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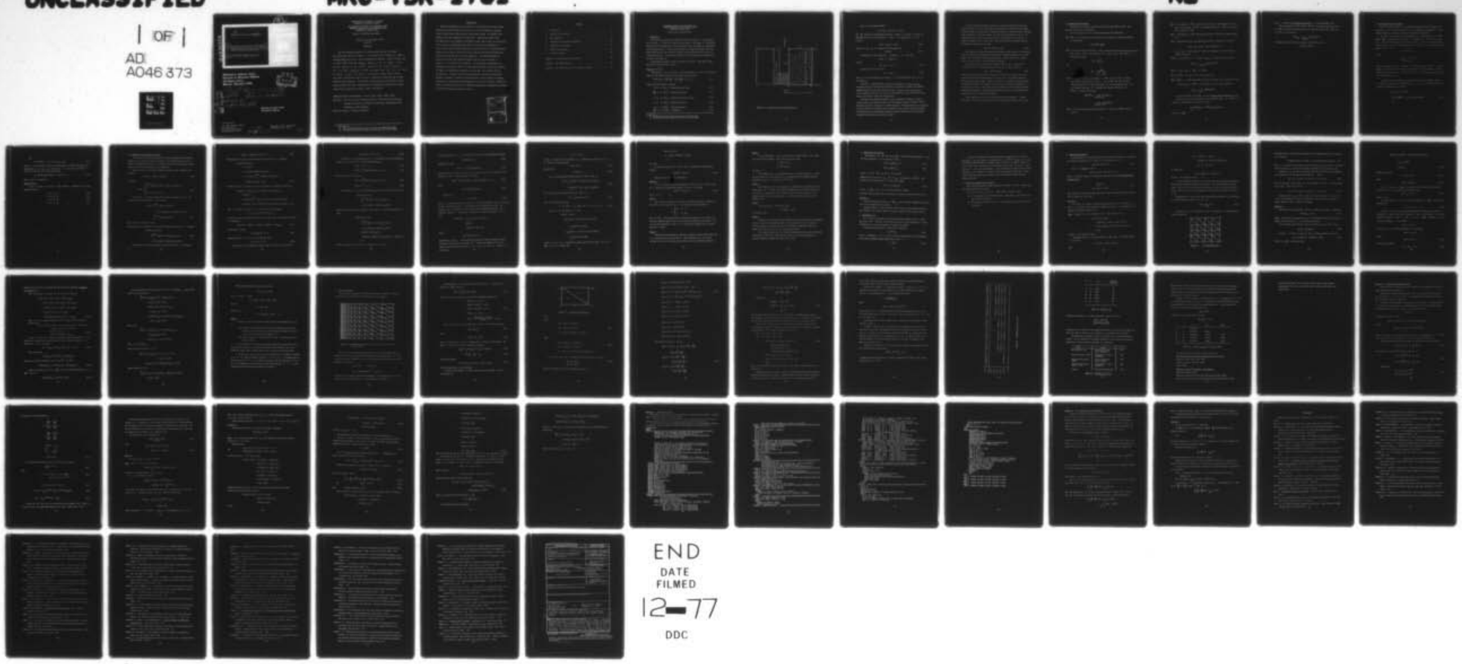

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THE NUMERICAL SOLUTION OF AXISYMMETRIC FREE BOUNDARY POROUS FLOW WELL PROBLEMS USING VARIATIONAL INEQUALITIES

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ABSTRACT

The free boundary problem for a fully penetrating well of radius r and filled with water to a depth h, in a layer of soil of depth H, radius R and permeability $k(x,y)$ can be formulated as follows: Find $\varphi \in C^1[r,R]$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $(xku_x)_x + (xku_y)_y = 0$ in Ω, $u(R,y) = H$ for $0 < y < H$, $u_n(x,0) = 0$ for $r < x < R$, $u(r,y) = h$ for $0 < y < h$, $u(r,y) = y$ for $h < y < \varphi(r)$, $u_n(x,\varphi(x)) = 0$ and $u(x,\varphi(x)) = \varphi(x)$ for $r < x < R$, where $\Omega = \{(x,y): r < x < R, \ 0 < y < \varphi(x)\}$. The results of Benci [Annali di Mat. 100 (1974), 191-209] are used to derive a variational inequality and to prove existence and uniqueness. The problem is approximated using piecewise linear finite elements and $O(h)$ convergence of the approximate solutions is proved using recent results due to Brezzi, Hager, and Raviart.

AMS(MOS) Subject Classification - 35-02, 35J25, 35N99, 35R99, 76S05

Key Words - Porous flow, Free boundary problems, Unconfined flow, Numerical methods, Variational inequalities, Existence, Uniqueness, Error estimates, Finite elements.

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EXPLANATION

When an axisymmetric well is sunk in a water aquifer the water flows through the ground towards the well where it can be removed by pumping. After a sufficiently long time the flow becomes steady. We analyse this problem using the assumptions of saturated - unsaturated porous flow: the upper part of the aquifer is denuded of water and is dry; the lower part of the aquifer is saturated (wet) and in it the flow is governed by a linear second order elliptic differential equation which is derived from the equation of continuity and Darcy's law (a linear relationship between the rate of flow and the water pressure gradient). The major mathematical difficulty is that the interface between the dry and wet regions is unknown - it is called a free boundary. We reformulate the problem as a variational inequality, that is, a variational problem in which the solution is subject to inequality constraints (in our case, the solution must be non-negative). We prove existence and uniqueness (which were previously unknown), introduce a numerical scheme based on finite elements, and prove convergence of the numerical approximation to the exact solution. Numerical results are given for the case of constant permeability (homogeneous isotropic soil) and the computer program is listed.

CONTENTS

THE NUMERICAL SOLUTION OF AXISYMMETRIC FREE BOUNDARY POROUS FLOW WELL PROBLEMS USING VARIATIONAL INEQUALITIES

Colin W. Cryer and Hans Fetter

1. Introduction

The steady state problem to be considered is shown in Figure 1.1. An axisymmetric well of radius r is sunk into a layer of soil of depth H and radius R. The bottom of the soil layer is impervious. The outer boundary of the soil adjoins a catchment area and the hydraulic head u is equal to the constant H along this boundary. The water seeps towards the well and a pump (not shown) maintains the water level in the well at a constant height h_w. The water-air interface is a free boundary which intersects the well wall at a height h_s.

The mathematical problem can now be formulated as follows (see Hantush [1964], Bear [1972], and Cryer [1976, p. 86]):

Problem A (Classical)

Find functions $\varphi(x)$ (the height of the free boundary) and $u(x,y)$ (the hydraulic head) such that (from the equation of continuity and Darcy's law):

$$\operatorname{div}(k \operatorname{grad} u) = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial u}{\partial y}\right) = 0, \quad \text{in } \Omega, \tag{1.1}$$

together with the boundary conditions,

$$u = H, \quad \text{on } AB(\Gamma_1), \quad \text{(constant hydraulic head)}, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } BC(\Gamma_4), \quad \text{(impervious boundary)}, \tag{1.3}$$

$$u = h_w, \quad \text{on } CD(\Gamma_2), \quad \text{(interface with water at rest)}, \tag{1.4}$$

$$u = y, \quad \text{on } DE(\Gamma_3'), \quad \text{(interface with air)}, \tag{1.5}$$

$$u = y, \quad \text{on } EA(\Gamma_0), \quad \text{(interface with air)}, \tag{1.6}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } EA(\Gamma_0), \quad \text{(streamline)}. \tag{1.7}$$

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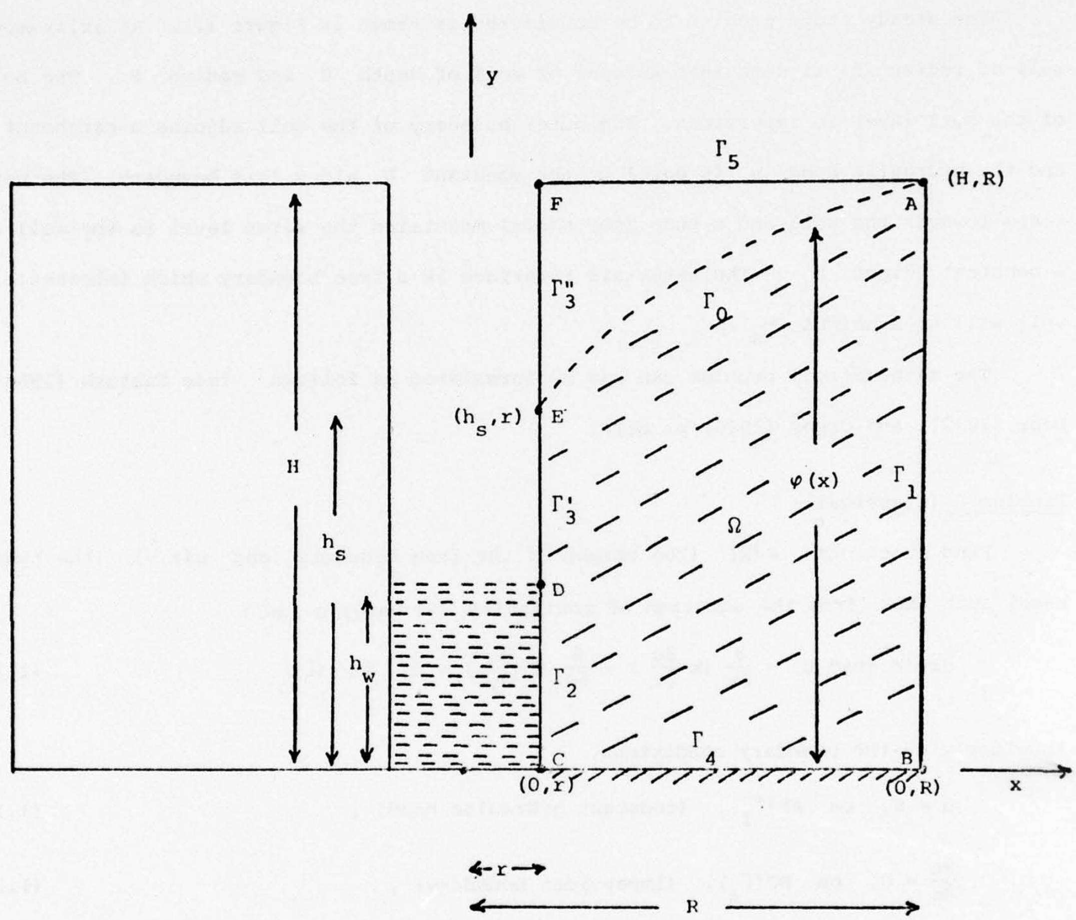

Figure 1.1: An axisymmetric fully penetrating well

Here, Ω is the (unknown) domain,

$$\Omega = \{(x,y): 0 < y < \varphi(x), \quad r < x < R\},$$

and $\frac{\partial}{\partial n}$ denotes the outward normal derivative. Finally, $k = k(x,y) = x\kappa$ where the permeability of the soil is denoted by $\kappa = \kappa(x,y)$. It is assumed, that k is of the form

$$k(x,y) = \exp[f(x) + g(y)] \tag{1.8}$$

where $f(x)$ and $g(y)$ are continuously differentiable and

$$g'(y) \geq 0 . \quad \square \tag{1.9}$$

In particular if the permeability κ is constant, $\kappa = 1$ say, then

$$k(x,y) = x = \exp[\ell n\ x]$$

so that

$$f(x) = \ell nx; \quad g(y) = 0 . \tag{1.10}$$

We will later use the fact that

$$u = y + p/\rho g \quad , \tag{1.11}$$

where g is the acceleration due to gravity, p is the fluid pressure, and ρ is the fluid density.

Ever since C. Baiocchi [1971] introduced a mathematically rigorous and from the numerical point of view efficient approach to the solution of various free boundary problems related to fluid flow through porous media, numerous studies have appeared in the literature which extend his results in many directions; Cryer [1976] and Baiocchi, Brezzi, and Comincioli [1976] give bibliographies.

The basic idea introduced by Baiocchi, can be summarized as follows: Through a suitable change of the unknown variable, the free boundary problem is reduced to that of minimizing a quadratic functional on a closed convex set. This reformulation of the problem not only enables one to determine various properties of the solution, but it also offers the advantage that the new problem can readily be solved numerically by several methods including finite differences and finite elements.

The first problem considered by Baiocchi was the "model problem" of porous flow between two water reservoirs of different levels separated by a homogeneous rectangular dam with horizontal impervious base. Subsequently, Baiocchi, Comincioli, Guerri, and Volpi [1973] and Baiocchi, Comincioli, Magenes, and Pozzi [1973] considered the case of a rectangular dam consisting of two homogeneous layers, either horizontal or vertical. A further development is due to Benci [1973, 1974] who considered a rectangular dam with a permeability coefficient of the form

$$\kappa(x,y) = \exp(f(x) + g(y)) \ . \tag{1.12}$$

The problem of a fully penetrating axially symmetric well (Figure 1) is the axially symmetric equivalent of the problem of flow through a rectangular porous dam. The well problem is of considerable technological importance. Hantush [1964] gives a lengthy survey, and Cryer [1976, p. 86] gives further references.

Polubarinova-Kochina [1962, p. 283], Hantush [1964, p. 362], and Bear [1972, p. 368] show that for constant permeability the rate of flow Q_w is given by

$$Q_w = \pi\kappa(H^2-h^2)/\ln(R/r) \ . \tag{1.13}$$

Mauersberger [1969] and Youngs [1971] obtain exact expressions in closed form for the rate of flow with variable permeability of the form (1.12). Mauersberger [1965] has derived a variational principle. However, apart from the not entirely rigorous results of Mauersberger [1965a], there appear to be no results on existence and uniqueness. In this paper the results of Benci [1973, 1974] are applied to the well problem to obtain existence and uniqueness. A finite element method is then used to obtain numerical approximations, and error estimates are obtained.

After completing this report we learned that the case of constant κ had been formulated as a variational inequality and solved numerically by Elliott [1976, p. 62].

2. Notation and preliminaries

We state here certain basic definitions and results about Sobolev spaces. This material can be found in Adams [1975].

If Ω is an open set in R^2 the following spaces may be constructed:

$C(\bar{\Omega}) = C^0(\bar{\Omega})$ is the Banach space of functions which are bounded and uniformly continuous on $\bar{\Omega}$ with norm

$$\|u;\ C(\bar{\Omega})\| = \max_{\bar{\Omega}} |u| \quad .$$

$C^m(\Omega)$ is the vector space of functions u which, together with all their partial derivatives $D^\alpha u$ of order $|\alpha| \leq m$ are continuous on Ω. Here $\alpha = (\alpha_1,\ldots,\alpha_n)$ with α_j a non-negative integer,

$$|\alpha| = \sum_{j=1}^{n} |\alpha_j| \quad ,$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad .$$

$C^m(\bar{\Omega})$ is the Banach space of those functions $u \in C^m(\Omega)$ for which $D^\alpha u$ is bounded and uniformly continuous for $0 \leq |\alpha| \leq m$. If $u \in C^m(\Omega)$ then $D^\alpha u$ can be extended continuously to $\bar{\Omega}$ and we may regard $D^\alpha u$ as being defined on $\bar{\Omega}$, and it is readily shown that $D^\alpha u$ is bounded and uniformly continuous on $\bar{\Omega}$. $C^m(\bar{\Omega})$ is equipped with the norm

$$\|u; C^m(\bar{\Omega})\| = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)| \quad ,$$

$$= \max_{0 \leq |\alpha| \leq m} \max_{x \in \bar{\Omega}} |D^\alpha u(x)| \quad .$$

$C^\infty(\Omega)$ is the vector space consisting of functions u such that $u \in C^m(\Omega)$ for all m, $0 \leq m < \infty$.

$C_0^m(\Omega)$ is the subspace of $C^m(\bar{\Omega})$ consisting of functions u which "vanish near $\partial\Omega$", that is, the functions $u \in C^m(\Omega)$ with compact support in Ω. $C_0^m(\Omega)$ is of course also a subspace of $C^m(\Omega)$.

$C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ consisting of functions $u \in C^\infty(\Omega)$ with compact support in Ω.

$L^p(\Omega)$ is the Banach space of real measurable functions u defined on Ω for which

$$\|u\|_p \equiv \|u;\, L_p(\Omega)\| = [\int_\Omega |u(x)|^p dx]^{1/p} < \infty \quad .$$

Here, $1 \leq p < \infty$. Two elements of $L^p(\Omega)$ are identified if they are equal a.e. (almost everywhere).

$H^{m,p}(\Omega)$ is the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$ with respect to the norm

$$\|u\|_{m,p} = [\sum_{0 \leq |\alpha| \leq m} (\|D^\alpha u\|_p)^p]^{1/p} \quad .$$

$H_0^{m,p}\,\Omega$ is the closure of $C_0^\infty(\Omega)$, in $H^{m,p}(\Omega)$.

$H^m(\Omega) = H^{m,2}(\Omega)$. The norm in H^m is denoted by $\|\cdot\|_m$.

$H_0^m(\Omega) = H_0^{m,2}(\Omega)$. By the generalization of Poincare's Lemma, if Ω is bounded then the norm $\|\cdot\|_{m,p}$ on $H_0^{m,p}(\Omega)$ is equivalent to the norm $|\cdot|_{m,p}$ defined by

$$|u|_{m,p} = [\sum_{|\alpha|=m} (\|D^\alpha u\|_p)^p]^{1/p} \quad .$$

The spaces $H^{m,p}(\Omega)$ and $H_0^{m,p}(\Omega)$ are called <u>Sobolev spaces</u> or <u>Beppo-Levi</u> spaces.

There is a useful alternative definition of the Sobolev spaces. If $u \in L^p(\Omega)$ and if for some α there exists $v_\alpha \in L^1(\Omega)$ such that

$$\int_\Omega u(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\, \phi(x)\, dx \quad ,$$

for all $\phi \in C_0^\infty(\Omega)$,

then v is called the <u>weak partial derivative</u> of u and is denoted by $D^{\alpha}u$.

We can now define the $W^{m,p}$ spaces: $W^{m,p}(\Omega)$ is the Banach space of functions $u \in L^p(\Omega)$ such that the weak derivatives $D^{\alpha}u$ exist and satisfy $D^{\alpha}u \in L^p(\Omega)$ for $1 \leq |\alpha| \leq m$ with the norm

$$\|u\|_{m,p} = [\sum_{0 \leq |\alpha| \leq m} (\|D^{\alpha}u\|_p)^p]^{1/p} \quad .$$

An important result due to Myers and Serrin states that for all Ω

$$H^{m,p}(\Omega) = W^{m,p}(\Omega) \quad .$$

3. Weak formulation of the problem

In the (classical) Problem A it is implicitly assumed that the free boundary Γ_0 and solution u are "sufficiently smooth". Here we reformulate the problem as a second problem, Problem B, in which these smoothness assumptions are relaxed.

Let $\{\varphi, u\}$ be a "smooth" solution of Problem A. We cannot assume that $u \in C^2(\bar{\Omega})$ because $\frac{\partial u}{\partial y}$ is discontinuous at the corner D (see Figure 1.1). It might also appear that u could be discontinuous at the corners B and C; this is not the case since by reflecting both u and Ω in the x-axis we obtain a solution, u' say, in the reflected domain, Ω' say. Let

$$\Omega'' = \Omega \cup \Omega' \cup \Gamma_4 \quad ,$$

and let

$$u'' = \begin{cases} u, & \text{on} \quad \Omega \cup \Gamma_4 \ , \\ u', & \text{on} \quad \Omega' \ . \end{cases}$$

Then $\partial\Omega''$ is smooth at B and C so that u'' is well-behaved at B and C. We thus assume that "sufficiently smooth" implies the following: the free boundary Γ_0 has a continuous outward normal $\underline{n}$; $u \in C^2(\Omega) \cap C(\bar{\Omega})$; and u is continuously differentiable in a neighborhood of $\Gamma_0 \cup \Gamma$.

To obtain the weak formulation of the problem we proceed as follows. For any $\psi \in C^\infty(\Omega)$ which vanishes in a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3'$ it follows from Green's theorem (see Appendix C) and (1.1) that

$$\begin{aligned} &\int_\Omega k \, \text{grad}\, u \, \text{grad}\, \psi \, dxdy \\ &= \int_{\partial\Omega} \psi k \frac{\partial u}{\partial n} ds - \int_\Omega \psi \, \text{div}(k \, \text{grad}\, u) dxdy \quad , \qquad (3.1) \\ &= 0 \quad . \end{aligned}$$

Let

$$V = \{v \in H^1(\Omega): \ v = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_2 \cup \Gamma_3'\} ; \tag{3.2}$$

that is, V is the closure in $H^1(\Omega)$ of the functions $\psi \in C^\infty(\Omega)$ which vanish in a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3'$. Since every $v \in V$ is the limit in $H^1(\Omega)$ of functions ψ for which (3.1) holds, we have that

$$\int_\Omega k \text{ grad } u \text{ grad } v \, dxdy = 0, \quad \text{for all } v \in V . \tag{3.3}$$

Thus, $\{\varphi, u\}$ is a solution of

<u>Problem B (Weak)</u>

Find $\varphi \in C[r,R]$ and $u \in H^1(\Omega) \cap C(\bar{\Omega})$ such that u satisfies (3.3) and the boundary conditions

$$u = H, \text{ on } \Gamma_1 , \tag{3.4}$$

$$u = h_w, \text{ on } \Gamma_2 , \tag{3.5}$$

$$u = y , \text{ on } \Gamma_3' , \tag{3.6}$$

$$u = y, \text{ on } \Gamma_0 . \quad \square \tag{3.7}$$

4. Formulation as a variational inequality

In this section we follow Benci [1973, 1974] and reformulate the weak problem, Problem B, as a variational inequality, Problem C. The derivation given here is intended to complement that of Benci: it is hoped that the derivation given here brings out more clearly the essential steps in the derivation.

The first step is to introduce a "Baiocchi function" $w(x,y)$ defined on the rectangle

$$D = \{(x,y): \ r \le x \le R, \quad 0 \le y \le H\} , \tag{4.1}$$

as follows:

$$w(x,y) = \begin{cases} \int_y^{\varphi(x)} \exp(g(t)) \, [u(x,t) - t]dt, & \text{for } (x,y) \in \Omega , \\ 0, \quad \text{for } (x,y) \in D - \Omega . \end{cases} \tag{4.2}$$

We show that w satisfies a differential equation at points $(x,y) \in \Omega$. Differentiating (4.2) with respect to x we find that

$$\begin{aligned} w_x(x,y) &= \int_y^{\varphi(x)} \exp[g(t)] \, u_x(x,t)dt + \\ &\quad + \varphi'(x) \exp[g(\varphi(x))] \, [u(x,\varphi(x)) - \varphi(x)] , \\ &= \int_y^{\varphi(x)} \exp[g(t)] \, u_x(x,t)dt , \end{aligned} \tag{4.3}$$

since, by (1.6), $u(x, \varphi(x)) = \varphi(x)$.

Multiplying (4.3) by $\exp[f(x)]$ and then differentiating with respect to x we obtain

$$\begin{aligned} &(\exp[f(x)] \, w_x(x,y))_x \\ &= \int_y^{\varphi(x)} [\exp[f(x) + g(t)]u_x(x,t)]_x \, dt + \\ &\quad + \varphi'(x) \exp[f(x) + g(\varphi(x))] \, u_x(x, \varphi(x)) . \end{aligned} \tag{4.4}$$

We now carry out similar computations for the derivatives of w with respect to y .

$$w_y(x,y) = -\exp[g(y)]\ [u(x,y) - y] \quad . \tag{4.5}$$

Multiplying by $\exp[-g(y)]$ and differentiating with respect to y we obtain,

$$(\exp[-g(y)]\ w_y(x,y))_y$$

$$= 1 - u_y(x,y) \quad ,$$

$$= 1 - \exp[-g(y)]\ [\exp[g(y)]\ u_y(x,y)] \quad ,$$

$$= 1 + \exp[-g(y)]\ \{\int_y^{\varphi(x)} (\exp[g(t)]\ u_t(x,t))_t\ dt -$$

$$- \exp[g(\varphi(x))]\ u_y(x, \varphi(x))\} \quad . \tag{4.6}$$

Multiplying (4.4) by $\exp[-g(y)]$ and (4.6) by $\exp[f(x)]$ and adding, we obtain that

$$(\exp[f(x) - g(y)]\ w_x(x,y))_x + (\exp[f(x) - g(y)]\ w_y(x,y))_y$$

$$= \exp[f(x)] +$$

$$+ \exp[-g(y)] \int_y^{\varphi(x)} [(k(x,t)\ u_x(x,t))_x + (k(x,t)\ u_t(x,t))_t]dt +$$

$$+ \exp[-g(y) + f(x) + g(\varphi(x))]\ [\varphi'(x)\ u_x(x,\varphi(x)) - u_y(x,\varphi(x))] \quad . \tag{4.7}$$

Since u satisfies $\operatorname{div}(k \operatorname{grad} u) = 0$, and since on the free boundary

$$\varphi' u_x - u_y = -[1 + (\varphi')^2]^{1/2}\ u_n = 0 \quad ,$$

the last two terms on the right hand side of (4.7) vanish. Thus, if the elliptic operator L is defined by

$$(Lw)(x,y) = (\exp[f(x) - g(y)]w_x)_x + (\exp[f(x) - g(y)]w_y)_y \quad , \tag{4.8}$$

we have from (4.7) that

$$Lw = \exp[f(x)], \quad \text{in } \Omega \; . \tag{4.9}$$

Since, from (4.2), $w \equiv 0$ in $D - \overline{\Omega}$, we also have that

$$Lw = 0, \quad \text{in } D - \overline{\Omega} \; . \tag{4.10}$$

At this point it is appropriate to observe that, from (4.2), (4.3), and (4.5) ,

$$w_x = w_y = w_n = 0, \quad \text{on} \quad \Gamma_0 . \tag{4.11}$$

We denote by Φ the boundary values of w. The function Φ can be determined as follows. From (1.2) and (4.2) it follows immediately that

$$\Phi(x,H) = 0, \quad \text{on} \quad \Gamma_5 , \tag{4.12}$$

$$\Phi(R,y) = \int_y^H \exp[g(t)]\,[H-t]dt, \quad \text{on} \quad \Gamma_1 . \tag{4.13}$$

From (1.4) and (1.5) we see that

$$\Phi(r,y) = \int_y^{h_w} \exp[g(t)]\,[h_w-t]dt, \quad \text{on} \quad \Gamma_2 , \tag{4.14}$$

$$\Phi(r,y) = 0, \quad \text{on} \quad \Gamma_3 . \tag{4.15}$$

To determine Φ on Γ_4 we must proceed more indirectly. From (4.4) we see that

$$\begin{aligned}
&[\exp[f(x)]\, w_x(x,0)]_x \\
&\quad = \int_0^{\varphi(x)} [\exp[f(x) + g(t)]\, u_x(x,t)]_x \, dt + \\
&\quad\quad + \varphi'(x) \exp[f(x) + g(\varphi(x))]\, u_x(x,\varphi(x)) .
\end{aligned}$$

Replacing the integrand using the equation div k grad u = 0 and then integrating, we obtain

$$\begin{aligned}
&[\exp[f(x)\, w_x(x,0)]]_x \\
&\quad = -\int_0^{\varphi(x)} [\exp[f(x) + g(t)]\, u_t(x,t)]_t \, dt + \\
&\quad\quad + \varphi'(x) \exp[f(x) + g(\varphi(x))]\, u_x(x,\varphi(x)) , \\
&\quad = \exp[f(x) + g(0)]\, u_y(x,0) + \\
&\quad\quad + \exp[f(x) + g(\varphi(x))]\, [\varphi'(x)\, u_x(x,\varphi(x)) - u_y(x,\varphi(x))] , \\
&\quad = 0 ,
\end{aligned}$$

since, by (1.3) and (1.7), $u_y = 0$ on Γ_4 and $u_n = 0$ on Γ_0.

We thus conclude that on Γ_4 Φ satisfies the linear second order differential equation

$$(\exp[f(x)]\Phi_x)_x = 0 \quad , \tag{4.16}$$

with general solution

$$\Phi(x,0) = A + B \int_r^x \exp[-f(t)]dt \quad . \tag{4.17}$$

Since the value of $\Phi(x,0)$ is given at $x = r$ by (4.14) and at $x = R$ by (4.13), the values of A and B can be determined. We obtain:

$$\Phi(x,0) = \Phi(r,0) + [\Phi(R,0) - \Phi(r,0)]\ F(x)/F(R), \text{ on } \Gamma_4 , \tag{4.18}$$

where

$$F(x) = \int_r^x \exp[-f(t)]dt \quad . \tag{4.19}$$

From (1.11),

$$u = y + p/\rho g \tag{4.20}$$

where g is the acceleration due to gravity, p is the fluid pressure, and ρ is the fluid density. The fluid pressure p is non-negative for physical reasons so that $u-y \geq 0$ and hence $w \geq 0$. Unfortunately, this cannot be proved mathematically without some additional assumptions. One approach is as follows. Consider the function $v = y - u$. Then

$$\begin{aligned} \mathrm{div}(k \text{ grad } v) &= \mathrm{div}\ k \text{ grad } y - \mathrm{div}\ k \text{ grad } u \quad , \\ &= \frac{\partial}{\partial y}(k) \quad , \\ &= \exp[f(x) + g(y)]\ g'(y) \quad . \end{aligned}$$

Hence

$$\mathrm{div}\ k \text{ grad } v \geq 0 \text{ in } \Omega$$

provided that $g'(y) \geq 0$. From the maximum principle for elliptic equations (Courant and Hilbert [1962, p. 326]) we conclude that v attains its maximum in $\bar{\Omega}$ on $\partial\Omega$. However, from the boundary conditions, $v \leq 0$ on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Also, on Γ_4 we have that

$$v_y = 1 - u_y = 1$$

so that v cannot attain its maximum on Γ_4. Consequently, we see that $v \leq 0$ in Ω. That is, we have shown that

$$w \geq 0 \quad \text{in} \quad \Omega . \tag{4.21}$$

provided that

$$g'(y) \geq 0 . \tag{4.22}$$

Let a be the bilinear operator defined on $H^1(D) \times H^1(D)$ by,

$$\begin{aligned} a(u,v) &= \int_D \exp[f(x) - g(y)] \operatorname{grad} u \operatorname{grad} v \, dxdy , \\ &\equiv \int_D \exp[f(x) - g(y)] \, [u_x v_x + u_y v_y] dxdy . \end{aligned} \tag{4.23}$$

Let j be the linear functional defined on $H^1(D)$ by,

$$j(v) = \int_D \exp[f(x)] v \, dx \, dy . \tag{4.24}$$

Let K be the closed convex set

$$K = \{v \in H^1(D) : v - w \in H^1_0(D) \text{ and } v \geq 0 \text{ a.e. in } D\} . \tag{4.25}$$

Now let v be any element in K. Then

$$\begin{aligned} & a(w,v-w) + j(v-w) \\ &= \int_D \exp[f(x) - g(y)] \operatorname{grad} w \operatorname{grad}(v-w) dxdy + \\ &\quad + \int_D \exp[f(x)] \, (v-w) dxdy , \\ &\geq \int_\Omega \exp[f(x) - g(y)] \operatorname{grad} w \operatorname{grad}(v-w) dxdy + \\ &\quad + \int_\Omega \exp[f(x)] \, (v-w) dxdy , \end{aligned}$$

since $w = 0$ in $D - \Omega$. Integrating by parts, and noting that either $v-w = 0$ or $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$, we obtain that

$$a(w,v-w) + j(v-w)$$

$$\geq \int_\Omega [-Lw + \exp[f(x)]] (v-w)dxdy,$$

$$= 0 ,$$

from (4.9) .

We have thus shown that if $g'(y) \geq 0$ then w satisfies the variational inequality

$$a(w,v-w) + j(v-w) \geq 0 . \tag{4.26}$$

More precisely, Benci [1973, 1974] proves

<u>Theorem 4.1</u>

If u is a solution of the weak problem and $g'(y) \geq 0$ then $w \in H^1(D) \cap C(\bar{D})$ and w satisfies the variational inequality: Find $w \in K$ such that

$$a(w,v-w) + j(v-w) \geq 0 , \tag{4.27}$$

for all $v \in K$. □

<u>Remark 1</u>

Benci [1974, p. 194] and, in a special case, Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 19] assert that if u is a solution of the weak problem and

$$\pi = \begin{cases} u-y, & \text{in } \Omega , \\ 0 , & \text{in } D-\Omega , \end{cases}$$

then $\pi \in H^1(D)$. We have been unable to follow their arguments, and it seems to us that their arguments require that Ω be such that trace theorems can be applied. This is not a serious difficulty, since if u is "sufficiently smooth" then we certainly have that $\pi \in H^1(D)$.

<u>Remark 2</u>

Both Benci [1974] and Baiocchi, Comincioli, Magenes and Pozzi [1973] assume that φ is a monotonically decreasing function. While undoubtedly true, this assumption does not seem to be necessary for the derivation of the variational inequality.

Remark 3

We have assumed that f and g are continuously differentiable. This assumption can be relaxed somewhat: Benci [1974] only requires that

$$f \in H^{1,2+\mu}[r,R] \quad ,$$

$$g \in H^{1,2+\mu}[0,H] \quad ,$$

for some $\mu > 0$.

The assumption that f be reasonably smooth is not unduly restrictive. However, the soil around a well often consists of N horizontal layers of different constant permeabilities, and g is then not in $H^{1,2+\mu}[0,H]$.

Remark 4

While condition (4.22) is not necessary, it appears that some condition must be imposed upon $g(y)$. Baiocchi and Friedman [to appear] refer to numerical solutions of the variational inequality (4.27) by Comincioli showing that some choices of g apparently lead to negative water pressures p, and this has recently been proved by Friedman [1977].

Remark 5

For results when k is not of the form

$$k = \exp[f(x) + g(y)]$$

see Baiocchi [1976].

Remark 6

Although it is not apparent from the above derivation of the variational inequality (4.27), the fact that the boundary values Φ of the Baiocchi function w can be explicitly computed is related to the fact that, as shown by Mauersberger [1969] and Youngs [1971], the flow rate Q_w can be explicitly computed.

If a problem is such that Q_w cannot be determined explicitly, then an additional complication is introduced, namely that the unknown Q_w must also be found; see Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 46].

5. Properties of the solution

By assumption $f(x)$ and $g(y)$ are bounded. Thus, the bilinear functional a and the linear functional j are continuous on $H^1(D)$:

$$a(v,v) \leq \alpha_2(\|v\|_{1,2})^2 \quad , \tag{5.1}$$

$$|j(v)| \leq \beta_2\|v\|_{1,2} \quad , \tag{5.2}$$

for all $v \in H^1(D)$, where α_2 and β_2 are constants.

Remembering that the norms $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$ are equivalent on $H_0^{m,p}(D)$ (see section 2) we can conclude that a is coercive on $H_0^1(D)$:

$$a(v,v) \geq \alpha_1 \, (\|v\|_{1,2})^2 \quad , \tag{5.3}$$

for all $v \in H_0^1(D)$, where α_1 is a strictly positive constant.

It follows from the basic theory of variational inequalities (Stampacchia [1964]) that

Theorem 5.1

There exists a unique solution $w \in H_0^1(D)$ of the variational inequality formulation (4.27) of the axisymmetric well problem. □

Although Theorem 5.1 answers the most basic questions, namely regarding existence and uniqueness, there remain a number of interesting questions which we now mention:

1. How smooth is w?

If w is to be a solution of the classical problem, Problem A, then w cannot just be in $H^1(D)$. Moreover, as will be seen in section 6, the smoothness of w plays an important role in the error analysis of approximation methods.

It follows from the results of Benci [1974, p. 200] that

$$w \in H^{2,p}(D) \quad , \tag{5.4}$$

for any p satisfying $1 < p < \infty$. Since D has the cone property it is a consequence of the Sobolev embedding theorem (Adams [1975, p. 97]) that

$$w \in C^1(\bar{D}) \quad . \tag{5.5}$$

It would be of considerable interest for numerical applications if it could be shown that w was even smoother. In particular, if $w \in H^{5/2-\varepsilon,p}$ for any $\varepsilon > 0$ then the quadratic approximations due to Brezzi, Hager, and Raviart [to appear] could be applied. At the time of writing we do not know whether this is true. Of course, we cannot expect w to be very smooth because w has a discontinuity across the free boundary. References on the regularity of solutions of variational inequalities include: Brezis [1971]; Brezis and Kinderlehrer [1973/74]; Brezis and Stampacchia [1968]; Lewy and Stampacchia [1969].

2. What are the properties of w and Ω?

Many interesting questions suggest themselves concerning w and Ω. In particular the following properties are known:

(a) $w_x \leq 0$ and $w_y \leq 0$ in D. (Benci [1974, p. 200 and p. 202])

(b) φ is continuous and strictly decreasing. (Benci [1974, p. 207] and Baiocchi and Friedman [to appear].)

(c) For other results for the case $k \equiv 1$ see Caffarelli [to appear] and Jensen [to appear].

6. Numerical approximation

It has been shown in the previous sections that the Baiocchi function w satisfies the variational inequality (4.27): Find $w \in K$ such that for all $v \in K$,

$$a(w,v-w) + j(v-w) \geq 0 . \tag{6.1}$$

Since a is symmetric, that is

$$a(v,w) = a(w,v), \quad \text{for all } v,w \in V ,$$

there is a connection between the variational inequality (6.1) and the unilateral minimization problem

$$\begin{aligned} &\min_{v \in K} J(v) , \\ &J(v) = a(v,v) + 2j(v) . \end{aligned} \tag{6.2}$$

This connection is given by the following theorem which is well-known but which we prove for the convenience of the reader.

Theorem 6.1

Let $a(v,w)$ be a symmetric bilinear form satisfying $a(v,v) \geq 0$ for all $v \in V$. Then w is a solution of the variational inequality (6.1) iff w is a solution of the unilateral minimization problem (6.2).

Proof. Suppose that u is a solution of (6.1). Then, for any $v \in K$,

$$\begin{aligned} J(v) &= a(v,v) + 2j(v) , \\ &= a(u+(v-u), u + (v-u)) + 2j(u + (v-u)) , \\ &= J(u) + 2[a(u,v-u) + j(v-u)] + a(v-u, v-u) , \\ &\geq J(u) \end{aligned}$$

so that u is a solution of (6.2).

Now suppose that u is a solution of (6.2). Then, since K is convex, for any $v \in K$ we have

$$u + t(v-u) \in K, \quad \text{for } 0 \leq t \leq 1 .$$

Thus

$$G(t) \equiv J(u+t(v-u)) - J(u,u) \quad ,$$

$$= 2t[a(u,v-u) + j(v-u)] + t^2 a(v-u,v-u) \quad ,$$

$$\geq 0, \quad \text{for} \quad 0 \leq t \leq 1 \quad .$$

It follows that

$$G'(0) = 2[a(u,v-u) + j(v-u)] \geq 0 \; ;$$

so that u is a solution of (6.1). □

We note that Theorem 6.1 does not assert that either the variational inequality or the unilateral minimization problem has a solution. In the present case the bilinear form is coercive (see (5.3)) and we know from Theorem 5.1 that a solution exists and is unique. (Theorem 6.1 with the added assumption of coercivity is given by Lions [1971, p. 9]).

We approximate w by choosing a finite-dimensional approximation K_h and solving the finite-dimensional problem: Find $w_h \in K_h$

$$J(w_h) = \min_{v_h \in K_h} J(v_h) \quad . \tag{6.3}$$

The convex set K_h is constructed as follows. The domain D is triangulated as shown in Figure 6.1.

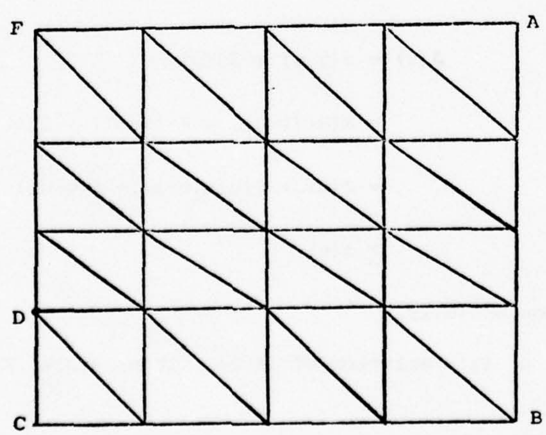

Figure 6.1: The triangulation of D

The subdivisions are not necessarily uniform, but it is assumed that there is a constant $\beta > 0$ such that

$$\frac{1}{\beta}\text{(maximum interval length)} \leq h \leq \beta \text{ (minimum interval length)}, \qquad (6.4)$$

where h is a measure of the fineness of the subdivision. The set of interior gridpoints will be denoted by D_h and the set of boundary gridpoints will be denoted by ∂D_h.

We denote by V_h the space of piecewise linear functions (linear finite elements) v_h corresponding to the triangulation in Figure 6.1. We set

$$K_h = \{v_h \in V_h: v_h \geq 0 \text{ in } D \text{ and } v_h = \Phi \text{ on } \partial D_h\} . \qquad (6.5)$$

Since the functions v_h are linear, v_h is non-negative in D iff v_h is non-negative on $D_h \cup \partial D_h$. Since $\Phi \geq 0$ on ∂D,

$$K_h = \{v_h \in V_h: v_h \geq 0 \text{ on } D_h \text{ and } v_h = \Phi \text{ on } \partial D_h\} . \qquad (6.6)$$

The approximation w_h is readily computed as is shown in section 7. Here, we derive an error estimate for $\|w-w_h\|$ by combining the ideas of Brezzi and Sacchi [to appear] and Brezzi, Hager, and Raviart [to appear].

<u>Theorem 6.2</u>

The piecewise linear approximate solution w_h exists and is unique. Furthermore,

$$\| w-w_h \|_{1,2} = O(h) \quad . \qquad (6.7)$$

<u>Proof</u>: The existence and uniqueness of w_h is an immediate consequence of Theorem 6.1 together with the fact that a is a coercive bilinear form.

We now introduce some notation. For any two functions $g_1, g_2 \in L^2(D)$ we set

$$(g_1, g_2) = \int_D g_1 g_2 \, dxdy \quad . \qquad (6.8)$$

From (5.4), $w \in H^2(D)$, so that w_x and $w_y \in H^1(D)$ and hence (see (4.8)),

$$Lw = \text{div} \exp[f(x) - g(y)] \text{grad } w \in L^2(D) \quad . \qquad (6.9)$$

For any $v_0 \in H_0^1(D)$ we thus have that

$$a(w,v_0) = \int_D \exp[f(x) - g(y)] \text{ grad } w \text{ grad } v_0 \, dxdy \ ,$$

$$= -\int_D v_0 \, Lw \, dxdy \ ,$$

$$= (-Lw, v_0) \ . \tag{6.10}$$

Finally, we note that

$$j(v) = (e,v) \tag{6.11}$$

where

$$e(x,y) = \exp[f(x)] \ . \tag{6.12}$$

If $v \in K$ then $v - w \in H_0^1(D)$ so that using (6.10) and (6.11) the variational inequality for w may be written in the equivalent form

$$(-Lw + e, v-w) \geq 0 \ , \tag{6.13}$$

for all $v \in K$.

In (6.13) we may set $v = w + v_0$ for any non-negative $v_0 \in H_0^1(D)$, from which we conclude that

$$-Lw + e \geq 0 \text{ a.e. in } D \ . \tag{6.14}$$

For any $\varepsilon > 0$ let φ_ε be a smooth non-negative function which is equal to 1 at points in D which are at least a distance 2ε from ∂D and equal to 0 at points less than a distance ε from D. Then, $v_\varepsilon = \varphi_\varepsilon w \in H_0^1(D)$. Setting $v = w + v_\varepsilon$ in (6.13) and letting $\varepsilon \to 0$ we obtain

$$(-Lw + e, w) \geq 0 \ .$$

Setting $v = w - v_\varepsilon$ in (6.13) and letting $\varepsilon \to 0$ we obtain

$$(-Lw + e, -w) \geq 0 \ .$$

Hence,

$$(-Lw + e, w) = 0 \ . \tag{6.15}$$

Finally, by assumption,

$$w \geq 0 \text{ a.e. in } D \ . \tag{6.16}$$

Inequalities (6.14) and (6.16) together with equality (6.15) constitute a complementarity problem for w .

Since $-a(w_h, v_h - w_h) \le (e, v_h - w_h)$ for all $v_h \in K_h$, we see that

$$a(w-w_h, w-w_h) = a(w-w_h, w-v_h) + a(w-w_h, v_h-w_h) \quad ,$$

$$= a(w-w_h, w-v_h) - a(w_h, v_h-w_h) + a(w, v_h-w_h) \quad ,$$

$$\le a(w-w_h, w-v_h) + (e, v_h-w_h) + (-Lw, v_h-w_h) \quad ,$$

$$= a(w-w_h, w-v_h) + (-Lw + e, v_h-w_h) \quad ,$$

$$\le \alpha_2 \|w-w_h\|_{1,2} \|w-v_h\|_{1,2} + (-Lw + e, v_h-w_h) \quad , \tag{6.17}$$

where α_2 is the constant introduced in (5.1). Using (6.14) and (6.15), we conclude that

$$(-Lw + e, v_h-w_h) = (-Lw + e, v_h-w) - (-Lw + e, w_h) + (-Lw + e, w) \quad ,$$

$$= (-Lw+e, v_h - w) - (-Lw + e, w_h) \quad ,$$

$$\le (-Lw + e, v_h - w) \quad ,$$

$$\le \| -Lw + e\|_{0,2} \| v_h - w\|_{0,2} \quad . \tag{6.18}$$

For any $v \in H^2(D)$ let v^I denote the piecewise linear interpolate to v . It follows from the work of Ciarlet and Raviart [1972] that there is a constant C independent of v and h such that

$$\|v - v^I\|_{m,2} \le C \|v\|_{2,2} h^{2-m}, \quad \text{for} \quad m = 0,1 \quad . \tag{6.19}$$

Next, we note that

$$\|w-w_h\|_{1,2} \le \|w-w^I\|_{1,2} + \|w^I-w_h\|_{1,2} \quad .$$

Squaring, and using the inequality $(a+b)^2 \le 2(a^2+b^2)$, we obtain

$$\frac{1}{2}(\|w-w_h\|_{1,2})^2 \le (\|w-w^I\|_{1,2})^2 + (\|w^I-w_h\|_{1,2})^2 \quad . \tag{6.20}$$

Finally, we observe that $w^I - w_h \in H_0^1(D)$ so that, from the coercivity of a on $H_0^1(D)$ (see 5.3),

$$\alpha_1 (\|w^I-w_h\|_{1,2})^2 \le a(w^I-w_h, w^I-w_h) \quad . \tag{6.21}$$

We can now begin the final computations. We set $E = \|w-w_h\|_{1,2}$. Using (6.19), (6.20), and (6.21) we obtain

$$\begin{aligned}
\frac{\alpha_1}{2} E^2 &\le \alpha_1 (C\|w\|_{2,2})^2 h^2 + a(w^I-w_h, w^I-w_h) \quad , \\
&= a(w^I-w_h, w^I-w_h) + O(h^2) \quad , \\
&= a(w^I-w, w^I-w) + 2a(w^I-w, w-w_h) + \\
&\quad + a(w-w_h, w-w_h) + O(h^2) \quad , \\
&\le \alpha_2 (\|w^I-w\|_{1,2})^2 + 2\alpha_2 \|w^I-w\|_{1,2} \|w-w_h\|_{1,2} + \\
&\quad + a(w-w_h, w-w_h) + O(h^2) \quad .
\end{aligned}$$

Using (6.19),

$$\begin{aligned}
\frac{\alpha_1}{2} E^2 &\le \alpha_2 (C\|w\|_{2,2})^2 h^2 + 2\alpha_2 C\|w\|_{2,2} h E + \\
&\quad + a(w-w_h, w-w_h) + O(h^2) \quad , \\
&= a(w-w_h, w-w_h) + 2C_1 h E + O(h^2) \quad ,
\end{aligned}$$

where $C_1 = \alpha_2 C \|w\|_{2,2}$.

Using (6.17) and (6.18) with $v_h = w^I$,

$$\begin{aligned}
\frac{\alpha_1}{2} E^2 &\le \alpha_2 (C\|w\|_{2,2}) h E + (-Lw + e, w^I - w_h) + \\
&\quad + 2C_1 h E + O(h^2) \\
&\le 3C_1 h E + \|-Lw + e\|_{0,2} \|w^I-w_h\|_{0,2} + O(h^2) \quad .
\end{aligned}$$

Using (6.19) with $m = 0$,

$$\begin{aligned}
\frac{\alpha_1}{2} E^2 &\le 3C_1 h E + \|-Lw+e\|_{0,2} C\|w\|_{2,2} h^2 + O(h^2) \quad , \\
&= 3C_1 h E + O(h^2) \quad .
\end{aligned}$$

Thus, multiplying through by $\frac{2}{\alpha_1}$ we have that

$$E^2 \leq 2C_2 h E + O(h^2)$$

with $C_2 = 6C_1/\alpha_1$. Hence,

$$(E - C_2 h)^2 \leq (C_2 h)^2 + O(h^2) = O(h^2) \quad ,$$

so that

$$E - C_2 h = O(h) \quad ,$$

and finally,

$$E = ||w-w_h||_{1,2} = O(h) \quad . \quad \square$$

Remarks

1. There are a number of interesting questions about the convergence of w_h to w:

   (a) How fast does the approximate free boundary converge to the true free boundary? In this connection see Brezzi, Hager, and Raviart [to appear, p. 21] and Brezzi and Sacchi [to appear, p. 9].

   (b) Can one obtain an L^∞ estimate for the error? In this connection see Baiocchi [1976a].

2. We draw attention to a number of related references on the numerical solution of variational inequalities: Falk [1974]; Glowinski [1976]; Glowinski, Lions, and Tremolieres [1976]; Mosco and Strang [1974]; Hager [1976]; Hager and Strang [1975].

3. The functions v_h must satisfy, at least approximately, the boundary conditions $v = \Phi$ on ∂D and the inequality restraints $v \geq 0$ a.e. in D . By formulating the problem in terms of $\tilde{v} = v - \Phi$, the boundary conditions take the simple form $\tilde{v} = 0$ on ∂D, but the inequality constraints take the more complicated form $\tilde{v} \geq -\Phi$. It was, therefore, decided to retain the formulation in terms of v .

7. A numerical example

As an example we consider the specific geometry shown in Figure 7.1, which was chosen because it had previously been considered by several authors.

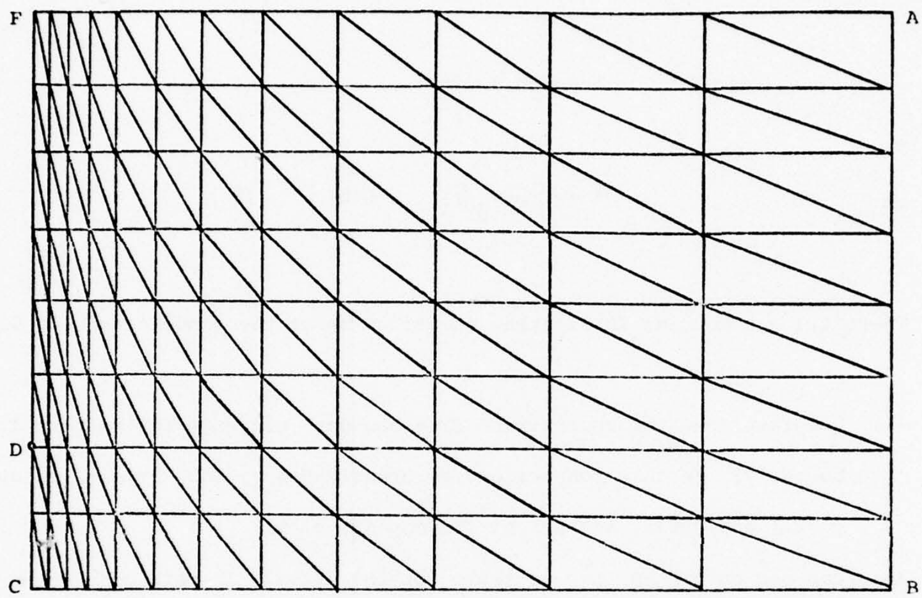

Figure 7.1: A numerical example ($r = 4.8$, $R = 76.8$, $h_w = 12$, $H = 48$, $m = 8$, $n = 12$.)

Because the solution changes most rapidly near the well, the subdivisions were taken to be uniform in the y-direction and logarithmic in the x-direction. If n and m denote the number of subdivisions in the x- and y-directions, the coordinates of the gridpoints were given by

$$y_j = j\,H/m, \qquad 0 \le j \le m\ ,$$

$$x_i = r\,\exp[(i/n)\ln(R/r)], \qquad 0 \le i \le n\ .$$

The integer m was always chosen to be a multiple of 4 so that the corner D was a gridpoint; this was advisable since w is not smooth at the corner D .

The permeability κ was taken to be one so that $f(x) = \ln x$ and $g(y) = 0$. From (4.23), (4.24), and (6.2),

$$J(v) = \int_D x[v_x^2 + v_y^2 + 2v]dxdy \quad . \tag{7.1}$$

From (4.12), (4.13), (4.14), (4.15), and (4.18), the boundary values Φ are

$$\begin{aligned}
&\Phi(x,H) = 0, \quad \text{on } AF\,, \\
&\Phi(R,y) = (H-y)^2/2, \quad \text{on } AB\,, \\
&\Phi(r,y) = (h_w-y)^2/2, \quad \text{on } CD\,, \\
&\Phi(r,y) = 0, \quad \text{on } DF\,, \\
&\Phi(x,0) = \frac{h_w^2 \ln(R/x) + H^2 \ln(x/r)}{2 \ln(R/r)}, \quad \text{on } BC\,.
\end{aligned} \tag{7.2}$$

For $v_h \in K_h$ let $V = \{V_{ij}\}$ denote the $N = (m+1) \times (n+1)$ vector such that

$$V_{ij} = v_h(x_i, y_j) \quad . \tag{7.3}$$

Then

$$J(v_h) = V^T AV + 2b^T V \tag{7.4}$$

where $A = \{A(i,j;\, i,j)\}$ is an $N \times N$ matrix and $b = \{b(i,j)\}$ is an N-vector. The matrix A and vector b are best obtained by computing

$$J_R(v_h) = \int_R x[(v_{h,x})^2 + (v_{h,y})^2 + 2v_h]dxdy \quad , \tag{7.5}$$

$$= V^T A_R V + 2b_R^T V\,, \quad \text{say}, \tag{7.6}$$

for a grid rectangle

$$R = \{(x,y):\ x_i \le x \le x_{i+1},\ \ y_j \le y \le y_{j+1}\}\,, \tag{7.7}$$

and then summing over all grid rectangles.

Consider such a rectangle R which we divide into two triangles, L and U . (See Figure 7.2).

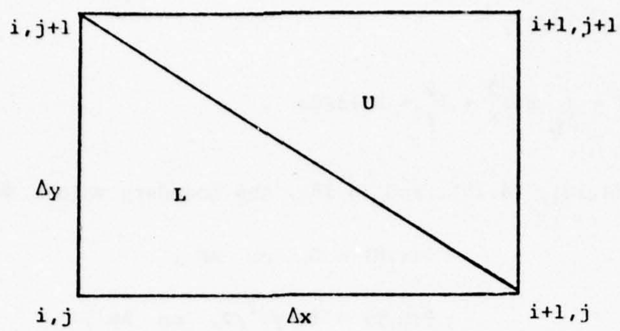

Figure 7.2: A typical grid rectangle R

Then,

On L:

$$v_{h,x} = (V_{i+1,j} - V_{i,j})/\Delta x \ ,$$

$$v_{h,y} = (V_{i,j+1} - V_{i,j})/\Delta y \ , \qquad (7.8)$$

$$v_h = V_{i,j} + (x-x_i)v_{h,x} + (y-y_j)v_{h,y} \ ,$$

On U:

$$v_{h,x} = (V_{i+1,j+1} - V_{i,j+1})/\Delta x \ ,$$

$$v_{h,y} = (V_{i+1,j+1} - V_{i+1,j})/\Delta y \ , \qquad (7.9)$$

$$v_h = V_{i+1,j+1} + (x-x_{i+1})v_{h,x} + (y-y_{j+1})\, v_{h,y} \ .$$

Let X_L and X_U be the x coordinates of the centroids of L and U :

$$x_L = x_i + \Delta x/3 \ , \qquad (7.10)$$

$$x_U = x_i + 2\Delta x/3 \ .$$

By direct computation, the non-zero components of Λ_R are:

$$A_R(i,j;i,j) = \frac{1}{2}\Delta x\,\Delta y\ \ x_L(\Delta x^{-2} + \Delta y^{-2})$$

$$A_R(i+1,j+1;i+1,j+1) = \frac{1}{2}\Delta x\,\Delta y\ x_U(\Delta x^{-2} + \Delta y^{-2}) \ ,$$

$$A_R(i+1,j;i+1,j) = \frac{1}{2}\Delta x\,\Delta y\ x_L\ \Delta x^{-2} + \frac{1}{2}\Delta x\,\Delta y\ x_U\ \Delta y^{-2} \ , \qquad (7.11)$$

$$A_R(i,j+1;i,j+1) = \frac{1}{2}\Delta x\,\Delta y\ x_L\ \ \Delta y^{-2} + \frac{1}{2}\Delta x\,\Delta y\ x_U\ \Delta x^{-2} \ ,$$

$$A_R(i,j;i+1,j) = -\frac{1}{2}\Delta x\,\Delta y\ \ x_L\ \Delta x^{-2} \ ,$$

$$A_R(i,j;i,j+1) = -\frac{1}{2}\Delta x\,\Delta y\ \ x_L\ \Delta y^{-2} \ ,$$

$$A_R(i,j+1;i+1,j+1) = -\frac{1}{2}\Delta x\,\Delta y\ \ x_U\ \Delta x^{-2} \ ,$$

$$A_R(i+1,j;i+1,j+1) = -\frac{1}{2}\Delta x\,\Delta y\ \ x_U\ \Delta y^{-2} \ ,$$

$$A_R(i+1,j;i,j) = A_R(i,j;i+1,j) \ ,$$

$$A_R(i,j+1;i,j) = A_R(i,j;i,j+1) \ ,$$

$$A_R(i+1,j+1;i,j+1) = A_R(i,j+1;i+1,j+1) \ ,$$

$$A_R(i+1,j+1;i+1,j) = A_R(i+1,j;i+1,j+1) \ .$$

The non-zero components of b_R are:

$$b_R(i,j) = \frac{1}{2}\Delta x\,\Delta y\ \ x_L - \frac{1}{2}\Delta x\,\Delta y\ [\frac{x_L}{3} + \frac{2\Delta x}{36}] -$$

$$-\frac{1}{2}\Delta x\,\Delta y[\frac{x_L}{3} - \frac{\Delta x}{36}] \ ,$$

$$b_R(i+1,j) = \frac{1}{2}\Delta x\,\Delta y[\frac{x_L}{3} + \frac{2\Delta x}{36}] +$$

$$+ \frac{1}{2}\Delta x\,\Delta y[\frac{x_U}{3} + \frac{\Delta x}{36}] \ , \qquad (7.12)$$

$$b_R(i,j+1) = \frac{1}{2}\Delta x\,\Delta y[\frac{x_L}{3} - \frac{\Delta x}{36}] +$$

$$+ \frac{1}{2}\Delta x\,\Delta y[\frac{x_U}{3} - \frac{2\Delta x}{36}] \ ,$$

$$b_R(i+1,j+1) = \frac{1}{2}\ \Delta x\ \Delta y\ x_U - \frac{1}{2}\ \Delta x\ \Delta y[\frac{x_U}{3} - \frac{2\Delta x}{36}] -$$

$$- \frac{1}{2}\ \Delta x\ \Delta y[\frac{x_U}{3} + \frac{\Delta x}{36}]\ .$$

The problem

$$\begin{aligned} &\text{Minimize:} && V^T AV + 2b^T V \\ &\text{Subject to:} && V_{ij} = \phi_{ij} \text{ on } \partial D_h, \\ & && V_{ij} \geq 0\ , \end{aligned} \tag{7.13}$$

is a quadratic programming problem which is equivalent to a complementarity problem. Many algorithms are available for the solution of this problem: see Cottle [1974, 1974a], Cottle, Golub, and Sacher [1974], Cottle and Pang [1976], Cottle and Sacher [1973].

Here, we use a variant of S.O.R. (systematic overrelaxation) to solve (7.13). Given an initial guess $V^{(0)}$ satisfying the boundary conditions, we generate a sequence of approximations $V^{(k)}$ using the following ALGOL segment:

```
For i := 1 step 1 until n-1 do
For j := 1 step 1 until m-1 do

        vt1 = [b(i,j) - V(i,j+1)* A(i,j;i,j+1) -                    (7.14)
              - V(i+1,j)* A(i,j;i+1,j) -
              - V(i-1,j)* A(i,j;i-1,j) -
              - V(i,j-1)* A(i,j;i,j-1)]/A(i,j;i,j) ;
        vt2 = V(i,j) + ω* (vt1 - V(i,j)) ;
        v(i,j) = if vt2 ≤ 0 then 0 else vt2 ;
        end;
```

Here, ω is an overrelaxation parameter which must be chosen to optimize the rate of convergence.

The algorithm (7.14) is known to converge. The algorithm, and related algorithms, have been considered by many authors: Hildreth [1957]; Merzljakov [1962]; Fridman and Chernina [1967]; Martinet [1967]; Durand [1968/1969, 1972]; Glowinski [1971,1973];

Cryer [1971]; Comincioli [1971]; Miellou [1971, 1971a, 1972]; Luong [1973]; Martinet and Auslender [1974]; Eckhardt [1974]; Mangasarian [1976].

To choose the parameter ω we proceeded as follows. For the case $m = 16$, $n = 24$ computations were made with several values of ω and it was found that the optimum value of ω was approximately 1·7 . For the general case we set

$$\omega = \frac{2}{1+\sin[\pi/nfict]}$$

where

$$nfict = \cdot 8590 \; [(n+1)(m+1)]^{1/2} \quad .$$

The expression for ω is a modification of the theoretical optimim value for S.O.R. on a square (Varga [1962, p. 203]). The constant ·8590 was chosen so that $\omega = 1{\cdot}7$ when $m = 16$ and $n = 24$.

The computations presented no difficulties. The solution of the smallest problem is given in Table 7.1.

In Table 7.1 the position of the approximate free boundary is shown by the first zero term in each column. The approximate solution is identically zero on the vertical line $x = r$ so that it is not possible to determine the height h_s at which the free boundary intersects the well. As an approximation to h_s we take the height h_s' of the free boundary at the vertical gridline adjacent to the well. For example, from Table 7.1 we obtain $h_s' = 36$.

In Table 7.2 the values of h_s' for a sequence of decreasing grid lengths are given. The iterations were terminated when,

$$\|V^{10\ell} - V^{10(\ell-1)}\|_\infty < 10^{-6} \quad .$$

In judging this accuracy criterion, it should be remembered that $\|V\|_\infty = 1152$, so that the relative error is 10^{-9} .

x

y	4.8000000,+00	7.6195250,+00	1.2095242,+01	1.9200000,+01	3.0478100,+01	4.8380968,+01	7.6800000,+01
48.00	0	0	0	0	0	0	0
36.00	0	0	0	0	3.0332990,+00	3.1593876,+01	7.2000000,+01
24.00	0	1.7554644,+01	4.3617047,+01	8.1333841,+01	1.3309081,+02	2.0406162,+02	2.8800000,+02
12.00	0	8.8585977,+01	1.8202316,+02	2.8314043,+02	3.9437102,+02	5.1710713,+02	6.4800000,+02
0.0000	7.2000000,+01	2.5200000,+02	4.3200000,+02	6.1200000,+02	7.9200000,+02	9.7200000,+02	1.1520000,+03

Table 7.1: Solution for $m = 4$, $n = 6$

m	n	h'_s	number of iterations
4	6	36.00	20
8	12	30.00	40
16	24	30.00	70
32	48	30.00	120
64	96	30.00	230

<u>Table 7.2</u>: <u>Values of h'_s</u>

The method of determining ω seemed satisfactory, although the ratios

$$\frac{\|V^{10k} - V^{10(k-1)}\|_2}{\|V^{10(k-1)} - V^{10(k-2)}\|_2}$$

oscillated so as to suggest that the dominant eigenvalue of the iteration was complex and hence that (from the theory of S.O.R.) reducing ω would improve the convergence.

For comparison, we compare in Table 7.3 the values of h_s obtained by different authors. With the exception of the present computation, all the results are presented graphically so that we have had to estimate h_s from graphs.

Author	Method	h_s
Hall [1955, p. 29]	trial-free-boundary; finite differences	34.0
Taylor and Luthin [1969]	time-dependent; finite differences	34.0
Neuman and Witherspoon [1970]	trial free-boundary; finite elements	30.0
Neuman and Witherspoon [1971, p. 620]	time-dependent; finite differences	30.0
Present	Variational inequalities	30.0

<u>Table 7.3</u>: <u>Computed values of h_s</u>

The differences in Table 7.3 may be explained by the fact that the physical assumptions differed: Hall [1955] assumed capillarity and a lined well; Taylor and Luthin [1969] assumed partially saturated flow; and Neuman and Witherspoon [1970, 1971] made the same assumptions as in the present paper.

Finally, in Table 7.4 we give the computed values of w at a typical point $x = 19.2$ $y = 24$ for different values of m and n. We also give the differences between successive approximations and the ratios of successive differences. It can be seen that the results are consistent with the hypothesis that

$$w-w_h = O(h^2) .$$

In Theorem 6.2 we, of course, only proved that

$$\|w-w_h\|_{1,2} = O(h) .$$

m	n	w_h	Δw_h	ratio
4	6	81.333841		
8	12	83.478767	2.144926	
16	24	84.150541	.671774	3.193
32	48	84.298332	.147791	4.545
64	96	84.337269	.038930	3.796

Table 7.4: Values of w_h at $x = 19.2$, $y = 24.0$

In conclusion we draw attention to some related work:

Numerical solution of variational inequalities for porous flow

Comincioli [1974, 1974a, 1974b, 1975],

Comincioli, Guerri, and Volpi [1971],

Cottle [1974].

Numerical solution of axisymmetric well problems

Babbit and Caldwell [1948];

Yang [1949]; Boulton [1951]; Kashef, Touloukham, and Fadum [1952];

Kashef [1953]; Schmidt [1956]; Murray [1960]; Kirkham [1964]; Taylor [1966];

Taylor and Brown [1967] (see also Kealy and Busch [1971]; Herbert [1968]; Mauersberger [1967, 1968, 1968a, 1968b, 1968c]; France, Parekh, Peters, and Taylor [1971].

Appendix A: An alternative numerical approach

Brezzi and Sacchi [to appear] have given a different error analysis for the case of plane flow through a rectangular dam.

The approach of Brezzi and Sacchi requires that K_h be chosen so that for every $v_h \in K_h$ there is a $v \in K$ with $v \le v_h$. In this appendix we show how one can choose K_h so that this condition is satisfied.

In section 6 the approximation was obtained by triangulating D and approximating w by piecewise linear functions, the boundary conditions being satisfied approximately by interpolation at the nodes. However, in the special case when

$$e^{f(x)} = x ,$$

we have from (7.2) that, on BC,

$$\Phi(x,0) = [\Phi(R,0) \ln(x/r) + \Phi(r,0) \ln(R/x)]/\ln(R/r) ,$$

so that

$$\Phi_{xx}(x,0) = -[\Phi(R,0) - \Phi(r,0]/x^2 \ln(R/r), \\ < 0 .$$

Thus, Φ is concave on BC and any linear interpolate v_h lies below any $v \in K$. Therefore, even in this case, the condition of Brezzi and Sacchi is not satisfied.

To overcome this difficulty, we introduce new coordinates α and β defined by

$$\alpha = \int_r^x e^{-f(t)} \, dt, \quad \beta = \int_0^y e^{g(t)} \, dt . \tag{A.1}$$

The rectangle D is transformed into the rectangle D':

$$\begin{aligned} 0 < \alpha < \int_r^R e^{-f(t)} \, dt = \alpha_R, \quad \text{say}, \\ 0 < \beta < \int_0^H e^{g(t)} \, dt = \beta_R, \quad \text{say}. \end{aligned} \tag{A.2}$$

We have that

$$\begin{aligned} w_x = w_\alpha \, \alpha_x = w_\alpha \, e^{-f(x)} , \\ w_y = w_\beta \, \beta_y = w_\beta \, e^{g(y)} . \end{aligned} \tag{A.3}$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{vmatrix},$$

$$= \begin{vmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{vmatrix}^{-1},$$

$$= \begin{vmatrix} e^{-f(x)} & 0 \\ 0 & e^{g(y)} \end{vmatrix}^{-1},$$

$$= e^{f(x) - g(y)} . \tag{A.4}$$

Thus the minimization problem (6.2) is equivalent to the problem:

$$\min_{v' \in K'} J'(v') , \tag{A.5}$$

where

$$K' = \{v' \in H^1(D') : \ v' - \Phi' \in H_0^1(D') \text{ and } v' \geq 0 \text{ a.e. in } D'\} , \tag{A.6}$$

$$J'(v') = a'(v',v') + 2\, j'(v') , \tag{A.7}$$

$$a'(w',v') = \iint_{D'} (e^{-2g(y)} w'_\alpha v'_\alpha + e^{2f(x)} w'_\beta v'_\beta) d\alpha d\beta , \tag{4.8}$$

$$j'(v') = \iint_{D'} e^{2f(x)-g(y)} v' \, d\alpha d\beta . \tag{A.9}$$

We note that $f(x)$ and $g(y)$ are continuous so that the mapping $(x,y) \to (\alpha,\beta)$ is 1-smooth and $H^1(D)$ is mapped homeomorphically onto $H^1(D')$ (Adams [1975, p. 63]).

We can obtain an approximate solution w'_h by proceeding as in section 6. The rectangle D' is triangulated in the same way as shown in Figure 6.1. The subdivisions are not uniform, but the ratio(interval length)/h is bounded both above and below. The interior gridpoints are denoted by D'_h and the boundary gridpoints by $\partial D'_h$. The space of piecewise linear functions on D'_h is denoted by V'_h.

The approximation w'_h is then given by:

$$J'(w'_h) = \min_{v'_h \in K'_h} J'(v'_h) , \tag{A.10}$$

where

$$K'_h = \{v'_h \in V'_h : \; v'_h \geq 0 \text{ in } D'_h$$

$$\text{and } v'_h = \Phi' \text{ on } \partial D'_h\} . \tag{A.11}$$

<u>Lemma A.1</u>

If $v'_h \in K'_h$ then $\Phi' - v'_h \leq 0$ on $\partial D'$.

<u>Proof</u>: On Γ'_4 we have from (4.18) that

$$\Phi'(\alpha,0) = \frac{1}{\alpha_R} [\alpha \, \Phi(R,0) + (\alpha_R - \alpha) \quad \Phi(r,0)] ,$$

so that Φ' is linear and hence $v'_h = \Phi'$ on Γ'_4.

On Γ'_1 we have from (4.13) that

$$\Phi'(\alpha_R,\beta) = \text{Constant} - \int_0^y e^{g(t)} (H-t)dt ,$$

$$= \text{Constant} - H\beta + \int_0^y e^{g(t)} \, t \, dt .$$

Let (α_R,β_1) and (α_R,β_2) be two adjacent nodes on Γ'_1 corresponding to (R,y_1) and (R,y_2) on Γ_1. We may assume that $\beta_2 > \beta_1$. Then, for $\beta_1 \leq \beta \leq \beta_2$,

$$\Phi'(\alpha_R,\beta) - v_h(\alpha_R,\beta) = \int_{y_1}^{y} e^{g(t)} (t - c)dt ,$$

$$= E(y), \text{ say} ,$$

where the constant c is such that $E(y_2) = 0$. We conclude that $y_1 < c < y_2$.

Thus, $E(y)$ decreases monotonically for $y_1 \le y \le c$ and increases monotonically for $c \le y \le y_2$, so that $E(y) \le 0$.

Similar arguments show that $\Phi' - v_h' \le 0$ on Γ_2', while $\Phi' - v_h' = 0$ on $\Gamma_3' \cup \Gamma_5'$. □

Theorem A.2

The approximate solution w_h' exists and is unique. Furthermore,

$$\|w'-w_h'\|_{1,2} = O(h) \ .$$

Proof: The existence and uniqueness of w_h' is a consequence of the fact that the bilinear form a' is coercive.

Remembering that

$$-a'(w',v'-w') \le j'(v'-w'), \text{ for all } v' \in K' ,$$

and

$$-a'(w_h',v_h'-w_h') \le j'(v_h'-w_h'), \text{ for all } v_h' \in K_h' ,$$

we see that for all $v' \in K'$ and $v_h' \in K_h'$,

$$\begin{aligned} a'(w'-w_h', w'-w_h') &= -a'(w',w_h'-w') - a'(w_h',w'-w_h') , \\ &= -[a'(w',v'-w') + a'(w',w_h'-v')] - \\ &\quad -[a'(w_h',v_h'-w_h') + a'(w_h',w'-v_h')] , \\ &\le j'(v'-w') - a'(w',w_h'-v') + \\ &\quad + j'(v_h'-w_h') - a'(w_h',w'-v_h') . \\ &= -j'(w'-v'+w_h'-v_h') - a'(w',w_h'-v') - \\ &\quad -a'(w_h', w'-v_h') . \end{aligned}$$

Adding and subtracting the term $a'(w',w'-v_h')$ from the right hand side we obtain (following Brezzi and Sacchi [to appear]),

$$\begin{aligned} a'(w'-w_h', w'-w_h') &\le -j'(w'-v' + w_h'-v_h') - \\ &\quad -a'(w', w'-v' + w_h' - v_h') + \\ &\quad +a'(w'-w_h', w'-v_h') , \end{aligned}$$

so that

$$a'(w'-w_h',w'-w_h') \leq [-j'(w'-v_h') - a'(w',w'-v_h')] +$$
$$+ [-j'(w_h'-v') - a'(w',w_h'-v')] + \quad (A.12)$$
$$+ a'(w'-w_h', w'-v_h') \quad ,$$

for all $v_h' \in K_h'$ and $v' \in K'$.

The remainder of the proof is very similar to the proof of Theorem 6.2.

By (5.4), $w \in H^2(D)$. We have assumed that f and g are continuously differentiable so that the mapping $(x,y) \to (\alpha,\beta)$ is 2-smooth and $H^2(D)$ is mapped homeomorphically onto $H^2(D')$ (Adams [1975, p. 63]). Thus,

$$w' \in H^2(D') \quad . \quad (A.13)$$

Let w'^I denote the piecewise linear interpolant to w' . From Theorem 5 of Ciarlet and Raviart [1972] we conclude that

$$\|w' - w'^I\|_{m,2} \leq C \|w'\|_{2,2} h^{2-m}, \quad \text{for} \quad m = 0,1 \quad . \quad (A.14)$$

Using the same arguments and notation as in Theorem 6.2 we obtain that

$$a'(w',v_0') = -(L'w',v_0') \quad , \quad (A.15)$$

for all $v_0' \in H_0^1(D')$, where

$$L'w' = \frac{\partial}{\partial\alpha}(e^{-2g(y)} w_\alpha') + \frac{\partial}{\partial\beta}(e^{2f(x)} w_\beta') \in L^2(D') \quad . \quad (A.16)$$

Let
$$j'(v') = (e',v') \quad , \quad (A.17)$$

where
$$e'(\alpha,\beta) = \exp[2f(x) - g(y)] \quad . \quad (A.18)$$

Then the identities (A.15) and (A.17) can be used to manipulate (A.12). We obtain

$$a'(w'-w_h',w'-w_h') \leq [-j'(w'-v_h') - a'(w',w'-v_h')] +$$
$$+ [-j'(w_h'-v') - a'(w',w_h'-v')] +$$
$$+ a'(w'-w_h', w'-v_h') \quad ,$$

$$= -[j'(w'-v'_h) + j'(w'_h-v')] -$$

$$- [a'(w',w'-v') + a'(w', w'_h-v'_h)] +$$

$$+ a'(w'-w'_h, w'-v'_h) ,$$

$$= -[(e',w'-v'_h) + (e',w'_h-v')] -$$

$$-[(-Lw',w'-v') + (-Lw',w'_h-v'_h)] +$$

$$+ a'(w'-w'_h, w'-v'_h) ,$$

$$= -(-Lw' + e', w'-v'_h) -$$

$$-(-Lw'+e', w'_h-v') +$$

$$+ a'(w'-w'_h, w'-v'_h) \tag{A.19}$$

where in the last step but one we have used the fact that $w'-v'$ and $w'_h-v'_h$ belong to $H^1_0(D')$.

In (A.19) we now set $v'_h = w'^I$. By Lemma A.1 we have that $\Phi' - v'_h \leq 0$ on $\partial D'$ so that $v' - v'_h \leq 0$ on $\partial D'$ for all $v' \epsilon K'$. Since v'_h is non-negative, we may choose $v' \epsilon K'$ so that $v' - v'_h \leq 0$ on D'. Analogously to (6.14) we can show that

$$-L'w' + e' \geq 0 \quad a.e. \; in \; D' .$$

Thus, in (A.19),

$$-(-L'w'+e', w'_h-v') = -(-Lw'+e', w'^I-v') \leq 0 .$$

Noting (A.13) and (A.14) we thus conclude that

$$a'(w'-w'_h, w'-w'_h) \leq \|-Lw'+e'\|_{1,0} \, C \, \|w'\|_{2,2} \, h^2 +$$

$$+ \alpha'_2 \, \|w'-w'_h\|_{1,2} \, C \, \|w'\|_{2,2} \, h ,$$

$$= C'_1 \|w'-w'_h\|_{1,2} \, h + O(h^2) , \tag{A.20}$$

where α'_2 is the *coercivity constant for* a' and

$$C'_1 = \alpha'_2 \|w'\|_{2,2} \, C .$$

As in (6.20) and (6.21) we have that

$$\frac{1}{2}\left(\|w'-w_h'\|_{1,2}\right)^2 \le \left(\|w'-w'^I\|_{1,2}\right)^2 + \left(\|w'^I-w_h'\|_{1,2}\right)^2 ,$$

$$\alpha_1'\left(\|w'^I-w_h'\|_{1,2}\right)^2 \le a'(w'^I-w_h',\ w'^I-w_h') .$$

Setting $E' = \|w'-w_h'\|_{1,2}$ and using the same approach as in the proof of Theorem 6.2 we find that

$$\begin{aligned}\frac{\alpha_1'}{2}(E')^2 &\le a'(w'^I-w_h',\ w'^I-w_h') + O(h^2) ,\\ &\le a'(w'-w_h',\ w'-w_h') + 2C_1'\, h\, E' + O(h^2) ,\\ &\le 3\, C_1'\, h\, E' + O(h^2) ,\end{aligned}$$

from which it follows that $E' = O(h)$. □

Appendix B: The computer program

The computer program used to obtain the numerical results quoted in Section 7 is listed below. Two minor remarks are perhaps necessary:

1. The main program is written in ALGOL. Since the ALGOL compiler available to us does not optimize, the inner S.O.R. loop is executed by a FORTRAN subroutine.

2. The computations were performed in double precision so that the asymptotic behavior of the error, as shown in Table 7.4, would not be contaminated by round-off errors.

```
@NUALG,ISZX
BEGIN
COMMENT ****************************************************************
          PROGRAM FOR THE NUMERICAL SOLUTION , BY MEANS OF THE
          VARIATIONAL METHOD PROPOSED BY BAIOCCHI ET AL.,
          OF THE FREE BOUNDARY PROBLEM RELATED TO THE STATIONARY FLOW
          THROUGH A FULLY PENETRATING WELL


          VARIABLE NAMES USED IN THE PROGRAM INCLUDE THE FOLLOWING :
          -------------------------------------------------------------
          *Y1* IS THE HEIGHT OF THE WATER AT X = B
          *Y2* IS THE HEIGHT OF THE WATER AT X = A
          *B* IS THE RADIUS OF THE CATCHMENT AREA OF THE WELL
          *A* IS THE RADIUS OF THE WELL
          *M* AND *N* ARE THE NUMBERS OF POINTS OF SUBDIVISION OF THE
          SIDES OF THE RECTANGLE *R*,
          *OMEGA* IS THE RELAXATION PARAMETER,
          *EPS* IS USED IN THE TEST FOR STOPPING THE ITERATION,
          ARRAY *U* SHALL CONTAIN THE FUNCTION VALUES OF THE DISCRETE
          SOLUTION.
          *************************************************************;
   EXTERNAL FORTRAN PROCEDURE ITERAT;
   COMMON COEF0 (REAL2 ARRAY C0 (0:48,0:32););
   COMMON COEF1 (REAL2 ARRAY C1 (0:48,0:32););
   COMMON COEF2 (REAL2 ARRAY C2 (0:48,0:32););
   COMMON COEF3 (REAL2 ARRAY C3 (0:48,0:32););
   COMMON UNK (REAL2 ARRAY U(0:48,0:32););
   COMMON PAR (INTEGER IMAX,N,M,MODIT;REAL2 OMEGA,TEST,TEST2;);
   REAL2 ARRAY R (0 : 48);
   REAL2 ARRAY S (0 : 32);
   REAL2 DELTAX,DELTAY;
   REAL2 NFICT;
   REAL2 A,Y1,Y2,EPS,H1,H2;
   REAL2 A1,B1,A1B1;
   REAL2 B,AB,DENOM;
   REAL2 MU,LAMBDA,PI;
   REAL2 TEST2P,RATIO;
   REAL2 T1,T2,R2,RS;
   REAL2 T1P,R2P,RSP;
   INTEGER I,J,K,ITER,LB,UB;
   INTEGER NOCOLS,NP;
COMMENT  ***************************************************************;
   FORMAT FMT1 (E1,X10,'NUMBER OF ITERATIONS = ',I4,A1.0,
             X11,7(R15.8,X1),A1.0,
             X11,7(15('-'),X1),A1.0),
      FMT2 (X2,D6.4,X3,7(R15.8,X1),A1.0),
      FMT4 (X5,'ITER = ',I3,'  TEST = ',R14.7,' L2 NORM = ',R14.7,
         ' RATIO = ',R14.7,A1.0),
      FMT0 (E1,X5,' RW = ',R12.4,' RE = ',R12.4,A1.0,
            X5,' HW = ',R12.4,' HE = ',R12.4,A1.0,
            X5,' NX = ', I12 ,' NY = ', I12 ,A1.0,
            X5,'EPS = ',R12.4,' OM = ',R12.4,A1.0),
```

```
        FMT3 (X5,'***',X2,'TEST = ', R14,7, '***',A1,0);
COMMENT  ------------ GIVEN QUANTITIES ------------------------------------;
  READ (CARDS,M,N,A,AB,Y1,Y2,EPS);
  IMAX := 48 + 1;
  PI := 4,0&&0 * ARCTAN (1,0&&0);
  A1 := LN (A/A);
  B := A + AB;
  B1 := LN (B/A);
  A1B1 := B1 - A1;
  H1 := A1B1/N;
  H2 := Y1 / M;
COMMENT  *H1* AND *H2* ARE THE STEPSIZES FOR THE DISCRETIZATION;
  NFICT := 0,8590&&0 * SQRT (1,0&&0 * (N+1) * (M+1));
  OMEGA := 2,0&&0/(1,0&&0 + SIN (PI/NFICT));
COMMENT  COMPUTE OVERRELAXATION PARAMETER OMEGA               ;
  FOR I := 0 STEP 1 UNTIL N    DO
  R(I) := A *EXP (I*H1);
  FOR J := 0 STEP 1 UNTIL M DO
  S (J) := J*Y1/M;
  WRITE (PRINTER,FMT0,A ,B ,Y2,Y1,N,M,EPS,OMEGA);
  ITER := 0;
  TEST := 1,0&&0;
  TEST2 := 0,0&&0;
  TEST2P := 1,0&&0;
COMMENT ****************************************************************
        INITIALIZE *U(I,J)*
        ON THE BOUNDARY OF THE RECTANGLE *R* , U(I,J) IS GIVEN BY
        THE FUNCTION G(X,Y) (CF, BAIOCCHI ET AL,)
        EVERYWHERE ELSE WE SIMPLY SET IT TO SOME CONSTANT GE 0 ;
COMMENT  ----------ON THE SEGMENT (A,B) -------------------------------;
  FOR I := 0 STEP 1 UNTIL N       DO
  U(I,0) := 0,5&&0*(Y1**2*LN(R(I)/A)+Y2**2*LN(B/R(I)))/B1;
COMMENT  ----------ON THE SEGMENT (B,C) -------------------------------;
  FOR J := 1 STEP 1 UNTIL M       DO
  U(N,J) := 0,5&&0 * (Y1 - Y1 * J / M) **2;
COMMENT  ----------ON THE SEGMENT (A,D) -------------------------------;
  K := ENTIER (Y2 / H2) ;
  WRITE (PRINTER,<<I3 ,A1>>,K);
  FOR J := 1 STEP 1 UNTIL K DO
  U (0,J) := 0,5&&0 * (Y2 - Y1 * J/ M) **2 ;
COMMENT  ---INITIALIZE U (I,J) EVERYWHERE BY LINEAR INTERPOLATION -----;
  FOR I := 1 STEP 1 UNTIL N-1 DO
  FOR J := 1 STEP 1 UNTIL M-1 DO
  BEGIN
    LAMBDA := (R(I) - R(0)) / (R(N) - R(0));
    U (I,J) := U(N,J) * LAMBDA + U(0,J) * (1,0&&0 - LAMBDA);
  END;
COMMENT  **************************************************************;
  FOR I := 0 STEP 1 UNTIL N - 1 DO
  FOR J := 0 STEP 1 UNTIL M - 1 DO
  BEGIN
    DELTAX := R(I+1) - R(I);
    DELTAY := S(J+1) - S(J);
COMMENT  (LOWER TRIANGLE)     ----------------------------------------;
```

```
      T1 := (R(I) + 1 * DELTAX / 3.0&&0) * DELTAX * DELTAY / 2;
      T2 := DELTAX **2 * DELTAY /36.0&&0;
      R2 := DELTAX * T2 + (R(I) + 1 * DELTAX / 3.0&&0) * T1;
      RS := - DELTAY * T2 / 2 + (S(J) + 1 * DELTAY / 3.0&&0) * T1;
    C0(I   ,J  )    :=C0(I   ,J  )    + T1/ DELTAX **2;
    C1(I   ,J  )    :=C1(I   ,J  )    - T1/ DELTAX **2;
    C0(I   ,J  )    :=C0(I   ,J  )    + T1/ DELTAY **2;
    C2(I   ,J  )    :=C2(I   ,J  )    - T1/ DELTAY **2;
    C3(I   ,J  )    :=C3(I   ,J  )    + T1 - (R2 - R(I)*T1)/ DELTAX
                       -(RS - S(J)*T1)/ DELTAY;
    C0(I+1,J  )     :=C0(I+1,J  )     + T1/ DELTAX **2;
    C3(I+1,J  )     :=C3(I+1,J  )     +(R2 - R(I)*T1)/ DELTAX;
    C0(I   ,J+1)    :=C0(I   ,J+1)    + T1/ DELTAY **2;
    C3(I   ,J+1)    :=C3(I   ,J+1)    +(RS - S(J)*T1)/ DELTAY;
COMMENT  (UPPER TRIANGLE)     ----------------------------------------;
      T1P:= (R(I)  +2 * DELTAX / 3.0&&0) * DELTAX * DELTAY / 2;
      R2P:= DELTAX * T2 + (R(I)  +2 * DELTAX / 3.0&&0) * T1P;
      RSP:= - DELTAY * T2 / 2 + (S(J)  +2 * DELTAY / 3.0&&0) * T1P;
    C0(I+1,J+1)     :=C0(I+1,J+1)     +T1P/ DELTAX **2;
    C0(I+1,J+1)     :=C0(I+1,J+1)     +T1P/ DELTAY **2;
    C3(I+1,J+1)     :=C3(I+1,J+1)     + T1P+ (R2P- R(I+1)*T1P)/DELTAX
                       +(RSP- S(J+1)*T1P)/DELTAY;
    C0(I   ,J+1)    :=C0(I   ,J+1)    + T1P / DELTAX**2;
    C1(I   ,J+1)    :=C1(I   ,J+1)    - T1P / DELTAX**2;
    C3(I   ,J+1)    :=C3(I   ,J+1)    -(R2P - R(I+1)*T1P) / DELTAX;
    C0(I+1,J  )     :=C0(I+1,J  )     + T1P / DELTAY**2;
    C2(I+1,J  )     :=C2(I+1,J  )     - T1P / DELTAY**2;
    C3(I+1,J  )     :=C3(I+1,J  )     -(RSP - S(J+1)*T1P) / DELTAY;
   END;
COMMENT  ***************************************************************;
   FOR ITER := ITER + 1 WHILE TEST GEQ EPS AND ITER LEQ 400 DO
   BEGIN
      MODIT := MOD (ITER,10);
      ITERAY;
      IF MODIT EQL 0 THEN
      BEGIN
         TEST2 := SQRT (TEST2/((N-1)*(M-1)));
         RATIO := (TEST2/TEST2P)**0.1;
         WRITE (PRINTER,FMT4,ITER,TEST,TEST2,RATIO);
         TEST2P := TEST2;
         TEST2 := 0.0&&0;
      END;
   END;
COMMENT  ***************************************************************;
   UB := 0;
   LB := 1;
   ITER := ITER - 1;
   NOCOLS := N + 8;
   FOR NOCOLS := NOCOLS - 7 WHILE NOCOLS GTR 0 DO
   BEGIN
      NP := MIN (NOCOLS,7);
      UB := UB + NP;
      WRITE (PRINTER,FMT1,ITER,FOR I := (LB,1,UB) DO R(I-1));
      FOR J := M STEP -1 UNTIL 0 DO
```

```
    WRITE (PRINTER,FMT2, S(J) ,FOR I := (LB,1,UB) DO U(I-1,J));
    LB := UB + 1 ;
  END;
  WRITE(PRINTER,FMT3,TEST);
END
@FOR,ISZX ITERAT
      SUBROUTINE ITERAT
      IMPLICIT DOUBLE PRECISION (C,D,T,U,V)
      COMMON/COEF0/C0(1)
      COMMON/COEF1/C1(1)
      COMMON/COEF2/C2(1)
      COMMON/COEF3/C3(1)
      COMMON/UNK/U(1)
      COMMON/PAR/IMAX,N,M,MODIT,OMEGA,TEST,TEST2
      IF (MODIT .EQ. 0 ) TEST = 0.0D0
      DO 1 J = 2,M
      DO 1 I = 2,N
      IJ = I + IMAX * (J-1)
      IM1J = IJ - 1
      IP1J = IJ + 1
      IJM1 = IJ - IMAX
      IJP1 = IJ + IMAX
      UOLD = U(IJ)
      UNEW = -(C3(IJ) + C1(IJ)*U(IP1J) + C2(IJ) *U(IJP1)
     1    + C1(IM1J)*U(IM1J) + C2(IJM1)*U(IJM1)) / C0(IJ)
      VINT = (1.0D0 - OMEGA) * UOLD + OMEGA * UNEW
      U (IJ) = DMAX1 (VINT,0.0D0)
      IF (MODIT .NE. 0) GO TO 1
      VABS = DABS (U(IJ) - UOLD)
      TEST = DMAX1 (TEST,VABS)
      TEST2 = TEST2 + VABS**2
    1 CONTINUE
      RETURN
      END
@XQT
  4  6  4.8&&0 72.0&&0 48.0&&0 12.0&&0 1.0&&-6
@XQT
  8 12  4.8&&0 72.0&&0 48.0&&0 12.0&&0 1.0&&-6
@XQT
 16 24  4.8&&0 72.0&&0 48.0&&0 12.0&&0 1.0&&-6
@XQT
 32 48  4.8&&0 72.0&&0 48.0&&0 12.0&&0 1.0&&-6
```

Appendix C: The strong form of Green's theorem

In deriving variational inequalities it is often necessary to integrate by parts (i.e. use Green's theorem). The weak form of Green's theorem (for domains bounded by piecewise smooth boundaries) is well known (Kellogg [1953, p. 84]). The strong form of Green's theorem (for domains bounded by Jordan curves) is less well known. Since it is desirable to make the weakest possible a-priori assumptions about the free boundary, the strong form of Green's theorem is of value and we therefore describe it here.

We recall that a closed Jordan curve is a mapping

$$s \to z(s) = (x(s), y(s))$$

of the interval $[0,1]$ into the xy-plane such that $z(0) = z(1)$ and $z(s_1) = z(s_2)$ iff either $s_1 = s_2$ or $s_1 = 0$ and $s_2 = 1$. The curve is rectifiable if the mapping is of bounded variation; that is, there exists a constant L such that for all subdivisions

$$0 = s_0 < s_1 < \ldots < s_n = 1 ,$$

$$\sum_{i=0}^{n-1} |z(s_{i+1}) - z(s_i)| = \sum_{i=0}^{n-1} [(x(s_{i+1}) - x(s_i))^2 + (y(s_{i+1}) - y(s_i))^2]^{1/2}$$
$$\leq L .$$

If J is a closed rectifiable Jordan curve then $x(s)$ and $y(s)$ *are of bounded variation* so that the Riemann-Stieltjes integrals

$$\int_J f\,dx \quad \text{and} \quad \int_J f\,dy$$

are defined for every continuous function f .

Now let Ω be a bounded domain in the xy-plane with boundary $\partial\Omega$ which is a closed rectifiable Jordan curve. Green's formula takes the form

$$\iint_\Omega \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)dxdy = \int_{\partial\Omega} Mdy + Ndx \quad ,$$

where the integral over $\partial\Omega$ is taken in the positive direction around $\partial\Omega$. If $x(s)$ and $y(s)$ are absolutely continuous then the formula takes the more familiar form

$$\iint_\Omega \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)dxdy = \int_{\partial\Omega} (M\dot{y} + N\dot{x})ds \quad ,$$

$$= \int_{\partial\Omega} (Mn_x - Nn_y)d\ell \quad .$$

where ℓ denotes length and (n_x, n_y) is the unit outward normal which is defined a.e.

To establish conditions under which Green's formula holds, it suffices to consider the case when one of the functions, N say, is zero.

Theorem C.1

Let M be continuous on $\overline{\Omega}$. Assume that

(i) $\frac{\partial M}{\partial x}$ exists a.e. and is Lebesgue summable ($\frac{\partial M}{\partial x}$ exists everywhere and is Riemann integrable.)

(ii) $$\iint_R \frac{\partial M}{\partial x}\, dxdy = \int_{\partial R} Mdy$$

for every rectangle R: $a < x < A$, $b < y < B$ contained in Ω, where the integral over R is taken in the sense of Lebesgue (Riemann).

Then Green's formula holds:

$$\iint_\Omega \frac{\partial M}{\partial x}\, dxdy = \int_{\partial\Omega} Mdy \quad ,$$

the integrals being in the sense of Lebesgue (Riemann). □

Proofs of the above theorem are given by Verblunsky [1949] for the Lebesgue case and Potts [1951] for the Riemann case; these authors give references to earlier proofs.

If condition (i) of the theorem is assumed then sufficient conditions for condition (ii) to hold are:

(α) M *is absolutely continuous in* $a \leq x \leq A$ *for almost all* y in (b,B) (Lebesgue case).

(β) The integral $\int_{\partial R} Mdy$ exists, (Riemann case).

In particular, if u is absolutely continuous on Ω and continuous on $\overline{\Omega}$, and and if $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ belong to $L^2(\Omega)$, then

$$\iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \int_{\partial\Omega} udy + udx \ .$$

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The free boundary problem for a fully penetrating well of radius r and filled with water to a depth h, in a layer of soil of depth H, radius R and permeability $k(x,y)$ can be formulated as follows: Find $\varphi \in C^1[r,R]$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $(xku_x)_x + (xku_y)_y = 0$ in Ω, $u(R,y) = H$ for $0 < y < H$, $u_n(x,0) = 0$ for $r < x < R$, $u(r,y) = h$ for $0 < y < h$, $u(r,y) = y$ for $h < y < \varphi(r)$, $u_n(x,\varphi(x)) = 0$ and $u(x,\varphi(x)) = \varphi(x)$ for $r < x < R$, where $\Omega = \{(x,y): r < x < R, 0 < y < \varphi(x)\}$. The results of Benci [Annali di Mat. 100(1974), 191-209] are used to derive a variational inequality and to prove existence and uniqueness. The problem is approximated using piecewise linear finite elements and O(h) convergence of the approximate solutions is proved using recent results due to Brezzi, Hager, and Raviart.

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