theta_1} & 0 & 0 & \frac{n_1}{j \omega C_0} \\
 0 & \frac{Z_0^{(2)}}{j \tan \theta_2} & 0 & 0 & \frac{Z_0^{(2)}}{j \sin \theta_2} & 0 & \frac{n_2}{j \omega C_0} \\
 0 & 0 & \frac{Z_0^{(3)}}{j \tan \theta_3} & 0 & 0 & \frac{Z_0^{(3)}}{j \sin \theta_3} & \frac{n_3}{j \omega C_0} \\
 \frac{Z_0^{(1)}}{j \sin \theta_1} & 0 & 0 & \frac{Z_0^{(1)}}{j \tan \theta_1} & 0 & 0 & \frac{n_1}{j \omega C_0} \\
 0 & \frac{Z_0^{(2)}}{j \sin \theta_2} & 0 & 0 & \frac{Z_0^{(2)}}{j \tan \theta_2} & 0 & \frac{n_2}{j \omega C_0} \\
 0 & 0 & \frac{Z_0^{(3)}}{j \sin \theta_3} & 0 & 0 & \frac{Z_0^{(3)}}{j \tan \theta_3} & \frac{n_3}{j \omega C_0} \\
 \frac{n_1}{j \omega C_0} & \frac{n_2}{j \omega C_0} & \frac{n_3}{j \omega C_0} & \frac{n_1}{j \omega C_0} & \frac{n_2}{j \omega C_0} & \frac{n_3}{j \omega C_0} & \frac{1}{j \omega C_0}
 \end{bmatrix}$$

Figure 3-13. Normal Coordinate Impedance Matrix of a TETM Plate

$$v^{(i)} = \sqrt{\frac{c^{(i)}}{\rho}} = \text{velocity of propagation} \quad (72)$$

$$Z_o^{(i)} = A \rho v^{(i)} = A \sqrt{\rho c^{(i)}} \quad (76)$$

$$C_o = A \epsilon_{33}^s / 2h \quad (94)$$

$$n_i = A e_{33i}^o / 2h \quad (102)$$

$$k^{(i)} = \frac{e_{33i}^o}{\sqrt{\epsilon_{33}^s c^{(i)}}} = \text{coupling coefficient} \quad (88)$$

$$e_{33i}^o = \sum_{j=1}^3 \beta_j^{(i)} e_{33j} = \beta_1^{(i)} e_{331} + \beta_2^{(i)} e_{332} + \beta_3^{(i)} e_{333} \quad (46)$$

2h = plate thickness

A = area of concern \perp to thickness

ρ = plate material density

In expressions (94), (102), (88) and (46) the choice of x_3 in the thickness direction selects the coefficients; for an arbitrary direction m they become

$$C_o = A \epsilon_{mm}^s / 2h \quad (94a)$$

$$n_i = A e_{mmi}^o / 2h \quad (102a)$$

$$k^{(i)} = A e_{mmi}^o / \sqrt{\epsilon_{mm}^s c^{(i)}} \quad (88a)$$

$$e_{mmi}^o = \beta_j^{(i)} e_{mmj} \text{ using the sign convention} \quad (46a)$$

A convenient expression for the turns ratio, n_i , can be obtained by using Eqs. (72), (76), (88), and (94);

$$(n_i)^2 = C_o Z_o (k^{(i)})^2 \frac{v^{(i)}}{2h} \quad (150)$$

From Eq. (149) it is obvious that all the non-zero terms in the normal coordinate impedance matrix are frequency-dependent. Thus it is not practical to calculate the elements of this matrix as a separate identity, since it would have to be carried out for each frequency of interest.

However, in working with equivalent circuits of piezoelectric plates and describing material, it is usual to describe their piezoelectric properties in terms of coupling coefficients, velocities of propagation, dielectric constant and density. But, in order to use the normal coordinate matrix of Figure 3-13 in terms of already available constants, the properties must be specified in terms of the mode eigenvalues and eigenvectors as well as the appropriate tensor piezoelectric stress constants and dielectric constant.

These constants are available from the running of the previous programs, CROT and SYMEIG. Hence it is worthwhile, at this point, to accumulate the required information for the characterization of a plate, in the normal coordinate framework, and convert it to the more usual form of coupling coefficients and velocities. The program VCOUP, listed in Table 3-12, accomplishes this task. Again, the program has been written so that it is self-explanatory.

Line 160 is an index to keep track of the plate under discussion, since a multimode filter consists of several plates, which may or may not be identical.

Lines 165 through 290 are the input points for entering the output from the appropriate running of SYMEIG. Lines 295 through 330 are the input points for the output from the last part of the printout of CROT, for the material corresponding to the run of SYMEIG. Line 325 is the input point for the density of the plate under consideration.

All the program does is use Eqs. (46), (72), and (88) to make the appropriate calculations.

Table 3-13 shows the output of VCOUP for two plates. In the example the material in both cases was AT-cut quartz, with x_3 in the thickness direction, so the only difference in data is the plate label. The output consists of all the information about the plates under consideration for the multimode stacked filter required; so, at this point, it is possible to discard all previous data.

TABLE 3-12. LISTING OF PROGRAM VC0UP FOR THE CALCULATION OF NORMAL MODE VELOCITIES AND COUPLING COEFFICIENTS

```

1000 PROGRAM TO CALCULATE THE VELOCITIES AND COUPLING
1010 COEFFICIENTS FOR THE MODES OF A PLATE FOR USE
1020 IN FORMING THE NORMAL MODE MATRIX
1100 IT USES AS INPUT THE EIGENVALUES (EV) AND
1110 EIGENVECTORS (B'S) WITH THE COMPONENTS OF
1120 VECTOR STORED BY COLUMNS FROM SYMEIG
1200 IT ALSO USES THE PIEZOELECTRIC STRESS CONSTANTS
1210 AND DIELECTRIC PERMITTIVITY APPROPRIATE TO THE
1220 DIRECTION OF PROPAGATION FROM CR01 AS WELL
1230 AS THE DENSITY OF THE PLATE UNDER CONSIDERATION
140 DIMENSION EV(3),B(3,3),E(3)
150 DIMENSION EV(3),C(3),XK(3)
1550 ENTER PLATE NUMBER HERE I=1 TO 9
160 I=1
1650 EIGENVALUES EV(MODE)
170 EV(1)=0.29239228E11
180 EV(2)=0.38256421E11
190 EV(3)=0.13012144E12
2000 EIGENVECTORS B(COMPONENT,MODE) WITH THE COMPONENTS
2010 FOR EACH MODE STORED BY COLUMNS
210 B(1,1)=0.1E1
220 B(2,1)=0.0
230 B(3,1)=0.0
240 B(1,2)=0.0
250 B(2,2)=0.99806542
260 B(3,2)=0.62172451E-1
270 B(1,3)=0.0
280 B(2,3)=-0.62172451E-1
290 B(3,3)=0.99806542
2950 PIEZOELECTRIC STRESS CONSTANTS FOR THE APPROPRIATE
2960 PROPAGATION DIRECTION J FROM CR01
2970 E(1)=EEE(J1),E(2)=EEE(J2),E(3)=EEE(J3)
300 E(1)=-0.94904867E-1
310 E(2)=0.0
320 E(3)=0.0
3250 DIELECTRIC CONSTANT EPSI(J) FROM CR01
330 EP=0.39816236E-10
3350 DENSITY OF THE PLATE MATERIAL
340 DI=2.649E3
3550 CALCULATE NORMAL MODE PIEZOELECTRIC STRESS
3560 CONSTANTS EN(MODE)
360 DO 2 J=1,3
370 EN(J)=0.0
380 DO 2 K=1,3
390 EN(J)=B(K,J)*E(K)+EN(J)
400 2 CONTINUE
4150 CALCULATE COUPLING COEFFICIENTS XK(MODE) AND
4160 VELOCITIES CI(MODE)
420 DO 3 J=1,3
430 XK(J)=SQRT(EN(J)*EN(J)/(EP*EV(J)))
440 CI(J)=SQRT(EV(J)/DI)
450 3 CONTINUE
460 WRITE(6,100) I
470 WRITE(6,110)
480 WRITE(6,120) I,XK(1),I,XK(2),I,XK(3)

```

TABLE 3-12 (Cont'd)

```

490 WRITE(6,130)
500 WRITE(6,140) I,C1(1),I,C1(2),I,C1(3)
510 WRITE(6,150) I,DT
520 WRITE(6,160) I,EP
530 WRITE(6,170)
540 WRITE(6,180) I,B(1,1),I,B(1,2),I,B(1,3)
550 WRITE(6,190) I,B(2,1),I,B(2,2),I,B(2,3)
560 WRITE(6,200) I,B(3,1),I,B(3,2),I,B(3,3)
600 STOP
610 100 FORMAT(20X,'FOR PLATE ',I1)
620 110 FORMAT(2X,'THE COUPLING COEFFICIENTS ARE XX(MODE,PLATE)',//)
630 120 FORMAT(6X,'XK1',I1,' = ',F13.9, '//,6X,'XK2',I1,
631& ' = ',F13.9, '//,6X,'XK3',I1,' = ',F13.9)
640 130 FORMAT(//,2X,'THE VELOCITIES ARE CT(MODE,PLATE)',//)
650 140 FORMAT(6X,'C11',I1,' = ',E15.8, '//,6X,'C12',I1,
651& ' = ',E15.8, '//,6X,'C13',I1,' = ',E15.8)
660 150 FORMAT(//,2X,'THE DENSITY IS',//,6X,'DT',I1,
661& ' = ',E15.8)
670 160 FORMAT(//,2X,'THE DIELECTRIC CONSTANT IS',//,
671& 6X,'EP',I1,' = ',E15.8)
680 170 FORMAT(//,2X,'THE EIGENVECTORS FOR THIS PLATE ',
681& 'ARE BC(PLATE)(COMPONENT,MODE)',//)
690 180 FORMAT(1X,'B',I1,'(1,1)=' ,E15.8,1X,'B',I1,
691& '(1,2)=' ,E15.8,1X,'B',I1,'(1,3)=' ,E15.8, //)
700 190 FORMAT(1X,'B',I1,'(2,1)=' ,E15.8,1X,'B',I1,
701& '(2,2)=' ,E15.8,1X,'B',I1,'(2,3)=' ,E15.8, //)
710 200 FORMAT(1X,'B',I1,'(3,1)=' ,E15.8,1X,'B',I1,
711& '(3,2)=' ,E15.8,1X,'B',I1,'(3,3)=' ,E15.8)
750 END

```

TABLE 3.13. RESULTS OF VCOP FOR 2 PLATES OF AT CUT QUARTZ WITH X_3 IN THE THICKNESS DIRECTION

*RUN1-01

74871 02/23/77 13.437

FOR PLATE 1
THE COUPLING COEFFICIENTS ARE $KK(\text{MODE}, \text{PLATE})$

$KK11 = 0.00795302$

$KK21 = 0.$

$KK31 = 0.$

THE VELOCITIES ARE $CT(\text{MODE}, \text{PLATE})$

$CT11 = 0.33223239E 04$

$CT21 = 0.3002414E 04$

$CT31 = 0.70066352E 04$

THE DENSITY IS

$DT1 = 0.26497000E 04$

THE DIELECTRIC CONSTANT IS

$EP1 = 0.39316236E-10$

THE EIGENVECTORS FOR THIS PLATE ARE $B(\text{PLATE})(\text{COMPONENT}, \text{MODE})$

$B(1,1) = 0.10000000E 01$ $B(1,2) = 0.$ $B(1,3) = 0.$

$B(2,1) = 0.$ $B(2,2) = 0.99806542E 00$ $B(2,3) = -0.62172451E-01$

$B(3,1) = 0.$ $B(3,2) = 0.62172451E-01$ $B(3,3) = 0.99806542E 00$

TABLE 3-13. (Cont'd)

*160 I=2

*RUNH

3198T *02/26/77 15.992

FOR PLATE 2

THE COUPLING COEFFICIENTS ARE XK(MODE, PLATE)

XK12 = 0.03795802

XK22 = 0.

XK32 = 0.

THE VELOCITIES ARE CT(MODE, PLATE)

CT12 = 0.33223239E 04

CT22 = 0.33002414E 04

CT32 = 0.70036352E 04

THE DENSITY IS

DT2 = 0.26490000E 04

THE DIELECTRIC CONSTANT IS

EP2 = 0.39315235E-10

THE EIGENVECTORS FOR THIS PLATE ARE B(PLATE)(COMPONENT, MODE)

B2(1,1)= 0.10000000E 01 B2(1,2)= 0. B2(1,3)= 0.

B2(2,1)= 0. B2(2,2)= 0.99306542E 00 B2(2,3)=-0.62172451E-01

B2(3,1)= 0. B2(3,2)= 0.62172451E-01 B2(3,3)= 0.99306542E 00

D. TRANSFORMATION TO ACTUAL PLATE COORDINATES FROM THE NORMAL COORDINATE SYSTEM

After the difficult trail in the previous section, to go from the actual plate coordinates to normal coordinates, we now return to actual plate coordinates.

The normal mode equivalent circuit of Figure 3-12 and the corresponding impedance matrix in Figure 3-13 have left the port variables expressed in terms of the normal coordinate stresses and displacement components. For practical use, an equivalent circuit and matrix, with the port variables expressed in terms of the actual stresses and displacement components applied to the plate, is needed.

Equations 44, 45, 47 and 48 provide the mechanism for carrying out this task.

The matrix equation pertinent to the normal coordinate impedance matrix of Figure 3-13 is shown in Figure 3-14 in symbolic form. In this representation the appropriate form of the normal coordinate impedance element can be determined by comparing it to Figure 3-13. In Figure 3-14 the correspondence between the normal coordinate equivalent circuit variables and normal coordinate plate variables expressed by Equations 111 and 112 is also shown, along with a symbolic representation of the matrix operation.

Figure 3-15 shows the matrix interpretation of the tensor equations for the Normal Coordinate transformation in Equations 44, 45, 47 and 48. In this figure $[\beta]_t$ represents the transpose of the matrix, $[\beta]$, which is obtained by interchanging rows and columns of the matrix. This terminology agrees with the way the eigenvectors from SYMEIG were stored in NORM and VCOUP.

Figure 3-16 shows how the augmented matrix of this transformation would look in terms of the Equivalent Circuit Port Voltages when the Actual Plate Voltage Coordinates are expressed in terms of the Normal Coordinate Plate Voltages. This transformation is carried out by evaluating Equation 47 twice, once at the right side (top of the plate) where $x_3 = +h$ and once at the left side (bottom of the plate) where $x_3 = -h$; then we apply the definitions in Equation 111 to the result. In carrying out this expression the obvious corollary to Equation 111 has been applied to the actual plate variables. After performing these two transformations they were written in a manner that yielded the desired 7×7 matrix. This resulted in the expression,

$$[V] = [B][V^0], \quad (151)$$

where $[B]$ is related to $[\beta]$ as shown in Figure 16. Figure 3-16 also shows the relationship of the components of $[\beta]$ to the b_{ij} components of $[B]$.

Figure 3-17 shows how the augmented matrix, for expressing the normal coordinate equivalent circuit port currents in terms of the actual coordinate equivalent circuit port currents, would look. A comparison of this figure with Figure 3-16 shows that the matrix in Figure 3-17 is equal to the matrix of Figure 3-16, with the rows and columns interchanged. However, this is the definition of a transpose of a matrix; hence the symbol $[B]_t$ is used in

$$\begin{bmatrix} V_1^0 = A T_{31}^0 (-h) \\ V_2^0 = A T_{32}^0 (-h) \\ V_3^0 = A T_{33}^0 (-h) \\ V_4^0 = A T_{31}^0 (+h) \\ V_5^0 = A T_{32}^0 (+h) \\ V_6^0 = A T_{33}^0 (+h) \\ V_7^0 = V_7^0 \end{bmatrix} = \begin{bmatrix} Z_{11}^0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_{17}^0 \\ 0 & Z_{22}^0 & 0 & 0 & 0 & 0 & 0 & Z_{27}^0 \\ 0 & 0 & Z_{33}^0 & 0 & 0 & Z_{36}^0 & 0 & Z_{37}^0 \\ Z_{14}^0 & 0 & 0 & Z_{11}^0 & 0 & 0 & 0 & Z_{17}^0 \\ 0 & Z_{25}^0 & 0 & 0 & Z_{22}^0 & 0 & 0 & Z_{27}^0 \\ 0 & 0 & Z_{36}^0 & 0 & 0 & Z_{33}^0 & 0 & Z_{37}^0 \\ Z_{17}^0 & Z_{27}^0 & Z_{37}^0 & Z_{17}^0 & Z_{27}^0 & Z_{37}^0 & Z_{37}^0 & Z_{77}^0 \end{bmatrix} \begin{bmatrix} I_1^0 = -j\omega u_1^0 (-h) \\ I_2^0 = -j\omega u_2^0 (-h) \\ I_3^0 = -j\omega u_3^0 (-h) \\ I_4^0 = +j\omega u_1^0 (+h) \\ I_5^0 = +j\omega u_2^0 (+h) \\ I_6^0 = +j\omega u_3^0 (+h) \\ I_7^0 = I_7^0 \end{bmatrix}$$

$$[V^0] = [Z^0] [I^0]$$

Figure 3-14. Matrix Equation of the Normal Coordinate System

$$\begin{bmatrix} T_{31}^0 \\ T_{32}^0 \\ T_{33}^0 \end{bmatrix} = \begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} \end{bmatrix} \begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} = \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = \begin{bmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{aligned}
 [T_3] &= [\beta] [T_3^0] & [u] &= [\beta] [u^0] \\
 [T_3^0] &= [T_3]_t [T_3^0] & [u^0] &= [T_3]_t [u] \\
 & & & \text{of } [\beta]
 \end{aligned}$$

Figure 3-15. Matrix Representation of Coordinate Transformation Equations

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} \beta_1^{(1)} = b_{11} & \beta_1^{(2)} = b_{12} & \beta_1^{(3)} = b_{13} & 0 & 0 & 0 \\ \beta_2^{(1)} = b_{21} & \beta_2^{(2)} = b_{22} & \beta_2^{(3)} = b_{23} & 0 & 0 & 0 \\ \beta_3^{(1)} = b_{31} & \beta_3^{(2)} = b_{32} & \beta_3^{(3)} = b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1^{(1)} = b_{11} & \beta_1^{(2)} = b_{12} & \beta_1^{(3)} = b_{13} \\ 0 & 0 & 0 & \beta_2^{(1)} = b_{21} & \beta_2^{(2)} = b_{22} & \beta_2^{(3)} = b_{23} \\ 0 & 0 & 0 & \beta_3^{(1)} = b_{31} & \beta_3^{(2)} = b_{32} & \beta_3^{(3)} = b_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} V_1^0 = AT_{31}^0 (-h) \\ V_2^0 = AT_{32}^0 (-h) \\ V_3^0 = AT_{33}^0 (-h) \\ V_4^0 = AT_{31}^0 (+h) \\ V_5^0 = AT_{32}^0 (+h) \\ V_6^0 = AT_{33}^0 (+h) \\ V_7^0 = V_7 \end{bmatrix}$$

$$[V] = [B] [V^0] \quad [B] = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3-16. Matrix Equation of Actual Coordinate Port Voltages in Terms of Normal Coordinate Port Voltages

$$\begin{bmatrix} I_1^0 = -j\omega u_1^0 (-h) \\ I_2^0 = -j\omega u_2^0 (-h) \\ I_3^0 = -j\omega u_3^0 (-h) \\ I_4^0 = j\omega u_1^0 (+h) \\ I_5^0 = j\omega u_2^0 (+h) \\ I_6^0 = j\omega u_3^0 (+h) \\ I_7^0 = I_7^0 \end{bmatrix} = \begin{bmatrix} \beta_1^{(1)} = b_{11} & \beta_2^{(1)} = b_{21} & \beta_3^{(1)} = b_{31} & 0 & 0 & 0 & 0 \\ \beta_1^{(2)} = b_{12} & \beta_2^{(2)} = b_{22} & \beta_3^{(2)} = b_{32} & 0 & 0 & 0 & 0 \\ \beta_1^{(3)} = b_{13} & \beta_2^{(3)} = b_{23} & \beta_3^{(3)} = b_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1^{(1)} = b_{11} & \beta_2^{(1)} = b_{21} & \beta_3^{(1)} = b_{31} & 0 \\ 0 & 0 & 0 & \beta_1^{(2)} = b_{12} & \beta_2^{(2)} = b_{22} & \beta_3^{(2)} = b_{32} & 0 \\ 0 & 0 & 0 & \beta_1^{(3)} = b_{13} & \beta_2^{(3)} = b_{23} & \beta_3^{(3)} = b_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I = b_{77} \end{bmatrix} \begin{bmatrix} I_1 = -j\omega u_1 (-h) \\ I_2 = -j\omega u_2 (-h) \\ I_3 = -j\omega u_3 (-h) \\ I_4 = +j\omega u_1 (+h) \\ I_5 = +j\omega u_2 (+h) \\ I_6 = +j\omega u_3 (+h) \\ I_7 = I_7 \end{bmatrix}$$

$$[I^0] = [B]_t [I] \qquad [B]_t = \begin{bmatrix} \beta_t & 0 & 0 \\ -\beta_t & \beta_t & 0 \\ 0 & -\beta_t & \beta_t \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3-17. Matrix Equation of the Normal Coordinate Port Currents in Terms of Actual Coordinate Port Currents

Figure 3-17. The operation shown in Figure 3-17 results in the expression,

$$[I^0] = [B]_t [I] \quad (152)$$

The matrix equation of Figure 3-14, for the Normal Coordinate Equivalent Circuit Port Variables, is

$$[V^0] = [Z^0] [I^0] \quad (153)$$

If the actual coordinate equivalent circuit port variables are represented by the matrix relation in Equation 154,

$$[V] = [Z] [I] \quad (154)$$

Equations 151, 152, and 153 can be used to find the relationship between the actual coordinate impedance matrix $[Z]$ of Equation 154 and the normal coordinate Impedance matrix $[Z^0]$ of Equation 153. Replacing $[I^0]$ of Equation 153 by its Equivalent in Equation 152, yields

$$[V^0] = [Z^0] [B]_t [I] \quad (155)$$

If $[V^0]$ in Equation 155 is now pre multiplied by $[B]$ the result is

$$[V] = [B] [V^0] = [B] [Z^0] [B]_t [I] = [Z] [I] \quad (156)$$

The identity in Equation 156 clearly establishes the relationship between the actual coordinate impedance matrix and the normal coordinate impedance matrix as

$$[Z] = [B] [Z^0] [B]_t \quad (157)$$

The operation expressed in Equation 155 is shown in Figure 3-18, where the appropriate matrix multiplication technique has been employed. This figure shows the matrix elements expressed in terms of the eigenvector components and the b_{ij} elements of the $[B]$ matrix in Figure 3-16. Figure 3-19 shows the operation expressed in Equation 156, carried out in terms of the b_{ij} elements of the $[B]$ matrix.

The symmetry inherent in this Actual Coordinate Impedance matrix is shown in Figure 3-20. The variables here, may be identified by comparing them with the corresponding terms in the matrix of Figure 3-19.

After performing the operation of multiplying the matrices, it is relatively easy to write expressions for the elements in the actual coordinate impedance matrix in terms of the elements of the normal coordinate impedance matrix elements defined in Figure 3.13.

$V_1^0 = AT_{31}^0 (-h)$	$Z_{11}^0 \beta_1^{(1)}$	$Z_{11}^0 \beta_2^{(1)}$	$Z_{11}^0 \beta_3^{(1)}$	$Z_{14}^0 \beta_1^{(1)}$	$Z_{14}^0 \beta_2^{(1)}$	$Z_{14}^0 \beta_3^{(1)}$	Z_{17}^0	$I_1 = -j\omega u_1 (-h)$
$V_2^0 = AT_{32}^0 (-h)$	$Z_{22}^0 \beta_1^{(2)}$	$Z_{22}^0 \beta_2^{(2)}$	$Z_{22}^0 \beta_3^{(2)}$	$Z_{25}^0 \beta_1^{(2)}$	$Z_{25}^0 \beta_2^{(2)}$	$Z_{25}^0 \beta_3^{(2)}$	Z_{27}^0	$I_2 = -j\omega u_2 (-h)$
$V_3^0 = AT_{33}^0 (h)$	$Z_{33}^0 \beta_1^{(3)}$	$Z_{33}^0 \beta_2^{(3)}$	$Z_{33}^0 \beta_3^{(3)}$	$Z_{36}^0 \beta_1^{(3)}$	$Z_{36}^0 \beta_2^{(3)}$	$Z_{36}^0 \beta_3^{(3)}$	Z_{37}^0	$I_3 = -j\omega u_3 (-h)$
$V_4^0 = AT_{31}^0 (+h)$	$Z_{14}^0 \beta_1^{(1)}$	$Z_{14}^0 \beta_2^{(1)}$	$Z_{14}^0 \beta_3^{(1)}$	$Z_{11}^0 \beta_1^{(1)}$	$Z_{11}^0 \beta_2^{(1)}$	$Z_{11}^0 \beta_3^{(1)}$	Z_{17}^0	$I_4 = j\omega u_1 (+h)$
$V_5^0 = AT_{32}^0 (+h)$	$Z_{25}^0 \beta_1^{(2)}$	$Z_{25}^0 \beta_2^{(2)}$	$Z_{25}^0 \beta_3^{(2)}$	$Z_{22}^0 \beta_1^{(2)}$	$Z_{22}^0 \beta_2^{(2)}$	$Z_{22}^0 \beta_3^{(2)}$	Z_{27}^0	$I_5 = j\omega u_2 (+h)$
$V_6^0 = AT_{33}^0 (+h)$	$Z_{36}^0 \beta_1^{(3)}$	$Z_{36}^0 \beta_2^{(3)}$	$Z_{36}^0 \beta_3^{(3)}$	$Z_{33}^0 \beta_1^{(3)}$	$Z_{33}^0 \beta_2^{(3)}$	$Z_{33}^0 \beta_3^{(3)}$	Z_{37}^0	$I_6 = j\omega u_3 (+h)$
V_7^0	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$		I_7
	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$		
	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$	Z_{77}^0	
	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$		
	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$		
	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$		
	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$	$Z_{17}^0 \beta_1^{(1)}$	$Z_{17}^0 \beta_2^{(1)}$	$Z_{17}^0 \beta_3^{(1)}$		
	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$	$+Z_{27}^0 \beta_1^{(2)}$	$+Z_{27}^0 \beta_2^{(2)}$	$+Z_{27}^0 \beta_3^{(2)}$		
	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$	$+Z_{37}^0 \beta_1^{(3)}$	$+Z_{37}^0 \beta_2^{(3)}$	$+Z_{37}^0 \beta_3^{(3)}$		

Figure 3-18. Matrix Expressing Normal Coordinate Voltage Variables in Terms of Actual Coordinate Current Variables

Z_{11}	Z_{12}	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}
Z_{12}	Z_{22}	Z_{23}	Z_{15}	Z_{25}	Z_{26}	Z_{27}
Z_{13}	Z_{23}	Z_{33}	Z_{16}	Z_{26}	Z_{36}	Z_{37}
Z_{14}	Z_{15}	Z_{16}	Z_{11}	Z_{12}	Z_{13}	Z_{17}
Z_{15}	Z_{25}	Z_{26}	Z_{12}	Z_{22}	Z_{23}	Z_{27}
Z_{16}	Z_{26}	Z_{36}	Z_{13}	Z_{23}	Z_{33}	Z_{37}
Z_{17}	Z_{27}	Z_{37}	Z_{17}	Z_{27}	Z_{37}	Z_{77}

$$[Z] = [B] [Z^0] [B]_t$$

Figure 3-20. Actual Coordinate Impedance Matrix

For j and m ranging from 1 to 6 the appropriate expression is,

$$Z_{jm} = \sum_{p=1}^3 b_{qp} Z_{pk}^0 b_{lp} = \sum_{p=1}^3 \beta_q^{(p)} Z_{pk}^0 \beta_\ell^{(p)},$$

with $Z_{pk}^0 = \frac{Z_o^{(p)}}{j \tan \theta_{(p)}} \text{ for } k = p$ (158)

and $Z_{pk}^0 = \frac{Z_o^{(p)}}{j \sin \theta_{(p)}} \text{ for } k = p + 3,$

where the variables take the values shown below.

j	m	q	ℓ	k
1 to 3	1 to 3	j	m	p
	4 to 6	j	$m - 3$	$p + 3$
4 to 6	1 to 3	$j - 3$	m	$p + 3$
	4 to 6	$j - 3$	$m - 3$	p

For j or m equal to seven, the equation is,

$$Z_{jm} = \sum_{p=1}^3 b_{qp} Z_{p7}^0 = \sum_{p=1}^3 \beta_q^{(p)} Z_{p7}^0, \quad (159)$$

with $Z_{p7}^0 = \frac{n^{(p)}}{C_0}$,

where the variables take the values shown below.

j	m	q	j	m	q
7	1 to 3	m	1 to 3	7	j
	4 to 6	m - 3	4 to 6		j - 3

For both j and m equal seven the equation is

$$Z_{77} = Z_{77}^0 = \frac{1}{j\omega C_0}. \quad (160)$$

This impedance matrix is all that is required to specify the plate in the actual coordinate framework; however it would be nice to represent the normal coordinate equivalent circuit of Figure 3-12 as an Actual Coordinate Equivalent circuit. To do this, a network representation of the orthogonal transformation of Equations 44, 45, 47 and 48 is needed. Such a representation is available. Carlin and Giordano [18] show that a congruent transformation of a Z matrix ($C_t Z C$, where C is an $n \times n$ array of real values) can be represented as a multi-winding ideal transformer interconnection of the ports of the network. The orthogonal transformations of Equations 44, 45, 47 and 48 satisfy this condition since $\beta_t = \beta^{-1}$ and, incidentally, so does the coordinate rotation shown in Figure 3-2 and Equations 29 and 30. For the case of a 3 by 3 array this multi-winding ideal transformer is shown in Figure 3-21. This figure may be reversed, with the primary labeled with the superscripted variables upon interchanging sub and superscripts on the components of β (replace the components of β by the corresponding components of β_t) for the transformer turns ratios.

Applying this transformation to both plate surfaces of the crystal leads to the actual coordinate equivalent circuit shown in Figure 3-22.

Again, all the terms in the Impedance matrix of the actual coordinate Equivalent Circuit shown in Figure 3-20 are frequency dependent and, hence, there is no point in calculating them as a separate identity. Note, also, that in the general case every element is present in this Impedance matrix, whereas a large percentage of the elements of the normal mode impedance matrix of Figure 3-13 are zero.

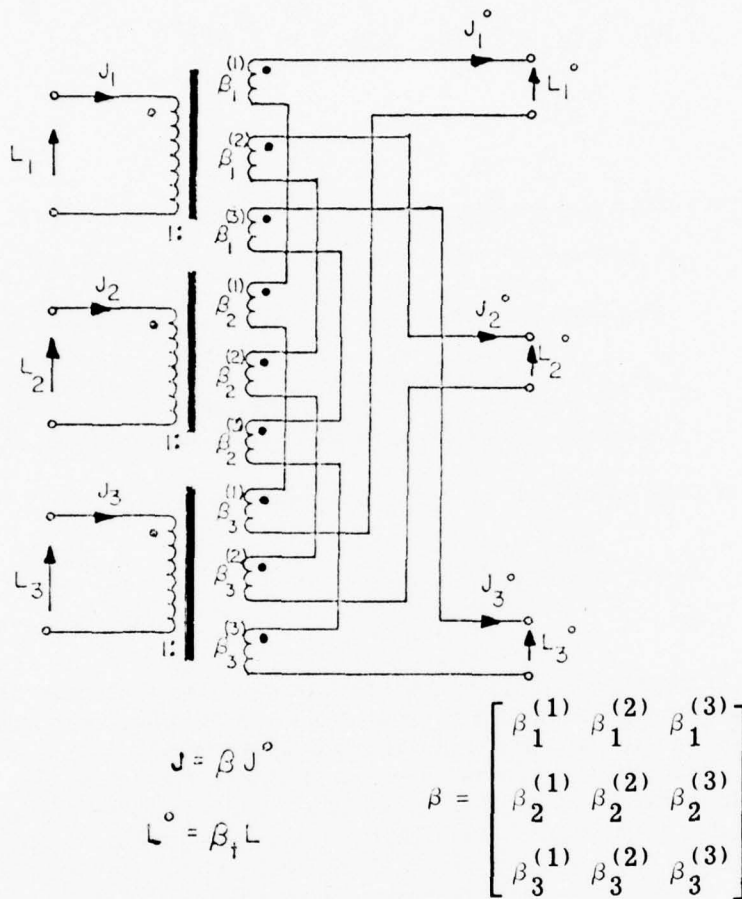


Figure 3-21. Ideal Transformer Realization of an Orthogonal Transformation

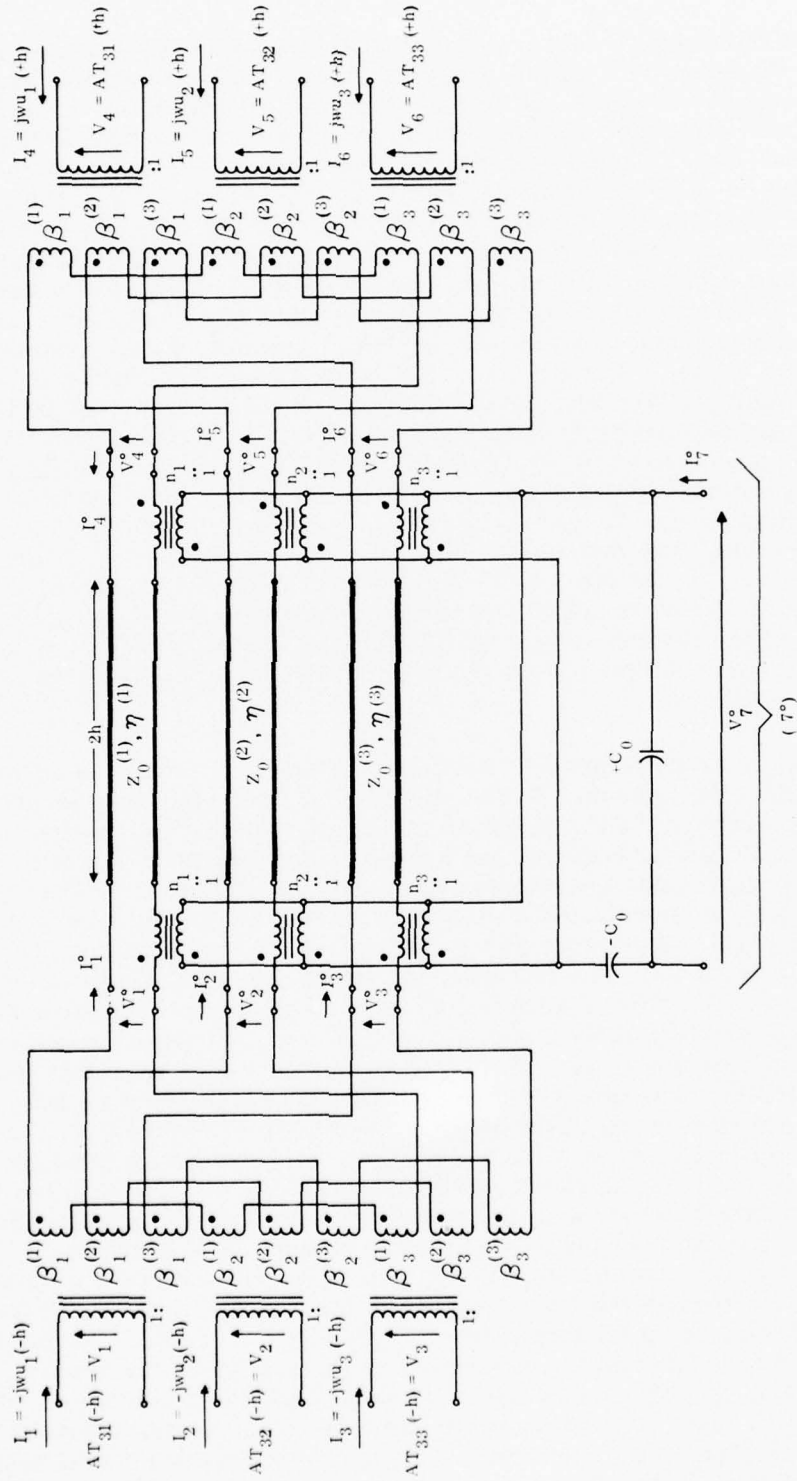


Figure 3-22. Seven Port Actual Coordinate Equivalent Circuit for a TETM Plate

E. MULTIMODE STACKED FILTERS AND MODE PROGRAMS

The actual coordinate impedance matrix for the TETM plate in Figure 3-20, or the equivalent circuit shown in Figure 3-22, are all that are required for an investigation of the multimode filters of concern in this program. Such a multimode filter consists of a group of such plates bonded together. If this is done with the actual coordinates of one plate, lined up with the actual coordinates of the preceding one, etc., the cascade consists of the equivalent circuit of Figure 3-22, where the ports 1, 2, 3 of the n^{th} plate (numbering left to right) are connected to ports 4, 5, 6, respectively, of the $n-1$ plate, with a prescribed network between them to account for the bond and to assure continuity of stress and displacement (velocity) across the boundary. The bond would be represented by a lumped constant T-version of the transmission line section shown in Figure 3-10. However, in general, bonds are lossy so that the j -multiplied trigonometric functions shown are replaced by hyperbolic functions and the propagation constant, θ , becomes complex. The values for $Z_0^{(i)}$ and θ_i also depend on the mode of propagation, so that a different circuit is required at each group of ports. The filter, in this case, is then completed by connecting, at least, one of the resulting electrical ports to a voltage generator with internal impedance, Z_g and, at least, one other of the ports to a load impedance, Z_l . The other electrical ports are connected as desired. Of course, in the above description there is no requirement for the n^{th} plate parameters to bear any relationship to the plate parameters of any other plate. However, this description does require that the actual coordinates of each plate are lined up.

Even in the more general case the thickness coordinate of each plate in the stack must be lined up. In this discussion, each plate is assumed to have its x_3 coordinate in the thickness direction, so that this coordinate will be lined up; however, the only requirement for the x_1 and x_2 axes are that they form a right-hand rectangular coordinate system with the x_3 axis. So, in the general case, the lateral coordinates of one plate do not have to coincide with those of the next plate. This more general situation is illustrated for a two layer stack in Figure 3-23. The relationship between the axes shown in this figure can be determined from Figure 3-2 and are obtained from Equation 27. Since a coordinate rotation about a common origin is an orthogonal transformation⁽¹³⁾ it can be represented by the ideal transformer network shown in Figure 3-21. In the particular case of coordinate rotation about a common x_3 axis, this general network reduces to the simpler one shown in Figure 3-24. Figure 3-25 shows the equivalent circuit for a two-layer stack, without a bond network between them, for two of the equivalent circuits for the plates shown in Figure 3-22, connected together with an arbitrary rotation about the assumed thickness direction. Figure 3-25 assumes that the coordinates of the plate on the left are the reference coordinates. In any multi-element structure of this type one plate must be used as the reference frame for the total system. If the structure is visualized as being built up in pieces, from left to right, it seems only logical to choose the first plate coordinates as the reference framework. The lack of a bond network in this figure assumes that the plates are in intimate contact with a rigid or welded contact between them, or that a lossless bond of negligible thickness has been used. One other observation should be made with regard to Figure 3-25. The electrical port variables have been turned around

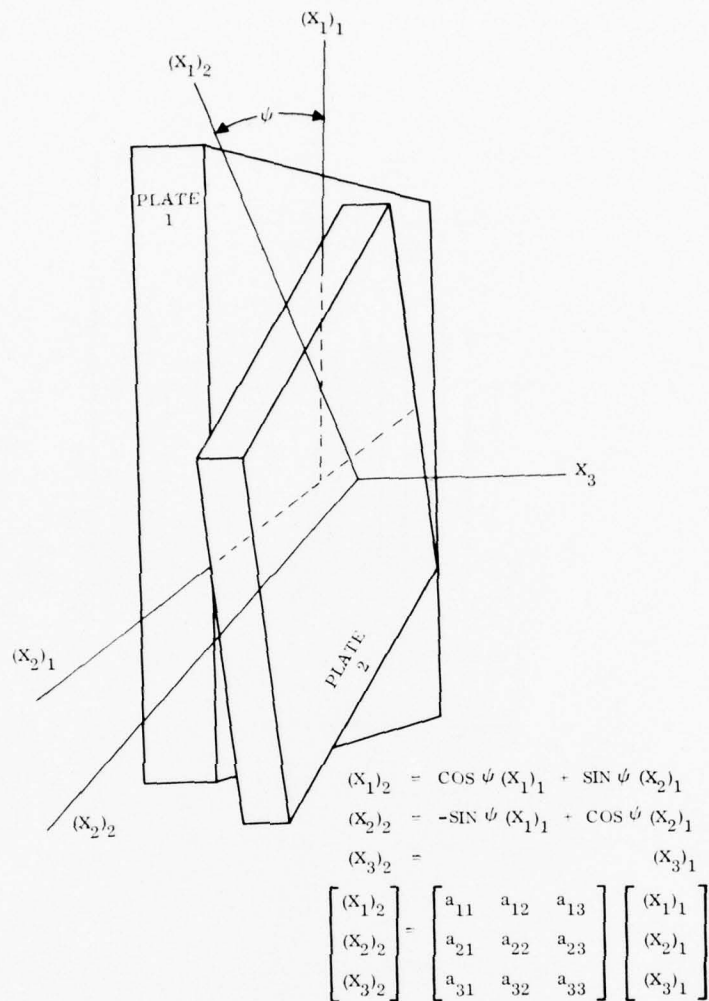


Figure 3-23. Two-Layer Stack of Crystal Plates Showing Relative Rotation About Common X_3 Axis

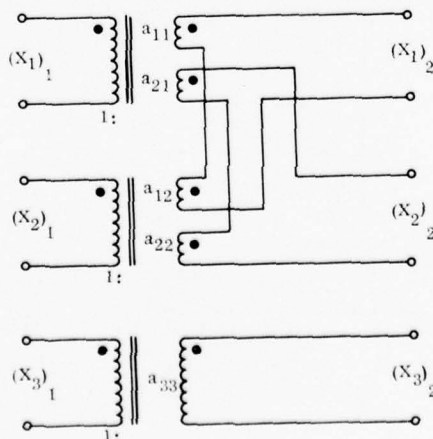


Figure 3-24. Network Realization of a Coordinate Rotation About X_3

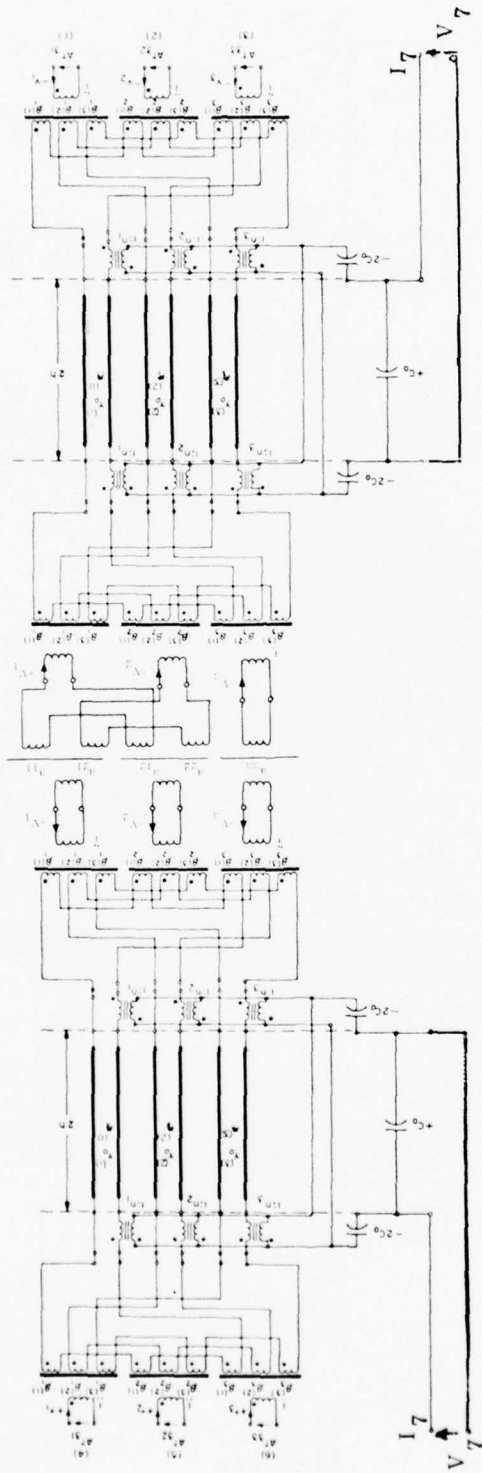


Figure 3-25. Equivalent Circuit of a Stacked Filter of Two Plates without Boundary Conditions Applied

between the two plates. This could be brought about in one of two ways: (1) either one plate could be reversed with respect to the other or, (2) if the port numbering system shown in Figure 3-25 is used, an ideal transformer with a 1 to 1 phase reversal would have to be inserted between the port variables shown and the plate connections, and then ignored.

In the work to follow it is this configuration in Figure 3-25 that is programmed, with the phase reversal transformer ignored and an intimate or rigid bond between plates assumed. As pointed out previously, conceptually, the inclusion of bonds is not a difficult task; however, they do complicate an already complicated circuit and tend to complicate the interpretation of the results. It, therefore, seems preferable to ignore them initially and concentrate on developing working programs. The programs are, likewise, limited to two plates but, again, procedures to include more are obvious.

In the course of this program three different two-plate, multimode programs were written, with varying assumptions. These are called MODE1, MODE2 and MODE3.

MODE1 assumes only one normal mode in each plate. This mode is either perpendicular or parallel to the plate direction and the plates in the filter stack are assumed to have their actual plate coordinates lined up. Thus, in essence, it is a transmission line representation of the lossless Mason Equivalent circuit. However, in arriving at the solution for this program, the same procedure was used, as in the MODE2 and MODE3 programs, instead of the common technique for the solution of a ladder network. It did not appear necessary to include plate rotation in this case since, unless the plates are oriented at right angles to each other (in which case no coupling between the mode in plate 1 and the mode in plate 2 takes place, so that the output is zero), the output will only be a reduced replica (more insertion loss) of the lined-up case.

MODE2 allows for two normal modes to be present; these are assumed to be at right angles to the plate thickness so that there is no difference between normal coordinates and actual plate coordinates (x_1 and x_2). This program does allow for an arbitrary rotation to occur between plates.

MODE3 is the program for the general case illustrated in Figure 3-25. All three normal modes are allowed and these may be at an angle to the actual plate coordinates; and one plate may be rotated with respect to the other.

In all three programs the two plates may have entirely different properties. In all the programs it is assumed that the electrodes for applying the fields are circular electrodes, on both the lateral surfaces of each plate. The area of these electrodes is also assumed to be the active area used in calculating the characteristic impedances. They also use the (common) convention of specifying a center frequency for each plate. In these programs the center frequency, f_0 , is defined as "the frequency at which the plate thickness ($2h$ in the previous discussion and TL in the programs) is a half wavelength thick at the velocity of propagation for mode1 (CT1)". This center frequency is used as input data rather than the actual plate thickness,

$$f_o = F\phi = \frac{v^{(1)}}{2(2h)} = \frac{CT1}{2(TL)} \quad (161)$$

The frequency range of interest is specified in terms of the initial frequency, (FINIT), frequency increment between desired frequencies, (FINC) and number of frequencies in the range of interest, (IFREQ). If FFINAL is the last frequency of interest,

$$IFREQ = ((FFINAL - FINIT)/FINC) + 1 \quad (162)$$

A listing of MODE1 is shown in Table 3-14. The MODE2 listing is given in Table 3-15.

These two programs do not, necessarily, require that information from CROT, SYMEIG and VCOUP be supplied to them. They only require, as input, the coupling coefficients, velocities of the modes of interest, the plate densities, center frequencies, dielectric constants and electrode diameters for each plate, as well as the frequency range of interest. This information is inputted at lines 230 through 470.

Again, these programs were written for operation on the Honeywell Information Systems Series 6000/600 Computer, as installed at the General Electric ESD facility in Syracuse, in the YFOR mode of time-sharing operation. They also use a time-sharing plot routine available at this facility in the permanent files. Lines 10, 130, 190, 200 and line 1960 all pertain to this particular plotting routine and, hence, do not apply to any other facility or mode of operation at this facility. However, it is relatively easy to convert or substitute other plotting programs for this particular one.

Most of the other statements are standard Fortran IV statements and should be adaptable to any other computing facility.

The free format statement at line 1810, in Mode 1 and 2 and at line 3020 in Mode 3, is a convenient way to write out the insertion loss versus frequency, and a great aid in trouble-shooting the programs. By inserting a write statement with this format reference at any appropriate place in the program, any desired variable can be printed out.

Again an attempt has been made to make the programs self-explanatory.

To aid in this Figure 3-26 shows the actual problem solved in the MODE2 program and the assumed impedance matrix for each plate, obtained from the matrix in Figure 3-13 by eliminating rows and columns 3 and 6, corresponding to mode 3. The actual solution of this problem is shown in Table 3-16, using the nomenclature of the MODE2 program. In general, for this case, the inherent symmetry of the plate matrices has been ignored, since this tends to make it easier to keep track of terms.

The IF conditions at lines 1675, 1677, 1707 and 1712 in the MODE 2 program allow for the handling of special cases. For example, if mode 2 is non-piezoelectrically coupled and the angle of plate rotation is zero, line 1712 is required. But this statement generally will not work at the resonant frequency of mode 2 in the case of identical plates; then line 1707 is required.

TABLE 3-14. LISTING OF PROGRAM MODE1 STACKED FILTER OF TWO PLATES, ONE MODE IN EACH PLATE PERPENDICULAR TO THICKNESS DIRECTION (NO ROTATION BETWEEN PLATES)

```

10**RUNH*:ADE576/RHYPL0T=(CORE=28)
100C PROGRAM TO CALCULATE THE TRANSFER FUNCTION OF A STACKED
101C FILTER OF TWO PLATES.
110C EACH PLATE HAS ONLY ONE PIEZOELECTRICALLY DRIVEN MODE.
130 PARAMETER NPLTS=1
140 COMPLEX ZL(5,5),ZK(5,5),ZL1(3,3),ZK1(3,3)
150 COMPLEX ZREI(2,2)
160 COMPLEX ZLE,ZG,EG,AMPI,AMP3
170 COMPLEX AMP0,VIN,V0UI,ZIN
190 DIMENSION X1(NPLTS),PIL0S(1000)
200 CHARACTER DA*(1(NPLTS))/'*'/
210 K=0
220 PI=3.1415926
230C INPUT DATA
232C CENTER FREQUENCIES OF PLATES 1 AND 2
240 F01=1.E7
250 F02=1.E7
255C DIAMETERS OF PLATES 1 AND 2
260 D1=10.E-3
270 D2=10.E-3
280C PLATE CONSTANTS
282C COUPLING COEFFICIENTS XK(MODE,PLATE)
290 XK11=0.088
310 XK12=XK11
325C VELOCITIES CI(MODE,PLATE)
330 CI11=3.32E3
350 CI12=CI11
365C DENSITIES OF PLATES 1 AND 2
370 D11=2.65E3
380 D12=D11
385C DIELECTRIC CONSTANTS OF PLATES 1 AND 2
390 EP1=4.58*8.85E-12
400 EP2=EP1
415C ELECTRICAL IMPEDANCES AND GENERATOR VOLTAGE
420 ZLE=CMPLX(1600.,0.)
430 ZG=ZLE
440 EG=CMPLX(1.,0.)
445C FREQUENCY RANGE OF INTEREST
447C INITIAL FREQUENCY,FREQUENCY INTERVAL,N0. OF FREQUENCIES
448C IN RANGE
450 FINI1=9.7E6
460 FINC=0.025E6
470 IFREQ=69
490C AREAS OF PLATES 1 AND 2
500 AREA1=PI*D1*D1/4.
510 AREA2=PI*D2*D2/4.
520C CHARACTERISTIC IMPEDANCES Z0(MODE,PLATE)
530 Z011=AREA1*D11*CI11
550 Z012=AREA2*D12*CI12
570C THICKNESSES OF PLATES 1 AND 2
580 IL1=CI11/(2.*F01)
590 IL2=CI12/(2.*F02)
600C CAPACITANCES C0 OF PLATES 1 AND 2
610 C01=EP1*AREA1/IL1
620 C02=EP2*AREA2/IL2

```

TABLE 3-14 (Cont'd)

```

630C TURNS RATIOS XN(MODE, PLATE)
640 XN11=X<11*SQRT(Z011*C01*2.*F01)
660 XN12=X<12*SQRT(Z012*C02*2.*F02)
680 FREQ=FINIT
690C LOOP THROUGH FREQUENCIES
700 DO1 I=1,IFREQ
710 K=K+1
720C PROPAGATION CONSTANTS THETA(MODE, PLATE)
730 TH11=2.*PI*FREQ*TL1/C111
750 TH12=2.*PI*FREQ*TL2/C112
770C COMPLEX IMPEDANCES
780 TGN11=SIN(TH11)/COS(TH11)
800 TGN12=SIN(TH12)/COS(TH12)
820 ZL(1,1)=CMPLX(0.,-Z011/TGN11)
830 ZL(1,2)=CMPLX(0.,0.)
840 ZL(1,3)=CMPLX(0.,-Z011/SIN(TH11))
850 ZL(1,4)=ZL(1,2)
860 ZL(1,5)=CMPLX(0.,-XN11/(2.*PI*FREQ*C01))
870 ZL(2,1)=ZL(1,2)
880 ZL(2,2)=ZL(1,2)
890 ZL(2,3)=ZL(1,2)
900 ZL(2,4)=ZL(1,2)
910 ZL(2,5)=ZL(1,2)
920 ZL(3,1)=ZL(1,3)
930 ZL(3,2)=ZL(1,2)
940 ZL(3,3)=ZL(1,1)
950 ZL(3,4)=ZL(1,2)
960 ZL(3,5)=ZL(1,5)
970 ZL(4,1)=ZL(1,2)
980 ZL(4,2)=ZL(2,4)
990 ZL(4,3)=ZL(1,2)
1000 ZL(4,4)=ZL(2,2)
1010 ZL(4,5)=ZL(2,5)
1020 ZL(5,1)=ZL(1,5)
1030 ZL(5,2)=ZL(2,5)
1040 ZL(5,3)=ZL(1,5)
1050 ZL(5,4)=ZL(2,5)
1060 ZL(5,5)=CMPLX(0.,-1./(2.*PI*FREQ*C01))
1070 ZR(1,1)=CMPLX(0.,-Z012/TGN12)
1080 ZR(1,2)=CMPLX(0.,0.)
1090 ZR(1,3)=CMPLX(0.,-Z012/SIN(TH12))
1100 ZR(1,4)=ZR(1,2)
1110 ZR(1,5)=CMPLX(0.,-XN12/(2.*PI*FREQ*C02))
1120 ZR(2,1)=ZR(1,2)
1130 ZR(2,2)=ZR(1,2)
1140 ZR(2,3)=ZR(1,2)
1150 ZR(2,4)=ZR(1,2)
1160 ZR(2,5)=ZR(1,2)
1170 ZR(3,1)=ZR(1,3)
1180 ZR(3,2)=ZR(1,2)
1190 ZR(3,3)=ZR(1,1)
1200 ZR(3,4)=ZR(1,2)
1210 ZR(3,5)=ZR(1,5)
1220 ZR(4,1)=ZR(1,2)
1230 ZR(4,2)=ZR(2,4)
1240 ZR(4,3)=ZR(1,2)
1250 ZR(4,4)=ZR(2,2)
1255 ZR(4,5)=ZR(2,5)
1260 ZR(5,1)=ZR(1,5)
1270 ZR(5,2)=ZR(2,5)
1280 ZR(5,3)=ZR(1,5)

```

TABLE 3-14. (Cont'd)

```

1290 ZR(5,4)=ZR(2,5)
1300 ZR(5,5)=CMPLX(0.,-1./(2.*PI*FREQ*C02))
1310C APPLY MECHANICAL BOUNDARY CONDITIONS TO LEFT PLATE
1311C FOR PLATE 1 V1=0
1320 ZL(1,1)=ZL(3,3)-ZL(3,1)*ZL(1,3)/ZL(1,1)
1330 ZL(1,2)=CMPLX(0.,0.)
1340 ZL(1,3)=ZL(3,5)-ZL(3,1)*ZL(1,5)/ZL(1,1)
1350 ZL(2,1)=ZL(1,2)
1360 ZL(2,2)=ZL(1,2)
1370 ZL(2,3)=ZL(1,2)
1380 ZL(3,1)=ZL(5,3)-ZL(5,1)*ZL(1,3)/ZL(1,1)
1390 ZL(3,2)=ZL(1,2)
1400 ZL(3,3)=ZL(5,5)-ZL(5,1)*ZL(1,5)/ZL(1,1)
1410C APPLY MECHANICAL BOUNDARY CONDITIONS TO RIGHT PLATE
1411C FOR PLATE 2 V3=0
1420 ZR(1,1)=ZR(1,1)-ZR(1,3)*ZR(3,1)/ZR(3,3)
1430 ZR(1,2)=CMPLX(0.,0.)
1440 ZR(1,3)=ZR(1,5)-ZR(1,3)*ZR(3,5)/ZR(3,3)
1450 ZR(2,1)=ZR(1,2)
1460 ZR(2,2)=ZR(1,2)
1470 ZR(2,3)=ZR(1,2)
1480 ZR(3,1)=ZR(5,1)-ZR(5,3)*ZR(3,1)/ZR(3,3)
1490 ZR(3,2)=ZR(1,2)
1500 ZR(3,3)=ZR(5,5)-ZR(5,3)*ZR(3,5)/ZR(3,3)
1510C TERMINATE PLATE 2 ON RIGHT IN ZLE
1520 ZREI(1,1)=ZR(1,1)-ZR(1,3)*ZR(3,1)/(ZLE+ZR(3,3))
1530 ZREI(1,2)=CMPLX(0.,0.)
1540 ZREI(2,1)=ZREI(1,2)
1550 ZREI(2,2)=ZREI(1,2)
1660C AT THE JUNCTION OF THE LEFT PLATE AND THE RIGHT
1661C PLATE, EQUATE VOLTAGES AND CURRENTS AND SOLVE FOR
1662C CURRENTS. V3(LEFT)=V1(RIGHT), I3(LEFT)=-I1(RIGHT)
1670C THIS GIVES I3 IN TERMS OF I1N.
1671C WHEN THIS VALUE IS PUT IN THE EXPRESSION FOR V5(VIN)
1672C THE INPUT IMPEDANCE ZIN CAN BE OBTAINED.
1680 ZIN=ZL(3,3)-ZL(3,1)*ZL(1,3)/(ZL(1,1)+ZREI(1,1))
1690 AMPI=EG/(ZG+ZIN)
1700 AMP3=-(ZL(1,3)/(ZL(1,1)+ZREI(1,1)))*AMPI
1720 VIN=ZL(3,1)*AMP3+ZL(3,3)*AMPI
1730 AMP0=(ZR(3,1)/(ZR(3,3)+ZLE))*AMP3
1740 VOUT=-ZLE*AMP0
1750C POWER AND INSERTION LOSS
1760 P0=REAL(VOUT*CONJG(-AMP0))
1770 PREF=REAL((EG*ZLE/(ZG+ZLE))*CONJG(EG/(ZG+ZLE)))
1780 PLOS(K)=10.*ALOG10(P0/PREF)
1785 WRITE(6,10) FREQ,PLOS(K)
1790 FREQ=FREQ+FINC
1800 I CONTINUE
1810 IO FORMAT(V)
1900 YMAX=PLOS(I)
1910 YMIN=PLOS(I)
1920 DO100 J=1,K
1930 IF(PLOS(J).GT.YMAX) YMAX=PLOS(J)
1940 IF(PLOS(J).LT.YMIN) YMIN=PLOS(J)
1950 IO CONTINUE
1960 CALL YPLT(PLOS,FINI,FINC,YMIN,YMAX,K,NPLTS,K1,DA,0)
1990 STOP
2000 END

```

TABLE 3-15. MODE2 PROGRAM LISTING STACKED FILTER OF TWO PLATES, TWO NORMAL MODES IN EACH PLATE PERPENDICULAR TO THICKNESS DIRECTION (ROTATION BETWEEN PLATES ALLOWED)

```

10*#RUNH*:ADE576/KHYPL0T=(CORE=2R)
100C PROGRAM TO CALCULATE THE TRANSFER FUNCTION OF A STACKED
101C FILTER OF TWO PLATES.
110C EACH PLATE HAS TWO PIEZOELECTRICALLY DRIVEN MODES.
120C PSI IS THE ANGLE OF ROTATION BETWEEN PLATES
130 PARAMETER NPLTS=1
140 COMPLEX ZL(5,5),ZR(5,5),ZLI(3,3),ZRI(3,3)
150 COMPLEX ZREI(2,2),ZRETR(2,2)
160 COMPLEX ZLE,ZG,EG,AMPI,AMP3,AMP4
170 COMPLEX AMP0,VIN,VOUT,ZIN
180 DIMENSION A(2,2)
190 DIMENSION K1(NPLTS),PIL0S(1000)
200 CHARACTER DA*(1(NPLTS)/**)/
210 K=0
220 PI=3.1415926
230C INPUT DATA
232C CENTER FREQUENCIES OF PLATES 1 AND 2
240 F01=1.E7
250 F02=1.E7
255C DIAMETER OF PLATES 1 AND 2
260 D1=10.E-3
270 D2=10.E-3
280C PLATE CONSTANTS
282C COUPLING COEFFICIENTS XK(MODE,PLATE)
290 XK11=0.088
300 XK21=2.*XK11
310 XK12=XK11
320 XK22=2.*XK12
325C VELOCITIES CI(MODE,PLATE)
330 CI11=3.32E3
340 CI21=1.1*CI11
350 CI12=CI11
360 CI22=1.1*CI12
365C DENSITIES OF PLATES 1 AND 2
370 DT1=2.65E3
380 DT2=DT1
385C DIELECTRIC CONSTANTS OF PLATES 1 AND 2
390 EP1=4.58*8.85E-12
400 EP2=EP1
405C ANGLE OF ROTATION PSI
407C PLATE 2 IS ROTATED BY PSI IN DEGREES IN RELATION TO 1
410 PSI=5.0
415C ELECTRICAL IMPEDANCES AND GENERATOR VOLTAGE
420 ZLE=CMPLX(1600.,0.)
430 ZG=ZLE
440 EG=CMPLX(1.,0.)
445C FREQUENCY RANGE OF INTEREST
447C INITIAL FREQUENCY,FREQUENCY INTERVAL,N0. OF FREQUENCIES
448C IN RANGE
450 FINI=9.7E6
460 FINC=0.025E6
470 IFREQ=69
490C AREAS OF PLATES 1 AND 2
500 AREA1=PI*D1*D1/4.

```

TABLE 3-15 (Cont'd)

```

510 AREA2=PI*D2*D2/4.
520C CHARACTERISTIC IMPEDANCES Z0(MODE, PLATE)
530 Z011=AREA1*DT1*CT11
540 Z021=AREA1*DT1*CT21
550 Z012= AREA2*DT2*CT12
560 Z022=AREA2*DT2*CT22
570C THICKNESSES OF PLATES 1 AND 2
580 TL1=CT11/(2.*F01)
590 TL2=CT12/(2.*F02)
600C CAPACITANCES C0 OF PLATES 1 AND 2
610 C01=EP1*AREA1/TL1
620 C02=EP2*AREA2/TL2
630C TURNS RATIOS XN(MODE, PLATE)
640 XN11=XK11*SQR1(Z011*C01*2.*F01)
650 XN21=XK21*SQR1(Z021*C01*CT21/TL1)
660 XN12=XK12*SQR1(Z012*C02*2.*F02)
670 XN22=XK22*SQR1(Z022*C02*CT22/TL2)
680 FREQ=FINIT
690C LOOP THROUGH FREQUENCIES
700 D01 I=1, IFREQ
710 K=K+1
720C PROPAGATION CONSTANTS THETA(MODE, PLATE)
730 TH11=2.*PI*FREQ*TL1/CT11
740 TH21=2.*PI*FREQ*TL1/CT21
750 TH12=2.*PI*FREQ*TL2/CT12
760 TH22=2.*PI*FREQ*TL2/CT22
770C COMPLEX IMPEDANCES
780 TGN11=SIN(TH11)/COS(TH11)
790 TGN21=SIN(TH21)/COS(TH21)
800 TGN12=SIN(TH12)/COS(TH12)
810 TGN22=SIN(TH22)/COS(TH22)
820 ZL(1,1)=CMPLX(0.,-Z011/TGN11)
830 ZL(1,2)=CMPLX(0.,0.)
840 ZL(1,3)=CMPLX(0.,-Z011/SIN(TH11))
850 ZL(1,4)=ZL(1,2)
860 ZL(1,5)=CMPLX(0.,-XN11/(2.*PI*FREQ*C01))
870 ZL(2,1)=ZL(1,2)
880 ZL(2,2)=CMPLX(0.,-Z021/TGN21)
890 ZL(2,3)=ZL(1,2)
900 ZL(2,4)=CMPLX(0.,-Z021/SIN(TH21))
910 ZL(2,5)=CMPLX(0.,-XN21/(2.*PI*FREQ*C01))
920 ZL(3,1)=ZL(1,3)
930 ZL(3,2)=ZL(1,2)
940 ZL(3,3)=ZL(1,1)
950 ZL(3,4)=ZL(1,2)
960 ZL(3,5)=ZL(1,5)
970 ZL(4,1)=ZL(1,2)
980 ZL(4,2)=ZL(2,4)
990 ZL(4,3)=ZL(1,2)
1000 ZL(4,4)=ZL(2,2)
1010 ZL(4,5)=ZL(2,5)
1020 ZL(5,1)=ZL(1,5)
1030 ZL(5,2)=ZL(2,5)
1040 ZL(5,3)=ZL(1,5)
1050 ZL(5,4)=ZL(2,5)
1060 ZL(5,5)=CMPLX(0.,-1./(2.*PI*FREQ*C01))
1070 ZR(1,1)=CMPLX(0.,-Z012/TGN12)
1080 ZR(1,2)=CMPLX(0.,0.)

```

TABLE 3-15. (Cont'd)

```

1090 ZR(1,3)=CMPLX(0.,-Z012/SIN(TH12))
1100 ZR(1,4)=ZR(1,2)
1110 ZR(1,5)=CMPLX(0.,-XN12/(2.*PI*FREQ*C02))
1120 ZR(2,1)=ZR(1,2)
1130 ZR(2,2)=CMPLX(0.,-Z022/TGN22)
1140 ZR(2,3)=ZR(1,2)
1150 ZR(2,4)=CMPLX(0.,-Z022/SIN(TH22))
1160 ZR(2,5)=CMPLX(0.,-XN22/(2.*PI*FREQ*C02))
1170 ZR(3,1)=ZR(1,3)
1180 ZR(3,2)=ZR(1,2)
1190 ZR(3,3)=ZR(1,1)
1200 ZR(3,4)=ZR(1,2)
1210 ZR(3,5)=ZR(1,5)
1220 ZR(4,1)=ZR(1,2)
1230 ZR(4,2)=ZR(2,4)
1240 ZR(4,3)=ZR(1,2)
1250 ZR(4,4)=ZR(2,2)
1255 ZR(4,5)=ZR(2,5)
1260 ZR(5,1)=ZR(1,5)
1270 ZR(5,2)=ZR(2,5)
1280 ZR(5,3)=ZR(1,5)
1290 ZR(5,4)=ZR(2,5)
1300 ZR(5,5)=CMPLX(0.,-1./(2.*PI*FREQ*C02))
1310C APPLY MECHANICAL BOUNDARY CONDITIONS TO LEFT PLATE
1311C FOR PLATE 1 V1=V2=0
1320 ZLT(1,1)=ZL(3,3)-ZL(3,1)*ZL(1,3)/ZL(1,1)
1330 ZLT(1,2)=CMPLX(0.,0.)
1340 ZLT(1,3)=ZL(3,5)-ZL(3,1)*ZL(1,5)/ZL(1,1)
1350 ZLT(2,1)=ZLT(1,2)
1360 ZLT(2,2)=ZL(4,4)-ZL(4,2)*ZL(2,4)/ZL(2,2)
1370 ZLT(2,3)=ZL(4,5)-ZL(4,2)*ZL(2,5)/ZL(2,2)
1380 ZLT(3,1)=ZL(5,3)-ZL(5,1)*ZL(1,3)/ZL(1,1)
1390 ZLT(3,2)=ZL(5,4)-ZL(5,2)*ZL(2,4)/ZL(2,2)
1400 ZLT(3,3)=ZL(5,5)-(ZL(5,1)*ZL(1,5)/ZL(1,1))
1401& -(ZL(5,2)*ZL(2,5)/ZL(2,2))
1410C APPLY MECHANICAL BOUNDARY CONDITIONS TO RIGHT PLATE
1411C FOR PLATE 2 V3=V4=0
1420 ZRT(1,1)=ZR(1,1)-ZR(1,3)*ZR(3,1)/ZR(3,3)
1430 ZRT(1,2)=CMPLX(0.,0.)
1440 ZRT(1,3)=ZR(1,5)-ZR(1,3)*ZR(3,5)/ZR(3,3)
1450 ZRT(2,1)=ZRT(1,2)
1460 ZRT(2,2)=ZR(2,2)-ZR(2,4)*ZR(4,2)/ZR(4,4)
1470 ZRT(2,3)=ZR(2,5)-ZR(2,4)*ZR(4,5)/ZR(4,4)
1480 ZRT(3,1)=ZR(5,1)-ZR(5,3)*ZR(3,1)/ZR(3,3)
1490 ZRT(3,2)=ZR(5,2)-ZR(5,4)*ZR(4,2)/ZR(4,4)
1500 ZRT(3,3)=ZR(5,5)-(ZR(5,3)*ZR(3,5)/ZR(3,3))
1501& -(ZR(5,4)*ZR(4,5)/ZR(4,4))
1510C TERMINATE PLATE 2 ON RIGHT IN ZLE
1520 ZRET(1,1)=ZRT(1,1)-ZRT(1,3)*ZRT(3,1)/(ZLE+ZRT(3,3))
1530 ZRET(1,2)=-(ZRT(1,3)*ZRT(3,2))/(ZLE+ZRT(3,3))
1540 ZRET(2,1)=-(ZRT(2,3)*ZRT(3,1))/(ZLE+ZRT(3,3))
1550 ZRET(2,2)=ZRT(2,2)-ZRT(2,3)*ZRT(3,2)/(ZLE+ZRT(3,3))
1560C PLATE 2 IS ROTATED ABOUT THE THICKNESS BY THE ANGLE
1561C PSI IN DEGREES IN RELATION TO PLATE 1
1562C THE DIRECTION COSINES BETWEEN THE NEW AND OLD AXIS
1563C OF PLATE 2 ARE
1570 A(1,1)=COS(PSI*PI/180.)
1580 A(1,2)=SIN(PSI*PI/180.)

```

TABLE 3-15. (Cont'd)

```

1590 A(2,1)=-A(1,2)
1600 A(2,2)=A(1,1)
1610C THE IMPEDANCES OF THE PLATE ON THE RIGHT AFTER
1611C ROTATION ARE
1620 ZRETR(1,1)=A(1,1)*A(1,1)*ZREI(1,1)
1621& +A(2,1)*A(1,1)*ZREI(2,1)+A(2,1)*A(1,1)*ZREI(1,2)
1622& +A(2,1)*A(2,1)*ZREI(2,2)
1630 ZRETR(1,2)=A(1,2)*A(1,1)*ZREI(1,1)
1631& +A(1,2)*A(2,1)*ZREI(2,1)+A(2,2)*A(1,1)*ZREI(1,2)
1632& +A(2,2)*A(2,1)*ZREI(2,2)
1640 ZRETR(2,1)=A(1,1)*A(1,2)*ZREI(1,1)
1641& +A(1,1)*A(2,2)*ZREI(2,1)+A(2,1)*A(1,2)*ZREI(1,2)
1642& +A(2,1)*A(2,2)*ZREI(2,2)
1650 ZRETR(2,2)=A(1,2)*A(1,2)*ZREI(1,1)
1651& +A(1,2)*A(2,2)*ZREI(2,1)+A(2,2)*A(1,2)*ZREI(1,2)
1652& +A(2,2)*A(2,2)*ZREI(2,2)
1660C AT THE JUNCTION OF THE LEFT PLATE AND THE ROTATED RIGHT
1661C PLATE, EQUATE VOLTAGES AND CURRENTS AND SOLVE FOR
1662C CURRENTS. V3=V1I, V4=V2I, I3=-I1I, I4=-I2I
1670C THIS GIVES I3 AND I4 IN TERMS OF I1I.
1671C WHEN THESE VALUES ARE PUT IN EXPRESSION FOR V5(V1I)
1672C THE INPUT IMPEDANCE ZIN CAN BE OBTAINED
1675 IF(ZRETR(2,2)+ZLI(2,2).EQ.(0.0,0.0).AND.ZRETR(1,2)
1676& .EQ.(0.0,0.0)) ZRETR(2,2)=(0.0,0.5)
1677 IF(ZRETR(2,2)+ZLI(2,2).EQ.(0.0,0.0).AND.ZRETR(1,2)
1678& .EQ.(0.0,0.0)) ZLI(2,2)=(0.0,0.5)
1680 ZIN=(ZLI(3,1)*(ZLI(2,3)*ZRETR(1,2)-ZLI(1,3)*(ZRETR(2,2)
1691& +ZLI(2,2)))+ZLI(3,2)*(ZRETR(2,1)*ZLI(1,3)-ZLI(2,3)
1692& *(ZRETR(1,1)+ZLI(1,1)))/(ZRETR(1,1)+ZLI(1,1))
1693& *(ZRETR(2,2)+ZLI(2,2))-ZRETR(2,1)*ZRETR(1,2))+ZLI(3,3)
1690 AMPI=EG/(ZG+ZIN)
1700 AMP3=((ZLI(2,3)*ZRETR(1,2)-ZLI(1,3)*(ZRETR(2,2)+ZLI(2,2)))
1701& /(ZRETR(1,1)+ZLI(1,1))*(ZRETR(2,2)+ZLI(2,2)
1702& -ZRETR(2,1)*ZRETR(1,2)))*AMPI
1707 IF(ZRETR(1,2).NE.(0.0,0.0)) AMP4=-(ZLI(1,3)/ZRETR(1,2)
1708& )*AMPI-(ZLI(1,1)+ZRETR(1,1))/ZRETR(1,2))*AMP3
1712 IF(ZRETR(1,2).EQ.(0.0,0.0)) AMP4=-(ZLI(2,3)/ZRETR(2,2)
1713& +ZLI(2,2))*AMPI-(ZRETR(2,1)/ZRETR(2,2)+ZLI(2,2))
1714& *AMP3
1720 VIN=ZLI(3,1)*AMP3+ZLI(3,2)*AMP4+ZLI(3,3)*AMPI
1730 AMP0=((A(1,1)*ZRT(3,1)+A(2,1)*ZRT(3,2))/(ZLE+ZRT(3,3))
1731& *AMP3+((A(1,2)*ZRT(3,1)+A(2,2)*ZRT(3,2))/(ZLE+
1732& ZRT(3,3)))*AMP4
1740 V0UI=-ZLE*AMP0
1750C POWER AND INSERTION LOSS
1760 P0=REAL(V0UI*CONJG(-AMP0))
1770 PREF=REAL((EG*ZLE/(ZG+ZLE))*CONJG(EG/(ZG+ZLE)))
1780 PLOSS(K)=10.*ALOG10(P0/PREF)
1785 WRITE(6,10) FREQ,PLOSS(K)
1790 FREQ=FREQ+FINC
1800 I CONTINUE
1810 IO FORMAT(V)
1900 YMAX=PLOSS(1)
1910 YMIN=PLOSS(1)
1920 D0100 J=1,K
1930 IF(PLOSS(J).GT.YMAX) YMAX=PLOSS(J)
1940 IF(PLOSS(J).LT.YMIN) YMIN=PLOSS(J)
1950 IO CONTINUE
1960 CALL YPLT(PLOSS,FINIT,FINC,YMIN,YMAX,K,NPLTS,K1,DA,0)
1990 STOP
2000 END

```

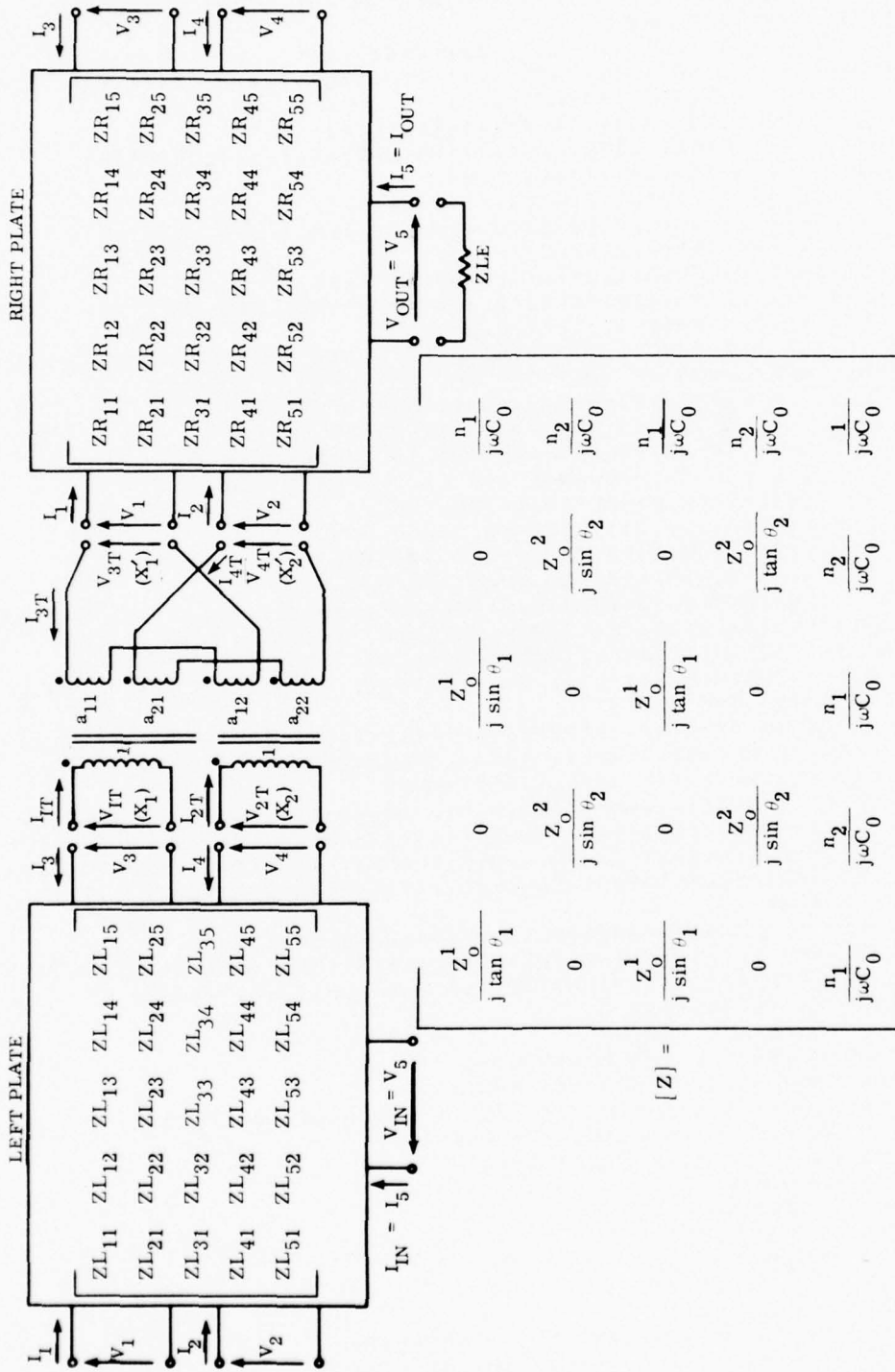


Figure 3-26. Mode-2 Problem

TABLE 3-16. SOLUTION OF MODE 2 PROBLEM

Apply mechanical boundary conditions to left plate $V_1 = V_2 = 0$

$$V_1 = ZL_{11}I_1 + ZL_{13}I_3 + ZL_{15}I_5 = 0$$

$$I_1 = -\frac{ZL_{13}}{ZL_{11}} I_3 - \frac{ZL_{15}}{ZL_{11}} I_5$$

$$V_2 = ZL_{22}I_2 + ZL_{24}I_4 + ZL_{25}I_5 = 0$$

$$I_2 = \frac{-ZL_{24}}{ZL_{22}} I_4 - \frac{ZL_{25}}{ZL_{22}} I_5$$

$$V_3 = ZL_{31}I_1 + ZL_{33}I_3 + ZL_{35}I_5$$

$$V_4 = ZL_{42}I_2 + ZL_{44}I_4 + ZL_{45}I_5$$

$$V_5 = ZL_{51}I_1 + ZL_{52}I_2 + ZL_{53}I_3 + ZL_{54}I_4 + ZL_{55}I_5$$

$$V_3 = \left\{ ZL_{33} - \frac{ZL_{31}ZL_{13}}{ZL_{11}} \right\} I_3 + \left\{ ZL_{35} - \frac{ZL_{31}ZL_{15}}{ZL_{11}} \right\} I_5$$

$$V_4 = \left\{ ZL_{44} - \frac{ZL_{42}ZL_{24}}{ZL_{22}} \right\} I_4 + \left\{ ZL_{45} - \frac{ZL_{42}ZL_{25}}{ZL_{22}} \right\} I_5$$

$$V_5 = \left\{ ZL_{53} - \frac{ZL_{51}ZL_{13}}{ZL_{11}} \right\} I_3 + \left\{ ZL_{54} - \frac{ZL_{52}ZL_{24}}{ZL_{22}} \right\} I_4$$

$$+ \left\{ ZL_{55} - \frac{ZL_{51}ZL_{15}}{ZL_{11}} - \frac{ZL_{52}ZL_{25}}{ZL_{22}} \right\} I_5$$

$$V_3 = ZLT_{11}I_3 + ZLT_{13}I_5$$

$$V_4 = ZLT_{22}I_4 + ZLT_{23}I_5$$

$$V_5 = ZLT_{31}I_3 + ZLT_{32}I_4 + ZLT_{33}I_5$$

$$I_5 = I_{IN} \quad , \quad V_5 = V_{IN}$$

TABLE 3-16. (CONT'D).

Apply mechanical boundary conditions to right plate $V_3 = V_4 = 0$

$$V_3 = ZR_{31}I_1 + ZR_{33}I_3 + ZR_{35}I_5 = 0$$

$$I_3 = -\frac{ZR_{31}}{ZR_{33}}I_1 - \frac{ZR_{35}}{ZR_{33}}I_5$$

$$V_4 = ZR_{42}I_2 + ZR_{44}I_4 + ZR_{45}I_5 = 0$$

$$I_4 = -\frac{ZR_{42}}{ZR_{44}}I_2 - \frac{ZR_{45}}{ZR_{44}}I_5$$

$$V_1 = ZR_{11}I_1 + ZR_{13}I_3 + ZR_{15}I_5$$

$$V_2 = ZR_{22}I_2 + ZR_{24}I_4 + ZR_{25}I_5$$

$$V_5 = ZR_{51}I_1 + ZR_{52}I_2 + ZR_{53}I_3 + ZR_{54}I_4 + ZR_{55}I_5$$

$$V_1 = \left\{ ZR_{11} - \frac{ZR_{13}ZR_{31}}{ZR_{33}} \right\} I_1 + \left\{ ZR_{15} - \frac{ZR_{13}ZR_{35}}{ZR_{33}} \right\} I_5$$

$$V_2 = \left\{ ZR_{22} - \frac{ZR_{24}ZR_{42}}{ZR_{44}} \right\} I_2 + \left\{ ZR_{25} - \frac{ZR_{24}ZR_{45}}{ZR_{44}} \right\} I_5$$

$$V_5 = \left\{ ZR_{51} - \frac{ZR_{53}ZR_{31}}{ZR_{33}} \right\} I_1 + \left\{ ZR_{52} - \frac{ZR_{54}ZR_{42}}{ZR_{44}} \right\} I_2$$

$$+ \left\{ ZR_{55} - \frac{ZR_{53}ZR_{35}}{ZR_{33}} - \frac{ZR_{54}ZR_{45}}{ZR_{44}} \right\} I_5$$

$$V_1 = ZRT_{11}I_1 + ZRT_{13}I_5$$

$$V_2 = + ZRT_{22}I_2 + ZRT_{23}I_5$$

$$V_5 = ZRT_{31}I_1 + ZRT_{32}I_2 + ZRT_{33}I_5$$

$$V_5 = V_{out} \quad , \quad I_5 = I_{out}$$

Terminate plate on right electrically in ZLE

$$V_{out} = - ZLE I_{out}$$

TABLE 3-16. (CONT'D).

$$V_{\text{out}} = ZRT_{31}I_1 + ZRT_{32}I_2 + ZRT_{33}I_{\text{out}} = -ZLEI_{\text{out}}$$

$$I_{\text{out}} = - \frac{ZRT_{31}}{ZLE + ZRT_{33}} I_1 - \frac{ZRT_{32}}{ZLE + ZRT_{33}} I_2$$

$$V_1 = \left\{ ZRT_{11} - \frac{ZRT_{13}ZRT_{31}}{ZLE + ZRT_{33}} \right\} I_1 - \left\{ \frac{ZRT_{13}ZRT_{32}}{ZLE + ZRT_{33}} \right\} I_2$$

$$V_2 = - \left\{ \frac{ZRT_{23}ZRT_{31}}{ZLE + ZRT_{33}} \right\} I_1 + \left\{ ZRT_{22} - \frac{ZRT_{23}ZRT_{32}}{ZLE + ZRT_{33}} \right\} I_2$$

$$V_1 = ZRET_{11}I_1 + ZRET_{12}I_2$$

$$V_2 = ZRET_{21}I_1 + ZRET_{22}I_2$$

Coordinates of plate on right are rotated about X_3 by the angle ψ in comparison to left plate coordinates (reference coordinates) to account for this

$$V_{3T} = a_{11}V_{1T} + a_{12}V_{2T} \quad - I_{1T} = a_{11}I_{3T} + a_{21}I_{4T}$$

$$V_{4T} = a_{21}V_{1T} + a_{22}V_{2T} \quad - I_{2T} = a_{12}I_{3T} + a_{22}I_{4T}$$

This is an orthogonal transformation so that

$$V_{1T} = a_{11}V_{3T} + a_{21}V_{4T} \quad - I_{3T} = a_{11}I_{1T} + a_{12}I_{2T}$$

$$V_{2T} = a_{12}V_{3T} + a_{22}V_{4T} \quad - I_{4T} = a_{21}I_{1T} + a_{22}I_{2T}$$

From Figure 3-26

$$V_{3T} = V_1 \quad I_{3T} = -I_1$$

$$V_{4T} = V_2 \quad I_{4T} = -I_2$$

$$V_{3T} = V_1 = ZRET_{11}(-I_{3T}) + ZRET_{12}(-I_{4T})$$

$$V_{4T} = V_2 = ZRET_{21}(-I_{3T}) + ZRET_{22}(-I_{4T})$$

$$V_{1T} = \left\{ a_{11}ZRET_{11} + a_{21}ZRET_{21} \right\} (-I_{3T}) \\ + \left\{ a_{11}ZRET_{12} + a_{21}ZRET_{22} \right\} (-I_{4T})$$

TABLE 3-16. (CONT'D).

$$\begin{aligned}
 V_{2T} &= \left\{ a_{12} ZRET_{11} + a_{22} ZRET_{21} \right\} (-I_{3T}) \\
 &+ \left\{ a_{12} ZRET_{12} + a_{22} ZRET_{22} \right\} (-I_{4T}) \\
 V_{1T} &= \left\{ a_{11} a_{11} ZRET_{11} + a_{21} a_{11} ZRET_{21} + a_{11} a_{21} ZRET_{12} + a_{21} a_{21} ZRET_{22} \right\} (I_{1T}) \\
 &+ \left\{ a_{11} a_{12} ZRET_{11} + a_{21} a_{12} ZRET_{21} + a_{11} a_{22} ZRET_{12} + a_{21} a_{22} ZRET_{22} \right\} (I_{2T}) \\
 V_{2T} &= \left\{ a_{12} a_{11} ZRET_{11} + a_{22} a_{11} ZRET_{21} + a_{22} a_{21} ZRET_{22} + a_{12} a_{21} ZRET_{12} \right\} (I_{1T}) \\
 &+ \left\{ a_{12} a_{12} ZRET_{11} + a_{22} a_{12} ZRET_{21} + a_{22} a_{22} ZRET_{22} + a_{12} a_{22} ZRET_{12} \right\} (I_{2T})
 \end{aligned}$$

$$V_{1T} = ZRETR_{11} I_{1T} + ZRETR_{12} I_{2T}$$

$$V_{2T} = ZRETR_{21} I_{1T} + ZRETR_{22} I_{2T}$$

but from Figure 3-26

$$V_3 = V_{1T} \quad I_3 = -I_{1T}$$

$$V_4 = V_{2T} \quad I_4 = -I_{2T}$$

$$V_3 = -ZRETR_{11} I_3 - ZRETR_{12} I_4$$

$$V_4 = -ZRETR_{21} I_3 - ZRETR_{22} I_4$$

$$V_3 = ZLT_{11} I_3 + ZLT_{13} I_{IN} = -ZRETR_{11} I_3 - ZRETR_{12} I_4$$

$$* -I_4 = \left\{ \frac{ZLT_{11} + ZRETR_{11}}{ZRETR_{12}} \right\} I_3 + \left\{ \frac{ZLT_{13}}{ZRETR_{12}} \right\} I_{IN}$$

$$V_4 = ZLT_{22} I_4 + ZLT_{23} I_{IN} = -ZRETR_{21} I_3 - ZRETR_{22} I_4$$

$$-I_4 = \left\{ \frac{ZRETR_{21}}{ZRETR_{22} + ZLT_{22}} \right\} I_3 + \left\{ \frac{ZLT_{23}}{ZRETR_{22} + ZLT_{22}} \right\} I_{IN}$$

$$\therefore \left\{ \frac{ZLT_{11} + ZRETR_{11}}{ZRETR_{12}} \right\} I_3 + \left\{ \frac{ZLT_{13}}{ZRETR_{12}} \right\} I_{IN}$$

$$= \left\{ \frac{ZRETR_{21}}{ZRETR_{22} + ZLT_{22}} \right\} I_3 + \left\{ \frac{ZLT_{23}}{ZRETR_{22} + ZLT_{22}} \right\} I_{IN}$$

TABLE 3-16. (CONT'D).

$$* I_3 = \left\{ \frac{ZLT_{23} ZRETR_{12} - ZLT_{13} (ZRETR_{22} + ZLT_{22})}{(ZLT_{11} + ZRETR_{11}) (ZRETR_{22} + ZLT_{22}) - ZRETR_{21} ZRETR_{12}} \right\} I_{IN}$$

From

$$V_3 = - ZRETR_{11} I_3 - ZRETR_{12} I_4 = ZLT_{11} I_3 + ZLT_{13} I_{IN}$$

$$-I_3 = \left\{ \frac{ZRETR_{12}}{ZLT_{11} + ZRETR_{11}} \right\} I_4 + \left\{ \frac{ZLT_{13}}{ZLT_{11} + ZRETR_{11}} \right\} I_{IN}$$

$$V_4 = -ZRETR_{21} I_3 - ZRETR_{22} I_4 = ZLT_{22} I_4 + ZLT_{23} I_{IN}$$

$$-I_3 = \left(\frac{ZLT_{22} + ZRETR_{22}}{ZRETR_{21}} \right) I_4 + \left(\frac{ZLT_{23}}{ZRETR_{21}} \right) I_{IN}$$

$$I_4 = \left\{ \frac{ZLT_{13} ZRETR_{21} - ZLT_{23} (ZLT_{11} + ZRETR_{11})}{(ZLT_{11} + ZRETR_{11}) (ZLT_{22} + ZRETR_{22}) - ZRETR_{12} ZRETR_{21}} \right\} I_{IN}$$

$$* V_{IN} = ZLT_{31} I_3 + ZLT_{32} I_4 + ZLT_{33} I_{IN}$$

$$* Z_{IN} = ZLT_{31} \left\{ \frac{ZLT_{23} ZRETR_{12} - ZLT_{13} (ZRETR_{22} + ZLT_{22})}{(ZLT_{11} + ZRETR_{11}) (ZRETR_{22} + ZLT_{22}) - ZRETR_{21} ZRETR_{12}} \right\} \\ + ZLT_{32} \left\{ \frac{ZLT_{13} ZRETR_{21} - ZLT_{23} (ZLT_{11} + ZRETR_{11})}{(ZLT_{11} + ZRETR_{11}) (ZLT_{22} + ZRETR_{22}) - ZRETR_{21} ZRETR_{12}} \right\} \\ + ZLT_{33}$$

For a generator with Impedance Z_G and voltage E_g

$$* I_{IN} = E_g / (Z_G + Z_{IN})$$

$$I_{out} = \frac{-ZRT_{31}}{ZLE + ZRT_{33}} I_1 - \frac{ZRT_{32}}{ZLE + ZRT_{33}} I_2$$

Retracing steps leads to

$$-I_1 = a_{11} I_3 + a_{12} I_4$$

$$-I_2 = a_{21} I_3 + a_{22} I_4$$

TABLE 3-16. (CONT'D).

$$* I_{\text{out}} = \left\{ \frac{a_{11} ZRT_{31} + a_{21} ZRT_{32}}{ZLE + ZRT_{33}} \right\} I_3$$

$$+ \left\{ \frac{a_{12} ZRT_{31} + a_{22} ZRT_{32}}{ZLE + ZRT_{33}} \right\} I_3$$

$$* V_{\text{out}} = - ZLE I_{\text{out}}$$

By the sign convention of Figure 3-26

$$* P_{\text{ower out}} = V_{\text{out}} \cdot (-I_{\text{out}})^{\text{conjugate}}$$

Equations with asterix are those used in MØDE 2 program.

The statements at lines 1675 and 1677 are one-way, to handle the behavior of identical plates, except that mode 2 in the left-hand plate is piezoelectrically coupled, while mode 2 in the right-hand plate is not. Time has not permitted a complete evaluation of these programs under all possible conditions, so there are probably other circumstances that also will not operate. Those that were corrected are those encountered in running the programs up to this time.

The listing for the *MODE3* program is shown in Table 3-17. The method of solution employed in the program follows that shown in Figure 3-26 and Table 3-16, where, in place of the normal mode matrix shown, the full actual coordinate matrix of Figure 3-19 is employed and the nomenclature is extended to allow for the inclusion of the additional terms. In writing this program the symmetry of the actual plate coordinate matrix has been employed, or neglected, depending on whether a simpler calculation could be done. The input data include all the data required for the *MODE2* program, extended to three modes as well as the eigenvectors for each plate. The general case of the *MODE3* program requires two runs, each, of *CRQT*, *SYMEIG* and *VC0UP*, one of each for each plate. The comments about the *MODE1* and *MODE2* programs, in general, apply to this program as well. Again, an attempt was made to make the program self-explanatory. Time did not permit a complete evaluation of this program so there are probably conditions that will not run. The *IF* at line 1950 handles the case of AT-cut quartz with no rotation between plates. For AT-cut quartz at the resonant frequency of mode 1, an exponential overflow, which has not been corrected, occurs; however, the results appear correct.

TABLE 3-17. LISTING OF MODE3 PROGRAM STACKED FILTER OF TWO PLATES; THREE NORMAL MODES ALLOWED IN EACH PLATE AT ARBITRARY ANGLES TO ACTUAL PLATE COORDINATES (ROTATION BETWEEN PLATES ALSO ALLOWED)

```

10*#RUNH*;ADE576/RHYPL0T=(CORE=28)
100C PROGRAM TO CALCULATE THE TRANSFER FUNCTION
101C OF A STACKED FILTER OF TWO PLATES
110C EACH PLATE CAN HAVE THREE PIEZOELECTRICALLY DRIVEN MODES
120C PSI IS THE ANGLE OF ROTATION BETWEEN PLATES
130 PARAMETER NPLTS=1
140 COMPLEX ZNL(7,7),ZNR(7,7),ZLC(7,7),ZRC(7,7)
150 COMPLEX ZLI(4,4),ZRI(4,4),ZREI(3,3),ZREIR(3,3)
160 COMPLEX D(36),R(36),PL1,PL2,PL3,PL4,PL5
170 COMPLEX PR1,PR2,PR3,PR4,PR5,ZS(3,3),ZSS,ZSS3
180 COMPLEX ZLE,ZG,EG
190 COMPLEX AMPI,AMP4,AMP5,AMP6,AMP0
200 COMPLEX VIN,VOUT,ZIN
210 DIMENSION B1(3,3),B2(3,3),A(3,3)
220 DIMENSION K1(NPLTS),PIL0S(1000)
230 CHARACTER DA*1(NPLTS)/*"/
240 K=0
250 PI=3.1415926
255 PI2=2.*PI
260C ***** INPUT DATA *****
262C PLATE 1 IS ON THE LEFT AND PLATE 2 IS ON THE RIGHT
265C CENTER FREQUENCIES OF PLATES 1 AND 2
270 F01=1.E7
280 F02=1.E7
285C DIAMETER OF PLATES 1 AND 2
290 DIA1=10.E-3
295 DIA2=10.E-3
300C *** PLATE CONSTANTS ***
305C COUPLING COEFFICIENTS XK(MODE,PLATE)
310 XK11=0.08795802
312 XK21=0.0
314 XK31=0.0
320 XK12=0.08795802
322 XK22=0.0
324 XK32=0.0
329C VELOCITIES CT(MODE,PLATE)
330 CT11=3.3223239E3
332 CT12=3.8002414E3
334 CT13=7.0086352E3
340 CT12=3.3223239E3
342 CT22=3.8002414E3
344 CT32=7.0086352E3
349C DENSITIES OF PLATES DT(PLATE)
350 DT1=2649.0
355 DT2=2649.0
359C DIELECTRIC CONSTANTS OF PLATES EP(PLATE)
360 EP1=0.39816236E-10
365 EP2=0.39816236E-10
370C *** EIGENVECTORS ***
371C THE COMPONENTS OF A MODE ARE STORED BY COLUMNS IN
372C ARRAY B B-PLATE-(COMPONENT,MODE)
380 B1(1,1)=1.0
381 B1(2,1)=0.0
382 B1(3,1)=0.0

```

TABLE 3-17. (Cont'd)

```

383 B1(1,2)=0.0
384 B1(2,2)=0.99806542
385 B1(3,2)=0.62172451E-1
386 B1(1,3)=0.0
387 B1(2,3)=-0.62172451E-1
388 B1(3,3)=0.99806542
400 B2(1,1)=1.0
401 B2(2,1)=0.0
402 B2(3,1)=0.0
403 B2(1,2)=0.0
404 B2(2,2)=0.99806542
405 B2(3,2)=0.62172451E-1
406 B2(1,3)=0.0
407 B2(2,3)=-0.62172451E-1
408 B2(3,3)=0.99806542
415C   *** ANGLE OF ROTATION PSI ***
416C   PLATE 2 IS ROTATED BY PSI DEGREES IN RELATION TO PLATE 1
420 PSI=5.0
428C   *** ELECTRICAL IMPEDANCES GENERATOR(ZG) LOAD(ZL) ***
429C   *** GENERATOR VOLTAGE(EG) ***
430 ZL=(1600.,0.)
434 ZG=ZL
438 EG=(1.0,0.)
440C   ***** FREQUENCY RANGE OF INTEREST *****
441C   INITIAL FREQUENCY FINIT
442C   FREQUENCY SPACING FINC
443C   NO. OF FREQUENCIES IN RANGE IFREQ=(FFINAL-FINIT)/FINC + 1
450 FINIT=9.7E6
454 FINC=0.025E6
458 IFREQ=69
460C   ***** CALCULATED DATA *****
469C   AREAS OF PLATES AREA(PLATE)
470 AREA1=PI*DIA1*DIA1/4.
475 AREA2=PI*DIA2*DIA2/4.
479C   CHARACTERISTIC IMPEDANCES Z0(MODE,PLATE)
480 Z011=AREA1*D11*CT11
484 Z021=AREA1*D11*CT21
488 Z031=AREA1*D11*CT31
500 Z012=AREA2*D12*CT12
504 Z022=AREA2*D12*CT22
508 Z032=AREA2*D12*CT32
519C   THICKNESSES OF PLATES IL(PLATE)
520 IL1=CT11/(2.*F01)
525 IL2=CT12/(2.*F02)
529C   CAPACITANCES OF PLATES C0(PLATE)
530 C01=EP1*AREA1/IL1
535 C02=EP2*AREA2/IL2
539C   TURNS RATIOS XN(MODE,PLATE)
540 XN11=SQRT(XK11*XK11*Z011*C01*CT11/IL1)
544 XN21=SQRT(XK21*XK21*Z021*C01*CT21/IL1)
548 XN31=SQRT(XK31*XK31*Z031*C01*CT31/IL1)
560 XN12=SQRT(XK12*XK12*Z012*C02*CT12/IL2)
564 XN22=SQRT(XK22*XK22*Z022*C02*CT22/IL2)
568 XN32=SQRT(XK32*XK32*Z032*C02*CT32/IL2)
570C   *** FROM NOW ON TERMS ARE FREQUENCY DEPENDENT ***
575 FREQ=FINIT
579C   LOOP THROUGH FREQUENCIES
580 DO 100 I=1,IFREQ

```

TABLE 3-17. (Cont'd)

```

585 K=K+1
588C PROPAGATION CONSTANTS TH(MODE, PLATE)=THETA(MODE, PLATE)
589 N=PI2*FREQ
590 TH11=W*IL1/CT11
594 TH21=W*IL1/CT21
598 TH31=W*IL1/CT31
610 TH12=W*IL2/CT12
614 TH22=W*IL2/CT22
618 TH32=W*IL2/CT32
629C NORMAL MODE MATRIX FOR PLATE 1 ZNL
630 TGN11=SIN(TH11)/COS(TH11)
634 TGN21=SIN(TH21)/COS(TH21)
636 TGN31=SIN(TH31)/COS(TH31)
640 ZNL(1,1)=CMPLX(0.,-Z011/TGN11)
641 ZNL(1,2)=(0.,0.)
642 ZNL(1,3)=(0.,0.)
643 ZNL(1,4)=CMPLX(0.,-Z011/SIN(TH11))
644 ZNL(1,5)=(0.,0.)
645 ZNL(1,6)=(0.,0.)
646 ZNL(1,7)=CMPLX(0.,-XN11/(W*C01))
650 ZNL(2,1)=(0.,0.)
651 ZNL(2,2)=CMPLX(0.,-Z021/TGN21)
652 ZNL(2,3)=(0.,0.)
653 ZNL(2,4)=(0.,0.)
654 ZNL(2,5)=CMPLX(0.,-Z021/SIN(TH21))
655 ZNL(2,6)=(0.,0.)
656 ZNL(2,7)=CMPLX(0.,-XN21/(W*C01))
660 ZNL(3,1)=(0.,0.)
661 ZNL(3,2)=(0.,0.)
662 ZNL(3,3)=CMPLX(0.,-Z031/TGN31)
663 ZNL(3,4)=(0.,0.)
664 ZNL(3,5)=(0.,0.)
665 ZNL(3,6)=CMPLX(0.,-Z031/SIN(TH31))
666 ZNL(3,7)=CMPLX(0.,-XN31/(W*C01))
670 ZNL(4,1)=ZNL(1,4)
671 ZNL(4,2)=ZNL(2,4)
672 ZNL(4,3)=ZNL(3,4)
673 ZNL(4,4)=ZNL(1,1)
674 ZNL(4,5)=(0.,0.)
675 ZNL(4,6)=(0.,0.)
676 ZNL(4,7)=ZNL(1,7)
680 ZNL(5,1)=ZNL(1,5)
681 ZNL(5,2)=ZNL(2,5)
682 ZNL(5,3)=ZNL(3,5)
683 ZNL(5,4)=ZNL(4,5)
684 ZNL(5,5)=ZNL(2,2)
685 ZNL(5,6)=(0.,0.)
686 ZNL(5,7)=ZNL(2,7)
690 ZNL(6,1)=ZNL(1,6)
691 ZNL(6,2)=ZNL(2,6)
692 ZNL(6,3)=ZNL(3,6)
693 ZNL(6,4)=ZNL(4,6)
694 ZNL(6,5)=ZNL(5,6)
695 ZNL(6,6)=ZNL(3,3)
696 ZNL(6,7)=ZNL(3,7)
700 ZNL(7,1)=ZNL(1,7)
701 ZNL(7,2)=ZNL(2,7)
702 ZNL(7,3)=ZNL(3,7)
703 ZNL(7,4)=ZNL(4,7)

```

TABLE 3-17. (Cont'd)

```

704 ZNL(7,5)=ZNL(5,7)
705 ZNL(7,6)=ZNL(6,7)
706 ZNL(7,7)=CMPLX(0.,-1./(W*C01))
799C   NORMAL MODE MATRIX FOR PLATE 2   ZNR
800 TGN12=SIN(TH12)/COS(TH12)
804 TGN22=SIN(TH22)/COS(TH22)
808 TGN32=SIN(TH32)/COS(TH32)
810 ZNR(1,1)=CMPLX(0.,-Z012/TGN12)
811 ZNR(1,2)=(0.,0.)
812 ZNR(1,3)=(0.,0.)
813 ZNR(1,4)=CMPLX(0.,-Z012/SIN(TH12))
814 ZNR(1,5)=(0.,0.)
815 ZNR(1,6)=(0.,0.)
816 ZNR(1,7)=CMPLX(0.,-XN12/(W*C02))
820 ZNR(2,1)=ZNR(1,2)
821 ZNR(2,2)=CMPLX(0.,-Z022/TGN22)
822 ZNR(2,3)=(0.,0.)
823 ZNR(2,4)=(0.,0.)
824 ZNR(2,5)=CMPLX(0.,-Z022/SIN(TH22))
825 ZNR(2,6)=(0.,0.)
826 ZNR(2,7)=CMPLX(0.,-XN22/(W*C02))
830 ZNR(3,1)=ZNR(1,3)
831 ZNR(3,2)=ZNR(2,3)
832 ZNR(3,3)=CMPLX(0.,-Z032/TGN32)
833 ZNR(3,4)=(0.,0.)
834 ZNR(3,5)=(0.,0.)
835 ZNR(3,6)=CMPLX(0.,-Z032/SIN(TH32))
836 ZNR(3,7)=CMPLX(0.,-XN32/(W*C02))
840 ZNR(4,1)=ZNR(1,4)
841 ZNR(4,2)=ZNR(2,4)
842 ZNR(4,3)=ZNR(3,4)
843 ZNR(4,4)=ZNR(1,1)
844 ZNR(4,5)=(0.,0.)
845 ZNR(4,6)=(0.,0.)
846 ZNR(4,7)=ZNR(1,7)
850 ZNR(5,1)=ZNR(1,5)
851 ZNR(5,2)=ZNR(2,5)
852 ZNR(5,3)=ZNR(3,5)
853 ZNR(5,4)=ZNR(4,5)
854 ZNR(5,5)=ZNR(2,2)
855 ZNR(5,6)=(0.,0.)
856 ZNR(5,7)=ZNR(2,7)
860 ZNR(6,1)=ZNR(1,6)
861 ZNR(6,2)=ZNR(2,6)
862 ZNR(6,3)=ZNR(3,6)
863 ZNR(6,4)=ZNR(4,6)
864 ZNR(6,5)=ZNR(5,6)
865 ZNR(6,6)=ZNR(3,3)
866 ZNR(6,7)=ZNR(3,7)
870 ZNR(7,1)=ZNR(1,7)
871 ZNR(7,2)=ZNR(2,7)
872 ZNR(7,3)=ZNR(3,7)
873 ZNR(7,4)=ZNR(4,7)
874 ZNR(7,5)=ZNR(5,7)
875 ZNR(7,6)=ZNR(6,7)
876 ZNR(7,7)=CMPLX(0.,-1./(W*C02))
997C   CONVERT FROM NORMAL MODE COORDINATES TO ACTUAL
998C   PLATE COORDINATES FOR BOTH PLATES 1 AND 2

```

TABLE 3-17. (Cont'd)

```

999C      ZL=(BB1)*ZNL*(BB1)I      ZR=(BB2)*ZNR*(BB2)I
1000 D0 20 M=1,7
1004 D0 20 N=1,7
1006 ZL(M,N)=(0.,0.)
1007 ZR(M,N)=(0.,0.)
1008 20 CONTINUE
1010 D0 30 M=1,3
1012 D0 30 N=1,3
1014 D0 30 J=1,3
1020 ZL(M,N)=B1(M,J)*ZNL(J,J)*B1(N,J)+ZL(M,N)
1022 ZL(M,N+3)=B1(M,J)*ZNL(J,J+3)*B1(N,J)+ZL(M,N+3)
1024 ZL(M+3,N)=ZL(M,N+3)
1026 ZL(M+3,N+3)=ZL(M,N)
1028 ZL(N,7)=B1(N,J)*ZNL(J,7)+ZL(N,7)
1030 ZL(N+3,7)=ZL(N,7)
1032 ZL(7,N)=ZL(N,7)
1034 ZL(7,N+3)=ZL(N,7)
1040 ZR(M,N)=B2(M,J)*ZNR(J,J)*B2(N,J)+ZR(M,N)
1042 ZR(M,N+3)=B2(M,J)*ZNR(J,J+3)*B2(N,J)+ZR(M,N+3)
1044 ZR(M+3,N)=ZR(M,N+3)
1046 ZR(M+3,N+3)=ZR(M,N)
1048 ZR(N,7)=B2(N,J)*ZNR(J,7)+ZR(N,7)
1050 ZR(N+3,7)=ZR(N,7)
1052 ZR(7,N)=ZR(N,7)
1054 ZR(7,N+3)=ZR(N,7)
1060 30 CONTINUE
1062 ZL(7,7)=ZNL(7,7)
1064 ZR(7,7)=ZNR(7,7)
1100C      APPLY MECHANICAL BOUNDARY CONDITIONS TO LEFT PLATE
1101C      FOR PLATE 1  V1=V2=V3=0
1102C      THIS CONVERSION IS MOST EASILY CARRIED OUT BY
1103C      DEFINING THE FOLLOWING PRODUCTS
1109 M=0
1110 D0 40 N=2,7
1112 D0 40 J=2,7
1114 M=M+1
1116 D(M)=ZL(1,1)*ZL(N,J)-ZL(N,1)*ZL(1,J)
1119 40 CONTINUE
1120C      IT IS ALSO CONVENIENT TO DEFINE THESE PRODUCTS
1130 PL1=ZL(1,1)*D(1)
1132 PL2=D(8)*D(1)-D(7)*D(2)
1134 PL3=D(14)*D(1)-D(13)*D(2)
1136 PL4=D(20)*D(1)-D(19)*D(2)
1138 PL5=D(26)*D(1)-D(25)*D(2)
1140 ZLT(1,1)=(D(15)*D(1)-D(13)*D(3))/PL1
1141& -(PL3*(D(9)*D(1)-D(7)*D(3))/(PL1*PL2))
1150 ZLT(1,2)=(D(16)*D(1)-D(13)*D(4))/PL1
1151& -(PL3*(D(10)*D(1)-D(7)*D(4))/(PL1*PL2))
1160 ZLT(1,3)=(D(17)*D(1)-D(13)*D(6))/PL1
1161& -(PL3*(D(11)*D(1)-D(7)*D(5))/(PL1*PL2))
1170 ZLT(1,4)=(D(18)*D(1)-D(13)*D(6))/PL1
1171& -(PL3*(D(12)*D(1)-D(7)*D(6))/(PL1*PL2))
1180 ZLT(2,1)=ZLT(1,2)
1190 ZLT(2,2)=(D(22)*D(1)-D(19)*D(4))/PL1
1191& -(PL4*(D(10)*D(1)-D(7)*D(4))/(PL1*PL2))
1200 ZLT(2,3)=(D(23)*D(1)-D(19)*D(5))/PL1
1201& -(PL4*(D(11)*D(1)-D(7)*D(5))/(PL1*PL2))
1210 ZLT(2,4)=(D(24)*D(1)-D(19)*D(6))/PL1

```

TABLE 3-17. (Cont'd)

1211& $-(PL4*(D(12)*D(1)-D(7)*D(6)))/(PL1*PL2)$
 1220 $ZLT(3,1)=ZLT(1,3)$
 1230 $ZLT(3,2)=ZLT(2,3)$
 1240 $ZLT(3,3)=((D(29)*D(1)-D(25)*D(5))/PL1)$
 1241& $-(PL5*(D(11)*D(1)-D(7)*D(5)))/(PL1*PL2)$
 1250 $ZLT(3,4)=((D(30)*D(1)-D(25)*D(6))/PL1)$
 1251& $-(PL5*(D(12)*D(1)-D(7)*D(6)))/(PL1*PL2)$
 1260 $ZLT(4,1)=ZLT(1,4)$
 1270 $ZLT(4,2)=ZLT(2,4)$
 1280 $ZLT(4,3)=ZLT(3,4)$
 1290 $ZLT(4,4)=((D(36)*D(1)-D(31)*D(6))/PL1)$
 1291& $-(D(32)*D(1)-D(31)*D(2))*(D(12)*D(1)-D(7)*D(6))/(PL1*PL2)$
 1300C APPLY MECHANICAL BOUNDARY CONDITIONS TO RIGHT PLATE
 1301C FOR PLATE 2 $V4=V5=V6=0$
 1302C THIS CONVERSION IS MOST EASILY CARRIED OUT BY
 1303C DEFINING THE FOLLOWING PRODUCTS
 1304C THE SYMMETRY OF ZR IS MADE USE OF IN THESE PRODUCTS
 1309 $M=0$
 1310 $D0\ 60\ N=2,7$
 1312 $D0\ 60\ J=2,7$
 1314 $M=M+1$
 1316 $R(M)=ZR(4,4)*ZR(N,J)-ZR(N,1)*ZR(1,J)$
 1319 60 CONTINUE
 1320C IT IS ALSO CONVENIENT TO DEFINE THESE PRODUCTS
 1330 $PR1=ZR(4,4)*R(1)$
 1332 $PR2=R(8)*R(1)-R(7)*R(2)$
 1334 $PR3=R(14)*R(1)-R(13)*R(2)$
 1336 $PR4=R(20)*R(1)-R(19)*R(2)$
 1338 $PR5=R(26)*R(1)-R(25)*R(2)$
 1340 $ZRT(1,1)=((R(15)*R(1)-R(13)*R(3))/PR1)$
 1341& $-(PR3*(R(9)*R(1)-R(7)*R(3)))/(PR1*PR2)$
 1350 $ZRT(1,2)=((R(16)*R(1)-R(13)*R(4))/PR1)$
 1351& $-(PR3*(R(10)*R(1)-R(7)*R(4)))/(PR1*PR2)$
 1360 $ZRT(1,3)=((R(17)*R(1)-R(13)*R(5))/PR1)$
 1361& $-(PR3*(R(11)*R(1)-R(7)*R(5)))/(PR1*PR2)$
 1370 $ZRT(1,4)=((R(18)*R(1)-R(13)*R(6))/PR1)$
 1371& $-(PR3*(R(12)*R(1)-R(7)*R(6)))/(PR1*PR2)$
 1380 $ZRT(2,1)=ZRT(1,2)$
 1390 $ZRT(2,2)=((R(22)*R(1)-R(19)*R(4))/PR1)$
 1391& $-(PR4*(R(10)*R(1)-R(7)*R(4)))/(PR1*PR2)$
 1400 $ZRT(2,3)=((R(23)*R(1)-R(19)*R(5))/PR1)$
 1401& $-(PR4*(R(11)*R(1)-R(7)*R(5)))/(PR1*PR2)$
 1410 $ZRT(2,4)=((R(24)*R(1)-R(19)*R(6))/PR1)$
 1411& $-(PR4*(R(12)*R(1)-R(7)*R(6)))/(PR1*PR2)$
 1420 $ZRT(3,1)=ZRT(1,3)$
 1430 $ZRT(3,2)=ZRT(2,3)$
 1440 $ZRT(3,3)=((R(29)*R(1)-R(25)*R(5))/PR1)$
 1441& $-(PR5*(R(11)*R(1)-R(7)*R(5)))/(PR1*PR2)$
 1450 $ZRT(3,4)=((R(30)*R(1)-R(25)*R(6))/PR1)$
 1451& $-(PR5*(R(12)*R(1)-R(7)*R(6)))/(PR1*PR2)$
 1460 $ZRT(4,1)=ZRT(1,4)$
 1470 $ZRT(4,2)=ZRT(2,4)$
 1480 $ZRT(4,3)=ZRT(3,4)$
 1490 $ZRT(4,4)=((R(36)*R(1)-R(31)*R(6))/PR1)$
 1491& $-(D(32)*R(1)-R(31)*R(2))*(R(12)*R(1)-R(7)*R(6))/(PR1*PR2)$
 1500C TERMINATE PLATE 2 ON RIGHT IN ZLE
 1501C THIS RESULTS IN $VOUT=-ZLE*AMP0$
 1510 $ZRETC(1,1)=ZRT(1,1)-ZRT(1,4)*ZRT(4,1)/(ZLE+ZRT(4,4))$

TABLE 3-17. (Cont'd)

```

1520 ZREI(1,2)=ZRI(1,2)-ZRI(1,4)*ZRI(4,2)/(ZLE+ZRI(4,4))
1530 ZREI(1,3)=ZRI(1,3)-ZRI(1,4)*ZRI(4,3)/(ZLE+ZRI(4,4))
1540 ZREI(2,1)=ZRI(2,1)-ZRI(2,4)*ZRI(4,1)/(ZLE+ZRI(4,4))
1550 ZREI(2,2)=ZRI(2,2)-ZRI(2,4)*ZRI(4,2)/(ZLE+ZRI(4,4))
1560 ZREI(2,3)=ZRI(2,3)-ZRI(2,4)*ZRI(4,3)/(ZLE+ZRI(4,4))
1570 ZREI(3,1)=ZRI(3,1)-ZRI(3,4)*ZRI(4,1)/(ZLE+ZRI(4,4))
1580 ZREI(3,2)=ZRI(3,2)-ZRI(3,4)*ZRI(4,2)/(ZLE+ZRI(4,4))
1590 ZREI(3,3)=ZRI(3,3)-ZRI(3,4)*ZRI(4,3)/(ZLE+ZRI(4,4))
1620C PLATE 2 IS ROTATED ABOUT THE THICKNESS BY THE ANGLE
1621C PSI IN DEGREES IN RELATION TO PLATE 1
1622C THE DIRECTION COSINES BETWEEN THE NEW AXES AND OLD AXES
1623C OF PLATE 2 (OR BETWEEN AXES OF PLATE 2 AND AXES
1624C OF PLATE 1) ARE
1630 A(1,1)=COS(PSI*PI/180.)
1634 A(1,2)=SIN(PSI*PI/180.)
1638 A(1,3)=0.0
1640 A(2,1)=-A(1,2)
1644 A(2,2)=A(1,1)
1648 A(2,3)=0.0
1650 A(3,1)=0.0
1654 A(3,2)=0.0
1658 A(3,3)=1.0
1670C THE IMPEDANCES OF THE PLATE ON THE RIGHT ( PLATE 2 )
1671C AFTER ROTATION ARE ZRETR=(A)T*ZREI*(A)
1680 D0 80 L=1,3
1682 D0 80 M=1,3
1684 ZRETR(L,M)=(0.,0.)
1686 D0 80 J=1,3
1688 D0 80 N=1,3
1690 ZRETR(L,M)=A(N,L)*ZREI(N,J)*A(J,M)+ZRETR(L,M)
1700 80 CONTINUE
1800C AT THE JUNCTION OF THE LEFT PLATE AND THE ROTATED
1801C RIGHT PLATE ,EQUATE VOLTAGES AND CURRENTS AND
1802C SOLVE FOR CURRENTS
1803C V4=V1I,V5=V2I,V6=V3I, I4=I1I,I5=I2I,I6=I3I
1804C THIS GIVES I4,I5,AND I6 IN TERMS OF I1I
1806C WHEN THESE VALUES ARE PUT IN THE EXPRESSION
1807C FOR V7(VIN) OF LEFT PLATE THE INPUT IMPEDANCE
1808C ZIN CAN BE OBTAINED
1810C IN CARRYING OUT THIS PROCEDURE IT IS CONVENIENT
1811C TO USE THE FOLLOWING VARIABLES
1820 ZS(1,1)=ZLI(1,1)+ZRETR(1,1)
1830 ZS(1,2)=ZLI(1,2)+ZRETR(1,2)
1840 ZS(1,3)=ZLI(1,3)+ZRETR(1,3)
1850 ZS(2,1)=ZLI(2,1)+ZRETR(2,1)
1860 ZS(2,2)=ZLI(2,2)+ZRETR(2,2)
1870 ZS(2,3)=ZLI(2,3)+ZRETR(2,3)
1880 ZS(3,1)=ZLI(3,1)+ZRETR(3,1)
1890 ZS(3,2)=ZLI(3,2)+ZRETR(3,2)
1900 ZS(3,3)=ZLI(3,3)+ZRETR(3,3)
1910 ZSS=ZS(1,1)*ZS(2,2)-ZS(2,1)*ZS(1,2)
1920 ZSS3= ZS(1,1)*ZS(3,2)-ZS(3,1)*ZS(1,2)
1950 IF(ZRETR(1,2).EQ.(0.0,0.0)) G0 T0 85
2000 ZIN=((ZLI(4,1)*ZS(1,2)-ZLI(4,2)*ZS(1,1))*(ZS(2,3)*ZS(1,2)
2001& -ZS(1,3)*ZS(2,2))+(ZLI(4,3)*ZS(1,2)-ZLI(4,2)*ZS(1,3))*ZSS)
2002& *((ZLI(3,4)*ZS(1,2)-ZLI(1,4)*ZS(3,2))*ZSS-(ZLI(2,4)*ZS(1,2)
2003& ))/(ZS(1,2)*ZSS)*((ZS(2,3)*ZS(1,2)-ZS(1,3)*ZS(2,2))*
2004& (ZS(1,1)*ZS(3,2)-ZS(3,1)*ZS(1,2))-(ZS(3,3)*ZS(1,2)-ZS(1,3)

```

TABLE 3-17. (Cont'd)

2005& *ZS(3,2))*ZSS))+(ZLT(4,1)*ZS(1,2)-ZLT(4,2)*ZS(1,1))
 2006& *(ZLT(2,4)*ZS(1,2)-ZLT(1,4)*ZS(2,2))+(ZLT(4,4)*ZS(1,2)
 2007& -ZLT(4,2)*ZLT(4,4))*ZSS)/(ZS(1,2)*ZSS))
 2100 AMPI=EG/(ZG+ZIN)
 2200 AMP6=((ZLT(3,4)*ZS(1,2)-ZLT(1,4)*ZS(3,2))*ZSS
 2201& -(ZLT(2,4)*ZS(1,2)-ZLT(1,4)*ZS(2,2))*ZSS3)
 2202& /(ZS(2,3)*ZS(1,2)-ZS(1,3)*ZS(2,2))*ZSS3
 2203& -(ZS(3,3)*ZS(1,2)-ZS(1,3)*ZS(3,2))*ZSS))*AMPI
 2300 AMP4=((ZS(2,3)*ZS(1,2)-ZS(1,3)*ZS(2,2))/ZSS)*AMP6
 2301& +((ZLT(2,4)*ZS(1,2)-ZLT(1,4)*ZS(2,2))/ZSS)*AMPI
 2400 AMP5=-(ZS(1,1)/ZS(1,2))*AMP4-(ZS(1,3)/ZS(1,2))*AMP6
 2401& -(ZLT(1,4)/ZS(1,2))*AMPI
 2410 G0 10 90
 2420 85 ZIN=((ZLT(4,1)*ZS(2,2)-ZLT(4,2)*ZS(2,1))
 2421& *(ZS(2,3)*ZS(1,2)-ZS(1,3)*ZS(2,2))
 2422& +(ZLT(4,4)*ZS(2,2)-ZLT(4,2)*ZS(2,1))*ZSS)
 2423& *((ZLT(2,4)*ZS(3,2)-ZLT(3,4)*ZS(2,2))*ZSS3
 2424& -(ZLT(3,4)*ZS(1,2)-ZLT(1,4)*ZS(3,2))
 2425& *(ZS(3,1)*ZS(2,2)-ZS(2,1)*ZS(3,2)))
 2426& /(ZSS*ZS(2,2))*((ZS(3,3)*ZS(1,2)-ZS(1,3)*ZS(3,2))
 2427& *(ZS(3,1)*ZS(2,2)-ZS(2,1)*ZS(3,2))
 2428& -(ZS(2,3)*ZS(3,2)-ZS(3,3)*ZS(2,2))*ZSS3))
 2429& +((ZLT(4,1)*ZS(2,2)-ZLT(4,2)*ZS(2,1))
 2430& *(ZLT(2,4)*ZS(1,2)-ZLT(1,4)*ZS(2,2))
 2431& +(ZLT(4,4)*ZS(2,2)-ZLT(4,2)*ZLT(2,4))*ZSS)
 2432& /(ZSS*ZS(2,2))
 2440 AMPI=EG/(ZG+ZIN)
 2450 AMP6=((ZLT(2,4)*ZS(3,2)-ZLT(3,4)*ZS(2,2))*ZSS3
 2451& -(ZLT(3,4)*ZS(1,2)-ZLT(1,4)*ZS(3,2))
 2452& *(ZS(3,1)*ZS(2,2)-ZS(2,1)*ZS(3,2))
 2453& /(ZS(3,3)*ZS(1,2)-ZS(1,3)*ZS(3,2))
 2454& *(ZS(3,1)*ZS(2,2)-ZS(2,1)*ZS(3,2))
 2455& -(ZS(2,3)*ZS(3,2)-ZS(3,3)*ZS(2,2))*ZSS3))*AMPI
 2460 AMP4=((ZS(3,3)*ZS(1,2)-ZS(1,3)*ZS(3,2))/ZSS3)*AMP6
 2461& +((ZLT(3,4)*ZS(1,2)-ZLT(1,4)*ZS(3,2))/ZSS3)*AMPI
 2470 AMP5=-(ZS(3,1)/ZS(3,2))*AMP4-(ZS(3,3)/ZS(3,2))*AMP6
 2471& -(ZLT(3,4)/ZS(3,2))*AMPI
 2500 90 VIN=ZLT(4,1)*AMP4+ZLT(4,2)*AMP5
 2501& +ZLT(4,3)*AMP6+ZLT(4,4)*AMPI
 2600 AMP0=((A(1,1)*ZRT(4,1)+A(2,1)*ZRT(4,2)+A(3,1)*ZRT(4,3))
 2601& /(ZLE+ZRT(4,4)))*AMP4
 2602& +((A(1,2)*ZRT(4,1)+A(2,2)*ZRT(4,2)+A(3,2)*ZRT(4,3))
 2603& /(ZLE+ZRT(4,4)))*AMP5
 2604& +((A(1,3)*ZRT(4,1)+A(2,3)*ZRT(4,2)+A(3,3)*ZRT(4,3))
 2605& /(ZLE+ZRT(4,4)))*AMP6
 2700 VOUT=-ZLE*AMP0

TABLE 3-17. (Cont'd)

```

2800C      **** POWER AND INSERTION LOSS ****
2900 P0=REAL(VOUT*CONJG(-AMP0))
2920 PREF=REAL((EG*ZLE/(ZG+ZLE))*CONJG(EG/(ZLE+ZG)))
2940 PLOSS(K)=10.*ALOG10(P0/PREF)
2960 WRITE(6,200) FREQ,PLOSS(K)
2980 FREQ=FREQ+FINC
3000 100 CONTINUE
3020 200 FORMAT(V)
3100 YMAX=PLOSS(I)
3110 YMIN=PLOSS(I)
3120 DO 300 J=1,K
3140 IF(PLOSS(J).GT.YMAX) YMAX=PLOSS(J)
3150 IF(PLOSS(J).LT.YMIN) YMIN=PLOSS(J)
3160 300 CONTINUE
3170 CALL YPLT(PLOSS,FINI,FINC,YMIN,YMAX,K,NPLTS,K1,DA,0)
3180 STOP
3200 END

```

F. COMPUTER GRAPHS OF TYPICAL MODE PROGRAM RESULTS

This section presents some of the results obtained to date from the MODE programs of the previous section. Obviously, in the time allowed, it has been impossible to completely evaluate these programs since the number of variables involved, and the number of possible combinations, are so enormous. Of these programs MODE2 is probably the most interesting from an experimental point of view, since it is relatively easy to generate sets of compatible variables for different cases (which may or may not be realizable in practice). However, in actual filter construction, MODE3 is more appropriate.

Figures 3-27 and 3-28 show two runs of the MODE1 program. The first is for two identical plates, each with a single mode called mode 1, with a coupling coefficient, $XK1 = .088$ velocity $CT1 = 3.32 \times 10^3$ meters/sec, density $DT1 = 2.65 \times 10^3$ kg/m³, dielectric constant $EP = 4.58 \times 8.85 \times 10^{-12}$ F/m, and electrode diameter = 1×10^{-2} m. Each plate is assumed to be a half wavelength thick at a center frequency of 10 MHz at this velocity of propagation. The filter, constructed of two of these plates, is driven from a generator of 1V and a resistance of 1600 ohms. The output voltage is also measured across a load resistor of 1600 ohms. These material parameters are appropriate for AT-cut quartz, as shown in Table 3-13.

The power insertion loss axis of these curves is in dB and the frequency axis is in hertz. These time sharing curves are crude but convenient for examining cases.

The curve of Figure 3-28 is called mode 2 and is characterized by a coupling coefficient that is twice that of mode 1 $XK2 = 2XK1$ and a velocity, $CT2 = 1.1 CT1$. Calculations for the response shown in Figure 3-28 were done for a half wavelength plate thickness at 10 MHz, determined by the velocity of propagation for mode 1 rather than mode 2. These curves, taken together, show the individual responses of the two modes most often used in the MODE2 program.

Figure 3-29 shows the response obtained when Modes 1 and 2 are combined in one plate, and two of these identical plates are bonded together with zero rotation between them. When the two modes are combined together in a single plate, and two of these plates are operated as a stacked filter, the most obvious change is the dip in the response curve, following the mode-1 response, over that obtained by just adding the two response curves shown previously. There is also a slight decrease in the frequency of peak response for mode 2 as compared to only having mode 2 alone.

Figure 3-30 shows the response of this same filter for a rotation angle of 5° between plates. Nothing much happens to the response except for a slight decrease in insertion loss at the peak response.

Figures 3-31, 3-32 and 3-33 show the response of this 5° -rotated stack of two 10 MHz plates about 5, 15 and 20 MHz. In all cases these curves are compatible with the type of response expected for a stacked filter of single-mode elements using Mason's equivalent circuit.

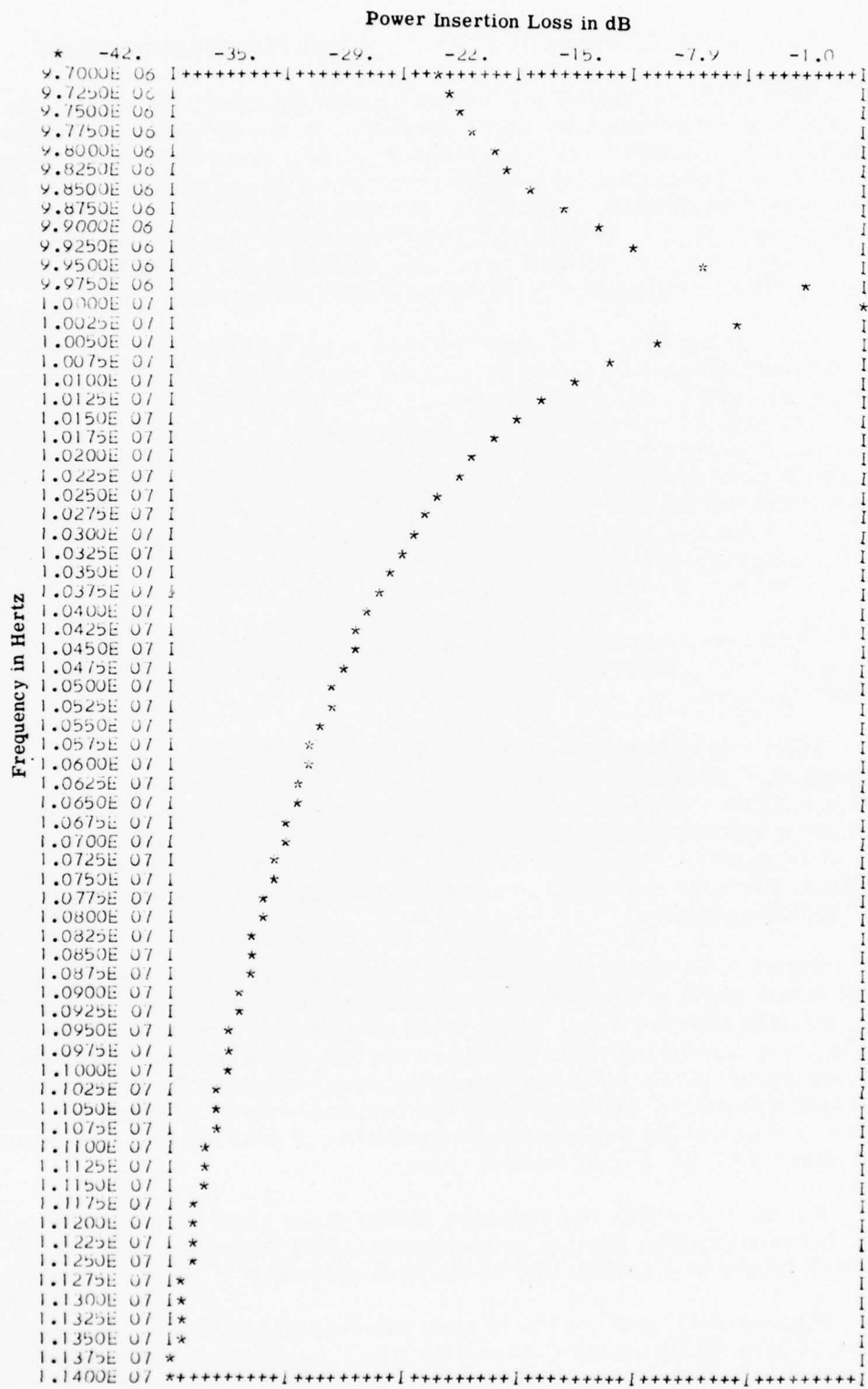


Figure 3-27. Mode 1 Response from MØDE 1 Program

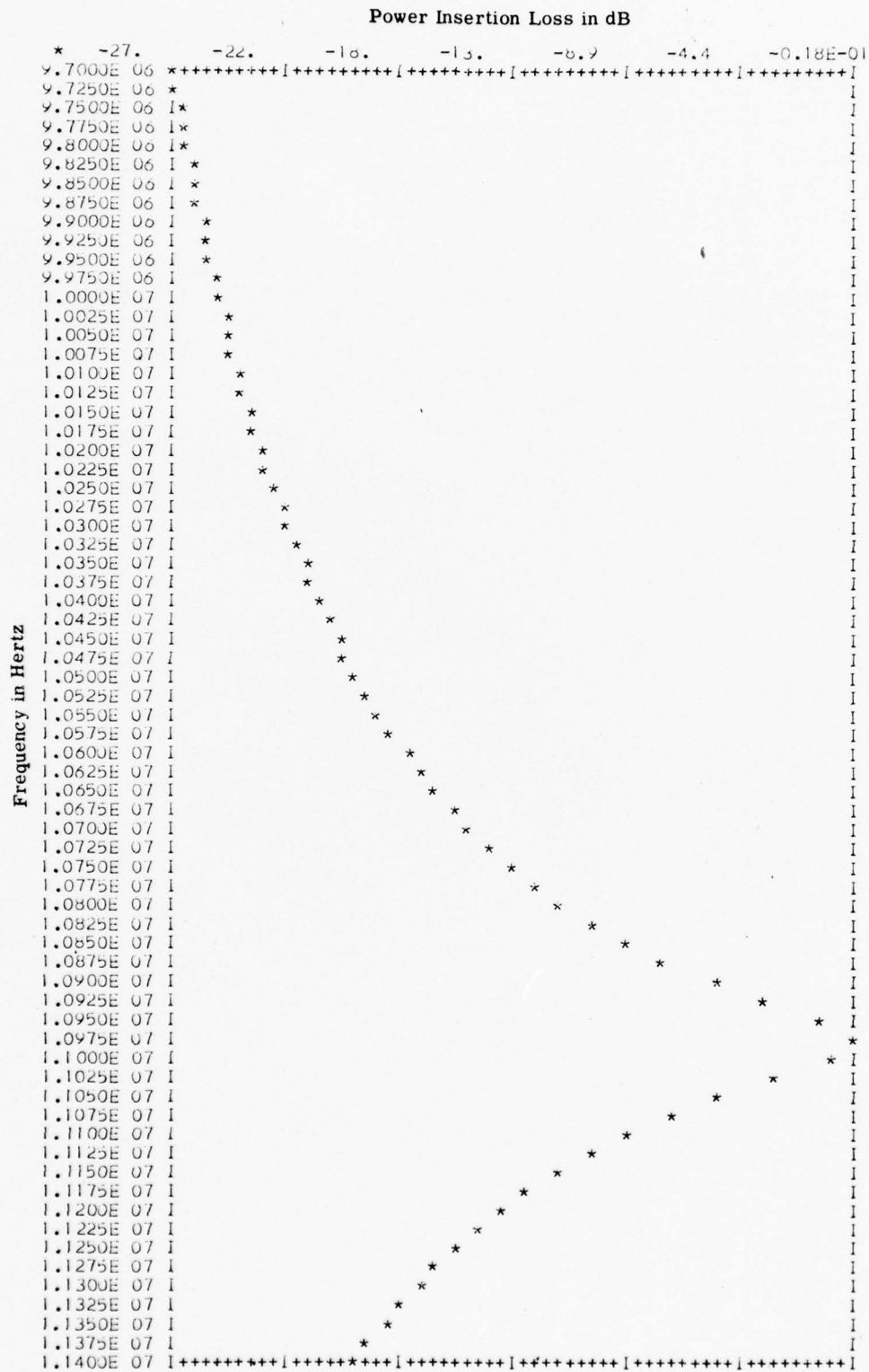


Figure 3-28. Mode 2 Response from MØDE 1 Program

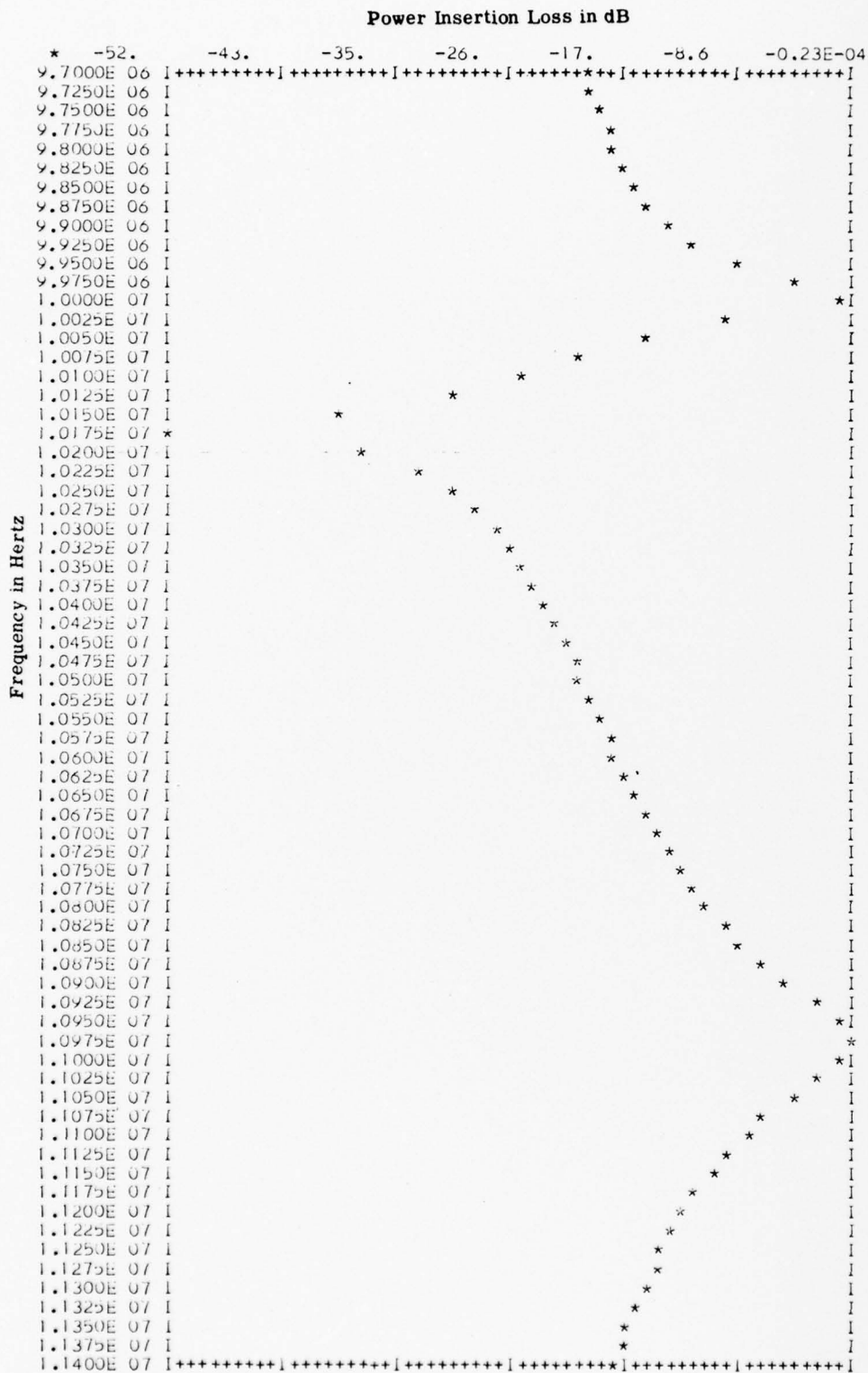


Figure 3-29. Two Element Stacked Filter (Mode 1 and Mode 2 in each plate)
 0° rotation between plates

$$\begin{array}{l} \text{Each} \\ \text{Plate} \end{array} \left\{ \begin{array}{ll} \text{XK1} = 0.088 & \text{CT1} = 3.32 \times 10^3 \\ \text{XK2} = 2\text{XK1} & \text{CT2} = 1.1 \text{CT1} \end{array} \right\}$$

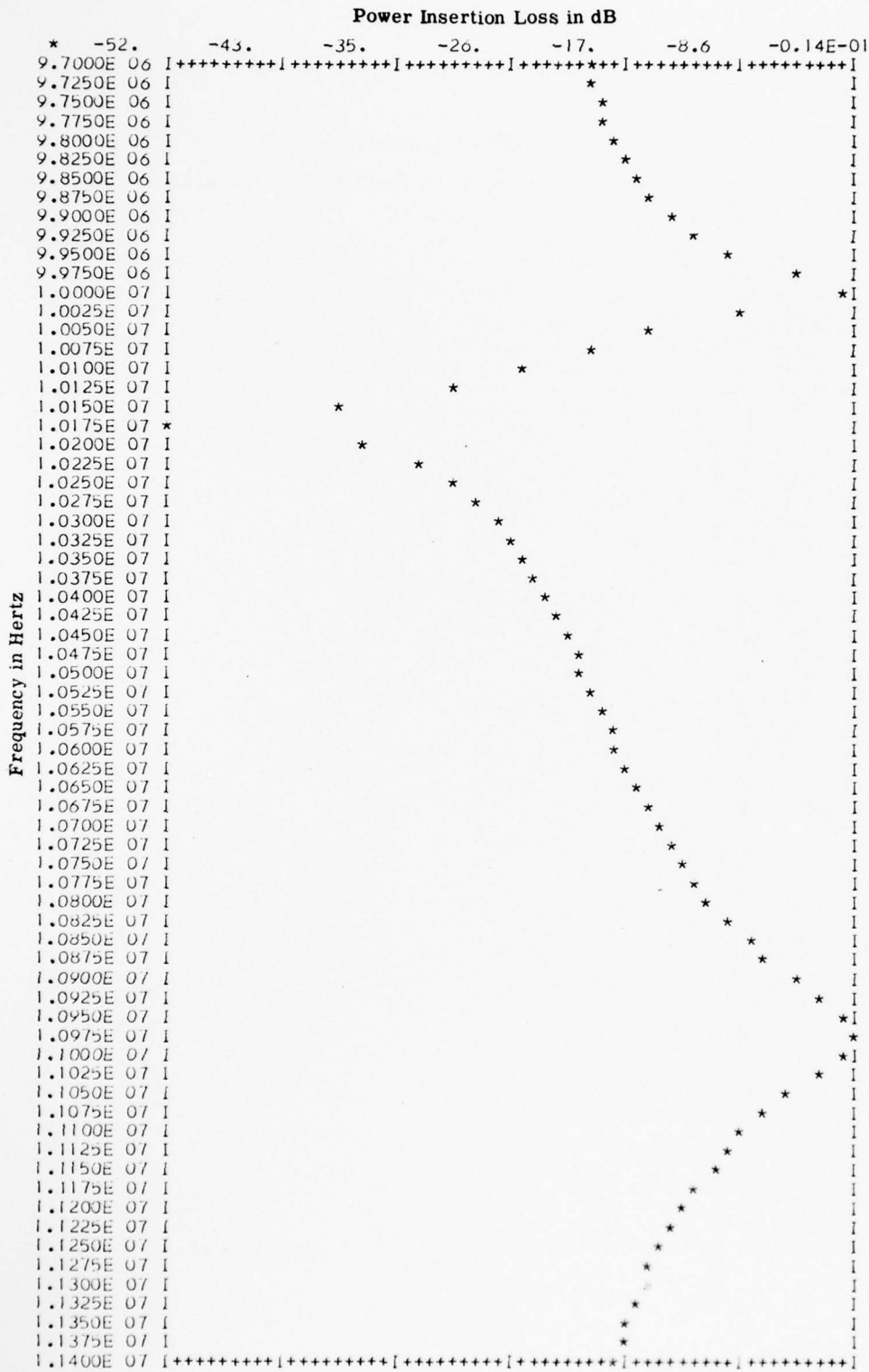


Figure 3-30. Two Element Stack +5° Rotation Between Plates Response at 10 Megahertz

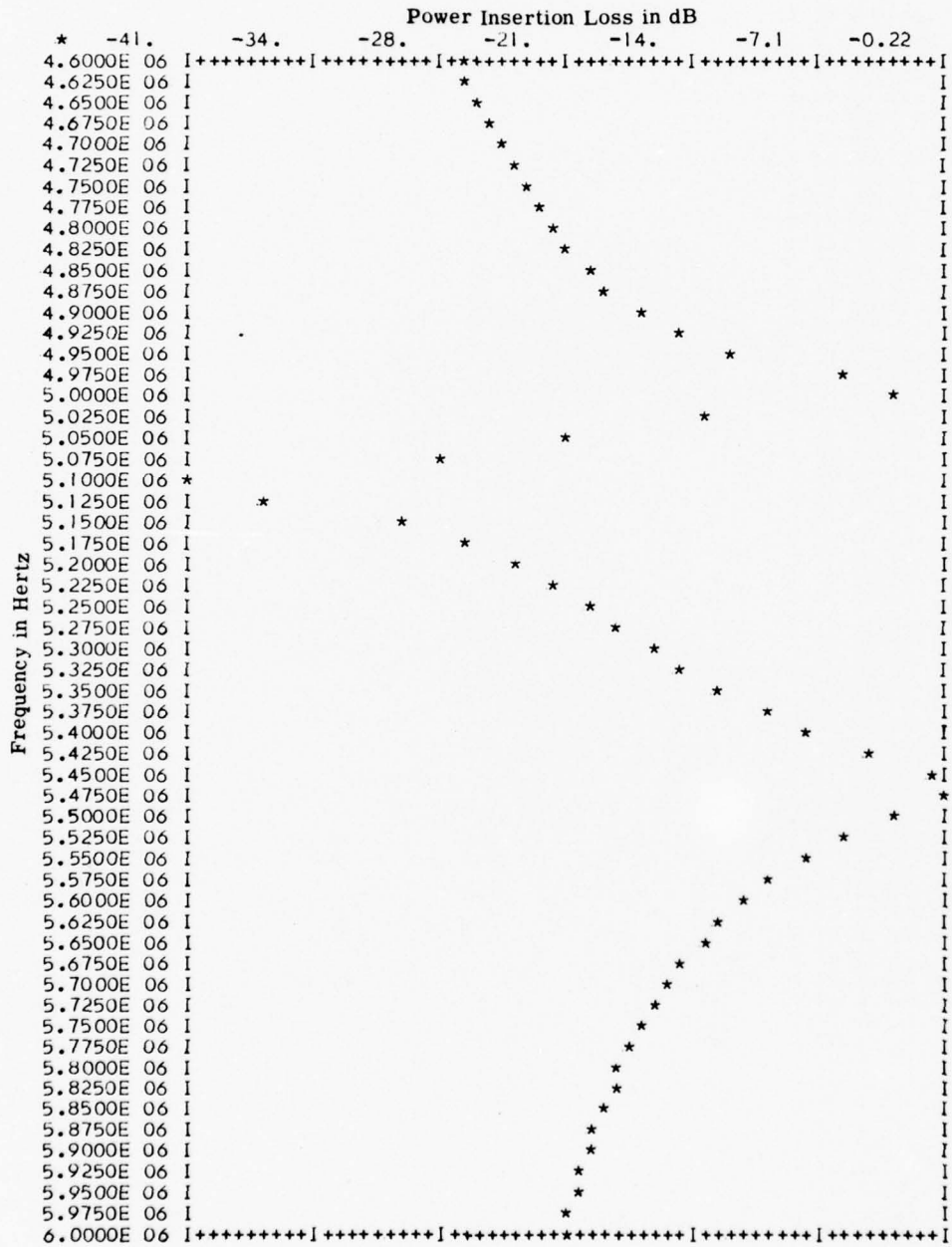


Figure 3-31. Two Element Stack $+5^\circ$ Rotation Between Plates Response at 5 Megahertz



Figure 3-32. Two Element Stack +5° Rotation Between Plates Response at 15 Megahertz

Power Insertion Loss in dB

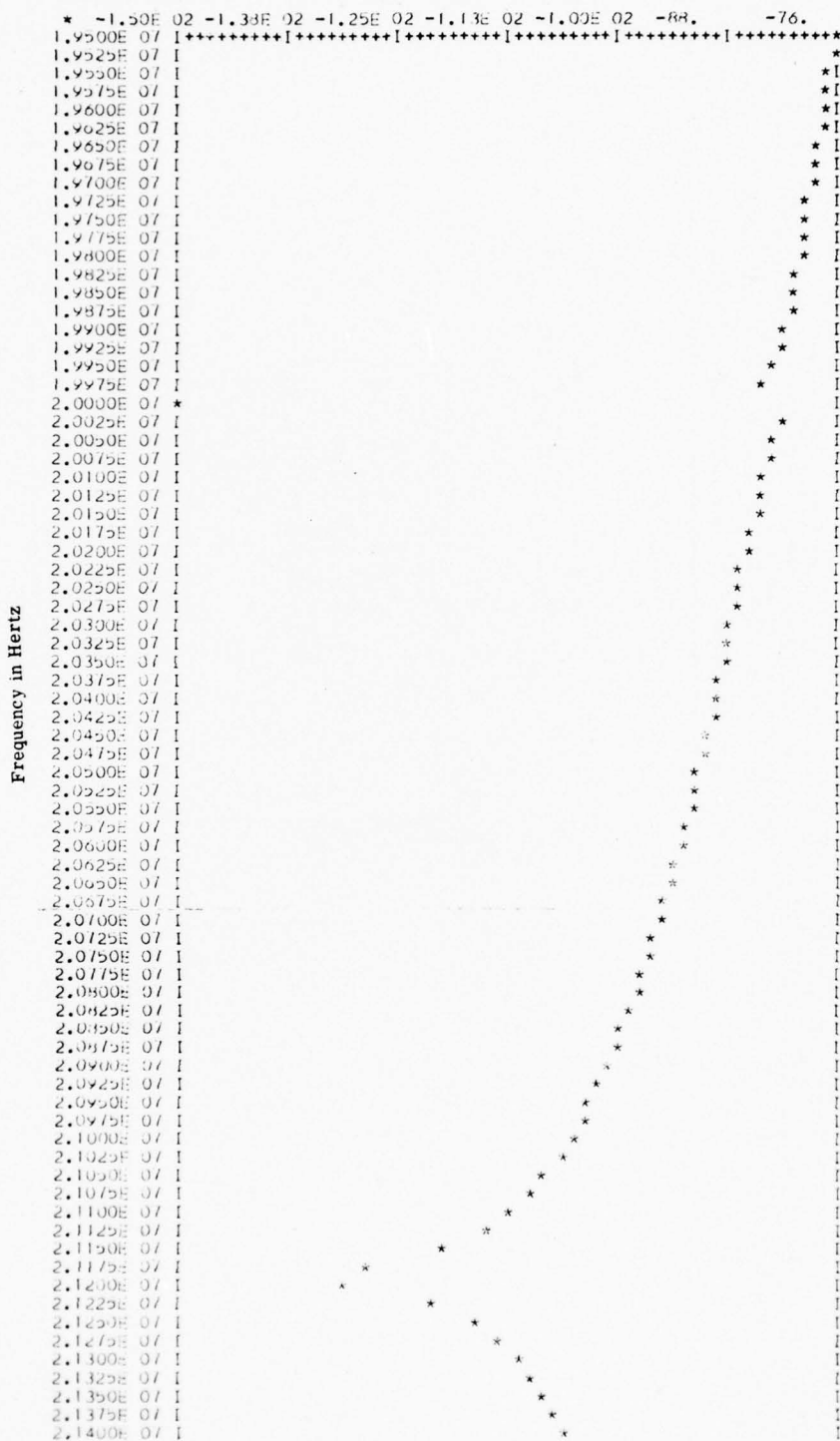


Figure 3-33. Two Element Stack +5° Rotation Response at 20 Megahertz

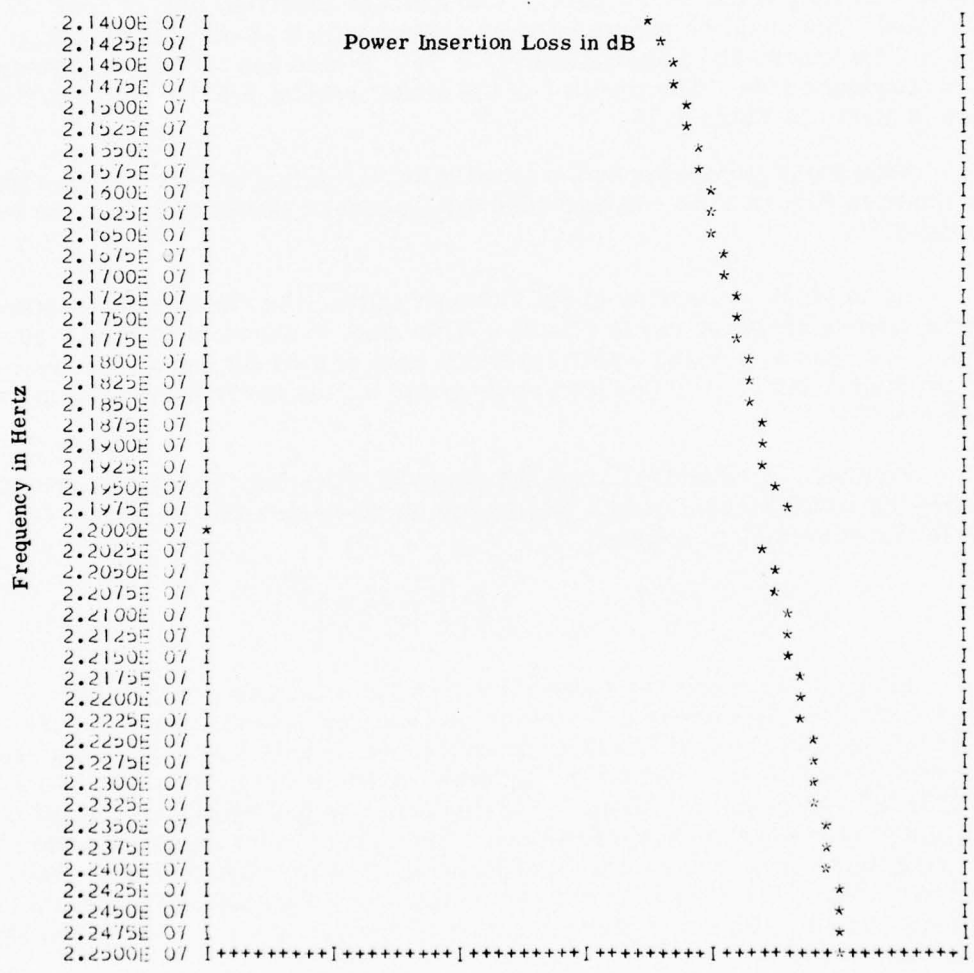


Figure 3-33. (Continued)

Figure 3-34 shows the response obtained with a -20° rotation between plates. Even at this angle of rotation, for these particular plates, nothing much happens. The response at a plate rotation angle of 45° is shown in Figure 3-35. Here, the response of mode 1 has decreased considerably and the notch following the mode 1 response has become shallower.

Figure 3-36 show the affect of rotating the two plates by 70° . A double-humped configuration occurs around the mode 2 response (actually a shifted down mode 2 response from that of Figure 3-29), and mode 1 has completely disappeared. Figure 3-37 continues the rotation of the two plates to 80° . The response now has a 5.73 dB peak at 10.4 MHz, a 7.1 dB dip at 10.475 MHz, and a 3.14 dB peak at 10.55 MHz. If an average insertion loss of 5 dB is assumed, this could be represented as a filter with a ± 2 dB passband ripple and a 3 dB bandwidth of approximately 2.5%. It also has a relatively steep low-frequency side. The printout of the actual results for this angle of rotation is shown in Table 3.18.

This looks promising so the results for a rotation of -82° between plates is shown in Figure 3-38. In this case the amount of passband ripple has been reduced.

At an angle of rotation of 85° between plates, the ripple has disappeared and a smooth response curve results. This case is shown in Figure 3-39. This curve has a midband power insertion loss of 3.67 dB and a 3 dB bandwidth of approximately 1.5%. The data appropriate to this curve are shown in Table 3-19.

Figures 3-40 and 3-41 show the effect of operating the MODE2 program, where the second mode in each plate is not piezoelectrically coupled. For these two curves in each plate,

$$\begin{array}{ll} \text{XK1} = 0.088 & \text{CT1} = 3.32 \times 10^3 \\ \text{XK2} = 0.0 & \text{CT2} = 1.1 \text{ CT1} \end{array}$$

Figure 3-40 shows the response curve for this case at a 0° angle of plate rotation. As expected, this curve is identical to the curve of Figure 3-27 for, in this case, there is no possible way to couple into the non piezoelectric second mode. Figure 3-41 shows the same case, but with plate 2 rotated 5° with respect to plate 1. In this case the plate boundary condition coupling to the non-piezoelectric mode, takes place, showing up as a dip in the response curve at the location of mode 2, shown in Figure 3-28. Incidentally, due to the method of solution employed in this MODE2 program, it will not operate with a non-piezoelectrically coupled mode 1. Mode 1 is always assumed present.

The case of two non-identical plates is illustrated in Figures 3-42 and 3-43. For these two curves the values for each plate are:

Plate 1		For Plate 2	
$\text{XK1} = 0.088$	$\text{CT1} = 3.32 \times 10^3$	$\text{XK1} = 0.088$	$\text{CT1} = 3.32 \times 10^3$
$\text{XK2} = 2\text{XK1}$	$\text{CT2} = 1.1 \text{ CT1}$	$\text{XK2} = 0$	$\text{CT2} = 1.1 \text{ CT1}$

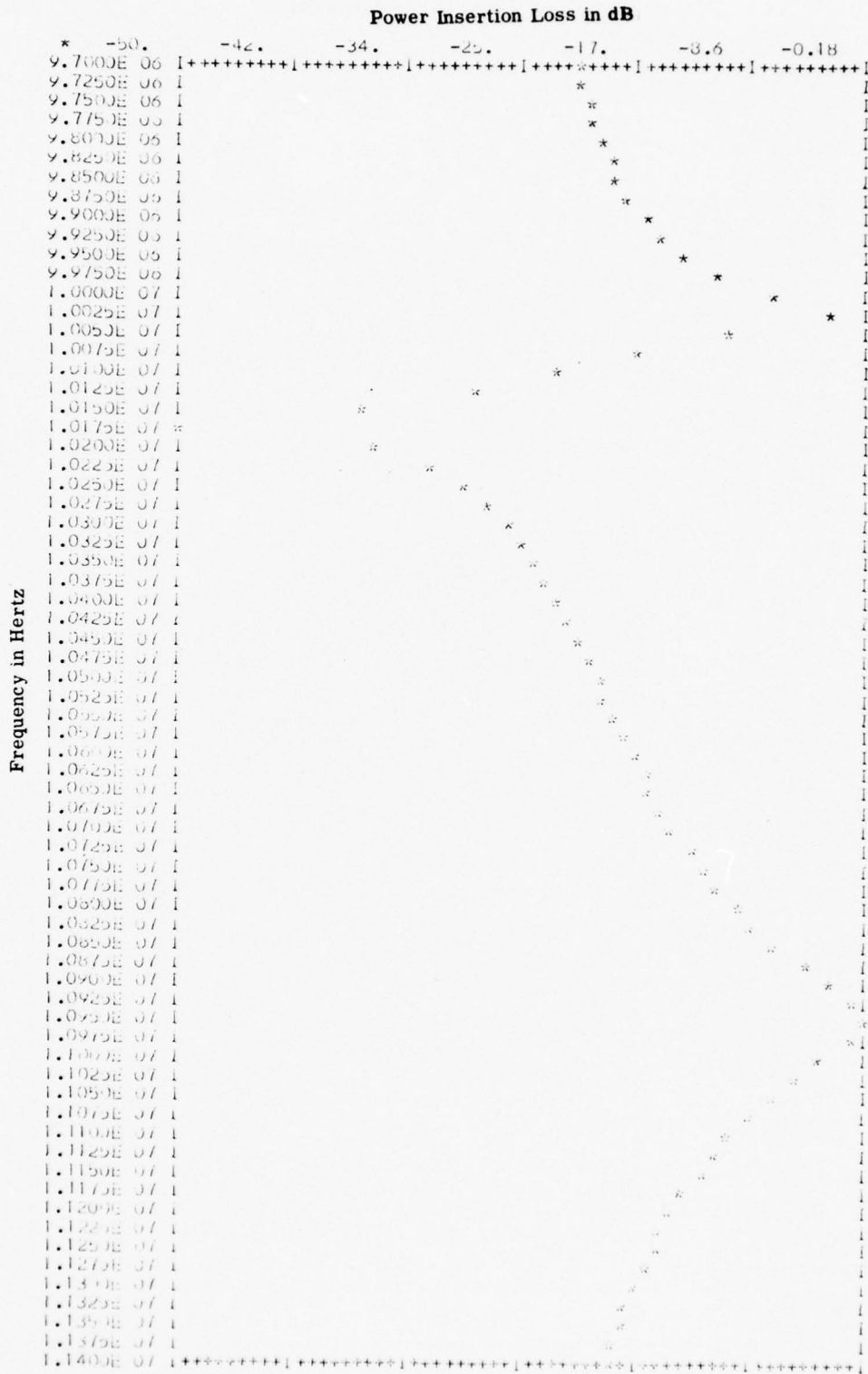


Figure 3-34. Two Element Stack -20° Rotation Between Plates

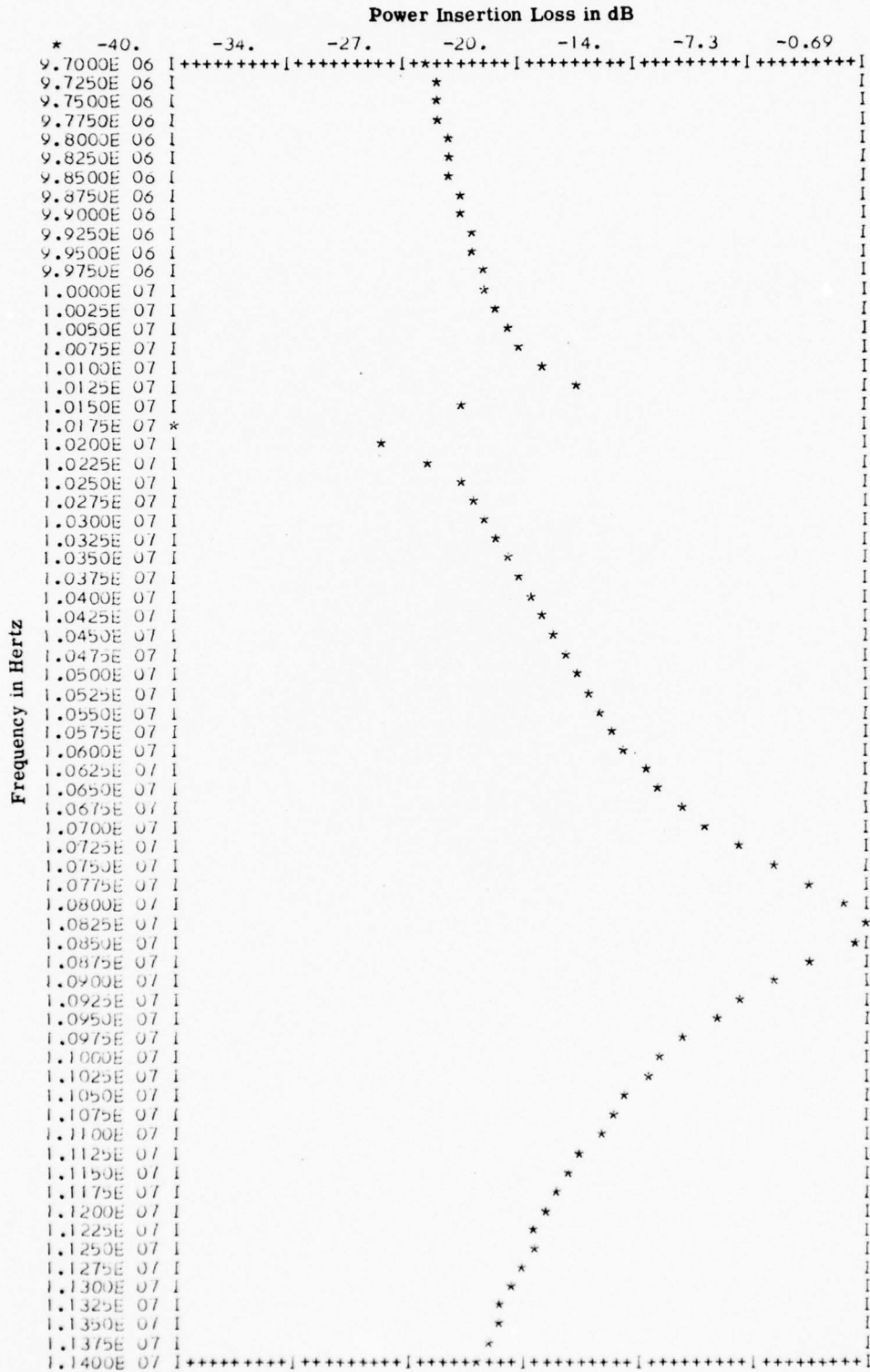


Figure 3-35. Two Element Stack 45° Rotation Between Plates

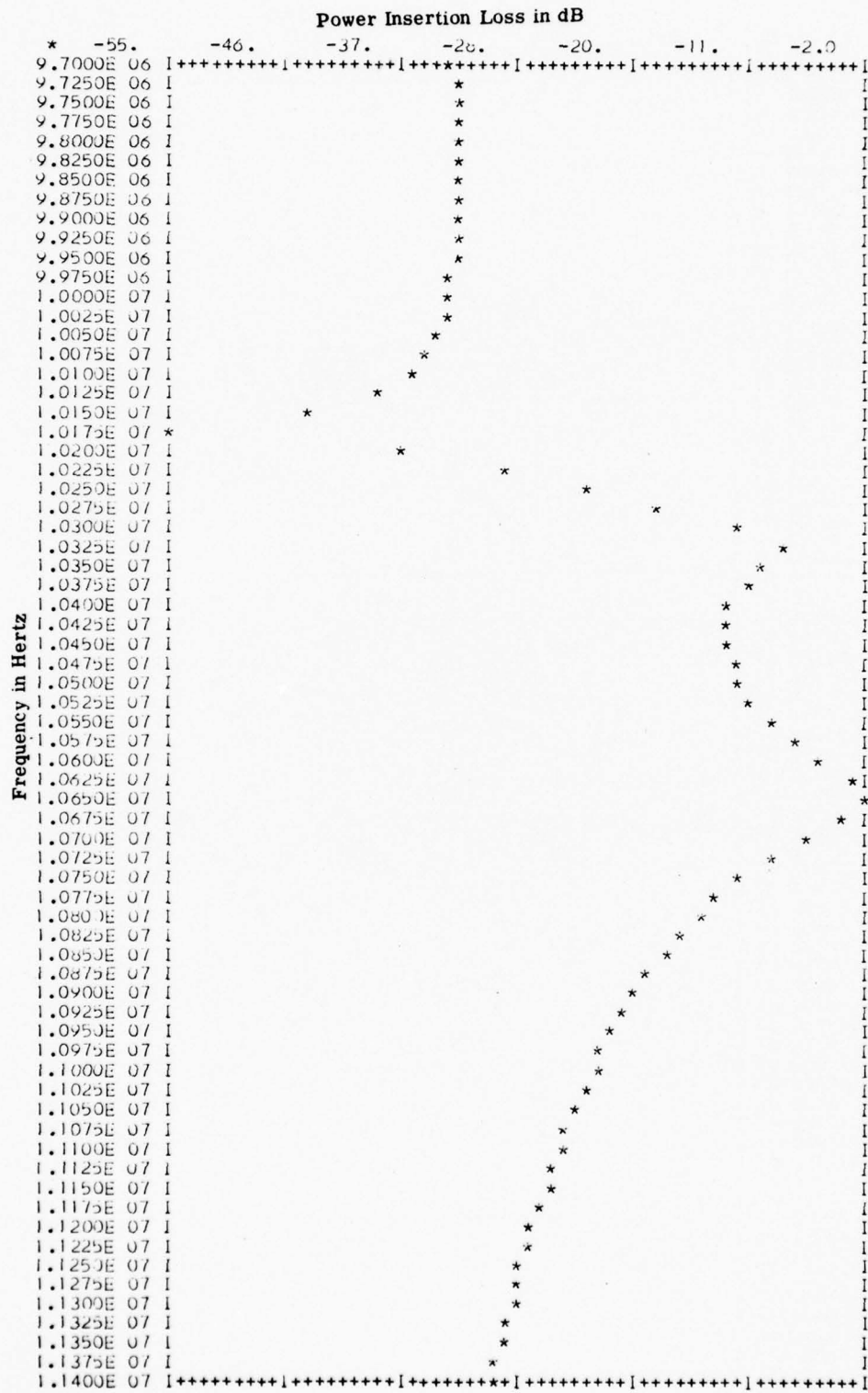


Figure 3-36. Two Element Stack 70° Rotation Between Plates

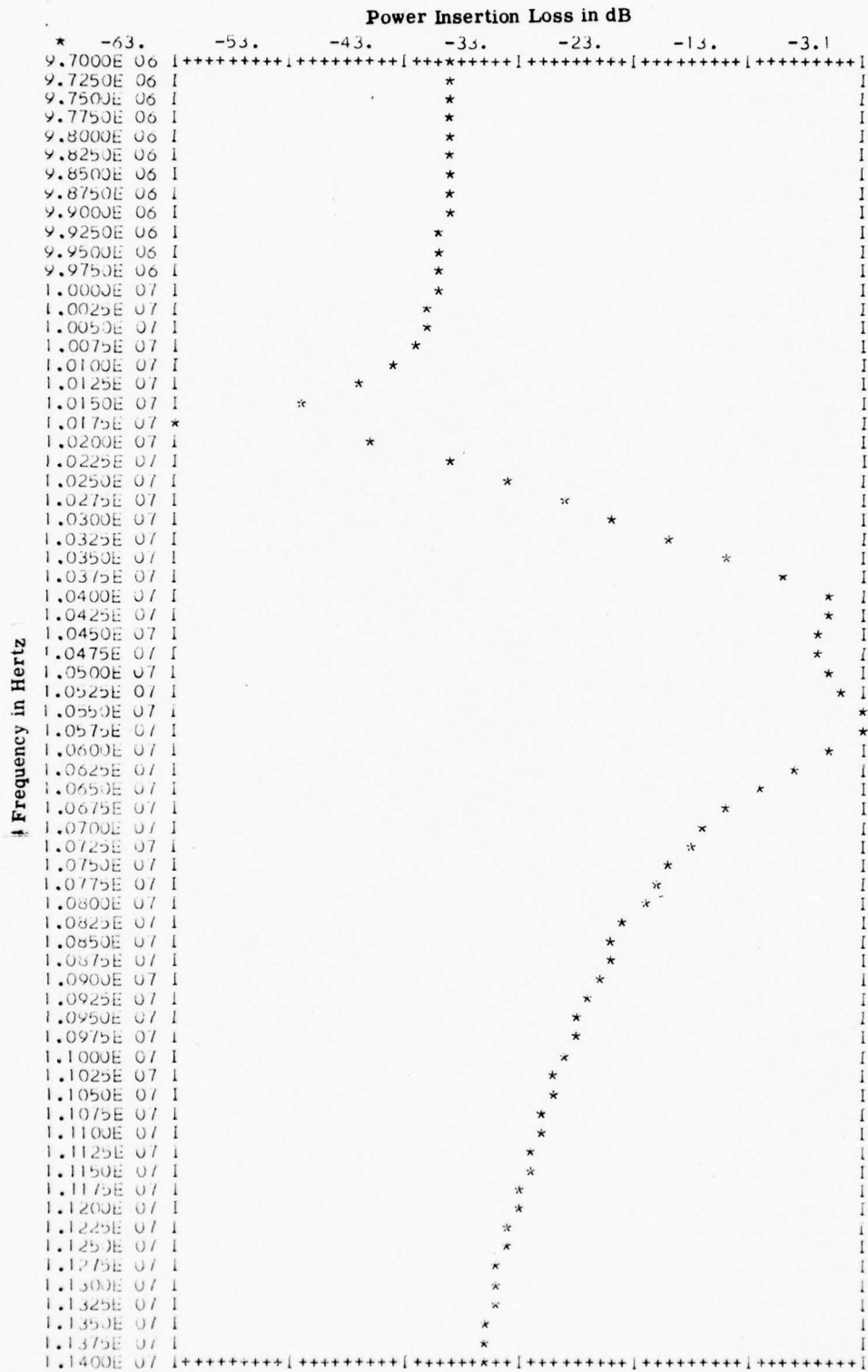


Figure 3-37. Two Element Stack 80° Rotation Between Plates

TABLE 3-18. DATA FOR STACKED FILTER OF TWO IDENTICAL PLATES
OF FIGURE 37

Each Plate	CT1 = 3.32×10^3	Angle of Plate
XK1 = 0.088	CT2 = 1.1 CT1	Rotation
XK2 = 2XK2		80°

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0.97000000E 07	-0.39546316E 02
0.97250000E 07	-0.39493949E 02
0.97500000E 07	-0.39449318E 02
0.97750000E 07	-0.39414415E 02
0.98000000E 07	-0.39391693E 02
0.98250000E 07	-0.39384327E 02
0.98500000E 07	-0.39396351E 02
0.98750000E 07	-0.39432996E 02
0.99000000E 07	-0.39501232E 02
0.99250000E 07	-0.39610388E 02
0.99500000E 07	-0.39773354E 02
0.99750000E 07	-0.40008307E 02
0.10000000E 08	-0.40342085E 02
0.10025000E 08	-0.40815485E 02
0.10050000E 08	-0.41494490E 02
0.10075000E 08	-0.42495325E 02
0.10100000E 08	-0.44047886E 02
0.10125000E 08	-0.46703523E 02
0.10150000E 08	-0.52477751E 02
0.10175000E 08	-0.63001627E 02
0.10200000E 08	-0.45983184E 02
0.10225000E 08	-0.39022934E 02
0.10250000E 08	-0.33859016E 02
0.10275000E 08	-0.29317022E 02
0.10300000E 08	-0.24923980E 02
0.10325000E 08	-0.20367315E 02
0.10350000E 08	-0.15366761E 02
0.10375000E 08	-0.98611822E 01
0.10400000E 08	-0.57309360E 01
0.10425000E 08	-0.58422754E 01
0.10450000E 08	-0.68777331E 01
0.10475000E 08	-0.70871861E 01
0.10500000E 08	-0.63929900E 01
0.10525000E 08	-0.43941731E 01
0.10550000E 08	-0.31440103E 01
0.10575000E 08	-0.33192564E 01
0.10600000E 08	-0.62390100E 01
0.10625000E 08	-0.95754723E 01
0.10650000E 08	-0.12433500E 02
0.10675000E 08	-0.14808603E 02
0.10700000E 08	-0.16810252E 02
0.10725000E 08	-0.18529768E 02
0.10750000E 08	-0.20032231E 02
0.10775000E 08	-0.21364023E 02
0.10800000E 08	-0.22558673E 02
0.10825000E 08	-0.23640826E 02
0.10850000E 08	-0.24629232E 02
0.10875000E 08	-0.25538499E 02
0.10900000E 08	-0.26379982E 02
0.10925000E 08	-0.27163029E 02
0.10950000E 08	-0.27894739E 02
0.10975000E 08	-0.28581625E 02

TABLE 3-18. (Cont'd)

0.11000000E 08	-0.29228710E 02
0.11025000E 08	-0.29340316E 02
0.11050000E 08	-0.30419316E 02
0.11075000E 08	-0.30970744E 02
0.11100000E 08	-0.31495515E 02
0.11125000E 08	-0.31996801E 02
0.11150000E 08	-0.32476065E 02
0.11175000E 08	-0.32935607E 02
0.11200000E 08	-0.33376869E 02
0.11225000E 08	-0.33801254E 02
0.11250000E 08	-0.34209992E 02
0.11275000E 08	-0.34604214E 02
0.11300000E 08	-0.34984924E 02
0.11325000E 08	-0.35353002E 02
0.11350000E 08	-0.35709313E 02
0.11375000E 08	-0.36054554E 02
0.11400000E 08	-0.36389422E 02

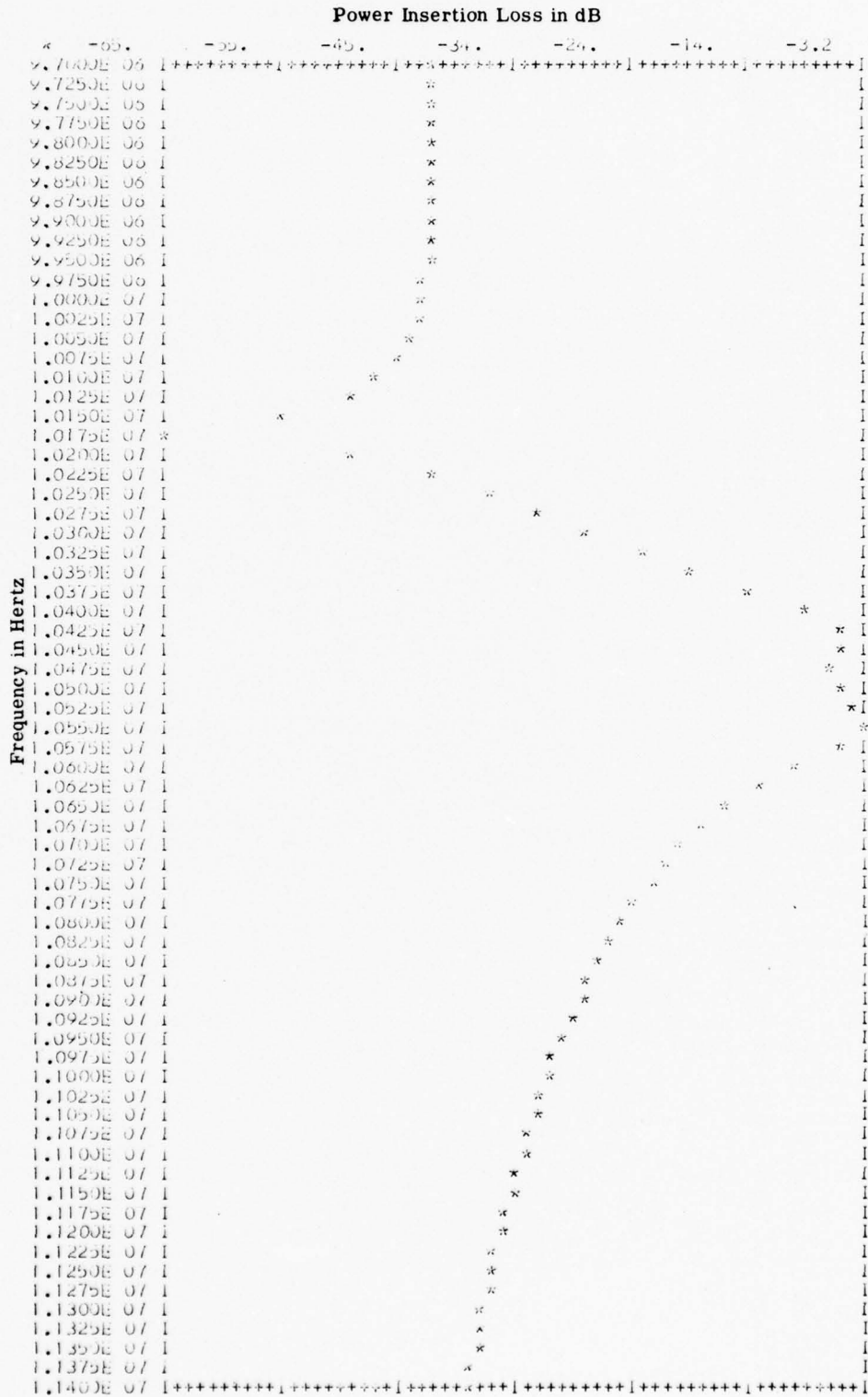


Figure 3-38. Two Element Stack -82° Rotation Between Plates

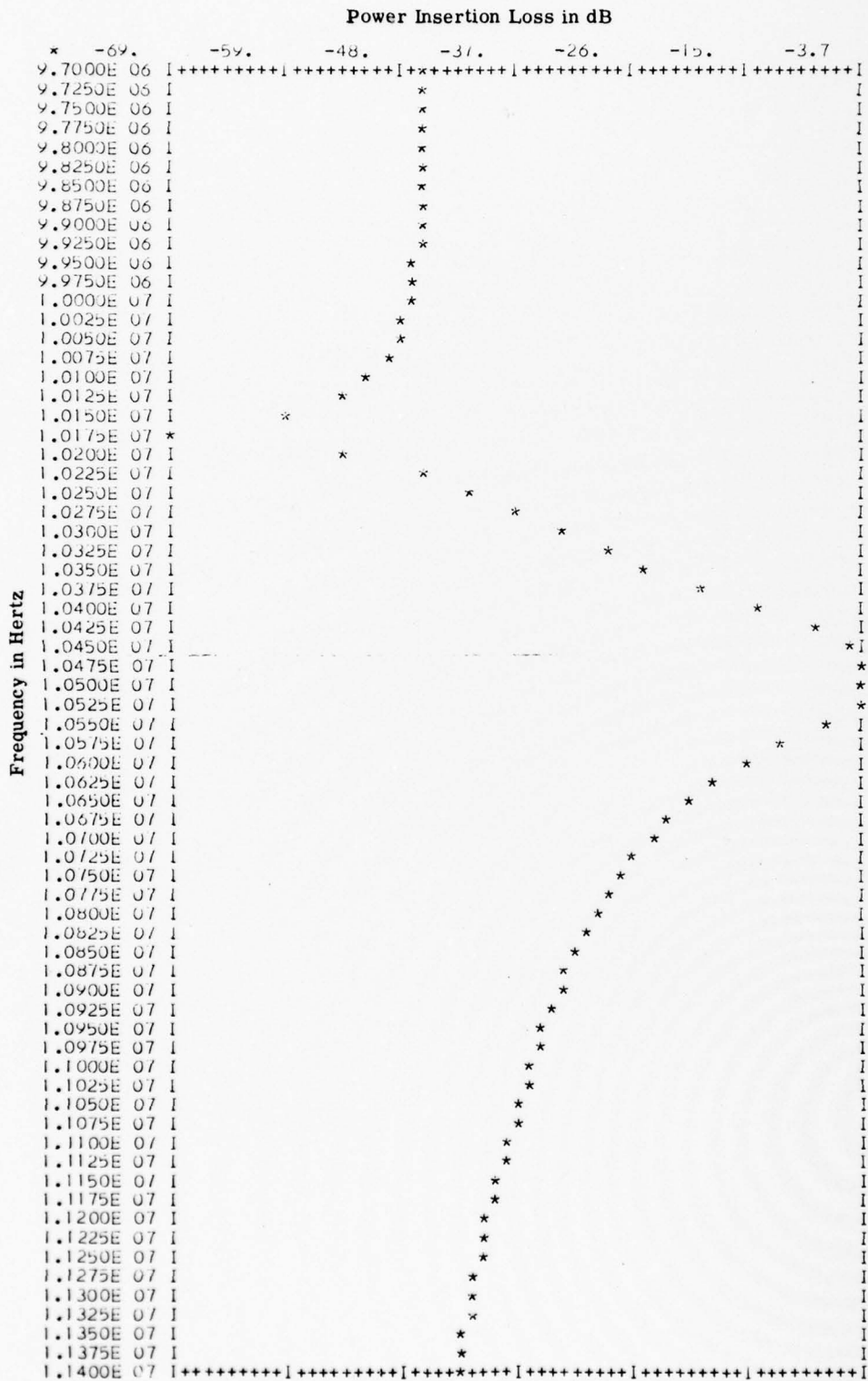


Figure 3-39. Two Element Stack 85° Rotation Between Plates

Table 3-19. Data for Stacked Filter of Two Identical Plates of Figure 39

Each Plate
 XK1 = 0.088 CT1 = 3.32×10^5 Angle of Plate Rotation
 XK2 = 2XK1 CT2 = 1.1CT1 85°

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0.97000000E 07	-0.45613794E 02
0.97250000E 07	-0.45566490E 02
0.97500000E 07	-0.45527452E 02
0.97750000E 07	-0.45498733E 02
0.98000000E 07	-0.45482907E 02
0.98250000E 07	-0.45483247E 02
0.98500000E 07	-0.45503904E 02
0.98750000E 07	-0.45550264E 02
0.99000000E 07	-0.45629547E 02
0.99250000E 07	-0.45751264E 02
0.99500000E 07	-0.45928622E 02
0.99750000E 07	-0.46180148E 02
0.10000000E 08	-0.46533184E 02
0.10025000E 08	-0.47029113E 02
0.10050000E 08	-0.47734707E 02
0.10075000E 08	-0.48767248E 02
0.10100000E 08	-0.50358026E 02
0.10125000E 08	-0.53060284E 02
0.10150000E 08	-0.56892118E 02
0.10175000E 08	-0.69486059E 02
0.10200000E 08	-0.52563060E 02
0.10225000E 08	-0.45724549E 02
0.10250000E 08	-0.40724295E 02
0.10275000E 08	-0.36408985E 02
0.10300000E 08	-0.32340657E 02
0.10325000E 08	-0.28263915E 02
0.10350000E 08	-0.23269077E 02
0.10375000E 08	-0.19234704E 02
0.10400000E 08	-0.13642220E 02
0.10425000E 08	-0.89800430E 01
0.10450000E 08	-0.46146222E 01
0.10475000E 08	-0.41721699E 01
0.10500000E 08	-0.36707944E 01
0.10525000E 08	-0.36846952E 01
0.10550000E 08	-0.70461994E 01
0.10575000E 08	-0.11140528E 02
0.10600000E 08	-0.14669737E 02
0.10625000E 08	-0.17560205E 02
0.10650000E 08	-0.19959005E 02
0.10675000E 08	-0.21991750E 02
0.10700000E 08	-0.23747145E 02
0.10725000E 08	-0.25287279E 02
0.10750000E 08	-0.26656160E 02
0.10775000E 08	-0.27836193E 02
0.10800000E 08	-0.29001707E 02
0.10825000E 08	-0.30021171E 02
0.10850000E 08	-0.30959112E 02
0.10875000E 08	-0.31827167E 02
0.10900000E 08	-0.32634569E 02
0.10925000E 08	-0.33389111E 02
0.10950000E 08	-0.34096759E 02
0.10975000E 08	-0.34763133E 02

TABLE 3-19. (Cont'd)

0.1100000E	03	-0.35392596E	02
0.11025000E	08	-0.35988952E	02
0.11050000E	08	-0.36555177E	02
0.11075000E	08	-0.37094472E	02
0.11100000E	08	-0.37603992E	02
0.11125000E	08	-0.38100995E	02
0.11150000E	08	-0.38572382E	02
0.11175000E	08	-0.39024711E	02
0.11200000E	03	-0.39459499E	02
0.11225000E	08	-0.39878054E	02
0.11250000E	08	-0.40281532E	02
0.11275000E	08	-0.40670977E	02
0.11300000E	03	-0.41047351E	02
0.11325000E	03	-0.41411469E	02
0.11350000E	03	-0.41764166E	02
0.11375000E	08	-0.42106088E	02
0.11400000E	03	-0.42437910E	02

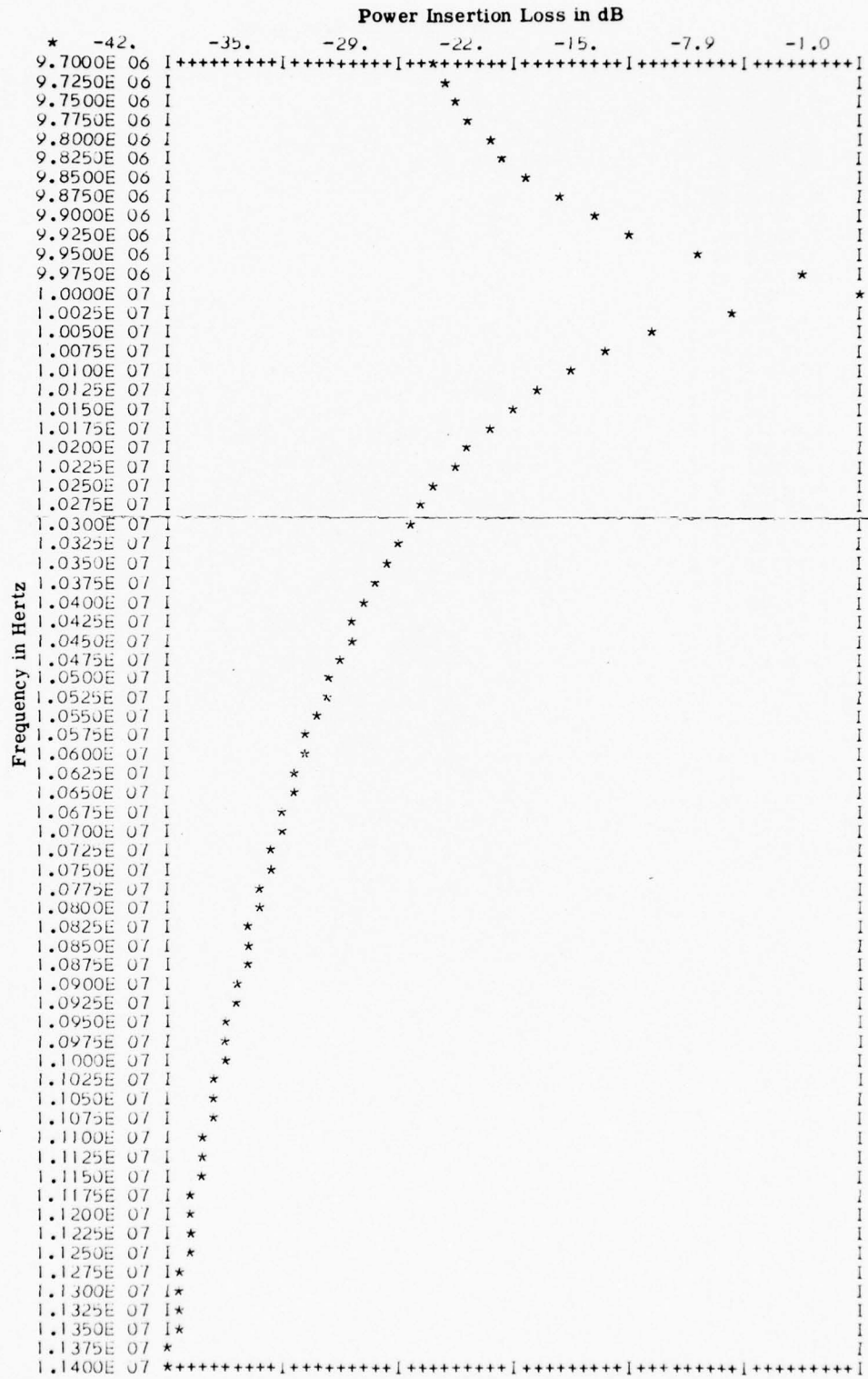


Figure 3-40. Two Element Identical Plate Stack Nonpiezoelectrically Coupled
 2nd Mode
 0° Rotation Between Plates
 Each Plate
 XK1 = 0.088 CT1 = 3.32 × 10³
 XK2 = 0.0 CT2 = 1.1 CT1

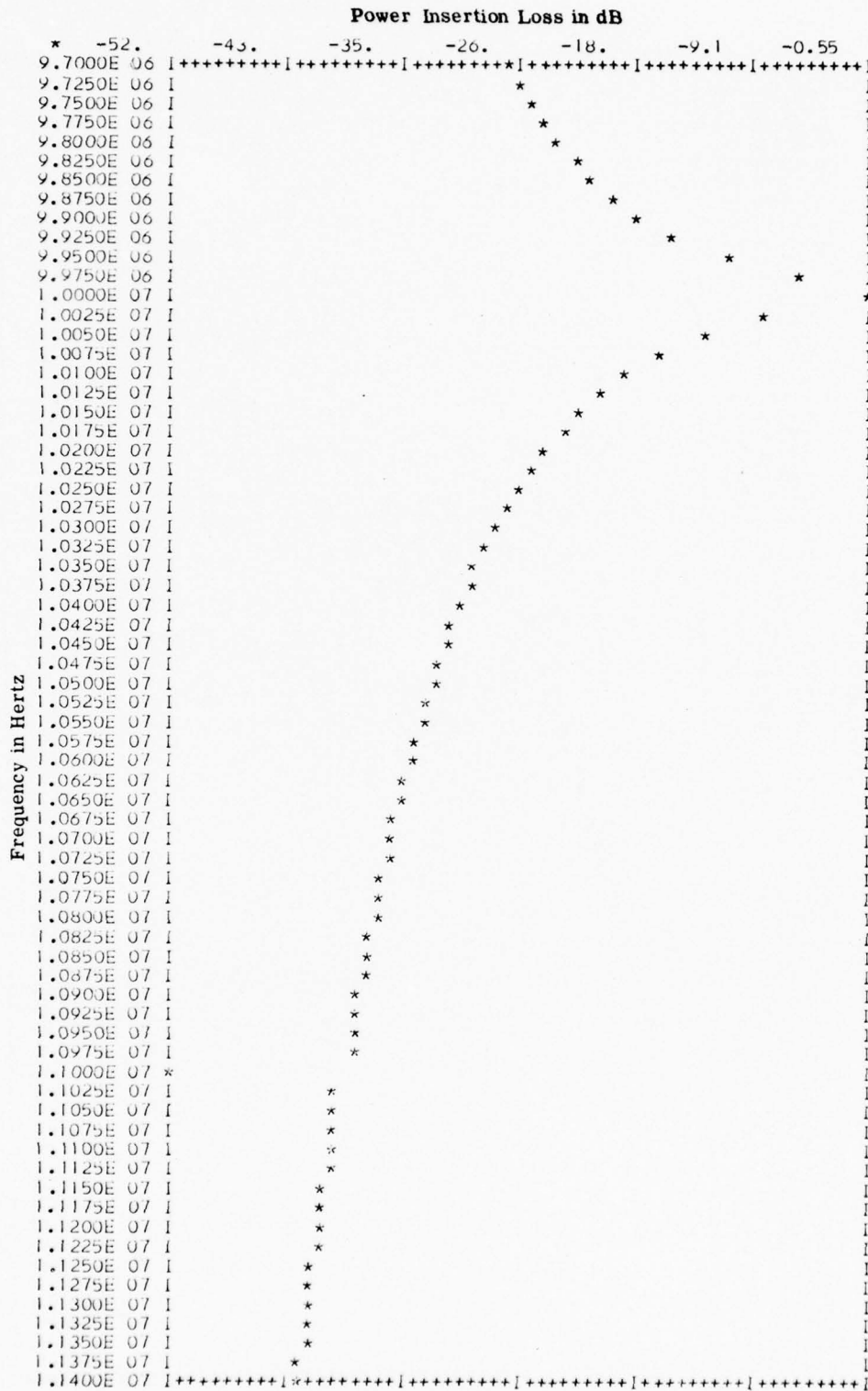


Figure 3-41. Two Element Identical Plate Stack Nonpiezoelectrically Coupled
2nd Mode 5° Rotation Between Plates

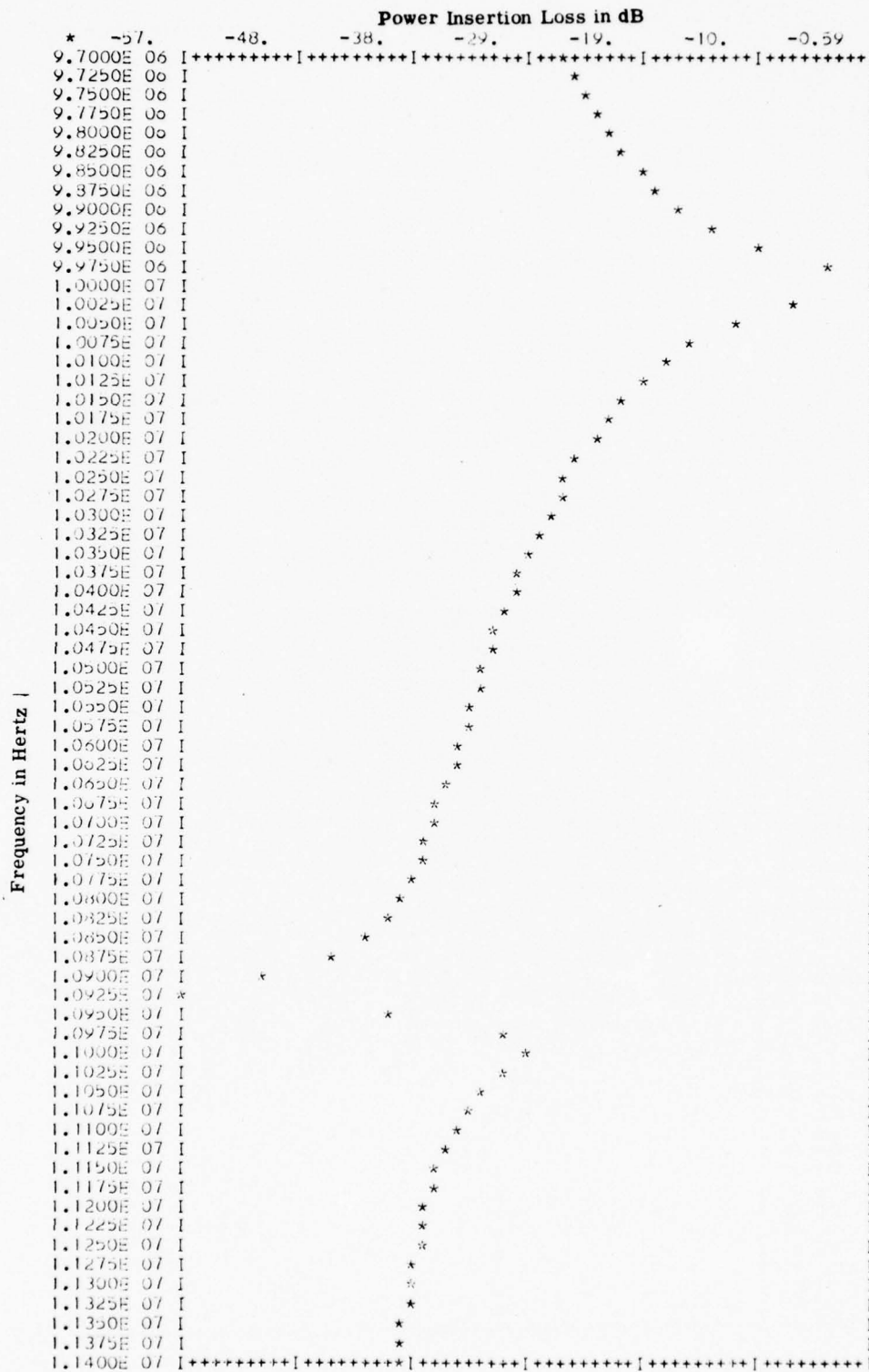


Figure 3-43. Two Element Non-Identical Plate Stack of Figure 3-42 with 5° Rotation Between Plates

Thus, in plate 2, the second mode is not piezoelectrically coupled. Except for this, the plate parameters are identical. Figure 3.42 shows the response of these plates when they are lined up. The response, in this case, will exhibit a notch at the mode-2 frequency, in contrast to Figure 3-41, since energy is coupled into this mode by plate 1, but is not available as output energy because of the lack of piezoelectric coupling to this mode in plate 2. Figure 3.43 shows the same two plates with a 5° rotation between them. In this case a small resonance is observed at mode 2, in contrast to Figure 3-42 since, now, due to plate boundary coupling, some of the piezoelectrically excited energy of plate 1 in mode 2 can be coupled to the output.

Figure 3-44 shows the response at 5° of plate rotation of a two-plate stack of identical elements similar to those shown in Figure 3-30 at a 5° rotation between the plates. However, for these plates,

$$\begin{aligned} \text{XK1} &= 0.088 & \text{CT1} &= 3.32 \times 10^3 \\ \text{XK2} &= 2\text{XK1} & \text{CT2} &= 1.15 \text{CT1}, \end{aligned}$$

so that here the velocity of mode 2 is faster than that shown earlier. Besides the obvious fact of making the mode 2 response occur at a higher frequency, this change also causes the notch following the mode-1 response to occur at a higher frequency. This means that the null in the response is due to the phase shift between the mode-1 and mode-2 responses. It also suggests that, in multi element stacked cascades, plates with different mode properties can be used to control stop-band rejection.

Figures 3-45, 3-46 and 3-47 show the effects on the frequency response of two identical element stacks, when the velocity of the second mode is slower than that of the first mode. For these plates

$$\begin{aligned} \text{XK1} &= 0.088 & \text{CT1} &= 3.32 \times 10^3 \\ \text{XK2} &= 1.1\text{XK1} & \text{CT2} &= 0.95 \text{CT1} \end{aligned}$$

Again, except for the change in velocity of the second mode, the plates are identical to those shown earlier. This change in velocity of mode 2 obviously makes the mode-2 response occur at a lower frequency than that of mode 1. (The plate is still one half wavelength thick at a frequency of 10 MHz and a velocity of propagation corresponding to CT1.)

Figure 3-45 shows the response for this case at a plate rotation angle of 5° . In this figure the lower-frequency response is the mode-2 response, and the higher-frequency response is the mode-1 response. Contrasting this figure to that shown in Figure 3-30 shows that the notch frequency now occurs below the mode-1 response.

Figure 3-46 shows the frequency response for this case at a plate rotation angle of 70° , similar to that shown in Figure 3-36. Again, a double-humped response begins to appear but, in this case, the notch is on the high side while, in Figure 3-36, it is on the low side.

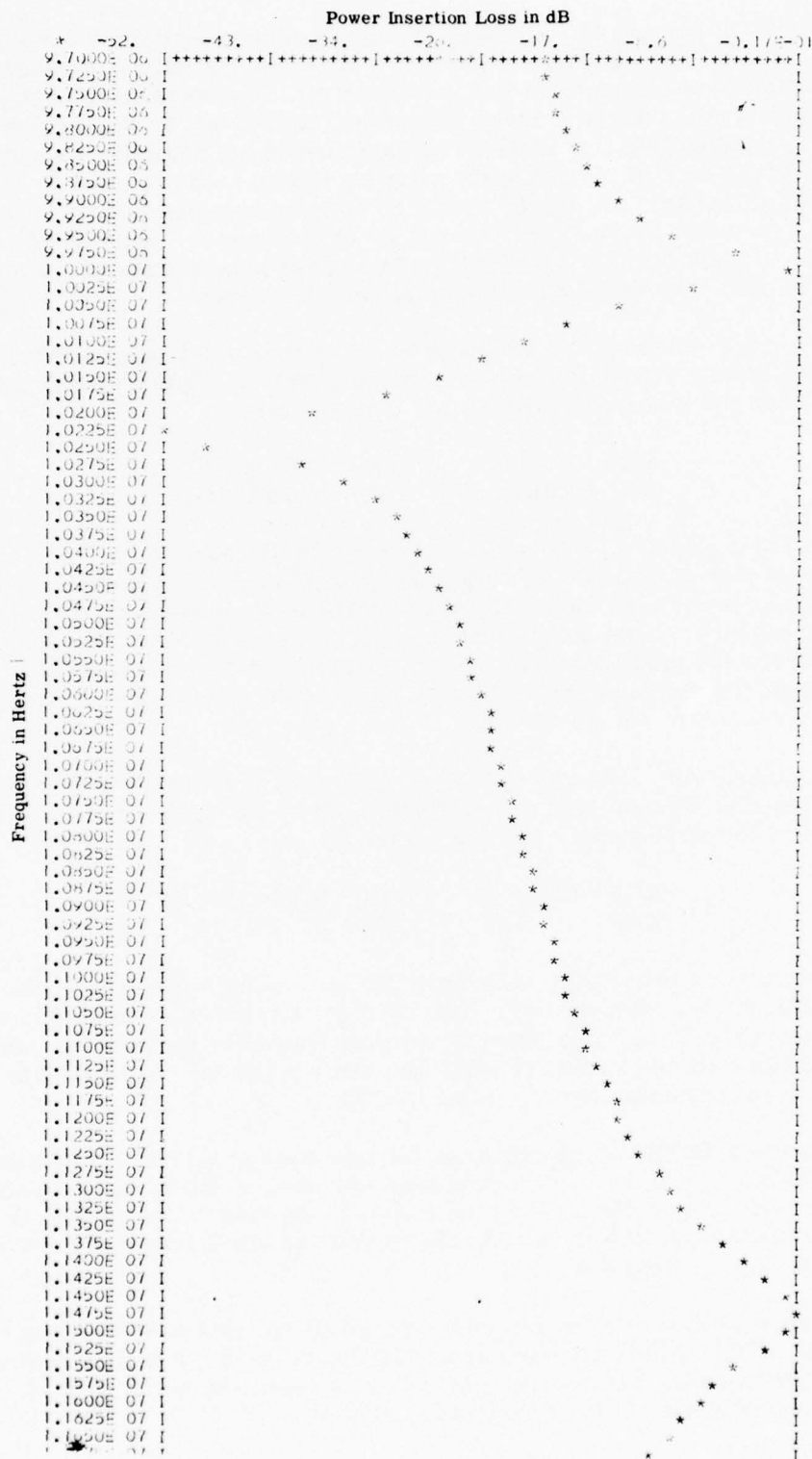


Figure 3-44. Two Element Identical Plate Stack 5° Rotation Between Plates
 Each Plate (Faster Mode2 Velocity)
 $XK1 = 0.088$ $CT1 = 3.32 \times 10^5$
 $XK2 = 2XK1$ $CT2 = 1.15 CT1$

Power Insertion Loss in dB

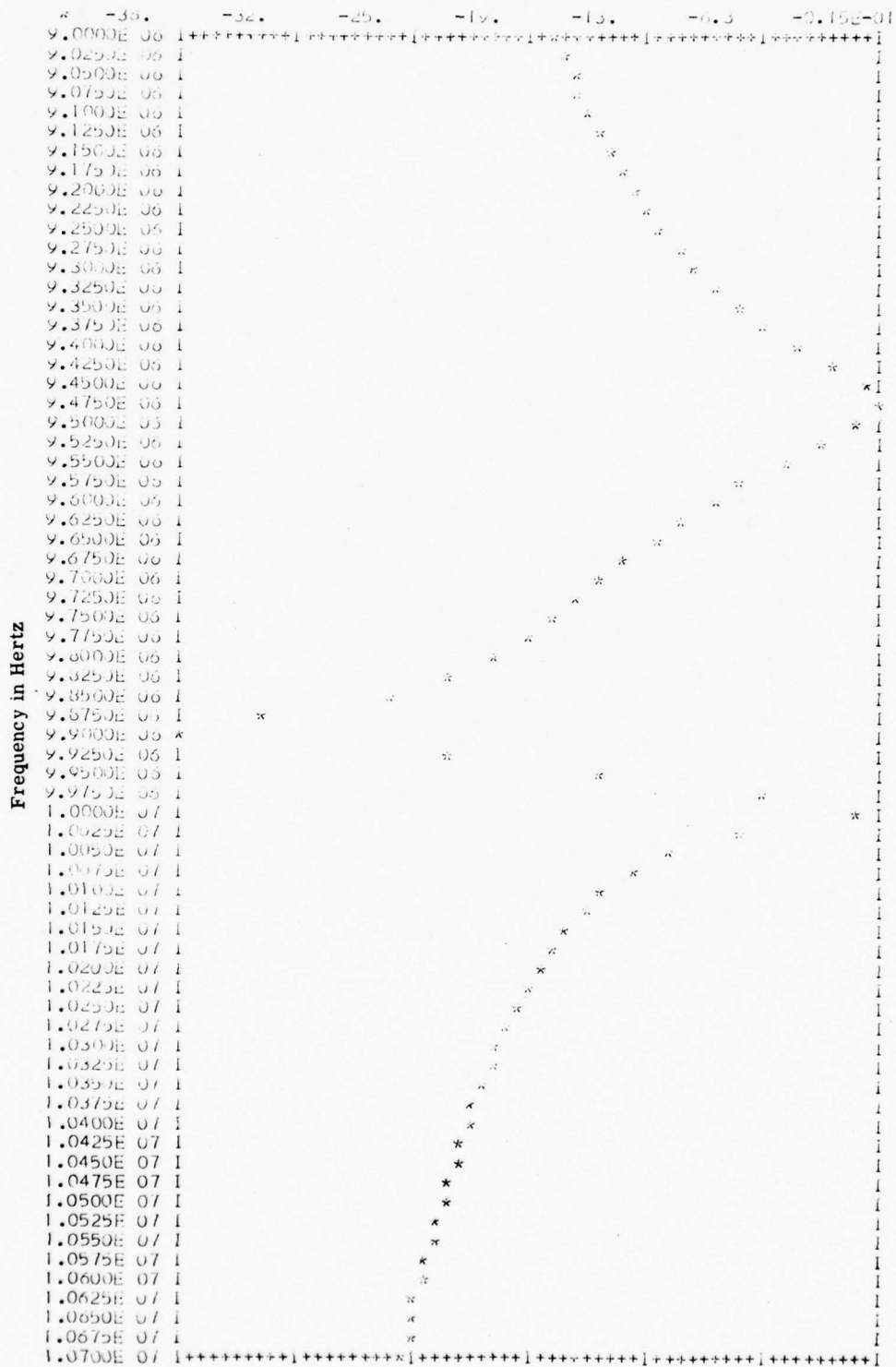


Figure 3-45. Two Element Identical Plate Stack 5° Rotation Between Plates
 Each Plate (Slower Mode 2 Velocity)
 XK1 = 0.088 CT1 = 3.32 × 10⁵
 XK2 = 2XK1 CT2 = 0.95 CT1



Figure 3-46. Two Element Identical Plate Stack Slower Mode 2 Velocity 70° Rotation Between Plates

Figure 3-47 shows a continuation of the plate rotation for this case to an angle of 83° . A single-peaked response curve, similar to that shown for the first case at an angle of 85° (shown in Figure 3-39), has occurred but, again, the notch occurs above the main response. This suggests the possibility of cascading, electrically, two (or more) element stacks of multimode filters to tailor the overall response. An electrical cascade of an element of the type shown in Figure 3-47, with that shown in Figure 3-39, after appropriately adjusting the plate thicknesses to make the main responses in each stack coincide, would yield an overall response curve with notches both above and below the main response.

Figures 3-48, 3-49, 3-50 and 3-51 show the response obtained from the MØDE3 program for AT-cut quartz. Input values for both plates are shown in Table 3-13, as the output of the VCØUP program. A sketch of the relationship of the normal modes of each plate to the plate axes are shown in Figure 3-3. For this crystal cut, mode 2 and mode 3 (those modes lying in the $\beta^{(2)}$ and $\beta^{(3)}$ directions of Figure 3-3) are non-piezoelectrically coupled.

Figure 3-48 shows the filter response of this two-plate stack of AT-cut quartz elements with the two plates lined up. A nice smooth response curve results (as expected from the mode-2 program results discussed in connection with Figure 3-40). Figure 3-49 shows the effects of rotating plate 2 by 5° with respect to plate 1. Again, similar to the discussion in connection with Figure 3-41, plate boundary condition coupling takes place and the response curve shows nulls and peaks due to the three modes.

Figure 3-50 shows the effects of continuing the plate rotation to 45° . Here an interchange of energy, from mode 1 to mode 2 occurs. The resonant effects of mode 2 appear, and are coupled back to the output through mode 1.

Figure 3-51 demonstrates the effects of continuing the plate rotation to 75° .

While not illustrated, the 3-mode equivalent of the MØDE2 program with the normal mode axes, lined up with the plate axes, can be achieved by setting $\beta_1^{(1)} = \beta_2^{(2)} = \beta_3^{(3)} = 1.0$, and the other components of the eigenvectors equal to zero in the MØDE3 program.

Also, while most of the results employed identical plates, the programs work equally well, if not better, for plates with non-identical parameters (since then many of the problems associated with impedance terms in both plates going to zero, at the same time, are avoided).

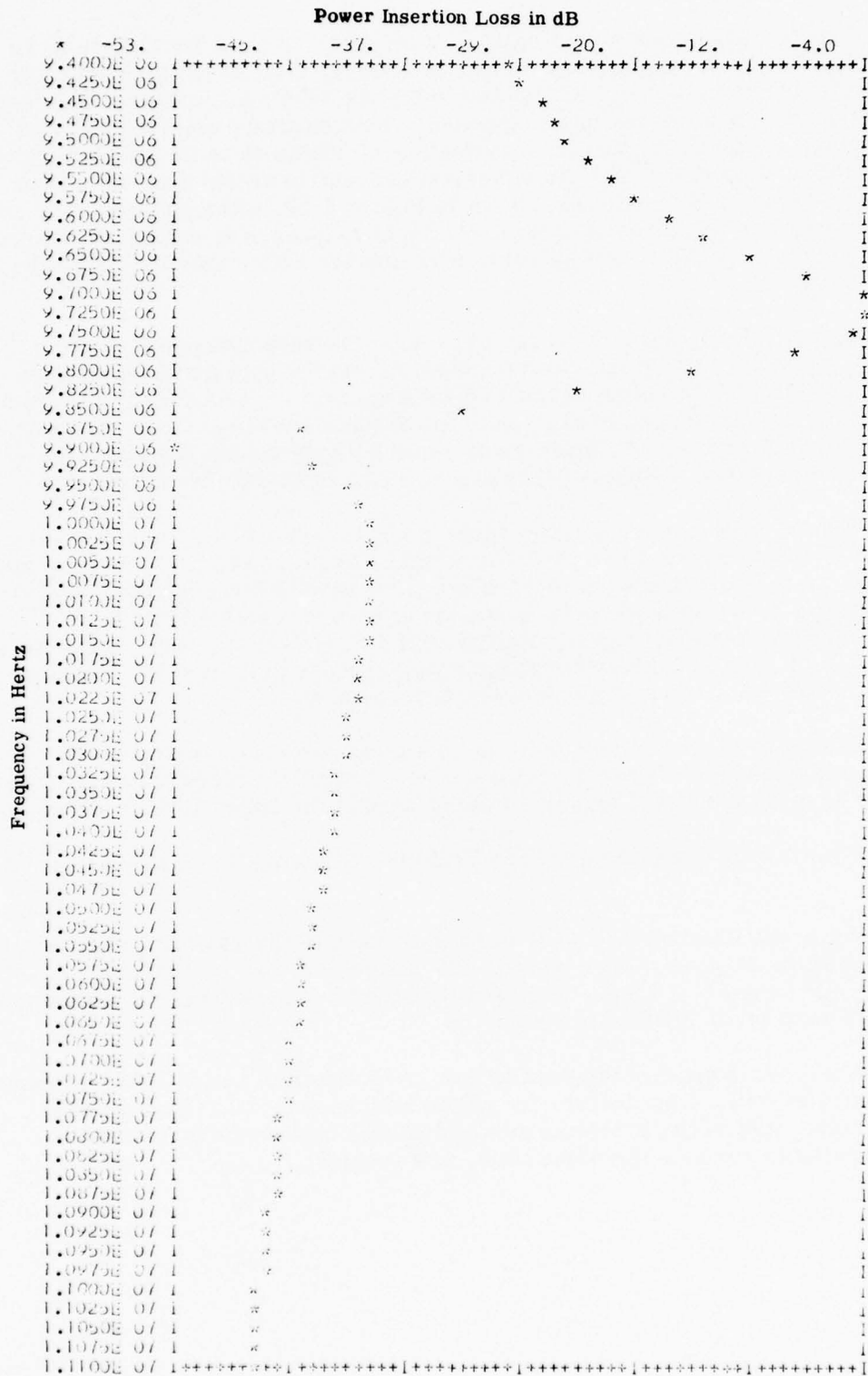


Figure 3-47. Two Element Identical Plate Stack Slower Mode 2 Velocity
83° Rotation Between Plates

Power Insertion Loss in dB

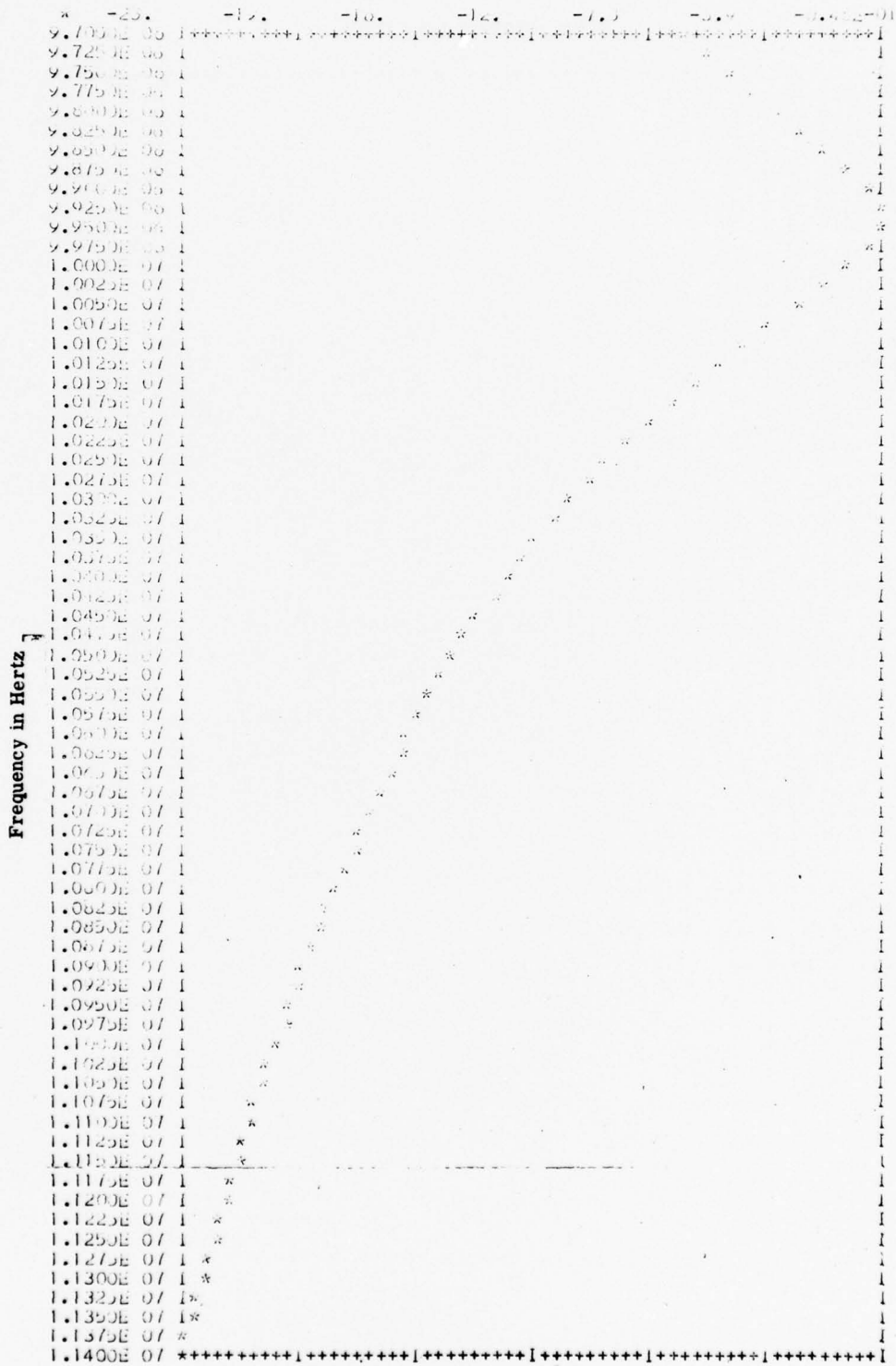


Figure 3-48. Two Element Identical Plate Stack of AT Cut-Quartz Plates
0° Rotation Between Plates

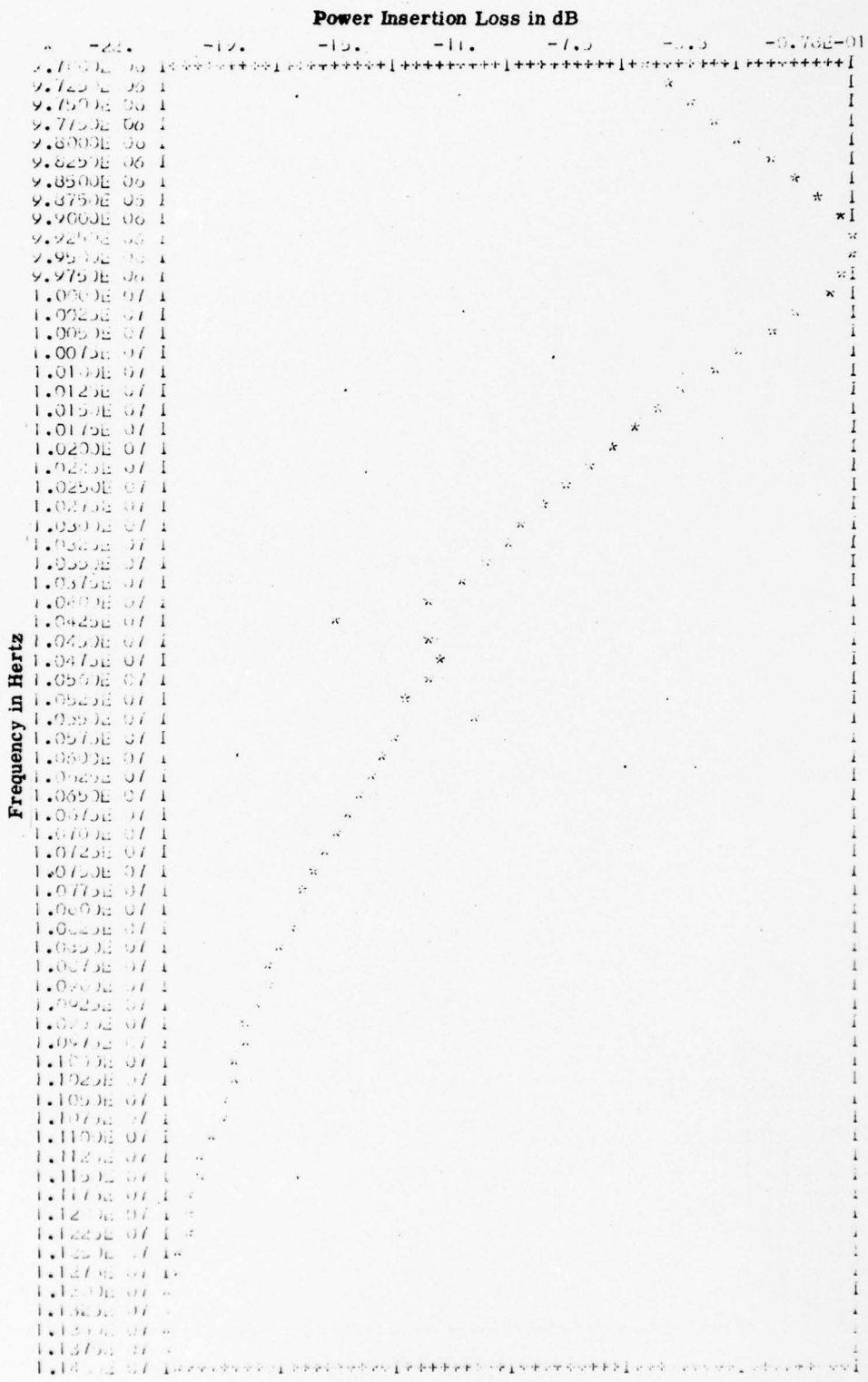


Figure 3-49. Two Element AT Cut Quartz Stack 5° Rotation Between Plates



Figure 3-50. Two Element AT Cut Quartz Stack 45° Rotation Between Plates

Power Insertion Loss in dB



Figure 3-51. Two Element AT Cut Quartz Stack 75° Rotation Between Plates

REFERENCES FOR SECTION 3

1. A.D. Ballato, "Transmission-Line Analogs for Piezoelectric Layered Structures", Ph.D. Dissertation, Polytechnic Institute of Brooklyn, 1972 ; also U.S. Army Electronics Command Report ECOM-4413, May, 1976.
2. H.F. Tiersten, "Thickness Vibrations of Piezoelectric Plates", J. Acoust. Soc. Amer., Vol 35, No. 1, Jan. 1963, pp 53-58.
3. H.F. Tiersten, "Linear Piezoelectric Plate Vibrations", Plenum Press, N.Y. 1969.
4. J.F. Nye, "Physical Properties of Crystals", Oxford University Press, London, 1957.
5. "Standards on Piezoelectric Crystals, 1949", Proc. IRE, Vol. 37, No. 12, Dec. 1949, pp 1378-1395.
6. W.P. Mason, "Piezoelectric Crystals and Their Application to Ultrasonics", D. Van Nostrand Co., Inc. N.Y., 1950.
7. R. Bechmann, "Elastic and Piezoelectric Constants of Alpha Quartz", Phys Rev., Vol., 110, No. 5, June 1958, pp 1060-1061.
8. A.J. Slobodnik, Jr. and J.V. O'Brien, "Complete Theory of Acoustic Bulk Wave Propagation in Anisotropic Piezoelectric Media", Air Force Systems Command Report AFCL-71-0601, November 1971.
9. F.B. Hildebrand, "Methods of Applied Mathematics", Prentice Hall, Inc, Englewood Cliffs, N.J., 1952.
10. E.A. Guillemin, "The Mathematics of Circuit Analysis", John Wiley & Sons, Inc, New York, N.Y., 1949.
11. E.A. Guillemin, "Theory of Linear Physical Systems", John Wiley & Sons, Inc, New York, N.Y., 1963.
12. J.L. Powell, "Quantum Mechanics", Addison Wesley Publishing Co., Inc, Reading, Mass, 1961.
13. H. Goldstein, "Classical Mechanics", Addison Wesley Publishing Co., Reading, Mass, 1959.
14. R.C. Weast, "Handbook of Tables for Mathematics", 3rd Ed., The Chemical Rubber Co., Cleveland, Ohio, 1964.

REFERENCES FOR SECTION 3 (CONT'D)

15. T. Yamada and N. Niizeki, "Admittance of Piezoelectric Plates Vibrating Under the Perpendicular Field Excitation", Proc IEEE, Vol. 58, No. 6, June 1970, pp 941-942.
16. S.A. Basri, "A Method for the Dynamical Determination of the Elastic, Dielectric and Piezoelectric Constants of Quartz", National Bureau of Standards Monograph 9, U.S. Department of Commerce, June 1960.
17. R.B. Adler, L.J. Chu and R.M. Fano, "Electromagnetic Energy Transmission and Radiation", John Wiley & Sons, Inc, New York, N.Y., 1960.
18. H.J. Carlin and A.B. Giordano, "Network Theory: An Introduction to Reciprocal and Nonreciprocal Circuits", Prentice Hall Inc, Englewood Cliffs, N.J., 1964.

4. DEVICE FABRICATION AND TEST

A. RESONATOR MATERIALS

The dominant material in the fabrication of crystal resonators has been single-crystal quartz. It has many advantages, including the very significant one of being relatively inexpensive. The low cost is well supported by the large number of commercially available source suppliers and their ability to process it with great accuracy. Quartz has excellent mechanical properties; it can be sawed, ground, lapped and highly optically polished in both small and large quantities. Crystalline quartz occurs in natural and in synthetic form. The latter can be grown with properties similar to and sometimes superior to that of the natural form. The orientations of quartz, for the application considered here, are the Y-cut, AC-cuts and the AT-cuts. All rotated Y-cut plates vibrate in pure thickness shear modes. The uniqueness of quartz for frequency control and filter applications results from the zero value of the first and second-order temperature coefficients for the AT-cut at room temperature. The AT-cut has a k value of about .088 compared to .14 for the Y-cut. However, the Y-cut has a temperature coefficient of $+90 \text{ ppm}/^\circ\text{C}$. The slightly smaller k value for the AT does not, therefore, detract from its value for resonators and filters. Quartz, itself, has a relatively high mechanical Q , both in the natural and synthetic form.

Although quartz has been the mainstay for frequency control and wave filters, several new crystals with high piezoelectric coupling and favorable mechanical properties are now available. Some of the more successful of these are the ferroelectrics, lithium niobate (LiNbO_3), lithium tantalate (LiTaO_3) and barium sodium niobate ($\text{Ba}_2\text{NaNb}_5\text{O}_{15}$), sometimes dubbed "banana". All three exhibit strong piezoelectric effects.

LiNbO_3 and LiTaO_3 have strong thickness shear mode X and Y-plates, but they are not pure. The Z-cuts contain pure extensional modes.

LiNbO_3 has several utilized rotated-Y-cuts, but the particle motion does not lie ideally for pure-mode orientation. Some of the cuts may have undesired mode coupling that is still low enough for some applications but, generally, they are second choices. The Z-cut is the only pure mode, its coupling being of the order of .17. The X-cut plate exhibits a large value of k , about .68. However, coupling exists to two shear modes – one strong and one weak.

LiTaO_3 is of the same crystal class, $3m$, as LiNbO_3 . The basic difference lies in values of coupling coefficient and temperature coefficient. LiTaO_3 has a cut (1) with a low temperature coefficient that may be useful for resonator applications. The X-cut has a parabolic frequency versus temperature characteristic.

$Ba_2NaNb_5O_{15}$ belongs to the crystal class, mm 2; it differs considerably from $LiNbO_3$ and $LiTaO_3$. The X- and Y-cut plates have pure thickness shear modes and the Z-cut plate has a pure thickness extensional mode. The Z-cut has a very high coupling factor, $k = .57$, which remains constant over a wide temperature range. This material has not been utilized as extensively as quartz, $LiNbO_3$ and $LiTaO_3$, and its present cost is relatively high.

Table 4-1 summarizes the various orientations and physical parameters of the crystals discussed here.

There are other materials that could qualify for use in multimode filter applications. $LiGaO_2$ and other new materials should be considered, but first a careful evaluation of their potential as crystal filter plates must be made. The often-used AT-cut quartz, along with some of the cuts from the various materials listed in Table 4-1, merit investigation. In particular, the recently measured material berlinite shows promise as a future useful crystal for filter applications. This material has very similar properties to crystalline quartz (including temperature-stable orientations) but has the added feature of a higher electromechanical coupling coefficient. As soon as a good quality supply of this material becomes available it will find application in various types of crystal filter configurations.

Other candidates for consideration in multimode filters, would be the doubly-rotated orientations of crystals as described by Ballato. (2) The doubly-rotated crystal plate has the advantage of two degrees of freedom with two selected angle cuts. As an example, this would lead to temperature stability and higher electromechanical coupling that cannot be achieved by some of the single-oriented crystal cuts.

B. BONDING TECHNIQUES

Bonding techniques can be divided in two general categories — 1) transparent organic materials that are used to form thin adhesive bonding layers and 2) metal surface films that are joined together to form a single bond.

The epoxy or polymer adhesive bonds have relatively low mechanical impedances and, therefore, layers (of such materials) of any appreciable thickness could have a significant effect on the shape of the frequency/response curves. Although these bonds possess good stability over wide temperature ranges, the low mechanical impedance has, in most applications, required that the thickness of the bonding layer be very thin — of the order of a few hundredths of an acoustic wavelength.

TABLE 4-1. PROPERTIES OF SOME FILTER MATERIALS.

Material	Cut	Mode	k	Temp. Coeff. $\times 10^{-6}/^{\circ}\text{C}$	Vel. $\times 10^3$ m/s	Density kg/m^3
Quartz	X	TE	.10	-20	5.70	2.65
Quartz	Y	TS	.14	+90	3.85	2.65
Quartz	AT	TS	.088	0	3.32	2.65
LiNbO ₃	X	TS	.68		4.75	4.7
LiNbO ₃	Z	E	.17		7.33	4.7
LiNbO ₃	36 ^o Y	E	.49		7.37	4.7
LiNbO ₃	163 ^o Y	S	.62		4.56	4.7
LiTaO ₃	X		.44		4.2	5.3
LiTaO ₃	Z		.19		6.08	5.3
LiTaO ₃	47 ^o Y		.29		7.40	5.3
LiTaO ₃	165 ^o Y		.41		4.56	5.3
BaNb ₅ O ₁₅	X	S	.21		1.82	5.3
	Y	S	.25		1.83	5.3
	Z	E	.57		3.07	5.3

Although the organic bonds are insulating, they can be used in very thin layers, provided no problems are encountered due to the ground surface layers of the resonators being separated by this thin layer. One advantage in using some of the organic materials is that the bond may be dissolved and the resonator discs recovered for additional experimental testing of other bonds or resonator configurations.

Table 4-2 is a tabulation of some of the materials (and their characteristics) that may be considered for use in bonding tasks. The bonds themselves could include one or more of the materials shown depending on the substrate and type of layer being attempted.

TABLE 2. PROPERTIES OF BONDING MATERIALS

Material	Velocity $\times 10^5$ m/s		$Z \cdot 10^6$ kg/s m ²	
Epoxy	2.60 Ext	1.22 Shear	2.86 Ext	1.34 Shear
Phenyl Benzoate			3.38	0.48
Indium	2.30	1.44	17.0	10.5
Gold	3.24	1.20	62.5	23.2
Silver	3.65	1.61	38.0	16.7
Copper	5.01	2.11	40.6	18.3
Aluminum	6.42	3.04	17.3	8.2
Quartz	5.70	3.32	15.1	8.8
LiNbO ₃	6.55	4.76	30.8	22.4
LiTaO ₃	5.55	4.21	29.4	22.3
BaNaNb ₅ O ₅	3.07	1.82	16.3	9.65

Most metallic bonds have been achieved using thermocompression bonding of indium or lead, singly or together with combinations of other metallic layers. The metallic bonds have larger characteristic mechanical impedances that more closely match that of the resonator material. Another advantage of using metallic-layer bonding is the resultant electrical conduction layer. While early techniques of metallic bonding involved only layers of indium or lead, these have been largely replaced by combinations of layers of several types of metals. Typical bonds consist of a flash of chromium, a thick layer of gold for adhesion and electrical conduction, and then the two surfaces to be bonded are plated with thin layers of indium. The two pieces are then thermocompression bonded together, so that the indium and gold alloy forms a good electrical and bond layer.

Various metallic-bonding techniques, have been used (3, 4, 5) which incorporate the evaporation of thin metallic layers to optically polished surfaces and are then welded by compression. These compression techniques involve the use of both thermocompression and room temperature compression in vacuum, air and inert gases.

The bonds themselves, fall into two categories, thin and thick. The thin films generally include organic, inorganic and metallic films. The thick film bonds will generally be metallic films. For precise thick bond layers, properly dimensioned glass or metal balls, wire or shims may be inserted together with the metallic films to obtain a specific bond thickness.

Since the main objective of the present study (Phase I) was the analytical modelling and programming of multimode stacked crystal filters, it was decided to use an organic lens bond for the preliminary studies of an experimental model. The lens bond has several advantages, the main one being that it can be dissolved and the resonator plates recovered for experimental testing of other bonds or resonators. A disadvantage is that it is insulating and can cause contact problems if the ground surface layers of the crystal stacks are separated by the thin insulating layer. The problem can be eliminated by some selective plating and offsetting of the crystal stack elements. These polymer bonds generally have low mechanical impedances.

Glass slide samples were first bonded together using the lens bond resin under pressure at room temperature. The bonds adhered well and did not separate when heated to 90°C. The test samples were successfully separated without damage using a decementing reagent. This lens bond was therefore selected for use in our initial studies of the stacked crystal filter elements.

Metallic bonds have large characteristic mechanical impedances that match the resonator impedances more closely. Bonds using epoxy and gold films were used in our analytical studies of two and three stacked crystal filters. In the past, indium was used as a metallic thermal compression bond. This method suffered in that the entire bonding operation, i.e., plating and pressure-bonding, had to be done in one pumpdown in a vacuum system using complicated fixtures and manipulators. Since bonding is a major consideration in the SCF, various procedures of thin metallic film bonds were studied. Some of these will be utilized in phase II of this program, which deals mainly with the fabrication of a variety of SCF representative models.

The bonding procedures include a wide variety of methods, including:

1. Cold-weld bonding of metallic films such as indium both in air and in vacuum.
2. Elevated temperature bonding of metallic films in both air and vacuum.
3. Room temperature non-indium metallic bond welding.
4. Ultrasonically welded layers using gold-aluminum and/or other metallic layers.
5. Field-assisted bonding of layers at low temperatures.

These metallic film techniques offer a well-matched bonding layer and some of these methods will be utilized as a permanently fixed bond for use in the Phase II study of experimental models of SCF's.

C. ELECTRODE CONSIDERATIONS

The characteristics of the electrodes, which have an effect on the electrical properties of the quartz resonator plate, depend on the surface finish of the plate, the electrode material, the geometry of the plate and the orientation of the electrodes with respect to the quartz plate. The electrodes are also affected by the size, shape, position and material properties of the connecting leads and the characteristics of the filter mounting supports. For purposes of orientation and rotation our electrode configurations were circular discs, with tabs extending from the active electrode area as shown in Figure 4-1 (Commonly referred to as a keyhole geometry). The latter can serve to locate the principal axis and act as a connector for external leads.

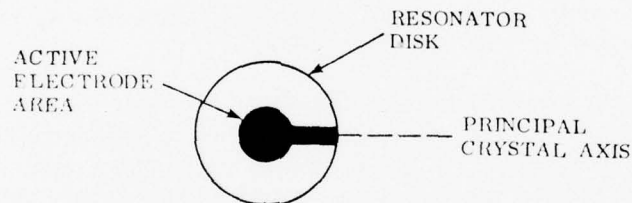
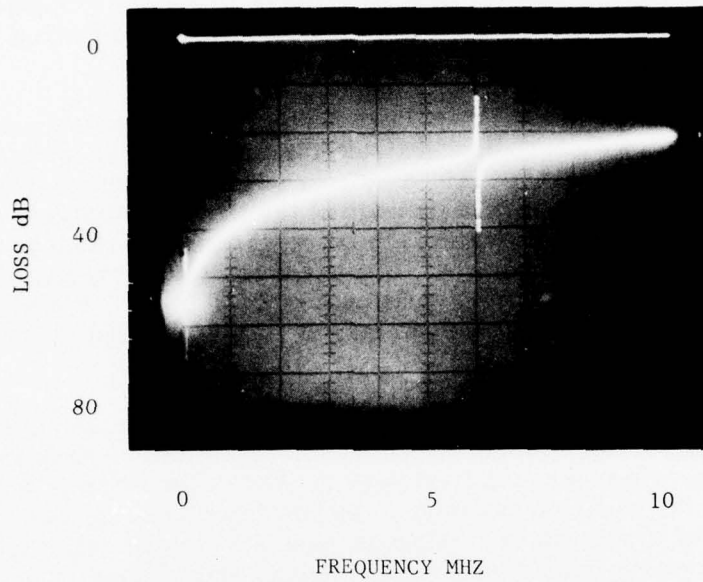


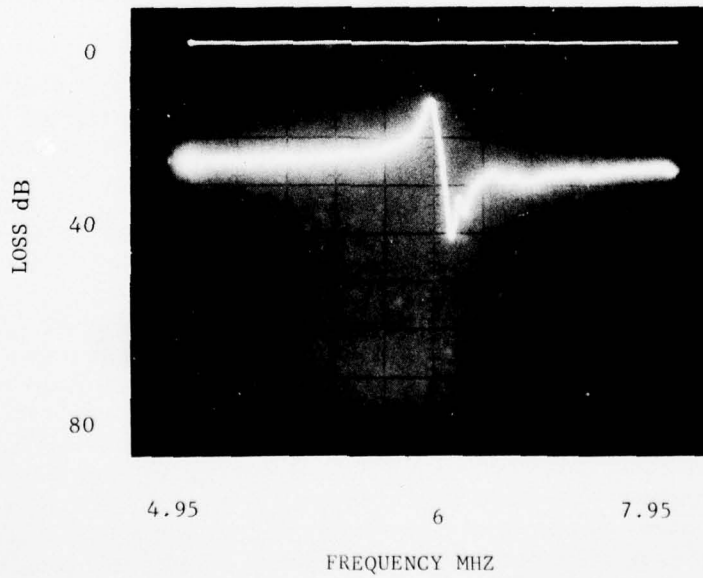
Figure 4-1. Electrode Configuration.

D. EXPERIMENTAL RESULTS

In order to verify some of the computer calculated results of Section 3.0 some cultured quartz crystal blanks were obtained from the J.K. Miller Co. These blanks were AT-cut with a 12 1/2 micron finish. Each blank was approximately 0.45" square and .011" thick. Gold electrodes of a keyhole design approximately 400 mils in diameter were then deposited on each of the broad faces of these crystals. A typical response curve for one of these electroded resonators is shown in Figure 4-2. This response curve was obtained by connecting the resonator in series with the input and output terminals of a 50 ohm H.P. Network Analyzer. The trace in 4-2a shows the plate response from 0 to 10 megahertz and indicates that the resonator is free of interfering modes in this region. Figure 4-2b shows a blown up version of the main plate response at the expected 6 MHz frequency, based on the velocity of AT-cut quartz and the plate thickness, and again exhibits a reasonably clean response.



(a)



(b)

Figure 4-2. Typical Response Curve for Range AT-Cut Quartz.

Two of these electroded plates were then bonded together with the crystalline axes or actual plate axes lined up. For this experiment a lens bond from Summers Laboratories, Inc. of Port Washington, Pennsylvania was used. This bonding material is a synthetic polyester adhesive that has worked well for applying transducers to acousto-optic delay lines. Using this bonding material and a pressure of approximately 80 lbs/sq inch, good adhering bonds with thicknesses in the order of 0.05 mils were achieved.

Figure 4-3 shows the Network Analyzer curves of this two-element stack of AT-cut quartz plates with zero degrees relative rotation between the plates of the stack. These traces were obtained by connecting the electrical terminals of one plate in the stack to the input terminals of the Network Analyzer and the electrical terminals of the other plate to the output terminals. Based on calculations of the clamped capacitances of the plates, the best termination impedance for these plates at 6 MHz is approximately 2000 ohms. However, the traces shown were measured in a 50 ohm circuit with no attempt at impedance matching.

Figure 4-3a shows the response from 0 to 110 MHz. The major peaks on this trace are the overtone responses of a single plate in the stack at 6, 18, 30, 42, and 54 MHz. However, the first peak visible on the trace is the fundamental response of the two-element stack. As discussed in conjunction with Figure 2-4 a stack of two crystals in intimate (welded) contact, each of which has a fundamental resonance at 6 MHz should display stack resonances at 3, 6 and 9 MHz.

Figure 4-3b shows an expanded view of the response of this stack from 0 to 40 MHz; Figure 4-3c shows a further expansion and covers only the range from 0 to 10 MHz.

It is obvious from this last curve that the stack bond departs from the ideal welded bond used in the computer simulation. Even allowing for the impedance mismatch between the optimum stack impedance and the 50 ohm measuring circuit impedance, it is obvious that the bond is not functioning well when it is located at a high stress point such as that required for resonances at half-wavelength and three-halves wavelength of the full stack. The two small responses visible in Figure 4-3c correspond to these resonances. The larger response is for the full stack, when it is wavelength thick. A comparison of this full wavelength resonance with the computer-generated response (Figure 3-48) for a two-element stack of AT-cut quartz plates, with 0° rotation between plates, shows that the measured response exhibits the predicted clean resonance.

This two-element stack was then disassembled. As stated earlier the ability of a lens bond to be dissolved with a heated solution of lens bond decentering fluid is an advantage in this type of investigation. The same set of stacked crystal elements can be used repeatedly to study the effect of plate rotation on the frequency response without introducing effects due to different crystals.

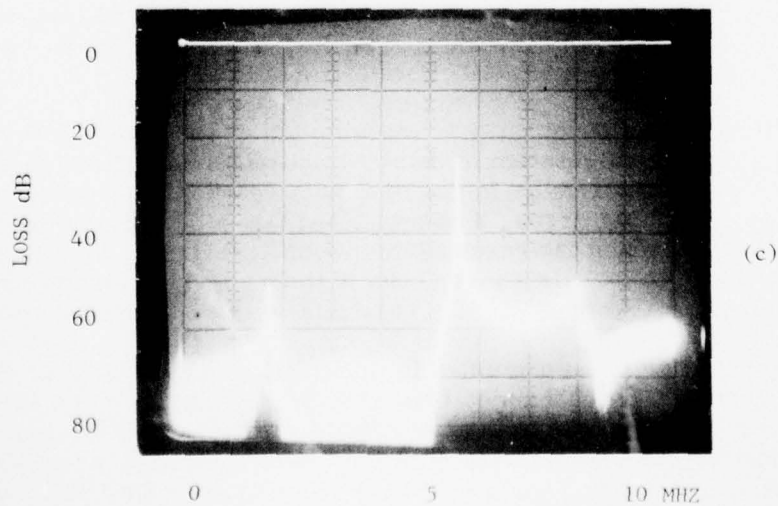
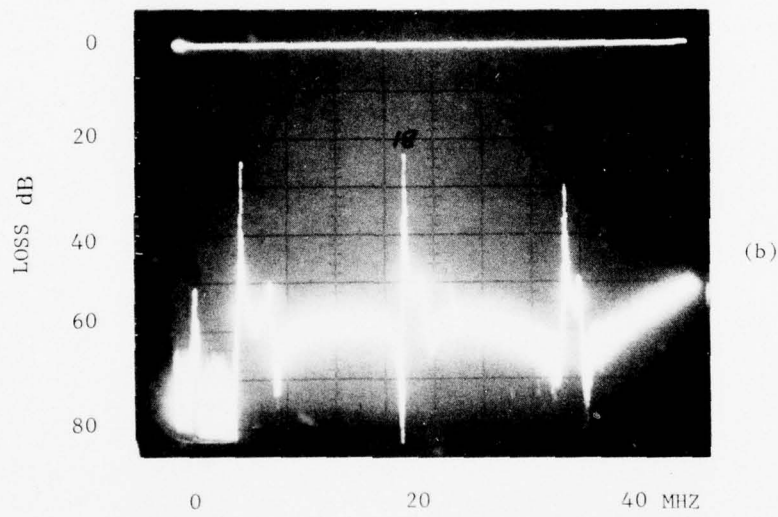
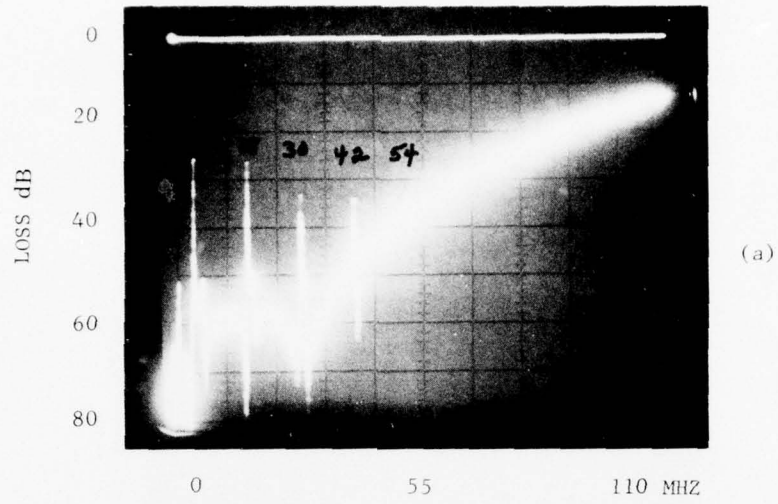


Figure 4-3. 2-Stack AT-Cut Crystals, 0° Rotation

The two elements were then reassembled with a relative plate rotation of approximately 45° . Figure 4-4 shows the H.P. Network Analyzer response of this configuration in a 50 ohm system about the full wavelength resonant frequency of the stack. Each horizontal scale division on this trace represents approximately 0.3 MHz and the left peak is centered at approximately 6 MHz.

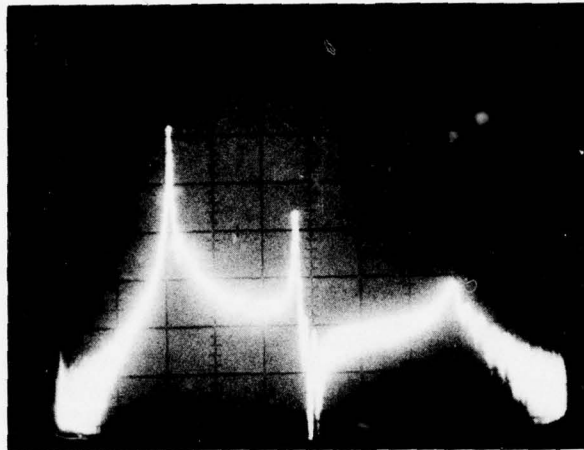


Figure 4-4. 2-Stack AT-Cut Crystals 45° Rotation.

The theoretical result for this case is shown in Figure 3-50. Scaling the frequencies shown there, from the assumed 10 MHz single-plate resonance to the 6 MHz used in the experimental crystals, places the first peak at 6 MHz, the center peak at 6.36 MHz and the right peak at 6.72 MHz in Figure 3-50. Comparison of this theoretical response with the experimental response of Figure 4-4, reveals a visible dip and peak in the vicinity of 6.36 MHz in the experimentally fabricated stacked response. The only responses visible in the trace are those at 6, 6.8 and 7.8 MHz. The latter response corresponds to the three-halves wavelength resonance of a filter stack and, again, shows that a satisfactory welded bond has not been totally achieved.

Without having an opportunity to examine either the theoretical response or the experimental configuration in detail, to conclusively establish whether the discrepancy between results is due to a plate rotation discrepancy, a calculation error, or a bond error, it is suggested that a better bond in the experimental device would bring the results into closer agreement. However, it might be necessary to include a lossy bond in the computer calculations to achieve satisfactory agreement at this particular angle of relative plate rotation.

The last configuration examined in this preliminary attempt to obtain a correlation between the experimental and theoretical results was a two-element stack of AT-cut quartz plates with approximately 75° of relative plate rotation between elements in the stack. The H.P. Network analyzer traces for this configuration are shown in Figure 4-5. Again, no attempt was made at impedance matching to the 2000 ohm impedance of the filter stack. Instead, the curves were measured in a 50-ohm circuit.

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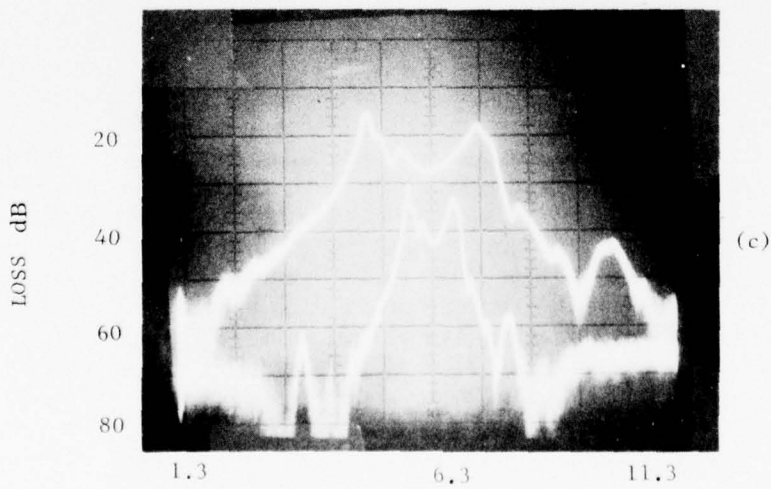
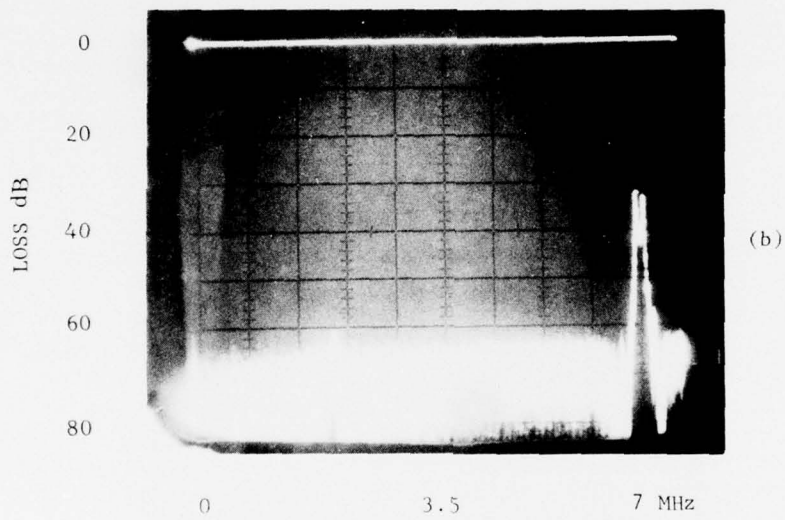
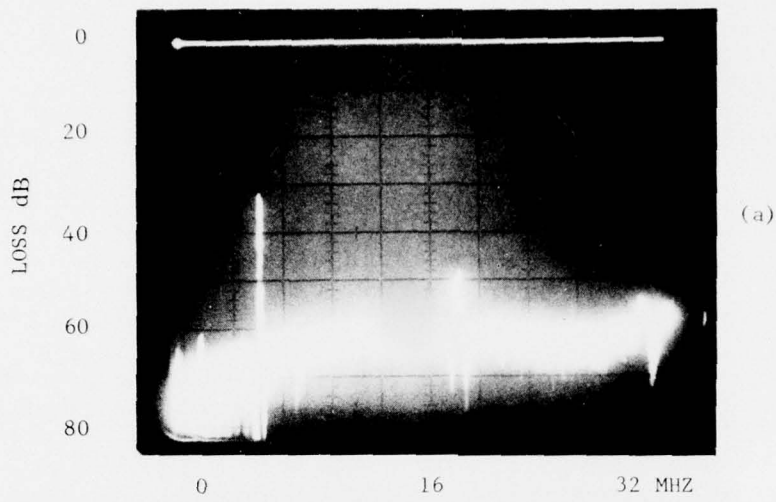


Figure 4-5. 2-Stack AT-Cut Crystal 75° Rotation.

Figure 4-5a shows the response from 0 to 32 MHz of the stack. Again, the overtone responses of the individual elements in the stack at 6, 18, and 30 MHz are the predominant responses with the half and three-halves wavelength responses of the stack barely visible. The half and full-wavelength responses of the stack are shown in Figure 4-5b trace. Here, again, the bond is not functioning properly when located at a high stress point. The double-humped nature of the 6 MHz response is also visible here. Figure 4-5c shows two expansions of this 6 MHz response. Both of these are centered at approximately 6.3 MHz. The lower of these two traces has a horizontal scale of approximately 1 MHz per division; the upper trace is about 0.3 MHz/division.

A comparison of these curves with Figure 3-51, for a computer response of a welded two-element stack of AT-cut quartz plates at 75° of relative rotation, shows excellent agreement in the response curve shapes. When the frequency scale shown in Figure 3-51 is multiplied by 0.6, the factor required to scale the frequency to the experimental filter frequency, there is almost complete agreement between this curve and the experimental curve over the frequency interval plotted in Figure 3-51. The two curves exhibit almost identical shapes with the dips and peaks at approximately the same frequencies. The major discrepancy is that the relative loss between peaks and valleys on the experimental curve is approximately one-half that predicted theoretically. This discrepancy is most likely due to the impedance mismatch of the experimental circuit.

This preliminary experimental investigation has demonstrated that there is a close correlation between the computer-generated response curves and the actual response curves. This means that it is realistic to use the simulation procedure to search for element configurations that exhibit the best response characteristics and, then, to fabricate only the most desirable configurations. It has been stressed throughout this section that the present experimental elements using lens bond to bond the elements together lack coupling of the half and three-halves wavelength resonances of the stack, and this condition must be equated to a poor bond. In actual practice, if a low-loss bond at the full wavelength resonance of a two-element stack is achieved, the lack of the other resonances would be immaterial and even desirable.

E. PROGRAM SIGNIFICANCE

The combined analytical results and preliminary experimental data obtained during Phase I of this program indicate that multimode crystal filters can be utilized in an effective manner to shape filter response characteristics. Together with recent advances in packaging⁽⁶⁾⁽⁷⁾, the SCF (which is compatible with integrated circuits) can yield a reasonably rugged device capable of achieving filter responses that cannot be achieved with single-plate crystal resonators. Whereas the various filter types, such as the single plate, monolithic and SAW devices, utilize a single orientation or cut for the crystal response, the SCF offers the advantage of a wide variety of crystal plates bonded together mechanically, so that both piezoelectric and mechanical effects can be used to achieve a great multitude of desired frequency responses.

The two-port SCF offers a wider bandwidth potential than other types of bulk-mode filters. SCF plates of different materials and unequal thicknesses

also offer narrowband filters. One main advantage of the stacked crystal filter should be in applications requiring a minimum of modulation and distortion effects for moderate-to-high signal levels. Nonlinear effects in front-end and output filtering can cause major problems in proximity to transmitters operating in adjacent bands. The SCF bulk-wave device with its high power-handling capability, greater selectivity and small size employs bulk acoustic waves and should have low nonlinear effects, thereby preserving frequency stability. Many types of filter responses are also required for frequency division multiplex systems. The SCF has the wide flexibility to deliver many different types of filter topologies that are useful in these and other frequency-control systems.

REFERENCES FOR SECTION 4.0

1. Warner, A.W., Ballman, A.A., "Low Temperature Coefficient of Frequency in a LiTaO₃ Resonator", Proc. IEEE, 55, 450 (1967).
2. "The Angular Dependence of Piezoelectric Plate Frequencies and Their Temperature Coefficients", A. Ballato and J. Iafrate; Proc. 30th Annual Frequency Control Symposium, June 1976, pp. 141-156.
3. Sittig, E.K., Cook, H.D., "A Method for Preparing and Bonding Ultrasonic Transducers Used in High Frequency Digital Delay Lines", Proc. IEEE, 56, (1375) - Aug. 1968.
4. Knox, J.D., "A Room Temperature Non-Indium Metallic Bond Tested by Welding Acoustic Shear-Wave Transducers to Paratellurite", RCA Review, Vol. 34, (369) June 1973.
5. Larson, J.D., Winslow, D.K., "Ultrasonic Welded Piezoelectric Transducers", IEEE Trans. on SU, Vol. SU-18, (142) July 1971.
6. "Packaging Precision Quartz Crystal Resonators", John R. Vig, Eric Hafner, R&D Technical Report, ECOM 4134, July 1973.
7. "Ceramic Flat Pack Enclosures for Precision Quartz Crystal Units", R. Donald Peters, General Electric Company, Neutron Devices Dept., St. Petersburg, Fla., ERDA Contract No. E-(29-2)-656, September 1976.

5. SUGGESTIONS FOR FURTHER WORK

With any work encompassing the number of independent variables employed in this project it is almost impossible to examine all cases. Hence the obvious recommendation is that the programs written during the nine month course of this project be exercised more fully. A catalog of results, for considerably more cases than time permitted examination of during the course of this contract, should be developed.

There is also no claim that the programs as presently listed in this report are in the best possible format. As written they are convenient for operating in a time sharing mode of operation where a limited number of complete changes in input data are to be carried out. In this mode of operation a few changes in input data can be conveniently carried out on-line, and the results of these changes are available almost instantaneously. For more extensive changes in data off-line preparation of an input data paper tape was employed.

This input data procedure probably should be changed to one employing Namelist and Data Statements. While the time-sharing programs presented here are convenient for the rapid examination of program changes the quality of the resulting graphs produced as output are poor. For better documentation of the results batch versions of these time-sharing programs should be developed so that the higher quality result obtainable with plotting routines like CalComp could be obtained.

In conjunction with this conversion to batch programs the use of Name-list and Data statements would also allow for the development of program object decks, so that it would not be necessary to compile the programs each time before execution of each case. This would result in some saving on computer costs if a large number of cases were to be examined.

Even with the programs as they now stand the special cases that result in programming difficulties or aborts should be examined more closely than was possible within the time framework of the project. The easiest way to discover these difficulties is to exercise the programs with various combination of variables until an unexpected result is encountered. The equations are then examined to determine the source of the difficulty and the appropriate course of action taken. With problems of the degree of complexity inherent in the multimode stacked filters it is almost impossible to determine in advance what these special cases are without encountering them.

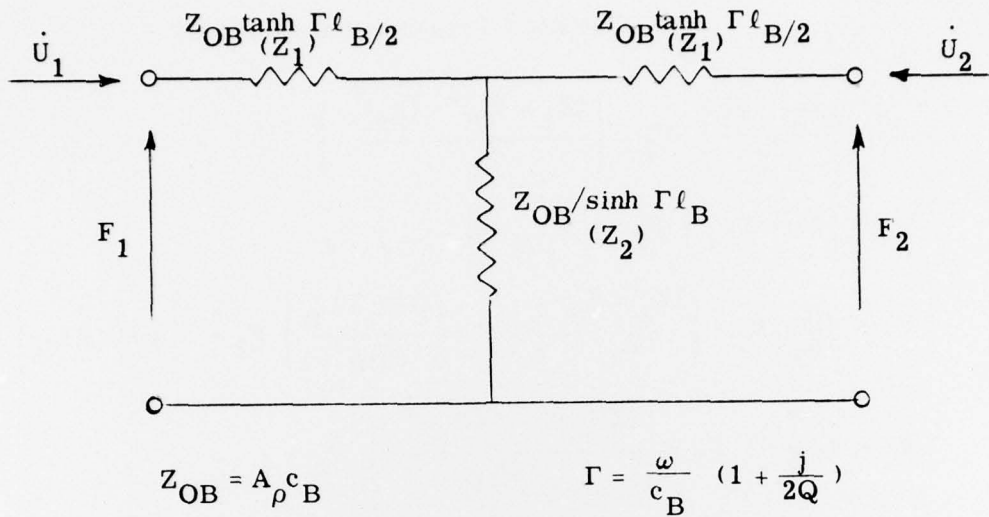
A more thorough experimental confirmation of multimode stacked filter theoretical results should also be carried out than was possible within the time limits of the present project.

So far, most of the suggestions made have dealt with matters pertaining to the programs as they were presented in the present report. Extensions to these programs should also be carried out.

The first of these is accounting for bonding layers between the plates in the multimode stack. As pointed out earlier, it is the authors' opinion that the appropriate time to incorporate bonds in the calculations is after fully operational thin bond cases have been obtained. Conceptually, the addition of bonds is relatively simple. Figure 5-1 shows the appropriate equivalent circuit of a lossy bond. In this equivalent circuit the parameters of the bond will depend on the type of wave propagating through the bond (shear waves and longitudinal waves will in general behave differently). Figure 5-1 also shows the Impedance Equations appropriate to this equivalent circuit along with the Transmission Equations of the bond. These latter equations are probably more appropriate for the inclusion of bonds in the multimode filter stack. Figure 5-2 shows the MODE2 problem of Figure 3-26 with a bond layer inserted between the plates in equivalent circuit form. Table 5-1 carries out the procedure required in this case for a determination of the output currents of the left plates in terms of the input current. Once this determination has been made these currents are then followed back through the equivalent circuit in a manner similar to that shown in Table 3-16 to obtain the comparable MODE2 solution with a bond layer.

The present programs are also limited to stacks of two multimode plates. While this is probably the most commonly employed configuration of stacked filters, it does not allow for an examination of the full potential of multimode stacked filters. Further work should, therefore, involve extending the multimode stack to a larger number of elements. It is again the authors' opinion that if this is to be carried out it should be done in terms of the augmented equivalent circuit of a TETM plate as shown in Figure 5-3. The use of this augmented circuit will allow for considerable flexibility in how the electrical input and output terminals in the stack are formed in relation to the electrical terminals of each plate in the stack. The augmented impedance matrix of this equivalent circuit is easily obtained from the impedance matrix for the actual plate coordinates shown in Figures 3-19 and 3-20, as shown in Figure 5-4.

It is also the authors' opinion that the problem of a multimode stack of more than two elements should be carried out in terms of the transmission matrix for this augmented plate. Table 5-2 shows schematically how this conversion of the augmented impedance matrix to the augmented transmission matrix can be carried out in terms of the submatrices of Figures 5-4. Figure 5-5 shows again schematically the transmission matrix associated with the augmented plate equivalent circuit of Figure 5-3. In this approach the stack of filter elements is built up and then the appropriate boundary conditions are applied.



a) Equivalent Circuit of a Lossy Bond.

where ω is the angular frequency
 Z_{OB} is the characteristic impedance of the bond
 Γ is the bond complex propagation constant
 ℓ_B is the bond thickness
 A is the cross sectional area of the bond
 ρ is the density of the bond material
 c_B is the velocity of propagation in the bond material
 Q is the mechanical Q of the bond

b) The Bond Impedance Equations

$$F_1 = (Z_1 + Z_2) \dot{U}_1 + Z_2 \dot{U}_2 = \left\{ \frac{Z_{OB}}{\tanh \Gamma \ell_B} \right\} \dot{U}_1 + \left\{ \frac{Z_{OB}}{\sinh \Gamma \ell_B} \right\} \dot{U}_2$$

$$F_2 = Z_2 \dot{U}_1 + (Z_1 + Z_2) \dot{U}_2 = \left\{ \frac{Z_{OB}}{\sinh \Gamma \ell_B} \right\} \dot{U}_1 + \left\{ \frac{Z_{OB}}{\tanh \Gamma \ell_B} \right\} \dot{U}_2$$

$$F_1 = Z_{B_{11}} \dot{U}_1 + Z_{B_{12}} \dot{U}_2$$

$$F_2 = Z_{B_{12}} \dot{U}_1 + Z_{B_{11}} \dot{U}_2$$

Figure 5-1. Equivalent Circuit of Bond.

c) The Bond Transmission Equations

$$F_1 = \left\{ \frac{Z_1 + Z_2}{Z_2} \right\} F_2 - \left\{ \frac{(Z_1 + Z_2)^2 - (Z_2)^2}{Z_2} \right\} \dot{U}_2$$

$$= \left\{ \cosh \Gamma \ell_B \right\} F_2 - \left\{ Z_{OB} \sin \Gamma \ell_B \right\} \dot{U}_2$$

$$\dot{U}_1 = \left(\frac{1}{Z_2} \right) F_2 - \left\{ \frac{(Z_1 + Z_2)}{Z_2} \right\} \dot{U}_2 = \left\{ \frac{\sinh \Gamma \ell_B}{Z_{OB}} \right\} F_2 - \left\{ \cosh \Gamma \ell_B \right\} \dot{U}_2$$

$$F_1 = \left(\frac{ZB_{11}}{ZB_{12}} \right) F_2 - \left\{ \frac{(ZB_{11})^2 - (ZB_{12})^2}{ZB_{12}} \right\} \dot{U}_2$$

$$\dot{U}_1 = \left(\frac{1}{ZB_{12}} \right) F_2 - \left(\frac{ZB_{11}}{ZB_{12}} \right) \dot{U}_2$$

Figure 5-1. (Cont'd).

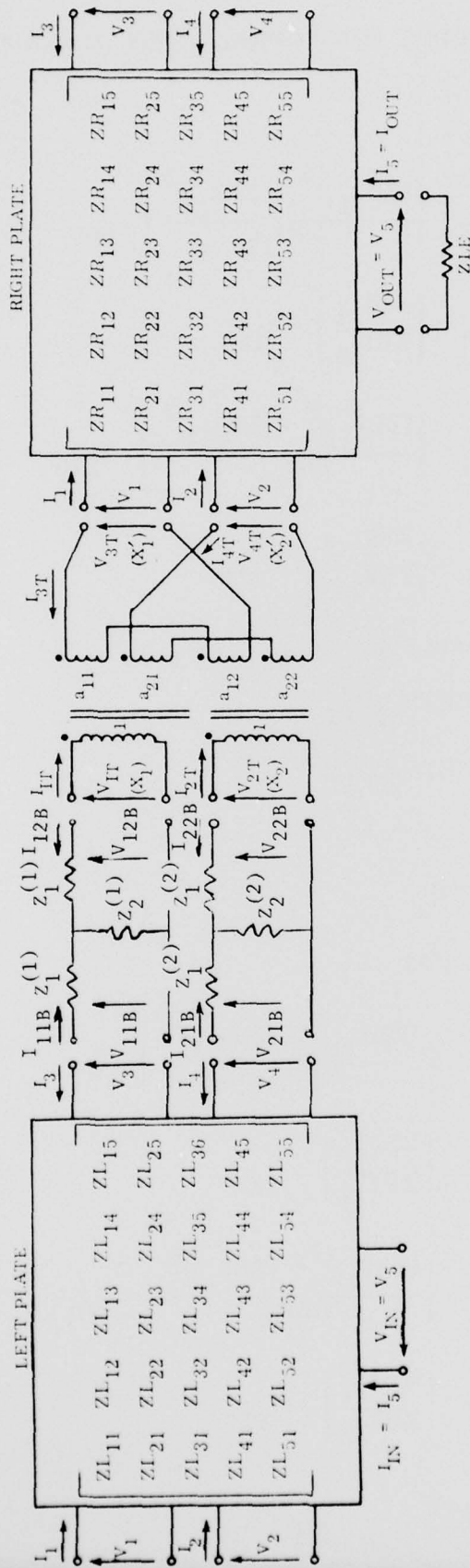


Figure 5-2. MODE2 Problem with Bonds

TABLE 5-1. PROCEDURE FOR ADDING BONDS TO MODE2 SOLUTION

From Figure 5.1

$$V_{11B} = \left\{ \frac{ZB1_{11}}{ZB1_{12}} \right\} V_{12B} - \left\{ \frac{(ZB1_{11})^2 - (ZB1_{12})^2}{ZB1_{12}} \right\} I_{12B}$$

$$I_{11B} = \left\{ \frac{1}{ZB1_{12}} \right\} V_{12B} - \left\{ \frac{ZB1_{11}}{ZB1_{12}} \right\} I_{12B}$$

$$V_{21B} = \left\{ \frac{ZB2_{11}}{ZB2_{12}} \right\} V_{22B} - \left\{ \frac{(ZB2_{11})^2 - (ZB2_{12})^2}{ZB2_{12}} \right\} I_{22B}$$

$$I_{21B} = \left\{ \frac{1}{ZB2_{12}} \right\} V_{22B} - \left\{ \frac{ZB2_{11}}{ZB2_{12}} \right\} I_{22B}$$

From Mode 2 solution Table 3-16

$$V_{1T} = ZRETR_{11} I_{1T} + ZRETR_{12} I_{2T}$$

$$V_{2T} = ZRETR_{21} I_{1T} + ZRETR_{22} I_{2T}$$

From Figure 5.2

$$V_{12B} = V_{1T} \quad I_{12B} = -I_{1T}$$

$$V_{22B} = V_{2T} \quad I_{22B} = -I_{2T}$$

$$V_{11B} = \left\{ \frac{ZB1_{11}}{ZB1_{12}} \right\} V_{1T} + \left\{ \frac{(ZB1_{11})^2 - (ZB1_{12})^2}{ZB1_{12}} \right\} I_{1T}$$

$$I_{11B} = \left\{ \frac{1}{ZB1_{12}} \right\} V_{1T} + \left\{ \frac{ZB1_{11}}{ZB1_{12}} \right\} I_{1T}$$

$$V_{21B} = \left\{ \frac{ZB2_{11}}{ZB2_{12}} \right\} V_{2T} - \left\{ \frac{(ZB2_{11})^2 - (ZB2_{12})^2}{ZB2_{12}} \right\} I_{2T}$$

$$I_{21B} = \left\{ \frac{1}{ZB2_{12}} \right\} V_{2T} + \left\{ \frac{ZB2_{11}}{ZB2_{12}} \right\} I_{2T}$$

TABLE 5-1. (CONT'D).

$$V_{11B} = \left\{ \left(\frac{ZB1_{11}}{ZB1_{12}} \right) ZRETR_{11} + \left[\frac{(ZB1_{11})^2 - (ZB1_{12})^2}{ZB1_{12}} \right] \right\} I_{1T}$$

$$+ \left\{ \left(\frac{ZB1_{11}}{ZB1_{12}} \right) ZRETR_{12} \right\} I_{2T}$$

$$I_{11B} = \left\{ \frac{ZRETR_{11} + ZB1_{11}}{ZB1_{12}} \right\} I_{1T} + \left\{ \frac{ZRETR_{12}}{ZB1_{12}} \right\} I_{2T}$$

$$V_{21B} = \left\{ \left(\frac{ZB2_{11}}{ZB2_{12}} \right) ZRETR_{21} \right\} I_{1T}$$

$$+ \left\{ \left(\frac{ZB2_{11}}{ZB2_{12}} \right) ZRETR_{22} + \left[\frac{(ZB2_{11})^2 - (ZB2_{12})^2}{ZB2_{12}} \right] \right\} I_{2T}$$

$$I_{21B} = \left\{ \frac{ZRETR_{21}}{ZB2_{12}} \right\} I_{1T} + \left\{ \frac{ZRETR_{22} + ZB2_{11}}{ZB2_{12}} \right\} I_{2T}$$

From Figure 5.2

$$V_3 = V_{11B} \quad I_3 = -I_{11B}$$

$$V_4 = V_{21B} \quad I_4 = -I_{21B}$$

$$V_3 = \left\{ \frac{(ZB1_{11}) ZRETR_{11} + (ZB1_{11})^2 - (ZB1_{12})^2}{ZB1_{12}} \right\} I_{1T}$$

$$+ \left\{ \frac{(ZB1_{11}) ZRETR_{12}}{ZB1_{12}} \right\} I_{2T}$$

$$-I_3 = \left\{ \frac{ZRETR_{11} + ZB1_{11}}{ZB1_{12}} \right\} I_{1T} + \left\{ \frac{ZRETR_{12}}{ZB1_{12}} \right\} I_{2T}$$

$$V_4 = \left\{ \frac{(ZB2_{11}) ZRETR_{21}}{ZB2_{12}} \right\} I_{1T}$$

$$+ \left\{ \frac{(ZB2_{11}) ZRETR_{22} + (ZB2_{11})^2 - (ZB2_{12})^2}{ZB2_{12}} \right\} I_{2T}$$

TABLE 5-1. (CONT'D).

$$-I_4 = \left\{ \frac{ZRETR_{21}}{ZB2_{12}} \right\} I_{1T} + \left\{ \frac{ZRETR_{22} + ZB2_{11}}{ZB2_{12}} \right\} I_{2T}$$

These can be solved for I_{1T} and I_{2T} in terms of I_3 and I_4

$$I_{1T} = \left\{ \frac{(ZB1_{12})(ZRETR_{22} + ZB2_{11})}{(ZRETR_{12})(ZRETR_{21}) - (ZRETR_{11} + ZB1_{11})(ZRETR_{22} + ZB2_{11})} \right\} I_3$$

$$- \left\{ \frac{(ZB2_{12})(ZRETR_{12})}{(ZRETR_{12})(ZRETR_{21}) - (ZRETR_{11} + ZB1_{11})(ZRETR_{22} + ZB2_{11})} \right\} I_4$$

$$I_{2T} = - \left\{ \frac{(ZB1_{12})ZRETR_{21}}{(ZRETR_{12})(ZRETR_{21}) - (ZRETR_{11} + ZB1_{11})(ZRETR_{22} + ZB2_{11})} \right\} I_3$$

$$+ \left\{ \frac{(ZB2_{12})(ZRETR_{11} + ZB1_{11})}{(ZRETR_{12})(ZRETR_{21}) - (ZRETR_{11} + ZB1_{11})(ZRETR_{22} + ZB2_{11})} \right\} I_4$$

Let

$$I_{1T} = BF_{11} I_3 - BF_{12} I_4$$

$$I_{2T} = -BF_{21} I_3 + BF_{22} I_4$$

$$V_3 = \left[\frac{BF_{11} \left\{ (ZB1_{11})(ZRETR_{11}) + (ZB1_{11})^2 - (ZB1_{12})^2 \right\} - BF_{12} (ZB1_{11})(ZRETR_{12})}{ZB1_{12}} \right] I_3$$

$$+ \left[\frac{BF_{22} (ZB1_{11})(ZRETR_{12}) - BF_{21} \left\{ (ZB1_{11})ZRETR_{11} + (ZB1_{11})^2 - (ZB1_{12})^2 \right\}}{ZB1_{12}} \right] I_4$$

$$V_4 = \left[\frac{BF_{11} (ZB2_{11})(ZRETR_{21}) - BF_{21} \left\{ (ZB2_{11})(ZRETR_{22}) + (ZB2_{11})^2 - (ZB2_{12})^2 \right\}}{ZB2_{12}} \right] I_3$$

$$+ \left[\frac{BF_{22} \left\{ (ZB2_{11})(ZRETR_{22}) + (ZB2_{11})^2 - (ZB2_{12})^2 \right\} - BF_{12} (ZB2_{11})(ZRETR_{21})}{ZB2_{12}} \right] I_4$$

but from Mode 2 solution Table 3-16

$$V_3 = ZLT_{11} I_3 + ZLT_{13} I_{IN}$$

$$V_4 = ZLT_{22} I_4 + ZLT_{23} I_{IN}$$

Solution then follows along the same pattern as shown for Mode 2 with additional steps involved.

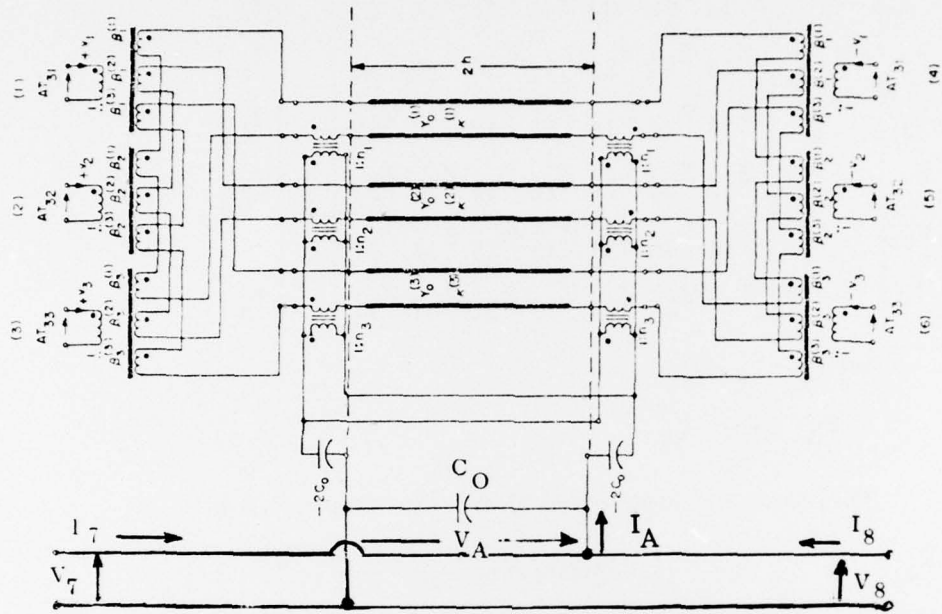


Figure 5-3. Augmented Equivalent Circuit of a TETM Plate.

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_A \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} \\ Z_{12} & Z_{22} & Z_{23} & Z_{15} & Z_{25} & Z_{26} & Z_{27} \\ Z_{13} & Z_{23} & Z_{33} & Z_{16} & Z_{26} & Z_{36} & Z_{37} \\ Z_{14} & Z_{15} & Z_{16} & Z_{11} & Z_{12} & Z_{13} & Z_{17} \\ Z_{15} & Z_{25} & Z_{26} & Z_{12} & Z_{22} & Z_{23} & Z_{27} \\ Z_{16} & Z_{26} & Z_{36} & Z_{13} & Z_{23} & Z_{33} & Z_{37} \\ Z_{17} & Z_{27} & Z_{37} & Z_{17} & Z_{27} & Z_{37} & Z_{77} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_A \end{bmatrix}$$

$$\text{but } I_A = I_7 + I_8$$

$$V_A = V_7 = V_8$$

Therefore the Augmented Matrix for Figure 5.3 is

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \hline V_4 \\ V_5 \\ V_6 \\ \hline V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & | & Z_{14} & Z_{15} & Z_{16} & | & Z_{17} & | & Z_{17} \\ Z_{12} & Z_{22} & Z_{23} & | & Z_{15} & Z_{25} & Z_{26} & | & Z_{27} & | & Z_{27} \\ Z_{13} & Z_{23} & Z_{33} & | & Z_{16} & Z_{26} & Z_{36} & | & Z_{37} & | & Z_{37} \\ \hline Z_{14} & Z_{15} & Z_{16} & | & Z_{11} & Z_{12} & Z_{13} & | & Z_{17} & | & Z_{17} \\ Z_{15} & Z_{25} & Z_{26} & | & Z_{12} & Z_{22} & Z_{23} & | & Z_{27} & | & Z_{27} \\ Z_{16} & Z_{26} & Z_{36} & | & Z_{13} & Z_{23} & Z_{33} & | & Z_{37} & | & Z_{37} \\ \hline Z_{17} & Z_{27} & Z_{37} & | & Z_{17} & Z_{27} & Z_{37} & | & Z_{77} & | & Z_{77} \\ \hline Z_{17} & Z_{27} & Z_{37} & | & Z_{17} & Z_{27} & Z_{37} & | & Z_{77} & | & Z_{77} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \hline I_4 \\ I_5 \\ I_6 \\ \hline I_7 \\ I_8 \end{bmatrix}$$

For later use subdivide as shown above

Figure 5-4. Augmented Impedance Matrix of Augmented Equivalent Circuit.

TABLE 5-2. CONVERSION OF AUGMENTED IMPEDANCE MATRIX TO AUGMENTED TRANSMISSION MATRIX

Define the subdivisions in Figure 5.4 as shown

$$V_{MI} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad V_{MO} = \begin{bmatrix} V_4 \\ V_5 \\ V_6 \end{bmatrix} \quad V_{EI} = [V_7] \quad V_{EO} = [V_8]$$

$$I_{MI} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad I_{MO} = \begin{bmatrix} I_4 \\ I_5 \\ I_6 \end{bmatrix} \quad I_{EI} = [I_7] \quad I_{EO} = [I_8]$$

$$Z_{MI} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12} & Z_{22} & Z_{23} \\ Z_{13} & Z_{23} & Z_{33} \end{bmatrix} \quad Z_{MT} = \begin{bmatrix} Z_{14} & Z_{15} & Z_{16} \\ Z_{15} & Z_{25} & Z_{26} \\ Z_{16} & Z_{26} & Z_{36} \end{bmatrix}$$

$$Z_{ME} = \begin{bmatrix} Z_{17} \\ Z_{27} \\ Z_{37} \end{bmatrix} \quad Z_{EI} = [Z_{77}] \quad Z_{ME}^T = [Z_{17} \ Z_{27} \ Z_{37}]$$

In these terms the augmented Impedance Matrix of Figure 5.4 is

$$\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{bmatrix} V_{MI} \\ V_{MO} \\ V_{EI} \\ V_{EO} \end{bmatrix} = \begin{bmatrix} Z_{MI} & Z_{MT} & Z_{ME} & Z_{ME} \\ Z_{MT} & Z_{MI} & Z_{ME} & Z_{ME} \\ Z_{ME}^T & Z_{ME}^T & Z_{EI} & Z_{EI} \\ Z_{ME}^T & Z_{ME}^T & Z_{EI} & Z_{EI} \end{bmatrix} \begin{bmatrix} I_{MI} \\ I_{MO} \\ I_{EI} \\ I_{EO} \end{bmatrix}$$

TABLE 5-2. (CONT'D).

Solve this set of four equations using algebraic matrix techniques for V_{MI} , I_{MI} , V_{EI} and I_{EI} in terms of V_{MO} , I_{MO} , V_{EO} , and I_{EO}

From 2

$$(5) \quad I_{MI} = Z_{MT}^{-1} V_{MO} - Z_{MT}^{-1} Z_{MI} I_{MO} - Z_{MT}^{-1} Z_{ME} I_{EI} - Z_{MT}^{-1} Z_{ME} I_{EO}$$

Substitute 5 into 4

$$\begin{aligned} V_{EO} = & Z_{ME}^T Z_{MT}^{-1} V_{MO} + (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) I_{MO} \\ & + (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME}) I_{EI} + (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME}) I_{EO} \end{aligned}$$

$$\begin{aligned} (6)* \quad I_{EI} = & (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} V_{EO} \\ & - (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) V_{MO} \\ & - (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) I_{MO} \\ & - I_{EO} \end{aligned}$$

Substitute 6 into 5

$$\begin{aligned} (7)* \quad I_{MI} = & \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) + Z_{MT}^{-1} \right\} V_{MO} \\ & + \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) \right. \\ & \left. - Z_{MT}^{-1} Z_{MI} \right\} I_{MO} \\ & - \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} V_{EO} \right. \end{aligned}$$

Substitute 6 and 7 into 1

$$\begin{aligned} (8)* \quad V_{MI} = & \left[Z_{MI} \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) + Z_{MT}^{-1} \right\} \right. \\ & \left. - Z_{ME} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) \right] V_{MO} \\ & + \left[Z_{MI} \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) \right. \right. \\ & \left. \left. - Z_{MT}^{-1} Z_{MI} \right\} \right. \end{aligned}$$

TABLE 5-2. (CONT'D).

$$\begin{aligned}
 & + Z_{MT} - Z_{ME} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) \} I_{MO} \\
 & + \{ Z_{ME} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} - Z_{MI} \} (Z_{MT}^{-1} Z_{ME}) \\
 & (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} \} V_{EO}
 \end{aligned}$$

Substituting 6 and 7 into 3

$$\begin{aligned}
 (9)^* \quad V_{EI} = & \{ Z_{ME}^T \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) + Z_{MT}^{-1} \right\} \\
 & - Z_{EI} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1}) \} V_{MO} \\
 & + \{ Z_{ME}^T \left\{ (Z_{MT}^{-1} Z_{ME}) (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) \right. \\
 & \left. - Z_{MT}^{-1} Z_{MI} \right\} + Z_{ME}^T - Z_{EI} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} \\
 & (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI}) \} I_{MO} \\
 & + \{ Z_{EI} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} - Z_{ME}^T (Z_{MT}^{-1} Z_{ME}) \\
 & (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} \} V_{EO}
 \end{aligned}$$

6, 7, 8 and 9 are the desired transmission matrix equations.

$$\begin{aligned}
E_E &= [Z_{EI} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} - Z_{ME}^T Z_{MT}^{-1} Z_{ME} (Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1}] \\
G_M &= - [(Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T Z_{MT}^{-1})] \\
H_M &= - [(Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1} (Z_{ME}^T - Z_{ME}^T Z_{MT}^{-1} Z_{MI})] \\
G_E &= [(Z_{EI} - Z_{ME}^T Z_{MT}^{-1} Z_{ME})^{-1}]
\end{aligned}$$

Figure 5-5. (Cont'd)