AD-A047 307 POLYTECHNIC INST OF NEW YORK BROOKLYN DEPT OF ELECTR--ETC F/G 14/4
CONTRIBUTION TO THE THEORY OF COMPONENT AVAILABILITY.(U)
FEB 76 C MARSHALL N00014-67-A-0438-0013
UNCLASSIFIED POLY-EE/EP-76-004 NL

1 OF 1
AD
A047 307

END
DATE
FILMED
1-78
DDC

Polytechnic
Institute
of New York

(11) Feb[redacted] [redacted]76

(12) 72p.

12 B.S.

(6)

CONTRIBUTION TO THE THEORY OF COMPONENT AVAILABILITY.

by
Clifford/Marshall

(9) Technical rept.,

Program in
System and Device Reliability

Prepared for
Office of Naval Research
(15) ~~Contract~~ NØØ14-67-A-Ø438-ØØ13

(14)
Report No. Poly-EE/EP-76-ØØ4,
POLY-EER-121

Polytechnic

Department of Electrical Engineering and
Electrophysics
Polytechnic Institute of New York
333 Jay Street
Brooklyn, New York 11201

CONTRIBUTIONS TO THE THEORY OF COMPONENT AVAILABILITY

By

Clifford Marshall

Polytechnic Institute of New York

333 Jay Street

Brooklyn, New York 11201

Technical Report No. POLY EE/EP 76-004

EER 121

Feb. 1976

ABSTRACT

This paper develops the concept of component availability as intrinsically dependent upon the system in which the component functions. The connection between the repair aspects of the system and component availability is developed in two ways. An integro-differential equation is derived that relates the two concepts. Alternatively the repair process is treated as a combinatorial queue in which each element of the system is tracked through its states of availability. Examples are given in some detail for small number of elements. Numerical solutions are given for illustrative purposes. Both time varying and steady state solutions are obtained. Some approaches to a trade off analysis for availability specification or design are given.

CONTENTS

1. Basic concepts in component availability.
   - 1.1 Definitions
   - 1.2 A fundamental relation
   - 1.3 Combinatorial queues
2. Two component queues
   - 2.1 One repairman
     - 2.1.1 Differential equations
       - 2.1.1.1 The general case
       - 2.1.1.2 All components the same
     - 2.1.2. Steady state
       - 2.1.2.1 The general case
       - 2.1.2.2. All components the same
     - 2.1.3. An illustration of the fundamental relation
   - 2.2 Summary results
3. Three component queues
   - 3.1 One repairman
     - 3.1.1. Differential equations
       - 3.1.1.1. The general case
       - 3.1.1.2. All components the same
     - 3.1.2. Steady state
       - 3.1.2.1. The general case
       - 3.1.2.2. All components the same
   - 3.2. Two repairmen
     - 3.2.1. Differential equations
       - 3.2.1.1. The general case
       - 3.2.1.2. All components the same
     - 3.2.2. Steady state
       - 3.2.2.1. The general case
       - 3.2.2.2. All components the same
   - 3.3 Summary results
4. The general m component, r repairmen case when all components are the same

CONTENTS (continued)

5. Application to system design

5.1 Two components

5.2 Three components

6. Directions for expanding research

6.1 Combinatorial queues

6.2 General service times

6.3 Waiting time as input to design analysis

6.4 Optimization and trade off analysis

References

Appendix: Numerical results and graphs

Chapter 1. Basis Concepts in Component Availability

In the study of reliability and availability two major points of view are widely considered. On the one hand, a complete operational system is considered and various aspects of system reliability or availability are addressed. Alternatively one may focus on the components which make up a system or indeed might be employed in a variety of different systems. A widely considered approach to these subjects is to express system reliability or availability in terms of component reliability or availability by means of the laws of probability and system logic. Implicit in much of this work is the concept of component reliability or availability defined independently of the system and use for which the component is intended. In the case of reliability, one can certainly give a meaningful, operational definition of component reliability which is completely independent of any system. When this is done, one must still consider special situations (such as stress and shelf-life) when applying specified component reliability to the formulation of a system composed of many parts. In the case of availability, there seems to be little or no justification for a concept of component availability independent of the system within which a component shall operate. The concept of repair which distinguishes availability from reliability carries with it a number of factors related to the repair process which are necessarily part of a system description.

This report takes the point of view that component availability must be defined within those aspects of a system which relate to the repair process. It develops a general relation between component availability and the system repair process. It then considers various repair process queues in detail, using them to obtain corresponding component availability. System availability is not considered directly in this work, such quantities would be obtained by using the component availabilities as derived here together with system logic and the laws of probability.

1.1 Definitions

In this report a system is assumed to consist of a total of N components with specific components designated by the index $i=1,\ldots,N$. The study deals with component availability and does not consider system logic. Therefore, component failures are treated as events taking place within the population of N components. Several of the quantities arising in the report are defined in this section, others will be introduced as they occur.

Most of the general material is developed for a single "typical" component. Since only that component is involved, it is not necessary to provide a subscript to quantities related to the component. However, a subscript is provided to clearly indicate the specific dependence on the component and to present results in a form directly applicable to more detailed investigations where more than one component is involved.

Several of the most basic quantities employed in this report are now defined and provided with notation:

$r_i(t)$ = the probability that component i has not failed up to time t. This is the component reliability as a function of time. One can consider a "steady state" value for component reliability, defined as $\lim_{t\to\infty} r_i(t) \equiv r_i$. In most cases $r_i = 0$ however, for completeness it is included among the defined quantities arising in this report.

$y_i(t)$ = the probability that component i is operational at time t. This is the component availability as a function of time. The most basic aspect of component availability is that repair (including replacement) is allowed. There is no limit, in the general situation, to the number of repair events that may have occurred up to time t. In fact the number of such events is a random variable which, though useful to study, falls outside the range of this report. The steady state availability is well defined and is most often a positive quantity $\lim_{t\to\infty} y_i(t) = y_i$, though in some special cases it can be zero.

In the initial work described in this report, it is useful to place analysis in the framework of exponential processes. Such processes often represent reasonable approximations to actual situations and at the same time are relatively simple to work with. Therefore, their use provides a meaningful starting point for availability studies. It is interesting to note that exponential service and exponential failure do not result from an assumption of Poisson arrival at the failed state from the operational state. One requires an infinite population of components, of the same kind, in order to make a Poisson arrival assumption. The assumption of exponential service (repair) time needs no comment. The assumption of exponential failure derives from the following general argument with f(x) a general failure density.

$$P\,[\text{a failure in } \Delta x \mid \text{operational at } x]\; P\,[\text{operational at } x]$$
$$= P\,[\text{a failure in } \Delta x \text{ and operational at } x] = f(x)\,\Delta x$$

When $f(x) = \lambda e^{-\lambda x}$,

$$P[\text{a failure in } \Delta x \mid \text{operational at } x] = \frac{\lambda e^{-\lambda x}\Delta x}{1-[1-e^{-\lambda x}]} = \lambda\,\Delta x$$

The exponential repair and failure rates for component i are denoted by μ_i and λ_i respectively.

The mathematical framework for the present study is a queue which represents repair. The total population of N components consists of operational components, failed components waiting for service, and failed components in service. Upon completion of service a component returns to the set of operational components. The queue itself may be of various forms representing the repair features of the system in which components under analysis are to operate. In particular the queue will involve specifications of service time distributions, number of repairmen, order of selection of down components for repair, and indeed the full range of possible variations to be found in queueing models. This report deals mostly with the number of repairmen and specifically allows each component to have its own form of repair time distribution (as specified by subscript i in the exponential case μ_i). In this report the queue concept is treated in two distinct ways.

- As a general approach to relate the repair aspects of a system, as reflected in the waiting time of the queue, to component availability.
- Detailed study of the queues themselves for very small (N = 2, 3) populations.

The waiting time distribution is basic to part of this work, it represents the probabilistic nature of the time a down component must wait before service begins (in a broader context the time to obtain a replacement or some other part may be included, with due care, in the waiting time). The waiting time is represented by the distribution function:

$W_i(s,t) = P$ [the time component i waits in queue before entering service is $\leq s$ if it joins the queue at time t]. It should be noted that a dynamic version of W_i is defined where its form is allowed to depend on t. In many cases W_i will not depend on t for one reason or another. In particular, interest may focus on the steady state value $\lim_{t\to\infty} W_i(s,t) \equiv W_i(s)$.

Though most of the present work is in terms of exponential service, some use is made of a general service time distribution denoted by:

$F_i(\tau) = P$ [repair time for component i $\leq \tau$], this is not considered to depend on the time component i enters the queue or enters service.

[illegible].2 A Fundamental Relation

In this section, a fundamental relation is developed between the component availability $y_i(t)$ and component waiting time behavior as represented by $W_i(s,t)$. That relation is felt to provide a bases for a wide range of investigations in the area of component availability some of which are addressed in detail in this report while some others are discussed briefly in the last section.

The fundamental relation is derived by considering how component i can be available at time $t+\Delta t$ from its possible situation at time t. In such a development, it is necessary to break the (possibly infinite) chain of events that can contribute to availability at time t. Though it is theoretically possible to introduce all possible situations of failure and repair, it is extremely cumbersome to do so. Therefore, the approach taken here is to subsume all possible previous events under the availability at the previous time values. The development follows.

$y_i(t+\Delta t)$ is the result of two distinct situations at time t:

1. Component i is available at time t and remains available over the subsequent interval Δt (independent events by assumption).

2. Component i is not available at time t and becomes available in interval Δt. Its failure within Δt upon becoming available is assumed of higher order in Δt and is omitted from the formulation.

If the probability of situation (2) is denoted by H, then

$$y_i\,(t+\Delta t) = (1-\lambda_i \Delta t)\, y_i\,(t) + H$$

The probability H includes all cases in which component i is in the service queue (unavailable) at time t and completes service by time t+Δt. A contribution is made to H by every possible service time for component i where that time represents the service event which prevents i from being available at time t (and hence part of situation (1) rather than H). For component i to be receiving service at time t, it must have failed at some previous time τ and entered the service system. Upon entry, it had to wait a time $s \geq 0$ (measured from τ) and at time $\tau + s$ its service started, to be completed between t and t+Δt. The possible cases for τ can be introduced into a sum of incremental terms which upon taking a limit process yields an integral formulation for H.

Let A_j = the probability that component i was operating at time τ_j and failed in the next $\Delta\tau$ interval.

$$A_j = y_i\,(\tau_j)\,\lambda_i \Delta\tau.$$

Let B_j = the probability that component i spent the remaining time period $t+\Delta t-\tau_j$ in the service system and completed service in the interval (t, t+Δt).

$$B_j = \lim_{\substack{\Delta s \to 0 \\ m \to \infty}} \Big(\sum_{k=1}^{m} P\,[\text{component waited till } s_k \text{ from } \tau_j$$

then entered service in interval $(s_k,\ s_k + \Delta s)$ and from that time on was in service, completing service in (t, t+Δt) measured from start of service at $\tau_j + s_k]\Big)$.

Then $$H = \lim_{\substack{\Delta\tau \to 0 \\ n \to \infty}} \sum_{j=1}^{n} A_j B_j.$$

The events involved in the detailed formulation of B_j can be appreciated by reference to Figure 1.

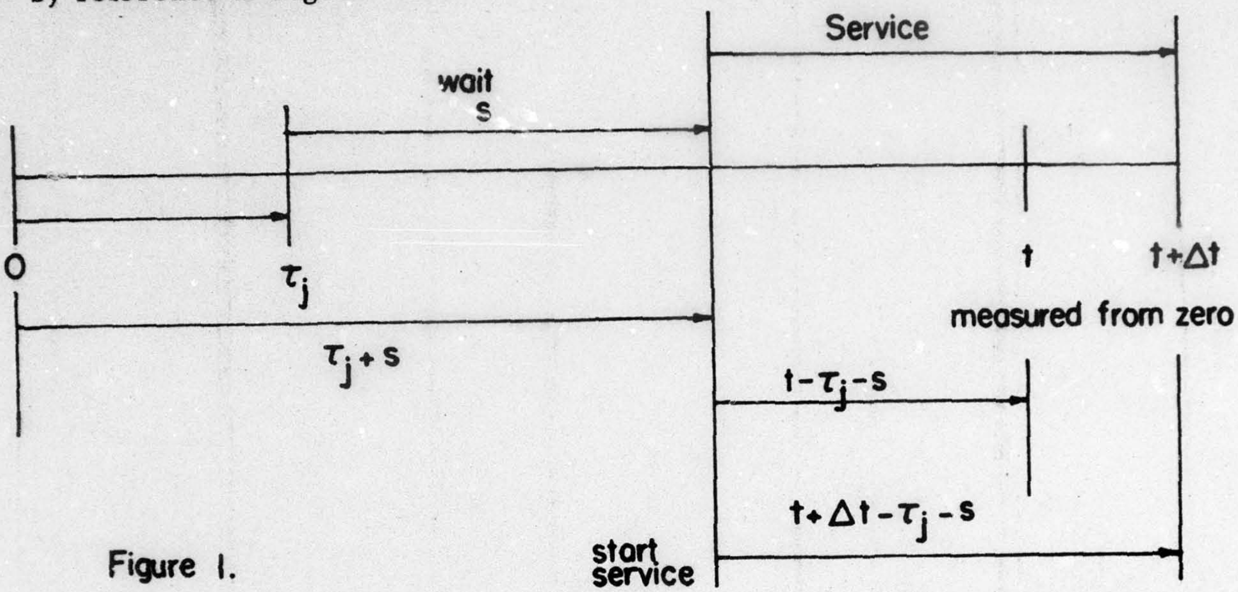


Figure 1.

By definition of F_i (t) and W_i (s, t) the contributions to the probability B_j are given by the following expression:

$$B_j = \lim_{\substack{\Delta \to 0 \\ m \to \infty}} \sum_{k=1}^{m} [W_i(s_k+\Delta s, \tau_j) - W_i(s_k, \tau_j)][F_i(t+\Delta t-\tau_j-s_k) - F_i(t-\tau_j-s_k)]$$

where the individual terms correspond to possible waiting times indicated by s_k.

Upon taking the indicated limits in the expression for B_j, one obtains the Stieltjes' integral:

$$B_j = \int_0^{t-\tau_j} [F_i(t+\Delta t-\tau_j-s) - F_i(t-\tau_j-s)\, dW_i(s, \tau_j).$$

Introduction of this expression into the equation for H and formulation of the limits in H yields:

$$H = \int_0^t y_i(\tau)\, \lambda_i \int_0^{t-\tau} [F_i\,(t+\Delta t-\tau-s) - F_i\,(t-\tau-s)]\, dW_i\,(s, \tau)\, d\tau.$$

Returning to the formulation of the equation for availability of component i and dividing through by Δt one obtains:

$$\frac{y_i\,(t+\Delta t) - y_i\,(t)}{\Delta t} = -\lambda_i\, y_i(t) + \int_o^t y_i\,(\tau)\,\lambda_i \int_o^{t-\tau} \frac{[F_i\,(t+\Delta t-s) - F_i(t-\tau-s)]}{\Delta t}\, dW_i(s,\tau)d\tau.$$

In the limit as $\Delta t \rightarrow o$, under the assumption that the service time distribution possesses a density function $f_i\,(t)$, the availability relation becomes:

$$\frac{dy_i(t)}{dt} = -\lambda_i y_i(t) + \int_o^t y_i(\tau)\lambda_i \int_o^{t-\tau} f_i\,(t-\tau-s)\, dW_i\,(s,\tau)\, d\tau \qquad (1)$$

Equation (1) is the fundamental relation that was desired. It relates component availability $y_i(t)$ to the waiting time distribution $W_i(s,t)$ for the same component. The waiting time represents the system in which the component is to operate. Features of the other components and of the system repair and service policies are all contained within the $W_i(s,t)$ expression.

Equation (1) utilizes the Stieltjes' integral employing waiting time distribution $W_i(s,t)$ because that distribution often has a finite jump at the s=0 value which presents the general use of a density function representation.

In this section, two extreme cases for equation (1) will be evaluated, an additional case will be illustrated in section 2.1.3. Potential applications and investigations involving equation (1) are discussed in Chapter 6.

Case 1 no repairmen. This is one extreme case in the concept of component availability. It means that a component is only available until it breaks down. Thus availability is the same as reliability for this case. The waiting time for repair is infinite so that $W_i(s,\tau)=0$ for all finite values of s. Thus the Stieltjes integral over $W_i(s,\tau)$ is zero and equation (1) becomes:

$$\frac{dy_i}{dt} = -\lambda_i\, y_i\,(t)$$

which is the well known equation for component reliability with solution:

$$y_i(t) \equiv r_i(t) = e^{-\lambda_i t}.$$

Case 2 repairman for each component. This is the extreme case opposite to Case 1. In this case each component has a repairman available to start repair whenever that component breaks down. Thus waiting time is always zero so that $W_i(s, t) = 1$ for $s \geq 0$. The Stieltjes integral contributes a single term at s=0 and gives zero values elsewhere.

It is convenient to specialize this case further by assuming exponential service time distribution with service rate μ_i. These observations yield the following form for equation (1):

$$\frac{dy_i(t)}{dt} = -\lambda_i y_i(t) + \mu_i \lambda_i e^{-\mu_i t} \int_0^t y_i(\tau) e^{\mu_i \tau} d\tau,$$

or alternatively:

$$\left[\frac{d y_i(t)}{dt} + \lambda_i y_i(t)\right]\frac{e^{\mu_i t}}{\mu_i \lambda_i} = \int_0^t y_i(\tau) e^{\mu_i \tau} d\tau.$$

Differentiation with respect to t (using Leibniz's rule on the right hand side) yields:

$$\frac{d^2 y_i(t)}{dt^2} + (\lambda_i + \mu_i) \frac{d y_i(t)}{dt} = 0,$$

with general solution.

$$y_i(t) = \frac{-c}{\lambda_i + \mu_i} e^{-(\lambda_i + \mu_i) t} + k.$$

It remains to evaluate the constants c and k. At t=0 it is assumed that $y_i(0) = 1$. To obtain another relation one observes that near t=0, before the first failure occurs availability is the same as reliability. Thus it is assumed that these quantities have the same slope (first derivative) at t=0. Since

$\frac{dr_i(t)}{dt} = -\lambda_i e^{-\lambda_i t}$ and $\frac{dy_i(t)}{dt} = ce^{-(\lambda_i+\mu_i)t}$ the assumption yields:

$\left.\frac{dr_i(t)}{dt}\right|_{t=0} = \left.\frac{dy_i(t)}{dt}\right|_{t=0}$ or $c=-\lambda_i$. One then obtains $k=\frac{\mu_i}{\lambda_i+\mu_i}$.

This leads to the well known result (reference 1,page 337) for availability when repair is undertaken at once with exponential service time distribution:

$$y_i(t) = \frac{\lambda_i}{\lambda_i+\mu_i} e^{-(\lambda_i+\mu_i)t} + \frac{\mu_i}{\lambda_i+\mu_i}$$

with steady state value

$$\tilde{y}_i = \lim_{t\to\infty} y_i(t) = \frac{\mu_i}{\lambda_i+\mu_i}$$

Though this is a well known result, it is derived here in a different way than elsewhere in the literature. It follows from and illustrates equation (1). In this context it is very clear how the availability and the system, represented by waiting time for service, are related.

The remainder of this report deals primarily with studying the details of the system service procedure which determines waiting time distribution $W_i(s,t)$. Such considerations also yield availability directly. In Chapter 6 some further consideration is given to the interplay between availability and waiting time as expressed by equation (1).

1.3 Combinational Queues

This report deals with component availability within the operational context of a system comprised of a number of components. The system has some facility to repair (or replace) components which provides the actual level of component availability. A relation between the system service capability, as expressed by component waiting time, and the component availability has been described above and given mathematical form in equation (1). More detailed study of the waiting time and the availability may be carried out by means of a queue model.

It is assumed that a system consists of m components and that each component is characterized by its failure rate λ_i and its repair rate μ_i. More general probability forms can be used but are difficult to work with (short of simulation) and this report limits itself to exponential processes. The system is provided with a service capability, specified in each case under study, which can be generally referred to as the service system, or service queue. When a component fails, it enters the service queue, waits if necessary for service, receives service, and upon completion of service returns to operation in the system. The history of failure and repair for a component represents the component availability. In this process of component failure, repair, and operation the system logic plays no role. From a broader point of view the interaction of components in system operation may effect individual properties such as failure rate λ but any such considerations are assumed to be accounted for by parameter values so far as component availability is concerned. System availability must consider the combination of component availabilities following the system logic. The system itself and the set of all components effect individual component availability only through the service queue.

Thus one deals with a queue system having a finite, fixed total population of m district elements, each of which has its own parameter values (probabilistic behavior) in general. There are three "states" possible for any member of the population: operational (available), waiting for service, or in service (the last two correspond to the unavailable condition).

Such a model can be thought of as a cyclic queue, a topic that has received attention in the literature (for example in reference 2). However, the interest in this report is to a special kind of detail and result not considered in the standard approaches to cyclic queues. For the kind of results desired here,it is necessary to follow each member of the population through the queue process. For m of any size,this becomes extremely involved to the point of practical infeasibility. However, for m=2 and 3, it is completely feasible and is done in the following chapters.

Because of the detailed nature of the queues treated here, it seems appropriate to give them the name "combinatorial queues". This implies full attention to the individual character of each element of a (small) finite population as it interacts with the other members through the queue model. The notation used for these models is $Q_{m,r}$ for m elements and r repairmen (channels). In each case first come first served (FIFO) service discipline is assumed and none of the possible variations of queue behavior are considered. Many generalizations suggest themselves for future study. An important special case is when all the components (population elements) are the same, i.e. λ_i ad μ_i do not depend on i. For these cases the special notation $Q^s_{m,r}$ may be used.

The cases for m=2 and m=3, though small size from a general point of view, are studied in detail for two reasons:

- These cases are small enough to be given through investigation indicating a variety of features in the nature of combinatorial queues as they relate to component availability.
- Components may be relatively complex and significant parts of a system so that actual systems may be encountered which are usefully considered as having only a few components. For such systems detailed studies as illustrated for the m=2 or 3 cases are appropriate. When this is the case, it is felt that the combinatorial queue approach is likely to be the most powerful treatment of system availability. By such methods, a complete analysis (without simulation) can be produced. The kind of results and their use is indicated by what is done for the cases in the next two chapters and also in Chapter 5.

Chapter 2. Two Component Queues.

This chapter deals with queues of the form $Q_{2,r}$. The cases r=0, no repairmen and r=2, full number of repairmen have been given in Section 1.2 where component availability is obtained from equation (1) for these cases in general. The only case to be considered here is when r=1, one repairman to service the components as they fail. Though one could formulate the $Q_{m,r}$ queue analysis in terms of Markov chains and the states of the system, the approach of this report is to use differential equations as the mathematical representation of the queue system.

2.1 One Repairman

The notation of this section is as follows:

$y_i(t)$ = availability of component i at time t, i=1, 2 .

$\tilde{y}_i$ = steady state availability of component i.

$P_{ij}(t)$ = probability that at time t component j is in service and component i is waiting for service. Only certain values of i and j lead to possible forms of P_{ij} namely P_{00}, P_{01}, P_{02}, P_{12}, P_{21}, where subscript zero indicates no component in that situation. For example P_{00} means both components are operational. When steady state values are intended, the same notation is used without the variable t.

In this notation, availability is the condition ofnot being in the service system so that:

$$y_1(t) = P_{00}(t) + P_{02}(t),$$

$$y_2(t) = P_{00}(t) + P_{01}(t).$$

Waiting time can also be expressed in terms of the queue probabilities $P_{ij}(t)$ or p_{ij} in the steady state. For simplicity consider the steady state waiting time distribution $W_i(s) = W_i(0) + \Pr$ [a completion in $\leq s$ | arrival finds one in service].P_a, where P_a=Pr [on arrival one is in service]. Thus one must consider the a posteriori probabilities.

Pr [component i completes service $\leq s$ | another customer finds i in service] = $1-e^{-\mu_i s}$ by the assumption of exponential service time.

Let $q_{00}(1)$ = Pr [component 1 arrives and finds none in the service system].

$q_{00}(2)$ = Pr [component 2 arrives and finds none in the service system]

q_{02} = Pr [on arrival component 1 finds 2 in service]

q_{01} = Pr [on arrival component 2 finds 1 in service]

Then:

$$W_1(s) = q_{00}(1) + (1-e^{-\mu_2 s})\, q_{02}$$

$$W_2(s) = q_{00}(2) + (1-e^{-\mu_1 s})\, q_{01}$$

where: $$q_{00}(1) = \frac{P_{00}}{P_{00} + P_{02}}, \quad q_{00}(2) = \frac{P_{00}}{P_{00} + P_{01}}$$

$$q_{01} = \frac{P_{01}}{P_{00} + P_{01}}, \qquad q_{02} = \frac{P_{02}}{P_{00} + P_{02}}.$$

These values, upon some simplification yield the following expressions for waiting time distributions:

$$\begin{cases} W_1(s) = 1 - \dfrac{P_{02}}{P_{00} + P_{02}} e^{-\mu_2 s} & , s>0 \\[2ex] W_1(0) = \dfrac{P_{00}}{P_{00} + P_{02}} & \end{cases}$$

$$\begin{cases} W_2(s) = 1 - \dfrac{P_{01}}{P_{00} + P_{01}} e^{-\mu_1 s} & , s>0 \\[2ex] W_2(0) = \dfrac{P_{00}}{P_{00} + P_{01}} & \end{cases}$$

The same logical cases contribute to the general time dependent waiting time distributions so that one has the following expressions:

$$\begin{cases} W_1(s,t) = 1 - \dfrac{P_{02}(t)}{P_{00}(t) + P_{02}(t)} e^{-\mu_2 s} & , s>0 \\[2ex] W_1(0,t) = \dfrac{P_{00}(t)}{P_{00}(t) + P_{02}(t)} & \end{cases}$$

$$\begin{cases} W_2(s,t) = 1 - \dfrac{P_{01}(t)}{P_{00}(t) + P_{01}(t)} e^{-\mu_1 s} & , s>0 \\[2ex] W_2(0,t) = \dfrac{P_{00}(t)}{P_{00}(t) + P_{01}(t)} & \end{cases}$$

2.1.1 Differential Equations

In the subsections of this section, the differential equations for the five service system probabilities P_{ij} (t) are presented. The first subsection gives the general case and the next subsection gives the case when both components are the same. Logical cases contributing to the probabilities are combined to form the equations, in most cases terms in $(\Delta t)^2$ or higher are never shown since they drop out upon forming differential limits. It is felt that the various cases are clearly indicated within the equations themselves and further discussion is therefore omitted.

2.1.1.1 General Case

The initial formulation for the combinatorial queue $Q_{2,1}$ gives the following system of equations:

$$P_{00}(t+\Delta t) = (1-\lambda_1\Delta t)(1-\lambda_2\Delta t)\ P_{00}(t) + (1-\lambda_2\Delta t)\mu_1\Delta t\ P_{01}(t) + (1-\lambda_1\Delta t)\ \mu_2\Delta t\ P_{02}(t)$$

$$P_{01}(t+\Delta t) = (1-\lambda_2\Delta t)\ (1-\mu_1\Delta t)\ P_{01}(t) + \lambda_1\Delta t\ P_{00}(t) + \mu_2\Delta t\ P_{12}(t)$$

$$P_{02}(t+\Delta t) = (1-\lambda_1\Delta t)(1-\mu_2\Delta t)\ P_{02}(t) + \lambda_2\Delta t\ P_{00}(t) + \mu_1\Delta t\ P_{21}(t)$$

$$P_{12}(t+\Delta t) = \lambda_1\Delta t\ (1-\mu_2\Delta t)\ P_{02}(t) + (1-\mu_2\Delta t)\ P_{12}(t)$$

$$P_{21}(t+\Delta t) = \lambda_2\Delta t\ (1-\mu_1\Delta t)\ P_{01}(t) + (1-\mu_1\Delta t)\ P_{21}(t)$$

Upon renewal of some higher order terms in Δt the system becomes:

$$P_{00}(t+\Delta t) = (1-\lambda_1\Delta t-\lambda_2\Delta t)\ P_{00}(t) + \mu_1\Delta t\ P_{01}(t) + \mu_2\Delta t\ P_{02}(t)$$

$$P_{01}(t+\Delta t) = (1-\lambda_2\Delta t-\mu_1\Delta t)\ P_{01}(t) + \lambda_i\Delta t\ P_{01}(t) + \mu_2\Delta t\ P_{12}(t)$$

$$P_{02}(t+\Delta t) = (1-\lambda_1\Delta t-\mu_2\Delta t)\ P_{02}(t) + \lambda_2\Delta t\ P_{00}(t) + \mu_1\Delta t\ P_{21}(t)$$

$$P_{12}(t+\Delta t) = \lambda_1\Delta t\ P_{02}(t) + (1-\mu_2\Delta t)\ P_{12}(t)$$

$$P_{21}(t+\Delta t) = \lambda_2\Delta t\ P_{01}(t) + (1-\mu_1\Delta t)\ P_{21}(t)$$

Forming differential quotients and taking the limit as $\Delta t \to 0$ results in the following system of difference differential equations. Since the difference is accounted for completely by explicit expression of each function, the system is in fact a system of differential equations.

$$\frac{dP_{00}(t)}{dt} = -(\lambda_1+\lambda_2) \; P_{00}(t) + \mu_1 \; P_{01}(t) + \mu_2 \, P_{02}(t)$$

$$\frac{dP_{01}(t)}{dt} = -(\lambda_2+\mu_1) \; P_{01}(t) + \lambda_1 \, P_{00}(t) + \mu_2 \, P_{12}(t)$$

$$\frac{dP_{02}(t)}{dt} = -(\lambda_1+\mu_2) \; P_{02}(t) + \lambda_2 \, P_{00}(t) + \mu_1 \; P_{21}(t)$$

$$\frac{dP_{12}(t)}{dt} = -\mu_2 \, P_{12}(t) + \lambda_1 \; P_{02}(t)$$

$$\frac{dP_{21}(t)}{dt} = -\mu_1 \; P_{21}(t) + \lambda_2 \, P_{01}(t)$$

One can in principle solve this linear system by transform or other "elementary" methods. Such solutions become extremely complicated. Therefore, it is felt that numerical solutions, for selected parameter values should be used to obtain desired solutions and to appreciate the nature of such solutions, some such results are given in the Appendix. Another approach is to study the steady state as done in a subsequent section. Still another approach is to consider the similar components case $Q^s_{2,1}$ which is done in the next subsection. Before turning to that topic, the "transformed"system is expressed below as a matter of general reference. To formulate that system, the notation $\tilde{P}_{ij}(s)$ is used for the (Laplace) transform of $P_{ij}(t)$ and assumed initial values are $P_{ij}(0)=0$ except for $i=j=0$ and $P_{00}(0)=1$. This results in (the dependence on s is not explicitly shown):

$$
\begin{array}{lllll}
(s+\lambda_1+\lambda_2)\,\tilde{p}_{00} & -\mu_1\tilde{p}_{01} & -\mu_2\tilde{p}_{02} & & =1 \\
-\lambda_1\tilde{p}_{00} & +(s+\lambda_2+\mu_1)\;\tilde{p}_{01} & & -\mu_2\tilde{p}_{12} & =0 \\
-\lambda_2\tilde{p}_{00} & & +(s+\lambda_1+\mu_2)\;\tilde{p}_{02} & -\mu_1\tilde{p}_{21} & =0 \\
 & & -\lambda_1\tilde{p}_{02} & (s+\mu_2)\,\tilde{p}_{12} & =0 \\
 & -\lambda_2\tilde{p}_{01} & & +(s+\mu_1)\tilde{p}_{21} & =0
\end{array}
$$

2.1.1.2 All Components Are the Same

In this case $\lambda_1=\lambda_2=\lambda$ and $\mu_1=\mu_2=\mu$ so that $P_{01}(t) = P_{02}(t)$ and $P_{12}(t) = P_{21}(t)$. The typical values $P_{01}(t)$, $P_{12}(t)$ and their transforms $\tilde{P}_{01}(s)$, $\tilde{P}_{12}(s)$ will be used. There are now only three quantities to be determined rather than five. The transformed system is:

$$
\begin{array}{llll}
(s+2_\lambda)\,\tilde{P}_{00} & -2\mu\tilde{P}_{01} & & =1 \\
-\lambda\tilde{P}_{00} + (s+\lambda+\mu) & \tilde{P}_{01} & \mu\tilde{P}_{12} & =0 \\
 & -\lambda\tilde{P}_{01} & +(s+\mu)\,\tilde{P}_{12} & =0
\end{array}
$$

From which one obtains:

$$\tilde{P}_{12}(s) = \frac{\lambda}{s+\mu}\;\tilde{P}_{01}(s)$$

$$\tilde{P}_{01}(s) = \frac{\lambda(s+\mu)}{(s+\mu)^2+\lambda s}\;\tilde{P}_{00}(s)$$

and

$$\tilde{P}_{00}(s) = \frac{s^2+\mu^2+(2\mu+\lambda)\,s}{s\,(s-r_1)\,(s-r_2)}$$

where $$r_1 = \frac{-(3\lambda + 2\mu) + \sqrt{\lambda^2 + 4\lambda\mu}}{2}$$

$$r_2 = \frac{-(3\lambda + 2\mu) - \sqrt{\lambda^2 + 4\lambda\mu}}{2}$$

Standard algebraic reduction and partial fraction expansion yields the solution in terms of the quantities:

$$a = \frac{1}{2\lambda^2 + 2\lambda\mu + \mu^2}, \quad b = \frac{1}{r_1\sqrt{\lambda^2 + 4\lambda\mu}}, \quad c = \frac{-1}{r_2\sqrt{\lambda^2 + 4\lambda\mu}}$$

The results are:

$$P_{00}(t) = \mu^2 a + (r_1^2 + r(2\mu+\lambda) + \mu^2) b\ e^{r_1 t} + (r_2^2 + r_2(2\mu+\lambda) + \mu^2) c\ e^{r_2 t}$$

$$P_{01}(t) = \mu\lambda a + \lambda(r_1+\mu)\ b e^{r_1 t} + \lambda\ (r_2+\mu)\ c\ \ e^{r_2 t}$$

$$P_{02}(t) = P_{01}(t)$$

$$P_{12}(t) = \lambda^2 a + \lambda^2\ b\ e^{r_1 t} + \lambda^2 c\ \ e^{r_2 t}$$

$$P_{21}(t) = P_{12}(t)$$

The values r_1 and r_2 are negative for all positive values of λ and μ (which are never negative, being rates of events). Thus limit values exist for all the $P_{ij}(t)$ quantities for all λ, μ values. In infinite population queues, this is often not the case but here all the probabilities are defined for any time.

Direct calculation establishes that the sum

$P_{00}(t) + 2\ P_{01}(t) + 2\ P_{12}(t) = 1$ as it should be.

The steady state values when the components are the same may be found from the above expressions to be:

$$P_{00} = \lim_{t\to\infty} P_{00}(t) = \mu^2 a = \frac{1}{1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2},$$

$$p_{01} = p_{02} = \lim_{t\to\infty} P_{00}(t) = \frac{\lambda}{\mu} p_{00},$$

$$p_{12} = p_{21} = \lim_{t\to\infty} P_{12}(t) = \left(\frac{\lambda}{\mu}\right)^2 p_{00}$$

This result will also be obtained as a special case of the general steady state results in Section 2.1.2.2.

2.1.2 Steady State

In this section the steady state values for $Q_{2,1}$ and $Q^s_{2,1}$ are obtained. This is done by supposing that the probabilities have become constant over time so that their derivatives are identically zero. A system of algebraic equations is obtained for the steady state probabilities. Lower case letters are used so that p_{ij} is the steady state version of $P_{ij}(t)$.

2.1.2.1 The General Case

The equations presented in Section 2.1.1.1 yield the following steady state system:

$$-(\lambda_1+\lambda_2)\, p_{00} + \mu_1\, p_{01} + \mu_2\, p_{02} = 0$$

$$-(\lambda_2+\mu_1)\, p_{01} + \lambda_1\, p_{00} + \mu_2\, p_{12} = 0$$

$$-(\lambda_1+\mu_2)\, p_{02} + \lambda_2\, p_{00} + \mu_1\, p_{21} = 0$$

$$-\mu_2\, p_{12} + \lambda_1\, p_{02} = 0$$

$$-\mu_1\, p_{21} + \lambda_2\, p_{01} = 0$$

Solution is (by successive substitution)

$$p_{02} = \lambda_2 \left[\frac{\lambda_1 + \lambda_2 + \mu_1}{\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2}\right] p_{00}$$

$$p_{01} = \lambda_1 \left[\frac{\lambda_1 + \lambda_2 + \mu_2}{\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2}\right] p_{00}$$

$$p_{12} = \frac{\lambda_1 \lambda_2}{\mu_2} \left[\frac{\lambda_1 + \lambda_2 + \mu_1}{\lambda_1 \mu_1 + \lambda_2 \mu_2 + \mu_1 \mu_2}\right] p_{00}$$

$$p_{21} = \frac{\lambda_1 \lambda_2}{\mu_1} \left[\frac{\lambda_1 + \lambda_2 + \mu_2}{\lambda_1 \mu_1 + \lambda_2 \mu_2 + \mu_1 \mu_2}\right] p_{00}$$

since the sum of the probabilities must be unity, we get p_{00} by forming that sum: $p_{00} = \frac{1}{S}$ where

$$S = 1 + \lambda_2 \left[\frac{\lambda_1+\lambda_2+\mu_1}{A}\right] + \lambda_1 \left[\frac{\lambda_1+\lambda_2+\mu_2}{A}\right] + \frac{\lambda_1 \lambda_2}{\mu_2} \left[\frac{\lambda_1+\lambda_2+\mu_1}{A}\right]$$

$$+ \frac{\lambda_1 \lambda_2}{\mu_1} \left[\frac{\lambda_1+\lambda_2+\mu_2}{A}\right], \text{ where } A=\lambda_1\mu_1+\lambda_2\mu_2+\mu_1\mu_2$$

The expression for S and indeed for any of the above values does not appear to simplify for general values of λ_1, λ_2, μ_1, and μ_2. To study them, one must employ numerical evaluation one form of which is discussed in Chapter 5. Other numerical results are presented in the Appendix.

2.1.2.2 Components The Same

When $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$ are introduced into the general steady state solution given in the previous section one obtains directly:

$$S = 1 + 2 \left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2$$

and $$p_{00} = \frac{1}{1+2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2}$$

$$p_{01} = p_{02} = \left(\frac{\lambda}{\mu}\right) p_{00}$$

$$p_{12} = p_{21} = \left(\frac{\lambda}{\mu}\right)^2 p_{00}$$

in agreement with the values previously obtained in section 2.1.1.2.

2.1.3 An Illustration of the Fundamental Relation

As described above, the $Q_{2,1}$ results can be used to formulate $y_i(t)$ and $w_i(s,t)$ for this case. It is interesting to introduce these values for one component, say i=1, into the fundamental relation, equation (1) of Section 1.2 and observe that the relation does indeed yield an identity under such substitution. This gives a kind of closure to the two distinct approaches developed in this report and indicates the complimentary role of each approach in the study of component availability.

Set i=1 in equation (1) of Section 1.2 and assume exponential service time distribution to obtain:

$$\frac{dy_1(t)}{dt} = -\lambda_1 y_1(t) + \int_0^t y_1(\tau)\lambda_1 \int_0^{t-\tau} \mu_1 \; e^{-\mu_1(t-\tau-s)} \, d \; w_1(s,\tau)\, d\tau$$

Let

$J = \int_0^{t-\tau} \mu_1 \; e^{-\mu_1(t-\tau-s)} \, d\, w_1(s,\tau)$, then using material introduced previously

for the waiting time distribution one obtains:

$$J = \mu_1 \; e^{-\mu_1(t-\tau)} \, w_1(0,\tau) + \int_0^{t-\tau} \mu_1 \; e^{-\mu_1(t-\tau-s)} \; \frac{\mu_2 \, P_{02}(\tau)}{P_{00}(\tau)+P_{02}(\tau)} \; e^{-\mu_2 s} \, ds$$

where $\dfrac{d \; w_1(s,\tau)}{ds} = \dfrac{\mu_2 P_{02}(\tau)}{P_{00}(\tau)+P_{02}(\tau)} \; e^{-\mu_2 s}$,

$$w_1(0,\tau) = \frac{P_{00}(\tau)}{P_{00}(\tau)+P_{02}(\tau)} \; .$$

Simplification and evaluation of the integral in J leads to:

$$J = \frac{\mu_1}{P_{00}(\tau)+P_{02}(\tau)} \left\{ P_{00}(\tau) \, e^{-\mu_1(t-\tau)} - \frac{\mu_2 \, P_{02}(\tau)}{\mu_1-\mu_2} \; e^{-\mu_1(t-\tau)} + \frac{\mu_2 P_{02}(\tau)}{\mu_1-\mu_2} \, e^{-\mu_2(t-\tau)} \right\}$$

From previous work $y_1(\tau)=P_{00}(\tau)+P_{02}(\tau)$ so the present version of equation (I) becomes;

$$\frac{d[P_{00}(\tau)+P_{02}(\tau)]}{dt} = -\lambda_1 [P_{00}(\tau)+P_{02}(\tau)]$$

$$+\int_o^t \frac{\mu_1\lambda_1}{\mu_1-\mu_2}\Big\{\mu_1 P_{00}(\tau) \quad e^{-\mu_1(t-\tau)} - \mu_2 P_{00}(\tau) \quad e^{-\mu_1(t-\tau)}$$

$$-\mu_2 P_{02}(\tau) \quad e^{-\mu_1(t-\tau)} + \mu_2 P_{02}(\tau) \quad e^{-\mu_2(t-\tau)}\Big\} d\tau$$

Alternatively:

$$\frac{d[P_{00}(\tau)+P_{02}(\tau)]}{dt} = -\lambda_1[P_{00}(\tau) + P_{02}(\tau)]$$

$$+\Big\{\mu_1\lambda_1 \quad e^{-\mu_1 t}\int_o^t P_{00}(\tau) \quad e^{\mu_1\tau} \, d\tau + \frac{\mu_1\lambda_1\mu_2}{\mu_1-\mu_2} \quad e^{-\mu_2 t}\int_o^t P_{02}(\tau) \quad e^{\mu_2\tau} d\tau$$

$$-\frac{\mu_1\lambda_1\mu_2}{\mu_1-\mu_2} \quad e^{-\mu_1 t}\int_o^t P_{02}(\tau) \quad e^{\mu_1\tau} \quad d\tau\Big\} . \qquad (\alpha)$$

Turning to the queue equations for $Q_{2,1}$ shows that:

$$\begin{aligned}\frac{d[P_{00}(\tau)+P_{02}(\tau)]}{dt} &= -(\lambda_1+\lambda_2) P_{00}(t) + \mu_1 P_{01}(t) + \mu_2 P_{02}(t) \\ &\quad -(\lambda_1+\mu_2) P_{02}(t) + \lambda_2 P_{00}(t) + \mu_1 P_{21}(t) \\ &= -\lambda_1[P_{00}(t) + P_{02}(t)] + \mu_1[P_{01}(t) + P_{21}(t)]\end{aligned}$$

Therefore the present form of equation (I) will be established as an identity if the bracket expression in equation (α) is shown to be $\mu_1[P_{01}(t)+P_{21}(t)]$.

For simplicity the μ_1 multiplier may be divided out at once since it is present in every term of the bracket.

From the differential equations, one obtains:

$$\frac{d[P_{01}(t) + P_{21}(t)]}{dt} = -\mu_1[P_{01}(t) + P_{21}(t)] + \lambda_1 P_{00}(t) + \mu_2 P_{12}(t)$$

where $$\frac{d\,P_{12}(t)}{dt} + \mu_2 P_{12}(t) = \lambda_1 P_{02}(t)$$

so that $P_{12}(t) = \lambda_1 \; e^{-\mu_2 t}\int_0^t e^{\mu_2 \tau} \; P_{02}(\tau)\, d\tau$ which may be introduced into the differential equation above to give

$$\frac{d[P_{01}(t)+P_{21}(t)]}{dt} + \mu_1[P_{01}(t)+P_{21}(t)] = \lambda_1 P_{00}(t) + \mu_2\lambda_1 \; e^{-\mu_2 t}\int_0^t e^{\mu_2 \tau} P_{02}(\tau)\, d\tau$$

which upon integration yields:

$$P_{01}(t) + P_{21}(t) = e^{-\mu_1 t}\int_0^t \lambda_1 \; e^{\mu_1 w} \; P_{00}(w)\, dw$$

$$+ \mu_2\lambda_1 \; e^{-\mu_1 t}\int_0^t e^{\mu_1 w} \; e^{-\mu_2 w}\int_0^w e^{\mu_2 \tau} \; P_{02}(\tau)\, d\tau\, dw$$

The first term on the right is equal to the first term in the bracket expression in (α) with μ_1 divided out. Integration of the second term by parts yields the remaining two terms of the bracket expression as follows:

Let $u = \int_0^w e^{\mu_2 \tau} \; P_{02}(\tau) d\tau$ and $dv = e^{(\mu_1 - \mu_2)\, w} dw$ then the technique of integration by parts yields:

$$\mu_2\lambda_1 \; e^{-\mu_1 t} \int_0^t e^{\mu_1 w} \; e^{-\mu_2 w}\int_0^w e^{\mu_2 \tau} P_{02}(\tau)\, d\tau dw =$$

$$\mu_2\lambda_1 e^{-\mu_1 t}\left\{\frac{e^{(\mu_1-\mu_2)t}}{\mu_1-\mu_2}\int_o^t e^{\mu_2\tau} P_{02}(\tau)\, d\tau - \int_o^t \frac{e^{\mu_1\tau} P_{02}(\tau)\, d\tau}{\mu_1-\mu_2}\right\}$$

$$= \frac{\mu_2\lambda_1 e^{-\mu_2 t}}{\mu_1-\mu_2}\int_o^t e^{\mu_2\tau} P_{02}(\tau)d\tau \quad - \frac{\mu_2\lambda_1 e^{-\mu_1 t}}{\mu_1-\mu_2}\int_o^t e^{\mu_1\tau} P_{02}(\tau)d\tau$$

as required.

This illustrates that introduction of the values $w_i(s,t)$ and $y_i(t)$ obtained from the solution of $Q_{2,1}$ satisfy the fundamental relation, equation (1), as in identity.

2.2 Summary Results.

Because of the relatively complex form of the solutions even in the two component case, general results require numerical calculation. Some such considerations are presented in Chapter 5 and the Appendix. In this section the case in which both components are the same is summarized in terms of the algebraic solution forms.

For two components, the only cases possible are $Q^s_{2,0}$, $Q^s_{2,1}$, and $Q^s_{2,2}$ so long as a FIFO service discipline is assumed. This report does not consider generalizations of the combinatorial queue problem. $Q^s_{2,0}$ has the (reliability case) solution

$y(t) = e^{-\lambda t}$ for component availability.

The steady state value $\bar{y} = \lim_{t\to\infty} y(t) = 0$ for this case. $Q^s_{2,1}$ has solution

$$y(t)=(\mu^2+\mu\lambda)a+[r_1^2 + r_1(2\mu+\lambda)+\mu^2+\lambda(r_1+\mu)]b \quad e^{r_1 t}+[r_2^2 +r_2(2\mu+\lambda)+\mu^2+\lambda(r_2+\mu)]c \quad e^{r_2 t}$$

where $r_1 = -\frac{(3\lambda+2\mu) + \sqrt{\lambda^2 + 4\lambda\mu}}{2}$, $r_2 = -\frac{(3\lambda + 2\mu)-\sqrt{\lambda^2 + 4\lambda\mu}}{2}$

$$a=\frac{1}{2\lambda^2 + 2\lambda\mu + \mu^2}, \quad b=\frac{1}{r_1\sqrt{\lambda^2 + 4\lambda\mu}}, \quad c= \frac{-1}{r_2\sqrt{\lambda^2 + 4\lambda\mu}}.$$

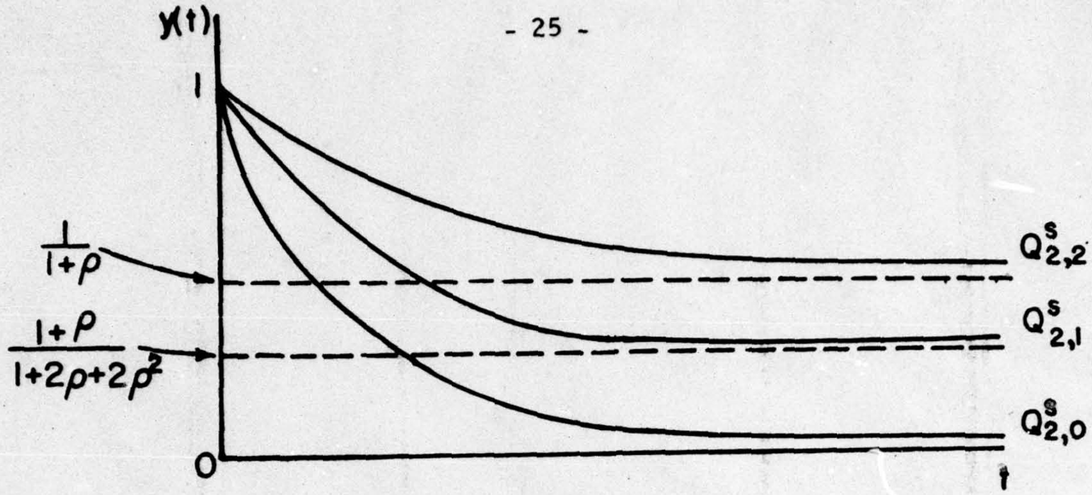


Figure 2

Chapter 3 Three Component Queues

This chapter deals with a system having three components. The FIFO assumption is made as is the assumption of exponential service and exponential breakdown distributions. Special cases for steady state and all components the same are considered. There are a total of four possible kinds of repair queue: $Q_{3,0}$, $Q_{3,1}$, $Q_{3,2}$, and $Q_{3,3}$. The first and last of these have already been treated in the general results of Section 1.2. This chapter treats $Q_{3,1}$ and $Q_{3,2}$. Results become complicated and can not readily be carried as far as the two component cases. It is felt that for more components special studies are called for and simulation of cases of interest might prove as (or more) effective as the differential equation approach. However, the material in this chapter should serve as a guide to the more involved cases. It is, of course, useful in its own right when a system consists (in some operational configuration) of three components (likely to be individually complex).

3.1 One Repairman

In this case the notation $P_{ijk}(t)$ is used for the probability that at time t, component k is in service, j is next in line waiting for service, and i is second in line waiting under the FIFO discipline. The value 0 means no component occupies the indicated position. Possible service queue probabilities are:

P_{000}, meaning none in the service system,

P_{001}, P_{002}, P_{003}, meaning one in the system and in service,

P_{021}, P_{031}, P_{012}, P_{032}, P_{013}, P_{023}, meaning one is waiting and one in service,

P_{123}, P_{213}, P_{132}, P_{312}, P_{231}, P_{321}, meaning two are waiting and one in service.

Availability is given by appropriate combinations of the service system probabilities, for example:

$$y_1(t) = P_{000}(t) + P_{002}(t) + P_{003}(t) + P_{032}(t) + P_{023}(t)$$

(i.e, component 1 is not in the service system). Waiting time can also be considered as illustrated in Chapter 2.

3.1.1 Differential Equations

In formulating the differential equations, higher order terms in Δt are either not included or are removed as limits are taken. The equations are formed by considering the logical cases that contribute to the probability values under the exponential distribution assumptions.

3.1.1.1 The General Case

The system of equations for queue probabilities at $t+\Delta t$ in terms of values at t follows:

$$P_{000}(t+\Delta t)=[1-(\lambda_1+\lambda_2+\lambda_3)\Delta t]\, P_{000}(t) + \mu_1\Delta t\, P_{001}(t)+\mu_2\Delta t\, P_{002}(t)+\mu_3\Delta t\, P_{003}(t)$$
$$P_{001}(t+\Delta t)=[1-\mu_1\Delta t][1-(\lambda_2+\lambda_3)\Delta t]\, P_{001}(t)+\lambda_1\Delta t\, P_{000}(t)+\mu_2\Delta t\, P_{012}(t)+\mu_3\Delta t P_{013}(t)$$
$$P_{002}(t+\Delta t)=[1-\mu_2\Delta t][1-(\lambda_1+\lambda_3)\Delta t]\, P_{002}(t)+\lambda_2\Delta t\, P_{000}(t)+\mu_1\Delta t\, P_{021}(t)+\mu_3\Delta t P_{023}(t)$$
$$P_{003}(t+\Delta t)=[1-\mu_3\Delta t][1-(\lambda_1+\lambda_2)\Delta t]\, P_{003}(t)+\lambda_3\Delta t P_{000}(t)+\mu_1\Delta t\, P_{031}(t)+\mu_2\Delta t P_{032}(t)$$
$$P_{021}(t+\Delta t)=(1-\mu_1\Delta t)(1-\lambda_3\Delta t)P_{021}(t)+\lambda_2\Delta t\, P_{001}(t) + \mu_3\Delta t\, P_{213}(t)$$
$$P_{031}(t+\Delta t)=(1-\mu_1\Delta t)(1-\lambda_2\Delta t)P_{031}(t) + \lambda_3\Delta t\, P_{001}(t) + \mu_2\Delta t\, P_{312}(t)$$
$$P_{012}(t+\Delta t)=(1-\mu_2\Delta t)(1-\lambda_3\Delta t)P_{012}(t)+\lambda_1\Delta t\, P_{002}(t) +\mu_3\Delta t\, P_{123}(t)$$
$$P_{032}(t+\Delta t)=(1-\mu_2\Delta t)(1-\lambda_1\Delta t)P_{032}(t)+\lambda_3\Delta t\, P_{002}(t) + \mu_1\Delta t\, P_{321}(t)$$
$$P_{013}(t+\Delta t)=(1-\mu_3\Delta t)(1-\lambda_2\Delta t)\, P_{013}(t)+\lambda_1\Delta t\, P_{003}(t)+ \mu_2\Delta t\, P_{132}(t)$$
$$P_{023}(t+\Delta t)=(1-\mu_3\Delta t)(1-\lambda_1\Delta t)\, P_{023}(t)+\lambda_2\Delta t\, P_{003}(t)+ \mu_1\Delta t\, P_{231}(t)$$
$$P_{123}(t+\Delta t)=(1-\mu_3\Delta t)\, P_{123}(t) + \lambda_1\Delta t\, P_{023}(t)$$
$$P_{213}(t+\Delta t)=(1-\mu_3\Delta t)\, P_{213}(t) + \lambda_2\Delta t\, P_{013}(t)$$

$P_{132}(t+\Delta t)=(1-\mu_2\Delta t)\,P_{132}(t)+\lambda_1\Delta t\,P_{032}(t)$
$P_{312}(t+\Delta t)=(1-\mu_2\Delta t)\,P_{312}(t)+\lambda_3\Delta t\,P_{012}(t)$
$P_{231}(t+\Delta t)=(1-\mu_1\Delta t)\,P_{231}(t)+\lambda_2\Delta t\,P_{031}(t)$
$P_{321}(t+\Delta t)=(1-\mu_1\Delta t)\,P_{321}(t)+\lambda_3\Delta t\,P_{021}(t)$

Upon division by Δt and passing to the limit as $\Delta t \to 0$, one obtains the following system of differential equations:

$$\frac{dP_{000}(t)}{dt} = -(\lambda_1+\lambda_2+\lambda_3)\,P_{000}(t)+\mu_1 P_{001}(t)+\mu_2\,P_{002}(t)+\mu_3\,P_{003}(t)$$

$$\frac{dP_{001}(t)}{dt} = -(\mu_1+\lambda_2+\lambda_3)\,P_{001}(t)+\lambda_1 P_{000}(t)+\mu_2\,P_{012}(t)+\mu_3\,P_{013}(t)$$

$$\frac{dP_{002}(t)}{dt} = -(\mu_2+\lambda_1+\lambda_3)\,P_{002}(t)+\lambda_2 P_{000}(t)+\mu_1\,P_{021}(t)+\mu_3\,P_{023}(t)$$

$$\frac{dP_{003}(t)}{dt} = -(\mu_3+\lambda_1+\lambda_2)\,P_{003}(t)+\lambda_3 P_{000}(t)+\mu_1 P_{031}(t)+\mu_2\,P_{032}(t)$$

$$\frac{dP_{021}(t)}{dt} = -(\mu_1+\lambda_3)\,P_{021}(t)+\lambda_2\,P_{001}(t)+\mu_3\,P_{213}(t)$$

$$\frac{dP_{031}(t)}{dt} = -(\mu_1+\lambda_2)\,P_{031}(t)+\lambda_3\,P_{001}(t)+\mu_2\,P_{312}(t)$$

$$\frac{dP_{012}(t)}{dt} = -(\mu_2+\lambda_3)\,P_{012}(t)+\lambda_1 P_{002}(t)+\mu_3\,P_{123}(t)$$

$$\frac{dP_{032}(t)}{dt} = -(\mu_2+\lambda_1)\,P_{032}(t)+\lambda_3\,P_{002}(t)+\mu_1\,P_{321}(t)$$

$$\frac{dP_{013}(t)}{dt} = -(\mu_3+\lambda_2)\,P_{013}(t)+\lambda_1\,P_{003}(t)+\mu_2\,P_{132}(t)$$

$$\frac{dP_{023}(t)}{dt} = -(\mu_3+\lambda_1)\,P_{023}(t)+\lambda_2\,P_{003}(t)+\mu_1\,P_{231}(t)$$

$$\frac{dP_{123}(t)}{dt} = -\mu_3\,P_{123}(t)+\lambda_1\,P_{023}(t)$$

$$\frac{dP_{213}(t)}{dt} = -\mu_3\,P_{213}(t)+\lambda_2\,P_{013}(t)$$

$$\frac{dP_{132}(t)}{dt} = -\mu_2\,P_{132}(t)+\lambda_1\,P_{032}(t)$$

$$\frac{dP_{312}(t)}{dt} = -\mu_2 P_{312}(t) + \lambda_3 P_{012}(t)$$

$$\frac{dP_{231}(t)}{dt} = -\mu_1 P_{231}(t) + \lambda_2 P_{031}(t)$$

$$\frac{dP_{321}(t)}{dt} = -\mu_1 P_{321}(t) + \lambda_3 P_{021}(t)$$

This system can only be treated by numerical solution methods. Some results are presented in the Appendix. The steady state can be considered along with some special cases. This is done in the present chapter.

3.1.1.2 All Components The Same

When the three components are the same $\mu_1=\mu_2=\mu_3=\mu$ and $\lambda_1=\lambda_2=\lambda_3=\lambda$. The sixteen probabilities reduce to four, which are taken to be (as typical cases) $P_{000}(t)$, $P_{001}(t)$, $P_{012}(t)$ and $P_{123}(t)$. The differential equations become:

$$\frac{dP_{000}(t)}{dt} = -3\lambda P_{000}(t) + 3\mu P_{001}(t)$$

$$\frac{dP_{001}(t)}{dt} = -(\mu+2\lambda) P_{001}(t) + \lambda P_{000}(t) + 2\mu P_{012}(t)$$

$$\frac{dP_{012}(t)}{dt} = -(\mu+\lambda) P_{012}(t) + \lambda P_{001}(t) + \mu P_{123}(t)$$

$$\frac{dP_{123}(t)}{dt} = -\mu P_{123}(t) + \lambda P_{012}(t)$$

Though considerably simpler than the full system when the components differ, this set of equations does not seem to yield a simple closed form solution with reasonable effort. One approach to solution is to take the Laplace transforms using initial conditions $P_{000}(0)=1$ and all other probability values zero at t=0. This results in the following set of equations in the transform variable s.

$$s \tilde{P}_{000}(s) - P_{000}(0) = -3\lambda \tilde{P}_{000}(s) + 3\mu \tilde{P}_{001}(s)$$

$$s \tilde{P}_{001}(s) - P_{001}(0) = -(\mu+2\lambda) \tilde{P}_{001}(s) + \lambda \tilde{P}_{000}(s) + 2\mu \tilde{P}_{012}(s)$$

$$s\,\tilde{P}_{012}(s) - P_{012}(0) = -(\mu+\lambda)\,\tilde{P}_{012}(s) + \lambda\,\tilde{P}_{001}(s) + \mu\,\tilde{P}_{123}(s)$$

$$s\,\tilde{P}_{123}(s) - P_{123}(0) = -\mu\,\tilde{P}_{123}(s) + \lambda\,\tilde{P}_{012}(s)$$

which become, upon substitution of initial values:

$$(s+3\lambda)\,\tilde{P}_{000}(s) = 1 + 3\mu\,\tilde{P}_{001}(s)$$

$$(s+\mu+2\lambda)\,\tilde{P}_{001}(s) = \lambda\,\tilde{P}_{000}(s) + 2\mu\,\tilde{P}_{012}(s)$$

$$(s+\mu+\lambda)\,\tilde{P}_{012}(s) = \lambda\,\tilde{P}_{001}(s) + \mu\,\tilde{P}_{123}(s)$$

$$(s+\mu)\,\tilde{P}_{123}(s) = \lambda\,\tilde{P}_{012}(s)$$

It should be kept in mind that there are in fact 6 values like P_{123}, 6 like P_{012} and 3 like P_{001}, these being representative values.

The above system yields:

$$\tilde{P}_{123}(s) = \frac{\lambda}{s+\mu} \quad \tilde{P}_{012}(s) = \frac{\lambda^2}{(s+\mu)^2 + \lambda s} \quad \tilde{P}_{001}(s)$$

$$\tilde{P}_{012}(s) = \frac{\lambda\,(s+\mu)}{(s+\mu)^2+\lambda s} \quad \tilde{P}_{001}(s)$$

$$\tilde{P}_{001}(s) = \frac{\lambda[(s+\mu)^2 + \lambda s]}{(s+\mu)^3 + 3\lambda s\,(s+\mu) + 2\lambda^2 s} \quad \tilde{P}_{000}(s)$$

So that:

$$\tilde{P}_{123}(s) = \frac{\lambda^3}{h(s)} \quad \tilde{P}_{000}(s)$$

$$\tilde{P}_{012}(s) = \frac{\lambda^2(s+\mu)}{h(s)} \quad \tilde{P}_{000}(s)$$

$$\tilde{P}_{001}(s) = \frac{\lambda[(s+\mu)^2+\lambda s]}{h(s)} \quad \tilde{P}_{000}(s)$$

where $h(s) = (s+\mu)^3 + 3\lambda s\,(s+\mu) + 2\lambda^2 s$

$$\tilde{P}_{000}(s) = \frac{h(s)}{s\left\{(s+\mu+\lambda)^3 + 3\lambda\,(s+\lambda)\,(s+\mu+\lambda) + 2\lambda^2(s+\lambda)\right\}}\,.$$

This does not seem to reduce to simple closed form expressions though of course the general nature of the solution is clear in terms of the roots of the denominator.

One can consider the limiting form (steady state) for the probabilities since:

$\lim_{s \to 0} s \tilde{P}_{000}(s) = P_{000}$, the steady state value, with similar expressions holding for the other probabilities.

Now $\tilde{P}_{000}(s) = \frac{a}{s}$ + terms corresponding to roots of the cubic expression in the denominator of $\tilde{P}_{000}(s)$. By setting $s=0$ in the algebraic formulation for the partial fraction expansion of $\tilde{P}_{000}(s)$ one obtains

$$a = \frac{1}{1+3\rho+6\rho^2+6\rho^3}, \quad \text{where } \rho = \frac{\lambda}{\mu}$$

Therefore, $p_{000} = \lim_{s \to 0} s \ \tilde{P}_{000}(s) = a$ (as given above)

and similarly $p_{001} = \rho \, p_{000}$,

$p_{012} = \rho^2 \, p_{000}$,

$p_{123} = \rho^3 \, p_{000}$.

These same values will be obtained as a special case of the steady state condition in Section 3. 1. 2. 2.

It is interesting to note that

$p_{000} + 3\, p_{001} + 6\, p_{012} + 6\, p_{123} = (1+3\rho+6\rho^2+6\rho^3)\, p_{000} = 1$ as, of course, should be the case.

3. 1. 2 The Steady State

The general steady state equations are obtained by setting the derivatives equal to zero in the equations of Section 3. 1. 1. 1. Steady state values are indicated by a probability notation without dependence on t, e.g., p_{000} is the probability that no components are in the service system.

3.1.2.1 The General Case

When each of the three components are allowed to have their own values of λ and μ, the one repairman steady state equations are as follows:

$$(\lambda_1+\lambda_2+\lambda_3)\,P_{000} = \mu_1 P_{001} + \mu_2 P_{002} + \mu_3 P_{003}$$
$$(\mu_1+\lambda_2+\lambda_3)\,P_{001} = \lambda_1 P_{000} + \mu_2 P_{012} + \mu_3 P_{013}$$
$$(\mu_2+\lambda_1+\lambda_3)\,P_{002} = \lambda_2 P_{000} + \mu_1 P_{021} + \mu_3 P_{023}$$
$$(\mu_3+\lambda_1+\lambda_2)\,P_{003} = \lambda_3 P_{000} + \mu_1 P_{031} + \mu_2 P_{032}$$

$$(\mu_1+\lambda_3)\,P_{021} = \lambda_2 P_{001} + \mu_3 P_{213}$$
$$(\mu_1+\lambda_2)\,P_{031} = \lambda_3 P_{001} + \mu_2 P_{312}$$
$$(\mu_2+\lambda_3)\,P_{012} = \lambda_1 P_{002} + \mu_3 P_{123}$$
$$(\mu_2+\lambda_1)\,P_{032} = \lambda_3 P_{002} + \mu_1 P_{321}$$
$$(\mu_3+\lambda_2)\,P_{013} = \lambda_1 P_{003} + \mu_2 P_{132}$$
$$(\mu_3+\lambda_1)\,P_{023} = \lambda_2 P_{003} + \mu_1 P_{231}$$
$$\mu_3 P_{123} = \lambda_1 P_{023}$$
$$\mu_3 P_{213} = \lambda_2 P_{013}$$
$$\mu_2 P_{132} = \lambda_1 P_{032}$$
$$\mu_2 P_{312} = \lambda_3 P_{012}$$
$$\mu_1 P_{231} = \lambda_2 P_{031}$$
$$\mu_1 P_{321} = \lambda_3 P_{021}$$

The last six equations yield six of the probability values directly in terms of another six values. One can write the other ten equations in terms of ten probability values by using the relations given in the last six equations. This procedure results in:

$$(\lambda_1+\lambda_2+\lambda_3)\,P_{000} = \mu_1 P_{001} + \mu_2 P_{002} + \mu_3 P_{003}$$

$$P_{001} = \frac{\lambda_1}{\alpha_1} P_{000} + \frac{\mu_2}{\alpha_1} P_{012} + \frac{\mu_3}{\alpha_1} P_{013}$$

$$p_{002} = \frac{\lambda_2}{\alpha_2} p_{000} + \frac{\mu_1}{\alpha_2} p_{021} + \frac{\mu_3}{\alpha_2} p_{023}$$

$$p_{003} = \frac{\lambda_3}{\alpha_3} p_{000} + \frac{\mu_1}{\alpha_3} p_{031} + \frac{\mu_2}{\alpha_3} p_{032}$$

$$\left(\frac{\mu_1 + \lambda_3}{\lambda_2}\right) p_{021} = p_{001} + p_{013}$$

$$\left(\frac{\mu_1 + \lambda_2}{\lambda_3}\right) p_{031} = p_{001} + p_{012}$$

$$\left(\frac{\mu_2 + \lambda_3}{\lambda_1}\right) p_{012} = p_{002} + p_{023}$$

$$\left(\frac{\mu_2 + \lambda_1}{\lambda_3}\right) p_{032} = p_{002} + p_{021}$$

$$\left(\frac{\mu_3 + \lambda_2}{\lambda_1}\right) p_{013} = p_{003} + p_{032}$$

$$\left(\frac{\mu_3 + \lambda_1}{\lambda_2}\right) p_{023} = p_{003} + p_{031}$$

Where $\alpha_1 = \mu_1 + \lambda_2 + \lambda_3$, $\alpha_2 = \mu_2 + \lambda_1 + \lambda_3$, and $\alpha_3 = \mu_3 + \lambda_1 + \lambda_2$.

In principle this system can be solved in algebraic terms to yield the probabilities as functions of the six parameters μ_i, λ_i, i=1, 2, 3. However, the results become extremely involved and it is not considered to be practical to proceed in this "direct" way.

Numerical solutions of the linear algebraic equations seem to be the most reasonable way to obtain particular results. Some examples are given in the Appendix. In addition, the special case of equal components is considered in the next section. An alternative approach to utilization of these equations for the study of component availability is discussed in Chapter 5.

3.1.2.2 All Components The Same

When the three components are the same $\mu_i = \mu$ and $\lambda_i = \lambda$ for i=1, 2, 3. Moreover, many of the probabilities are necessarily equal, e.g., p_{001} =

$p_{002}=p_{003}$, $p_{012}=p_{021}=p_{013}=p_{031}$ and so forth. Typical probabilities used are p_{000} (unique), p_{001} (three like this), p_{012} (six like this), and p_{123} (six like this). The equations defining these probabilities are:

$$\lambda p_{000} = \mu p_{001}$$

$$(\mu+2\lambda) p_{001} = \lambda p_{000} + 2\mu p_{012}$$

$$(\mu+\lambda) p_{012} = \lambda p_{001} + \mu p_{123}$$

$$\mu p_{123} = \lambda p_{012}$$

Let $\rho = \frac{\lambda}{\mu}$, then the solution is:

$$p_{001} = \rho p_{000}$$

$$p_{012} = \rho^2 p_{000}$$

$$p_{123} = \rho^3 p_{000}$$

Since the sum of all probabilities is unity one finds:

$$p_{000} = \frac{1}{1 + 3\rho + 6\rho^2 + 6\rho^3}$$

These results agree with the special limit case given in Section 3.1.1.2.

3.2 Two Repairmen

When there are three components and two repairmen, one must consider a total of ten service system probabilities $P_{ijk}(t)$ where k and j specify components in service and i specifies a component waiting for service with both repairmen occupied. The quantities to be considered are (all are fractions of time in the general case): p_{000}, p_{001}, p_{002}, p_{003}, p_{012}, p_{013}, p_{023}, p_{123}, p_{213}, p_{312}. Note that the indices specifying service are unordered, they simply tell which one or which two components are in service.

3.2.1 The differential equations

The usual procedure is followed in this section. Values of the service probabilities at a time $t+\Delta t$ are expressed in terms of appropriate values at time t. Higher order terms in Δt may be omitted in the formulation, or if included as part of the initial concept stage, fall out upon forming limits as $\Delta t \to 0$. Somewhat more care is required in the formulations for this case than in the cases delt with in previous sections, as discussed below.

3.2.1.1 The General Case

Because of the two servicemen, it is possible (as a logical case) that two distinct service activities could be completed in the same time interval Δt. Though a logical possibility, this is in fact an event of higher order when using independent, exponential service time distributions. Due to the somewhat complicated nature of this case, an initial formulation of the equations is given below, followed by the resulting system of differential equations. Some higher order terms are included within the initial formulation.

$$P_{000}(t+\Delta t)=[1-(\lambda_1+\lambda_2+\lambda_3)\Delta t]\,P_{000}(t)+\mu_1\Delta t\,P_{001}(t)+\mu_2\Delta t\,P_{002}(t)+\mu_3\Delta t\,P_{003}(t)$$
$$+\text{ higher order terms in }\Delta t.$$

$$P_{001}(t+\Delta t)=(1-\lambda_2\Delta t)(1-\lambda_3\Delta t)(1-\mu_1\Delta t)P_{001}(t)+\lambda_1\Delta t(1-\lambda_2\Delta t)(1-\lambda_3\Delta t)P_{000}(t)+$$
$$\mu_2\Delta t(1-\mu_1\Delta t)(1-\lambda_3\Delta t)\,P_{012}(t)+\mu_3\Delta t\,P_{013}(t)$$

$$P_{002}(t+\Delta t)=(1-\lambda_1\Delta t)(1-\lambda_3\Delta t)(1-\mu_2\Delta t)P_{002}(t)+\lambda_2\Delta t\,P_{000}(t)+\mu_1\Delta t\,P_{012}(t)+$$
$$\mu_3\Delta t\,P_{023}(t)$$

$$P_{003}(t+\Delta t)=(1-\lambda_1\Delta t)(1-\lambda_2\Delta t)(1-\mu_3\Delta t)P_{003}(t)+\lambda_3\Delta t\,P_{000}(t)+\mu_1\Delta t\,P_{013}(t)+\mu_2\Delta t\,P_{023}(t)$$

$$P_{012}(t+\Delta t)=(1-\lambda_3\Delta t)(1-\mu_1\Delta t)(1-\mu_2\Delta t)P_{012}(t)+\lambda_2\Delta t(1-\mu_1\Delta t)\,P_{001}(t)+$$
$$\lambda_1\Delta t(1-\mu_2\Delta t)\,P_{002}(t)+\mu_3\Delta t\,P_{123}(t)+\mu_3\Delta t\,P_{213}(t)$$

$$P_{013}(t+\Delta t)=(1-\lambda_2\Delta t)(1-\mu_1\Delta t)(1-\mu_3\Delta t)P_{013}(t)+\lambda_3\Delta t(1-\mu_1\Delta t)\,P_{001}(t)+$$
$$\lambda_1\Delta t(1-\mu_3\Delta t)\,P_{003}(t)+\mu_2\Delta t\,P_{312}(t)+\mu_2\Delta t\,P_{123}(t)$$

$$P_{023}(t+\Delta t)=(1-\lambda_1\Delta t)(1-\mu_2\Delta t)(1-\mu_3\Delta t)P_{023}(t)+\lambda_3\Delta t\,(1-\mu_2\Delta t)\,P_{002}(t)+$$
$$\lambda_2\Delta t\,(1-\mu_3\Delta t)\,P_{003}(t)+\mu_1\Delta t\,P_{213}(t)+\mu_1\Delta t\,P_{312}(t)$$

$$P_{123}(t+\Delta t)=(1-\mu_2\Delta t)(1-\mu_3\Delta t)\,P_{123}(t)+\lambda_1\Delta t\,P_{023}(t)$$

$$P_{213}(t+\Delta t)=(1-\mu_1\Delta t)(1-\mu_3\Delta t)\,P_{213}(t)+\lambda_2\Delta t\,P_{013}(t)$$

$$P_{312}(t+\Delta t)=(1-\mu_1\Delta t)(1-\mu_2\Delta t)\,P_{312}(t)+\lambda_3\Delta t\,P_{012}(t)$$

Upon taking the limits as $\Delta t \to 0$ after first dividing by Δt and forming appropriate differential quotients one obtains the following system of differential equations for the service system probabilities.

$$\frac{dP_{000}(t)}{dt} = -(\lambda_1+\lambda_2+\lambda_3)\, P_{000}(t) + \mu_1 P_{001}(t) + \mu_2 P_{002}(t) + \mu_3 P_{003}(t)$$

$$\frac{dP_{001}(t)}{dt} = -(\mu_1+\lambda_2+\lambda_3)\, P_{001}(t) + \lambda_1 P_{000}(t) + \mu_2 P_{012}(t) + \mu_3 P_{013}(t)$$

$$\frac{dP_{002}(t)}{dt} = -(\mu_2+\lambda_1+\lambda_3)\, P_{002}(t) + \lambda_2 P_{000}(t) + \mu_1 P_{012}(t) + \mu_3 P_{023}(t)$$

$$\frac{dP_{003}(t)}{dt} = -(\mu_3+\lambda_1+\lambda_2)\, P_{003}(t) + \lambda_3 P_{000}(t) + \mu_1 P_{013}(t) + \mu_2 P_{023}(t)$$

$$\frac{dP_{012}(t)}{dt} = -(\mu_1+\mu_2+\lambda_3)\, P_{012}(t) + \lambda_2 P_{001}(t) + \lambda_1 P_{002}(t) + \mu_3 P_{123}(t) + \mu_3 P_{213}(t)$$

$$\frac{dP_{013}(t)}{dt} = -(\mu_1+\mu_3+\lambda_2)\, P_{013}(t) + \lambda_3 P_{001}(t) + \lambda_1 P_{003}(t) + \mu_2 P_{312}(t) + \mu_2 P_{123}(t)$$

$$\frac{dP_{023}(t)}{dt} = -(\mu_2+\mu_3+\lambda_1)\, P_{023}(t) + \lambda_3 P_{002}(t) + \lambda_2 P_{003}(t) + \mu_1\, P_{213}(t) + \mu_1 P_{312}(t)$$

$$\frac{dP_{123}(t)}{dt} = -(\mu_2+\mu_3)\, P_{123}(t) + \lambda_1\, P_{023}(t)$$

$$\frac{dP_{213}(t)}{dt} = -(\mu_1+\mu_3)\, P_{213}(t) + \lambda_2\, P_{013}(t)$$

$$\frac{dP_{312}(t)}{dt} = -(\mu_1+\mu_2)\, P_{312}(t) + \lambda_3 P_{012}\,(t)$$

This system is complicated for a general solution in terms of closed form expressions depending on the six parameter values. Special cases are considered and some numerical solutions are given in the Appendix

3.2.1.2. All Components The Same

In this case $\mu_i=\mu$ and $\lambda_i=\lambda$ for i=1, 2, 3. It is also true that $P_{001}(t)=P_{002}(t)$, $=P_{003}(t)$, $P_{012}(t)=P_{013}(t)=P_{023}(t)$, and $P_{123}(t)=P_{213}(t)=P_{312}(t)$. These correspond to one in the service system, two in that system, and three in that system (with one waiting) respectively. Typical quantities are used for each case, the probabilities used are $P_{001}(t)$, $P_{012}(t)$, and $P_{123}(t)$.

The differential equations for this case become:

$$\frac{dP_{000}(t)}{dt} = -3\lambda\ P_{000}(t) + 3\mu\ P_{001}(t)$$

$$\frac{dP_{001}(t)}{dt} = -(\mu+2\lambda)\ P_{001}(t) + \lambda\ P_{000}(t) + 2\mu\ P_{012}(t)$$

$$\frac{dP_{012}(t)}{dt} = -(2\mu+\lambda)\ P_{012}(t) + 2\lambda\ P_{001}(t) + 2\mu\ P_{123}(t)$$

$$\frac{dP_{123}(t)}{dt} = -2\mu\ P_{123}(t) + \lambda P_{012}(t)$$

As observed in Section 3. 1. 1. 2 for the one repairman case, it does not seem possible to get useful direct solutions to this system. Numerical values can be computed and the steady state limit considered (as was done, using the transformed system for the one repairman case). It is felt that these techniques are sufficiently well illustrated for other cases and no further consideration is given to this case in this report.

3. 2. 2 Steady State

The steady state equations for the three component, two repairmen case are obtained in the usual way from the differential equations of Section 3. 2. 1. 1. The notation p_{ijk} is the same as in the time dependent case except that the service system probabilities do not depend on time in the steady state.

3. 2. 2. 1 The General Case

The linear algebraic system which defines the service system steady state probabilities is as follows:

$$(\lambda_1+\lambda_2+\lambda_3)\ p_{000} = \mu_1\ p_{001} + \mu_2\ p_{002} + \mu_3 p_{003}$$

$$(\mu_1+\lambda_2+\lambda_3)\ p_{001} = \lambda_1\ p_{000} + \mu_2\ p_{012} + \mu_3 p_{013}$$

$$(\mu_2+\lambda_1+\lambda_3)\ p_{002} = \lambda_2\ p_{000} + \mu_1\ p_{012} + \mu_3\ p_{023}$$

$$(\mu_3+\lambda_1+\lambda_2)\ p_{003} = \lambda_3\ p_{000} + \mu_1\ p_{013} + \mu_2\ p_{023}$$

$$(\mu_1+\mu_2+\lambda_3)\ p_{012} = \lambda_2\ p_{001} + \lambda_1\ p_{002} + \mu_3\ (p_{123} + p_{213})$$

$$(\mu_1+\mu_3+\lambda_2)\ p_{013} = \lambda_3\ p_{001} + \lambda_1\ p_{003} + \mu_2\ (p_{312} + p_{123})$$

$$(\mu_2+\mu_3+\lambda_1)\ p_{023} = \lambda_3\ p_{002} + \lambda_2\ p_{003} + \mu_1\ (p_{213} + p_{312})$$

$$(\mu_2+\mu_3)\, p_{123} = \lambda_1\, p_{023}$$

$$(\mu_1+\mu_3)\, p_{213} = \lambda_2\, p_{013}$$

$$(\mu_1+\mu_2)\, p_{312} = \lambda_3\, p_{012}$$

It does not seem reasonable to use this system to express the service system probabilities directly as functions of the six parameters μ_i, λ_i $i=1,2,3$. Numerical solution of the system for various parameter values is a much more reasonable approach to the generation of specific results. Some numerical results are given in the Appendix. An alternative approach is discussed in Chapter 5. The special case with all components equal is treated in the next section.

3.2.2.2 All Components The Same

In this case $\mu_i=\mu$ and $\lambda_i=\lambda$ for $i=1,2,3$ and $p_{001}=p_{002}=p_{003}$, $p_{012}=p_{013}$ $=p_{023}$, and $p_{123}=p_{213}=p_{312}$. The probabilities represent one in the service system, two in service system and three in service system with one waiting for service. The quantities p_{001}, p_{012}, and p_{123} are used as typical quantities. The governing equations for this case are:

$$\lambda\, p_{000} = \mu\, p_{001}$$

$$(\mu+2\lambda)\, p_{001} = \lambda\, p_{000} + 2\mu\, p_{012}$$

$$(2\mu+\lambda)\, p_{012} = 2\lambda\, p_{001} + 2\mu\, p_{123}$$

$$2\mu\, p_{123} = \lambda\, p_{012}$$

The solution is:

$$p_{001} = \rho\, p_{000}$$

$$p_{012} = \rho^2\, p_{000}$$

$$p_{123} = \frac{1}{2}\rho^3\, p_{000}$$

where there are three probabilities of each type and

$$p_{000} = \frac{1}{1 + 3\rho + 3\rho^2 + \frac{3}{2}\rho^3} .$$

Some general results in the case of all components equal are given in Chapter 4.

3.3 Summary Results

Because the solutions for general parameter values are so difficult to express in terms of μ_i and λ_i, summary results are discussed for the all components equal case as was done for two components in Section 2.2.

Three components, with all components the same, allow four cases without generalization of the basic queue assumptions (e.g., requiring FIFO conditions). These are denoted by $Q^s_{3,0}$, $Q^s_{3,1}$, $Q^s_{3,2}$, and $Q^s_{3,3}$. The extreme cases $Q_{3,0}$ and $Q_{3,3}$ have been treated in general in Section 1.2.

Unlike the situation for two components in Section 2.2, the time varying expressions for service system probabilities and hence component availability have not been obtained for the cases $Q^s_{3,1}$ and $Q^s_{3,2}$. Some numerical results can be found in the Appendix. However, the steady state values can be summarized here.

$Q^s_{3,0}$ steady state value $\tilde{y}^o = 0$

$Q^s_{3,1}$ $\tilde{y}^1 = p_{000} + 2\,p_{001} + 2\,p_{013}$

(based on component 2 for example)

$$\tilde{y}^1 = \frac{1 + 2\rho + 2\rho^2}{1 + 3\rho + 6\rho^2 + 6\rho^3}$$

$Q^s_{3,2}$ $\tilde{y}^2 = p_{000} + 2\,p_{001} + p_{013}$

Note that the notation p_{ijk} differs between cases $Q_{3,1}$ and $Q_{3,2}$, in the latter when two are in the service system only one case results (both in service) in the former two cases (since either customer can be the one in service).

$$\tilde{y}^2 = \frac{1 + 2\rho + \rho^2}{1 + 3\rho + 3\rho^2 + \frac{3}{2}\rho^3}$$

$Q^s_{3,3}$ $\tilde{y}^3 = \frac{1}{1+\rho}$

Direct algebraic comparison establishes that $\tilde{y}^o < \tilde{y}^1 < \tilde{y}^2 < \tilde{y}^3$ for all (positive) values of $\rho = \frac{\lambda}{\mu}$ as one would expect.

The spread between values like $\tilde{y}^1$, $\tilde{y}^2$ or $\tilde{y}^2$, $\tilde{y}^3$ serves as a potentially useful measure of what is achieved by introducing an additional serviceman into the service system. Figure 3 allows such a comparison as a function of ρ. Of course one repairman gives meaningful values of availability in relation to the zero value in the no repairman case. An additional repairman is seen to be increasingly worthwhile as ρ increases. However, going on to a third repairman produces limited effect for the values of ρ shown. As ρ increases, the three repairman case will be a real improvement. However, such values yield rather low availability in any case due to the tendency of components to fail at a faster rate than they are repaired ($\rho > 1$). Such component availability values should be avoided in design unless there is no chance to avoid them.

Chapter 4 The General m Component, r Repairmen Case when All Components are the Same.

The cases of two components with one repairman, three components with one repairman, and three components with two repairmen have been treated in previous sections for all components the same. In this chapter, the all components the same cases are given general treatment for m components and $r \leq m$ repairmen. The cases of no repairmen and m repairmen were treated in Section 1.2 by means of the fundamental relation, equation (1). However, the combinatorial queue model can be used to obtain the result for m repairmen as well. An example of such a development for m=3 is given by the equations.

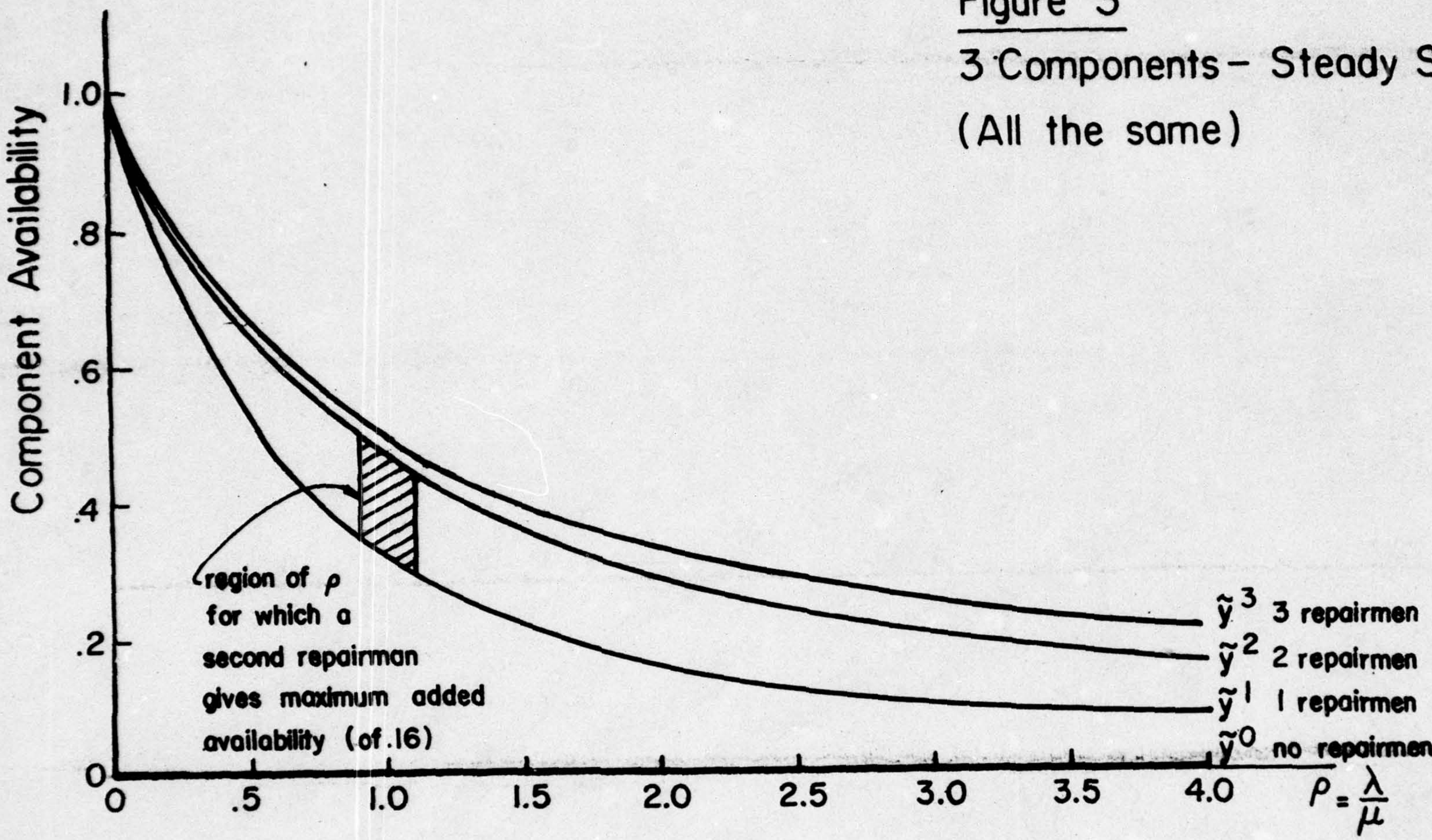
$$\frac{dP_0(t)}{dt} = -3\lambda P_0(t) + 3\mu P_1(t)$$

$$\frac{dP_1(t)}{dt} = -(\mu + 2\lambda) P_1(t) + \lambda P_0(t) + 2\mu P_2(t)$$

Figure 3

3 Components – Steady State

(All the same)



$$\frac{dP_2(t)}{dt} = -(2\mu+\lambda)\,P_2(t) + 2\lambda\,P_1(t) + \mu\,P_3(t)$$

$$\frac{dP_3(t)}{dt} = 3\mu\,P_3(t) + 3\lambda\,P_2\,(t)$$

where $P_i(t)$ = probability that there are i components in the 3 repairman queue at time t. The basic assumptions of the combinatorial queue model, previously specified, should be kept in mind.

The steady state values are determined by the following system with $\rho=\lambda/\mu$.

$$p_1=\rho\,p_0$$

$$(\mu+2\lambda)\,p_1=\lambda\,p_0+2\mu\,p_2$$

$$(2\mu+\lambda)\,p_2=2\lambda\,p_1+\mu\,p_3$$

$$p_3=\rho\,p_2$$

with solution $p_1=\rho\,p_0$, $p_2=\rho^2\,p_0$, $p_3=\rho^3\,p_0$ and $p_0=\dfrac{1}{1+3\rho+3\rho^2+\rho^3}$

The availability is $\tilde{y}^{\,3} = p_0+2p_1+p_2 = \dfrac{1+2\rho+\rho^2}{1+3\rho+3\rho^2+\rho^3} = \dfrac{1}{1+\rho}$

in agreement with the result from Section 1.2.

As a general approach to the same components case one considers $P_k(t)$ as the probability that there are k components in the service system at time t, out of a total population of m components. The steady state expression is p_k. In the case of $Q^s_{m,r}$, p_k indicates k receiving service and some waiting if $k\leq r$, otherwise r are in service and k-r are waiting. The availability for a component is the probability that the component is not in the service system. If k are in the service system and m-k are in the remaining (available) population, the probability that a particular individual component is available is (m-k)/m on the assumption that all the components are the same. Thus:

y=Pr (component 'a' is available)

$= \sum_{k=0}^{m}$ Pr (' a' is in non-service population | population size is m-k)

Pr (population size m-k)

$= \sum_{k=0}^{m} \frac{(m-k)}{m} p_k$.

Consider the case $Q^s_{m,1}$ (m=2, 3 have already been discussed), for which the defining equations are:

$$P_0(t+\Delta t) = (1-m\lambda\Delta t)\, P_0(t) + \mu\Delta t\, P_1(t)$$

$$P_k(t+\Delta t) = [1-(m-k)\lambda\Delta t]\,(1-\mu\Delta t)\, P_k(t) + \mu\Delta t\, P_{k+1}(t) + (m-k+1)\lambda\Delta t\, P_{k-1}(t)$$

for k=1, 2,..., (m-1)

$$P_m(t+\Delta t) = (1-\mu\Delta t)\, P_m(t) + \lambda\Delta t\, P_{m-1}(t)$$

These yield the difference differential equations:

$$\frac{dP_0(t)}{dt} = -m\lambda P_0(t) + \mu P_1(t)$$

$$\frac{dP_k(t)}{dt} = -[(m-k)\lambda+\mu]\, P_k(t) + \mu P_{k+1}(t) + (m-k+1)\lambda P_{k-1}(t), \quad k=1,2,\ldots,(m-1)$$

$$\frac{dP_m(t)}{dt} = -\mu P_m(t) + \lambda P_{m-1}(t)$$

The steady state equations are:

$$p_1 = m\rho\, p_0$$

$$p_{k+1} = [1+(m-k)\rho]\, p_k - (m-k+1)\rho\, p_{k-1}, \quad k=1,2,\ldots,(m-1)$$

$$p_m = \rho\, p_{m-1}$$

Where $\rho=\lambda/\mu$ as usual.

Induction yields the solution:

$$p_k = \frac{m!}{(m-k)!}\rho^k p_0, \quad k=1,2,\ldots,m$$

$$\text{and } p_0 = \frac{1}{1+\sum_{k=1}^{m}\frac{m!}{(m-k)!}\rho^k}$$

This result can be used to obtain the m=2 and m=3 results with one repairman obtained previously. Thus

$$m=2\ , \quad p_0 = \frac{1}{1 + 2\rho + 2\rho^2}$$

$$m=3\ , \quad p_0 = \frac{1}{1 + 3\rho + 6\rho^2 + 6\rho^3}$$

The availability values may also be considered.

$$m=2\ , \quad y = \sum_{k=0}^{2} \frac{(2-k)}{2} \, p_k = p_0 + \frac{1}{2} p_1$$

$$y = \frac{1 + \rho}{1 + 2\rho + 2\rho^2}$$, as previously obtained in Section 2.1.2.2.

$$m=3\ , \quad y = \sum_{k=0}^{3} \frac{(3-k)}{3} \, p_k = p_0 + \frac{2}{3} \, p_1 + \frac{1}{3} \, p_2$$

$$y = p_0 + \frac{2}{3}(3\rho)\, p_0 + \frac{1}{3}(6\rho^2)\, p_0$$

$$y = \frac{1 + 2\rho + 2\rho^2}{1 + 3\rho + 6\rho^2 + 6\rho^3}$$, as previously obtained.

Now consider the $Q^s_{m,2}$ case which only has meaning for $m \geq 3$ (except for the full repairman situation when m=2 which is not under discussion here). The service situation can have one in service and none waiting, or two in service with zero or more waiting, or of course none in service.

The equations governing this case follow:

$$P_0(t+\Delta t) = (1 - m\lambda\,\Delta t)\, P_0(t) + \mu \Delta t\, P_1(t)$$

$$P_1(t+\Delta t) = [1-(m-1)\,\lambda\,\Delta t]\,(1-\mu\Delta t)\, P_1(t) + m\lambda\,\Delta t\, P_0(t) + 2\mu\Delta t\, P_2(t)$$

$$P_k(t+\Delta t) = [1-(m-k)\lambda\,\Delta t]\,(1-2\mu\,\Delta t)\, P_k(t) + (m-k+1)\lambda\,\Delta t\, P_{k-1}(t) + 2\mu\Delta t\, P_{k+1}(t)$$

$$\text{for } 2 \leq k \leq (m-1)$$

$$P_m(t+\Delta t) = (1-2\mu\Delta t)\, P_m(t) + \lambda\,\Delta t\, P_{m-1}(t)$$

This leads to the difference-differential equations:

$$\frac{dP_0(t)}{dt} = -m\lambda\, P_0(t) + \mu\, P_1(t)$$

$$\frac{dP_1(t)}{dt} = -[(m-1)\lambda+\mu]\,P_1(t) + m\lambda P_0(t) + 2\mu\,P_2(t)$$

$$\frac{dP_k(t)}{dt} = -[(m-k)\,\lambda + 2\mu]\,P_k(t) + (m-k+1)\,\lambda\,P_{k-1}(t) + 2\mu\,P_{k+1}\,(t)$$

for $2 \leq k \leq m-1$

$$\frac{dP_m(t)}{dt} = -\,2\mu\,P_m(t) + \lambda\,P_{m-1}\,(t)$$

The steady state equations are:

$$p_1 = m\rho\,p_0$$

$$p_2 = \left[\,\frac{m-1}{2}\,\rho + 1\,\right] p_1 - \frac{m}{2}\,\rho\,p_0$$

$$p_{k+1} = \left[\,\frac{(m-k)}{2}\,\rho + 1\right] p_k - \frac{(m-k+1)}{2}\,\rho\,p_{k-1}\,, \quad 2 \leq k \leq m-1$$

$$p_m = \frac{\rho}{2}\,p_{m-1}$$

Induction yields the solution

$$p_k = \frac{m!}{(m-k)!\;2^{k-1}}\,\rho^k\,p_0 \quad \text{for } k=1,2,\ldots,m.$$

With $p_0 = \dfrac{1}{1 + \sum_{k=1}^{m} \dfrac{m!}{(m-k)!\;2^{k-1}}\,\rho^k}$.

When m=3 this result gives $p_0 = \dfrac{1}{1+3\rho+3\rho^2+\frac{3}{2}\,\rho^3}$

and availability.

$$y = p_0 + \frac{m-1}{m}\,p_1 + \frac{m-2}{m}\,p_2 = p_0 + \frac{2}{3}\,p_1 + \frac{1}{3}\,p_2$$

$$= p_0 + 2\rho\, p_0 + \rho^2\, p_0 = \frac{1+2\rho+\rho^2}{1+3\rho+3\rho^2+\frac{3}{2}\rho^3}\text{, as previously}$$

obtained.

For general r the situation is somewhat more involved than for the r=1, 2 cases shown above. The major feature contributing to increased complexity is the need for more individual equations. The defining equations are:

$$P_0\,(t+\Delta t) = (1-m\lambda\Delta t)\,P_0(t) + \mu\Delta t\,P_1\,(t)$$

$$P_1\,(t+\Delta t) = [1-(m-1)\lambda\Delta t](1-\mu\Delta t)P_1(t) + m\lambda\Delta t\,P_0(t) + 2\mu\Delta t\,P_2(t)$$

$$P_2\,(t+\Delta t) = [\,1-(m-2)\lambda\Delta t](1-2\mu\,\Delta t)P_2(t) + (m-1)\lambda\,\Delta t\,P_1(t) + 3\mu\Delta t\,P_3(t)$$

$$\vdots$$

$$P_{r-1}(t+\Delta t)=[1-(m-r+1)\lambda\,\Delta t]\,[1-(r-1)\,\mu\Delta t]\,P_{r-1}(t)$$

$$+\,(m-r+2)\lambda\,\Delta t\,P_{r-2}(t) + r\mu\Delta t\,P_r(t)$$

$$P_k(t+\Delta t) = [1-(m-k)\lambda\,\Delta t](1-r\mu\Delta t)\,P_k(t)+(m-k+1)\lambda\,\Delta t\,P_{k-1}(t)+r\mu\Delta t\,P_{k+1}(t)$$

for $k=r, r+1, \ldots, m-1$

$$P_m(t+\Delta t) = (1-r\mu\Delta t)\,P_m(t) + \lambda\,\Delta t\,P_{m-1}(t)$$

These yield the difference-differential equations:

$$\frac{dP_0(t)}{dt} = -\,m\lambda\,P_0(t) + \mu\,P_1(t)$$

$$\frac{dP_1(t)}{dt} = -[(m-1)\lambda + \mu]\,P_1(t) + m\lambda\,P_0(t) + 2\mu\,P_2(t)$$

$$\frac{dP_2(t)}{dt} = -[(m-2)\lambda+2\mu]\,P_2(t) + (m-1)\lambda\,P_1(t) = 3\mu\,P_3\,(t)$$

$$\vdots$$

$$\frac{dP_{r-1}(t)}{dt} = -[(m-r+1)\lambda+(r-1)\mu]\ P_{r-1}(t) + (m-r+2)\lambda\ P_{r-2}(t) + r\mu\ P_r(t)$$

$$\frac{dP_k(t)}{dt} = -[(m-k)\ \lambda+r\mu]\ P_k(t) + (m-k+1)\lambda P_{k-1}(t) + r\mu\ P_{k+1}(t)$$

$$k=r, r+1, \ldots, m-1$$

$$\frac{dP_m(t)}{dt} = -\ r\mu\ P_m(t) + \lambda\ P_{m-1}(t)$$

The steady state equations are:

$p_1 = m\rho\ p_0$

$2p_2=[(m-1)\ \rho+1]\ p_1 - m\rho\ p_0$

$3p_3=[(m-2)\ \rho+2]\ p_2 - (m-1)\ \rho\ p_1$

.

.

.

$rp_r=[(m-r+1)\ \rho + r-1]\ p_{r-1} - (m-r+2)\ \rho\ p_{r-2}$

$rp_{k+1}=[(m-k)\ \rho+r]\ p_k - (m-k+1)\ \rho\ p_{k-1}$, $k=r, r+1, \ldots, m-1$

$rp_m = \rho\ p_{m-1}$

This system does not lend itself to a simple general solution in terms of r. One can develop a solution by induction, however, it does not seem to be worthwhile in view of the resulting complexity.

Particular cases of interest can of course be treated as needed from the general equations. For example the r=3 case arrises in a four component situation. One can consider the r=3, m=4 case, for which the equations become:

$$p_1 = 4\rho\ p_0$$

$$2\ p_2 = (3\rho+1)\ p_1 - 4\rho\ p_0$$

$$3\ p_3 = (2\rho+2)\ p_2 - 3\rho\ p_1$$

$$3\ p_4 = \rho\ p_3$$

Which has the solution:

$$p_1 = 4\rho\, p_0$$

$$p_2 = 6\rho^2\, p_0$$

$$p_3 = 4\rho^3\, p_0$$

$$p_4 = \frac{4}{3}\rho^4\, p_0$$

where $p_0 = \dfrac{1}{1 + 4\rho + 6\rho^2 + 4\rho^3 + \frac{4}{3}\rho^4}$

The availability expression in this case is:

$$y = \sum_{k=0}^{4} \frac{4-k}{4}\; p_k = p_0 + \frac{3}{4}\, p_1 + \frac{1}{2}\, p_2 + \frac{1}{4}\, p_3$$

$$y = p_0 + 3\rho\, p_0 + 3\rho^2\, p_0 + \rho^3\, p_0 = \frac{1 + 3\rho + 3\rho^2 + \rho^3}{1 + 4\rho + 6\rho^2 + 4\rho^3 + \frac{4}{3}\rho^4}$$

It may be observed that $y < \frac{1}{1+\rho}$ which is the full repairman case. This should be the case since there are only three repairmen.

Chapter 5 Application to System Design

This report deals in detail with the concept of component availability within the context of various system service procedures. When the components operate in a system, various levels of availability are required for individual components. It is therefore of value to consider the different component availabilities together in some form to which a trade off analysis can be applied. Such a formulation is presented in this chapter. In a particular system design of course, the required level of system availability drives the rest of the availability analysis. System logic is utilized to relate system availability to component availability levels. The consideration of this chapter relates the λ and μ values (design parameters) and component availability levels for two component and three component systems. There is no reason that the procedure could not be extended to more components. However, the underlying queue problems and numerical analyses become involved. The procedure will be introduced for a two component system then generalized for the three component case.

5.1 Two Components

In a two component system, two component availabilities are to be considered, they are denoted by y_1 and y_2. The design parameters are the failure rates λ_1, λ_2 and repair rates μ_1, μ_2. Because of the difficulty in combining all these values into a relatively simple comparison form the following procedure has been developed.

One component is considered to be the reference or major component, though this does not really have any firm significance, and the other component is related to it by ratio value. In addition failure is related to repair by a ratio value, assumed to be the same for both components. These assumptions restrict the possible cases for study but result in a very managable presentation which still allows a wide range of options .

Following the procedure outlined above, the following assumptions are made, taking component 1 to be the reference (or major) component: $\mu_2=\beta\mu_1$, $\lambda_1 = \alpha\mu_1$, $\lambda_2=\alpha\mu_2=\alpha\beta\mu_1$. One can consider the differential equations for queue probabilities and the resulting component availabilities as functions of time. Only detailed computation is required to do this. However, for purpose of illustration only the steady state availabilities are treated here.

Making the substitutions for λ_i and μ_i in the results for $Q_{2,1}$ steady state (the cases $Q_{2,0}$ and $Q_{2,2}$ are not of great interest here), one obtains:

$$p_{02} = A\, p_{00}$$

$$p_{01} = B\, p_{00}$$

$$p_{12} = C\, p_{00}$$

$$p_{21} = D\, p_{00}$$

where $A = \alpha\beta \left(\dfrac{\alpha + \alpha\beta + 1}{\alpha + \alpha\beta^2 + \beta}\right)$, $B = \alpha \left(\dfrac{\alpha + \alpha\beta + \beta}{\alpha + \alpha\beta^2 + \beta}\right)$,

$C = \frac{\alpha}{\beta} A$, $D = \alpha\beta B$. Let $S = 1 + A + B + C + D$ then $y_1 = (1+A)/S$, $y_2 = (1+B)/S$. Now one considers curves that relate y_1 and y_2 as shown in Figure 4. Curves are shown for fixed α and for fixed β. The $\beta=1$ case is for both components the same. Constant β curves are symmetric about $\beta=1$ as can be established by the substitution $\frac{1}{\beta}$ for β in the above formulation. Such a substitution interchanges y_1 and y_2.

For fixed α as $\beta \to 0$ one obtains $y_1(\beta = 0) = \frac{1}{(\alpha+1)^2}$ and $y_2(\beta=0) = \frac{1}{1+\alpha}$ for example when $\alpha=1$, $y_1 \to .25$, $y_2 \to .5$ for $\beta=0$ which agrees with the numerical results shown in the figure.

As $\beta \to \infty$ one obtains:

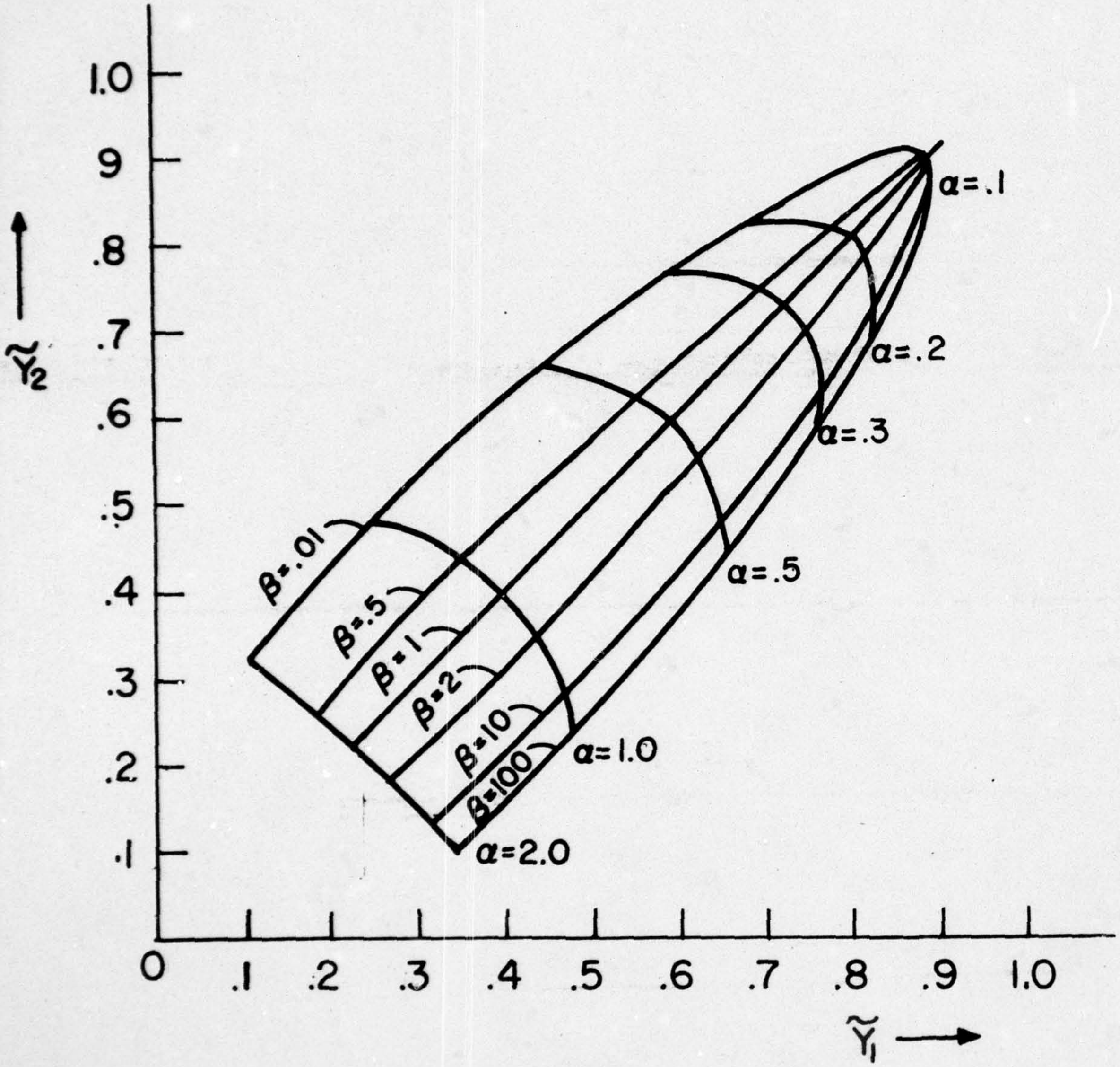
$$\lim_{\beta\to\infty} A = \alpha, \quad \lim_{\beta\to\infty} B = \lim_{\beta\to\infty} C = 0$$

$$\lim_{\beta\to\infty} D = \alpha(\alpha+1), \quad \text{then} \lim_{\beta\to\infty} S = (1+\alpha)^2$$

so as $\beta \to \infty$, $y_1 \to \frac{1}{1+\alpha}$ and $y_2 \to \frac{1}{(1+\alpha)^2}$ which shows the interchange of y_1 and y_2 under the transformation $\beta \to \frac{1}{\beta}$ (by comparison with the $\beta=0$ case). For example as $\beta \to \infty$ for $\alpha=1$ one finds $y_1 = .5$ and $y_2 = .25$ (in the limit).

In this formulation, an increase in α means that failure rate is increasing relative to the ability of the system to repair components. Of course, this reduces component availability. For a fixed value of α the availability of component 1, namely y_1 increases with increasing β. An increase in β means that component 2 has increasing repair rate relative to component 1 (when $\beta > 1$, the repair rate is greater for component 2 than for component 1). Thus it is not the repair rate that produces the effect on availability but the waiting time. Even though the repair time for component 2 decreases in this instance, its availability decreases because it must wait while component 1 undergoes service with relatively longer service time.

The α, β curves of Figure 4 can be used in a number of ways in design trade off studies. For example if system availability analysis shows that $y_1 \geq .75$ and $y_2 \geq .65$ then only those α, β values yielding such pairs could



<u>Figure 4</u> $Q_{2,1}$

Steady State – 2 Component Availability Values

$\mu_2 = \beta\mu_1$

$\lambda = \alpha\mu$ (all cases)

λ = failure

μ = repair

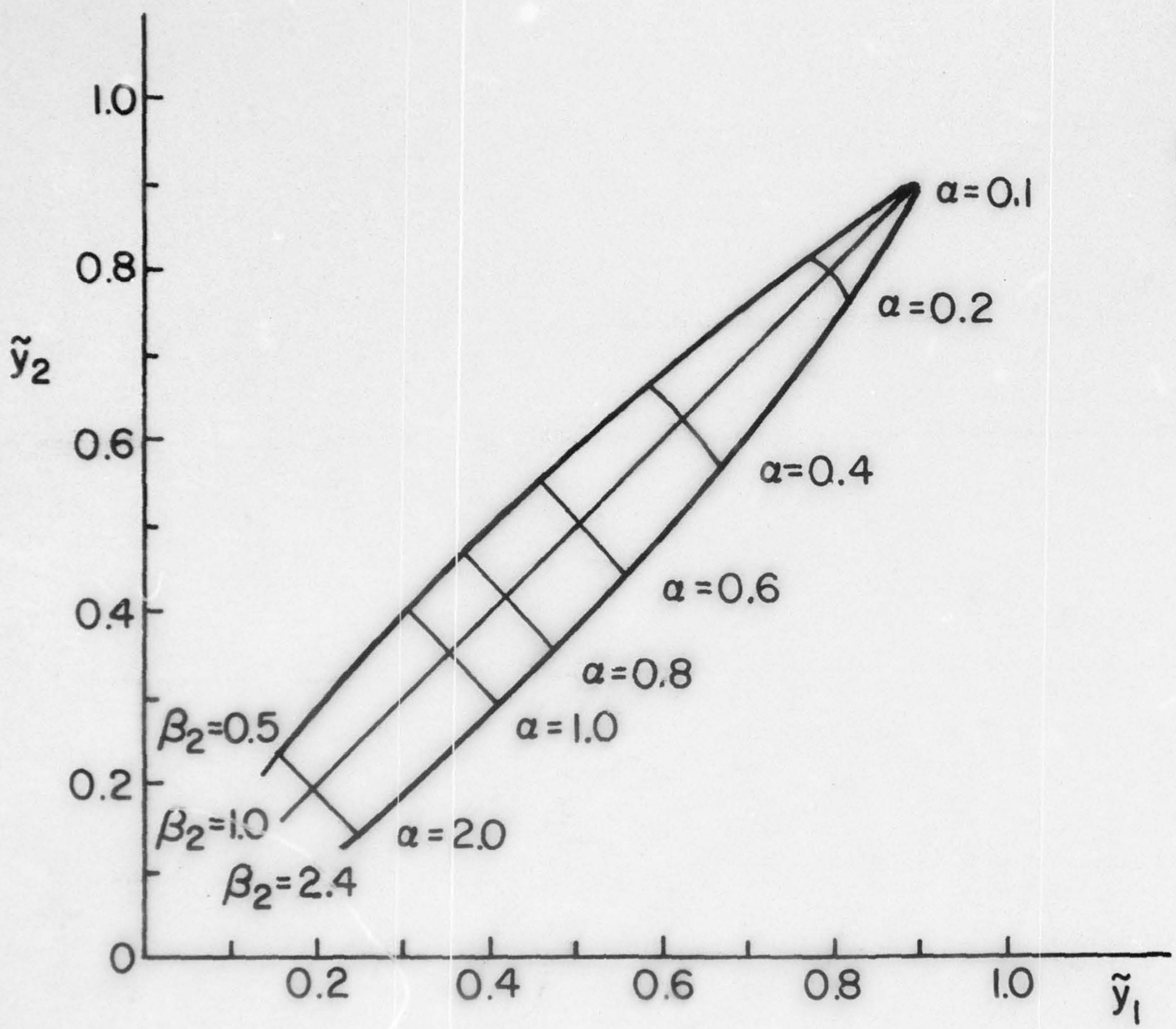


Figure 5

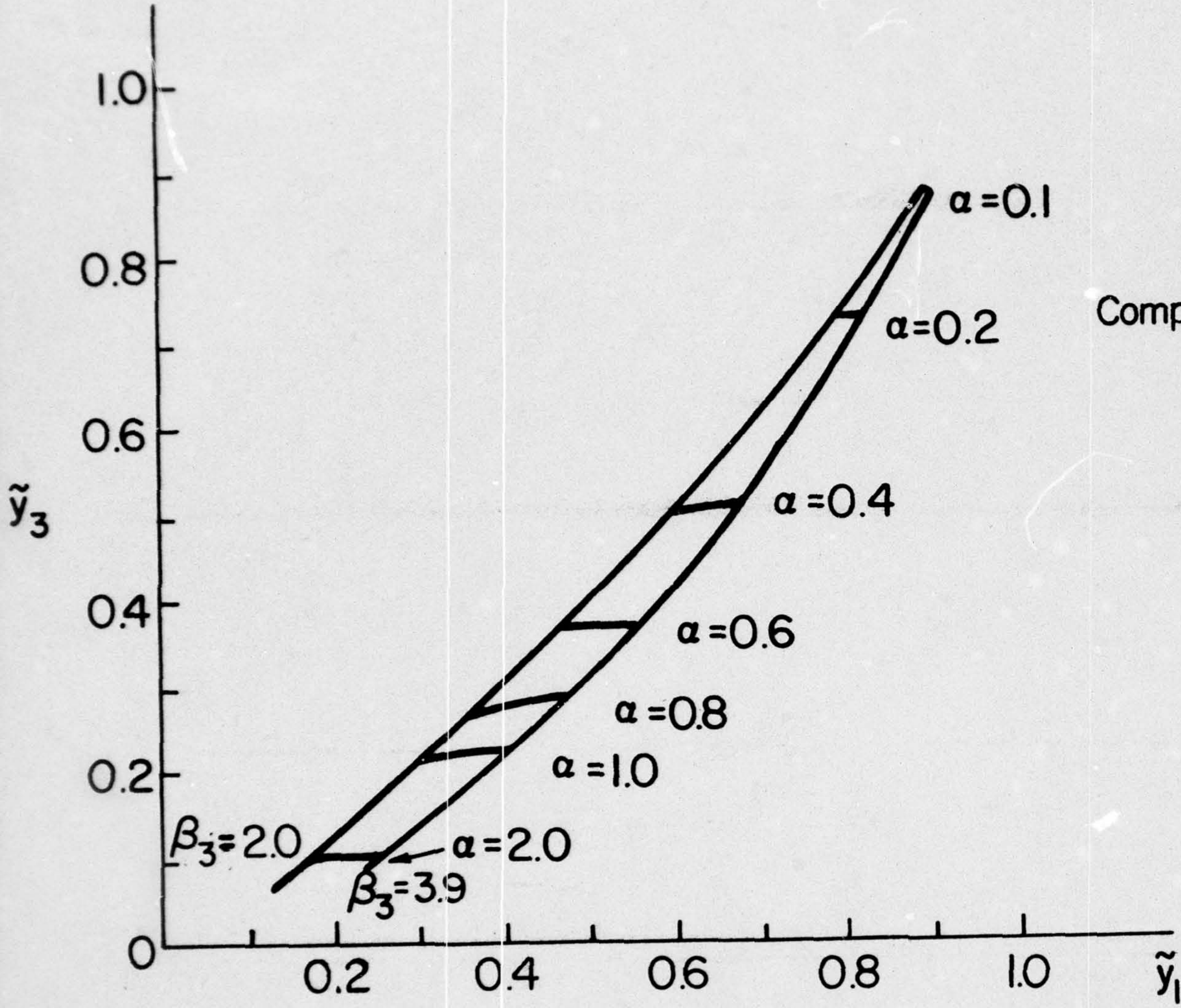
STEADY STATE $Q_{3,1}$

Component Availability Values $\tilde{y}_2, \tilde{y}_1$

$\mu_2 = \beta_2 \mu_1$, $\mu_3 = \beta_3 \mu_1$

$\lambda = \alpha \mu$ (all cases)

$\beta_3 = \beta_2 + 1.5$



<u>Figure 6</u> STEADY STATE $Q_{3,1}$

Component Availability Values $\tilde{y}_3, \tilde{y}_1$

$\mu_2 = \beta_2 \mu_1$, $\mu_3 = \beta_3 \mu_1$

$\lambda = \alpha \mu$ (all cases)

$\beta_3 = \beta_2 + 1.5$

Chapter 6 Directions for Expanding Research

6.1 Combinatorial queues

The concept of a queuing system in which the individual members of the population (components) are identified and followed through the stages of the system has been called a combinatorial queue in this report. Such queues were utilized to study two and three component systems with a FIFO service descipline and exponential arrival and service distributions. In addition to more general service times, discussed in the next section, a number of alternative investigations can be considered for the study of combinatorial queues as models for availability calculations.

If a component can be replaced with some short (or zero) wait then one can bring in the concept of an inventory for each part. The down part may undergo repair then enter its inventory to wait until needed as a replacement. This sort of situation, widely studied in the literature by various methods, has not been treated here. However it can be incorporated into a broader description of the combinatorial queue concept. In doing so it seems likely that a more general definition of component availability would result. One way to do this would be to define the component as the set of components that can perform a particular role within a system. Such a general component is only unavailable when the set becomes empty or during the transition period when a failed member of the set is replaced by one that is operable. There seems to be an area for useful research in this direction.

Another potentially useful generalization is to introduce priorities to replace the FIFO type of service. Different components can be assigned different service priorities on the basis of their importance to the system of which the components are a part. These priorities would represent one more system input to the component availability concept. A closely related idea arises when there are several repairmen available to the service function. Repairmen can be assigned to components in such a way that when a particular component fails it must be repaired by a particular repairman (or set of repairmen). An extreme case of this for, say two repairmen, would be having one repairman dedicated to one component and the other repairman responsible for the repair of all the other components. Such situations are likely to have useful application such as when one component is particularly complex or when its service requires special safety features.

Rather than have each component distinct in its descriptive characteristics it might be appropriate to have several components the same forming a class of similar components. The combinatorial queue would then deal with a number of classes where the particular members of one class have no special features. In the extreme cases where all components are the same there is only one class. This case has been treated in various sections of the present study. In many respects the generalization to classes of components shows potential utility. In particular it might lead to an ability to treat larger numbers of components than can be handled directly by the basic combinatorial queue models.

6.2 General Service Times

It is felt that a major restriction of the work described in this report is the use of exponential service time distributions. A considerable body of research has been carried out on service times and a number of distributions have been found based on combinations of statistical analysis and theoretical probability arguments. Therefore, it would clearly be of value to generalize the concepts of this report to a number of service time distributions. The ideas have been formulated and illustrated here for the exponential case. To generalize one would have to employ numerical solution techniques and simulation. It is unlikely that any kind of "closed form" solution approach would be possible.

One aspect of general service time analyses is to consider the service time as a sum of several random times (possibly dependent) each having its own characteristic type of probability distribution. Those characteristics arise from the nature of the service being carried out, possibly depending upon the relative amount of "search" activity involved in contrast to "repair" or "replace" activity. Indeed it may be conjectured that service time distributions can be formulated by proper mixes of such basic distributions. Such studies are rather distinct from the ideas of this report but combine with them in a more general description of component availability.

Of course one may also wish to generalize the way in which components fail. Exponential failure has been used in this report. This kind of generalization can also be introduced. However, it is felt that the service time is the more significant effect on any broader study of component availability.

6.3 Waiting Time as Input to Design Analysis

The basic result expressed as equation (1) in Section 1.2 relates component availability and waiting time in the repair service queue for components. Most of the present report has addressed the service queue directly and attained waiting time and availability for special cases. An alternative approach is to consider various waiting time distributions and use equation (1) to obtain the resulting availability values. In many cases this would be accomplished by numerical integration of equation (1) once the waiting time distribution was specified. Several procedures suggest themselves for the formulation of waiting time distributions:

. Various large population queue problems can be solved to give reasonable limit type values for waiting time distribution,

. Waiting time distributions may be stipulated in either numerical form or as specified distribution functions,

. Results for simple cases with known solutions for waiting time can be used as a basis for assuming the form of the waiting time distribution in more general cases.

The second of these approaches introduces a synthesis problem into queuing theory. One postulates a desirable waiting time distribution and uses equation (1) to obtain the corresponding availability. However, it is necessary to consider whether or not a service queue structure exists that would yield the postulated waiting time. If such a queue or queues exist the next problem is to construct an appropriate queue. These questions of queue synthesis seem to hold potential for considerable amounts of useful research.

The third procedure can be illustrated by considering some results obtained in this report. An outline of the arguments involved and an initial try at obtaining the form of a waiting time distribution are given below. This material is intended only to illustrate an approach, it has not been developed beyond initial stages.

A large class of waiting time distributions may be considered as steady state, not depending on the time at which a component enters the service queue. For the two components, one repairman case the steady state waiting time expression was found to be of the form (for FIFO service):

$$W_i(0) = a_i$$

$$W_i(s) = 1 - b_i e^{-\mu_j s}$$

where μ_j is the repair rate of the "other" component and a_i, b_i are normalized probabilities of none in the queue and the "other" in the queue respectively. Normalization was achieved by dividing by the sum of the unnormalized probabilities which represents the probability of component i not in the queue, that is the availability y_i of component i.

One may generalize from the above result and consider the three components, one repairman case. In that case, for a typical component, say i = 1 one may consider:

$y_1 = p_{000} + p_{002} + p_{003} + p_{023} + p_{032}$, and assume the form:

$$W_1(0) = \frac{p_{000}}{y_1}$$

$$W_1(s) = \frac{p_{000}}{y_1} + (1-e^{-\mu_2 s})\frac{p_{002}}{y_1} + (1-e^{-\mu_3 s})\frac{p_{003}}{y_1} + A\frac{p_{023}}{y_1} + B\frac{p_{032}}{y_1},$$

where A and B depend on the joint distribution of service for components 2 and 3.

The joint density for two independent exponential distributions is:

$$\frac{\mu_2 \mu_3}{\mu_2 - \mu_3}(e^{-\mu_3 s} - e^{-\mu_2 s})$$

where component 2 is in service first then component 3.

A reasonable way to generalize this is in terms of the moment generating function which for two variables is:

$$\frac{\mu_2 \mu_3}{(\mu_2 - t)(\mu_3 - t)} = \frac{a}{\mu_2 - t} + \frac{b}{\mu_3 - t} = \frac{-\mu_2 \mu_3}{(\mu_2 - \mu_3)(\mu_2 - t)} + \frac{\mu_2 \mu_3}{(\mu_2 - \mu_3)(\mu_3 - t)}.$$

For three joint exponential variables, supposedly related to components 1, 2 and 3 one has the generating function:

$$\frac{\mu_1}{\mu_1 - t} \cdot \frac{\mu_2}{\mu_2 - t} \cdot \frac{\mu_3}{\mu_3 - t} = \frac{\mu_2 \; \mu_3}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)} \left(\frac{\mu_1}{\mu_1 - t}\right)$$

$$+ \frac{\mu_1 \; \mu_3}{(\mu_1 - \mu_2)(\mu_3 - \mu_2)} \left(\frac{\mu_2}{\mu_2 - t}\right) + \frac{\mu_1 \; \mu_2}{(\mu_1 - \mu_3)(\mu_2 - \mu_3)} \left(\frac{\mu_3}{\mu_3 - t}\right).$$

This yields the density function:

$$\frac{\mu_1 \; \mu_2 \; \mu_3}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)} \, e^{-\mu_1 s} + \frac{\mu_1 \mu_2 \mu_3}{(\mu_1 - \mu_2)(\mu_3 - \mu_2)} \, e^{-\mu_2 s} + \frac{\mu_1 \mu_2 \mu_3}{(\mu_1 - \mu_3)(\mu_2 - \mu_3)} \, e^{-\mu_3 s}.$$

In general for m joint exponentials the density function has the form

$$a \sum_{i=1}^{m} \frac{e^{-\mu_i s}}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{m} (\mu_j - \mu_i)} , \quad a = \prod_{i=1}^{m} \mu_i$$

These kind of density functions may be used in combination with parameter values for the steady state queue probabilities to form waiting time distributions. One can retain the availability factor y in the waiting time expression. Such a procedure would turn equation (1) of Section 1.2 into an integro-differential equation for component availability which could be subjected to analysis.

6.4 Optimization and Trade Off Analysis

The work reported on in this report has delt with component availability in the context of a repair facility present as part of the total operational system. The point of view considered was taken from the operations end of the system and the desirability of available components. An alternative point of view is in terms of cost, either in actual expenditure or modified performance, of desired levels of availability. The effect of one, two or more repairmen has been investigated as regards the resulting levels of component availability. However, the cost increments have not been considered. For example it may be better relative to some measure of cost (or effectiveness) to have the kind of availability one repairman can supply rather than have two repairmen at increased expense. One can formulate an appropriate objective function or goal of system operation. These quantitative expressions combine system requirements in terms of component availability and the cost of repair operations. They should also include the cost of availability levels representing the value of increased availability and the penalty for reduced availability. A study of goal seeking or optimization of effective objectives can be made subject to the constraints imposed by relations between component availability and the repair function. Such relations have been formulated in the present work so that the initial steps have already been taken toward trade off analyses of available systems in terms of component availability. Every system has its own particular characteristic so that a general theory of trade off optimization can not be made. However, some general methodological procedure can be formulated and illustrated using special system configurations.

Appendix

Some typical numerical results are presented here in graph form. These are meant to be illustrative of the kind of results one may obtain from both the differential equation systems and the steady state formulations. Typical parameter values have been used as specified on the various graphs.

Differential equation systems are solved using a subroutine recently developed by R. Flynn at the Polytechnic.

The systems of linear algerbraic equations representing steady state values are of course redundant. Thus one must utilize a subset in each case and our procedure was to omit the initial equation in each formulation. This worked very well. A standard subroutine from the Scientific Subroutine Package was employed. Similar calculations were used to produce Figures 4, 5, and 6 of Chapter 5.

The following examples are presented in the Appendix:

Figure Number.	Case.
7 --------------------	$Q_{2,1}$ time variation
8 --------------------	$Q_{2,1}$ steady state
9 --------------------	$Q_{3,1}$ time variation
10 --------------------	$Q_{3,1}$ steady state
11 --------------------	$Q_{3,2}$ time variation
12 --------------------	$Q_{3,2}$ steady state

References

1. M. Shooman *Probabilistic Reliability*, Mc Graw-Hill, 1968

2. D. Gross and C. Harris *Fundamentals of Queuing Theory*, Wiley, 1974

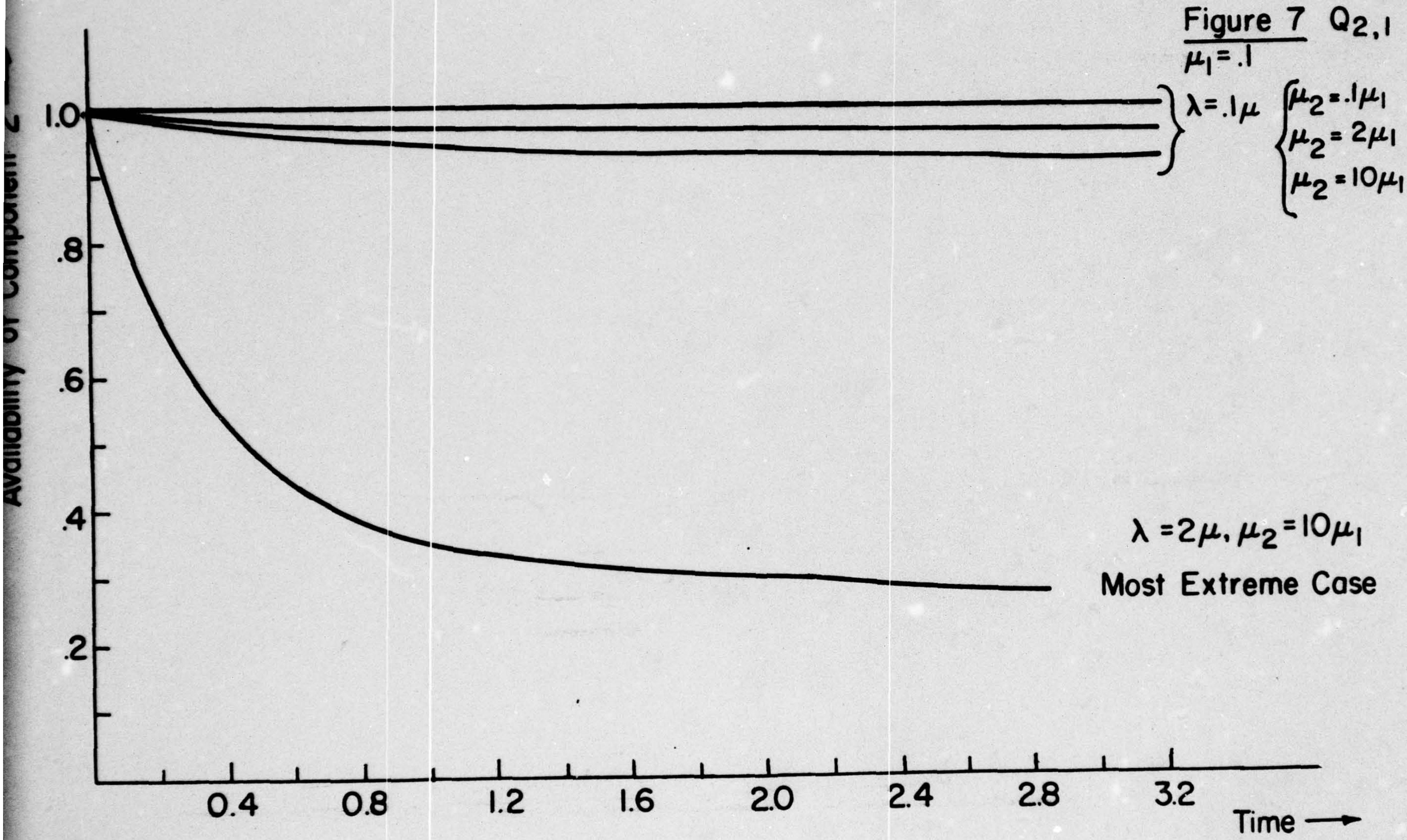


Figure 7 $Q_{2,1}$
$\mu_1 = .1$
$\lambda = .1\mu$
$\mu_2 = .1\mu_1$
$\mu_2 = 2\mu_1$
$\mu_2 = 10\mu_1$
$\lambda = 2\mu, \mu_2 = 10\mu_1$
Most Extreme Case
Availability of Component 2
1.0
.8
.6
.4
.2
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
Time

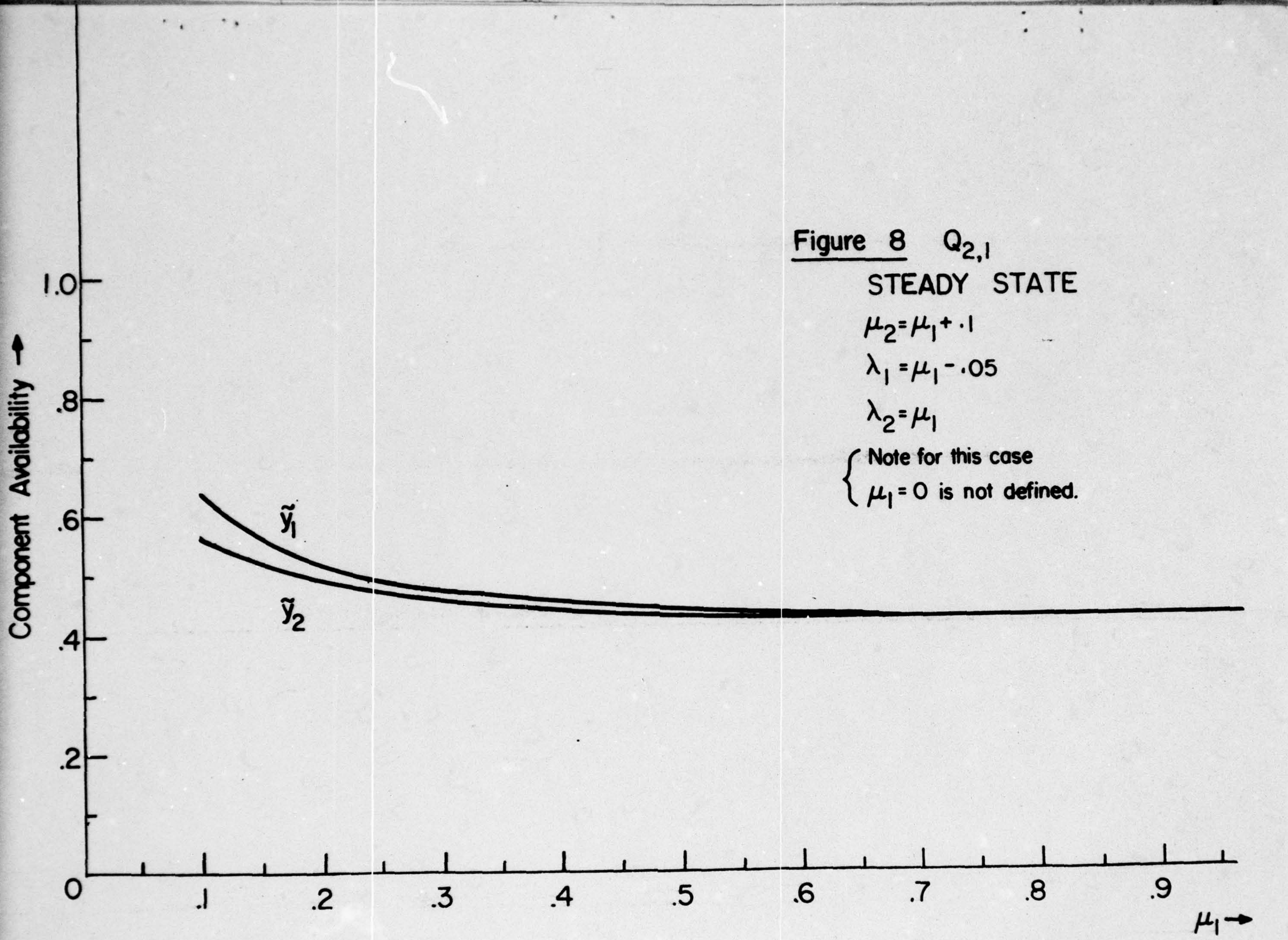
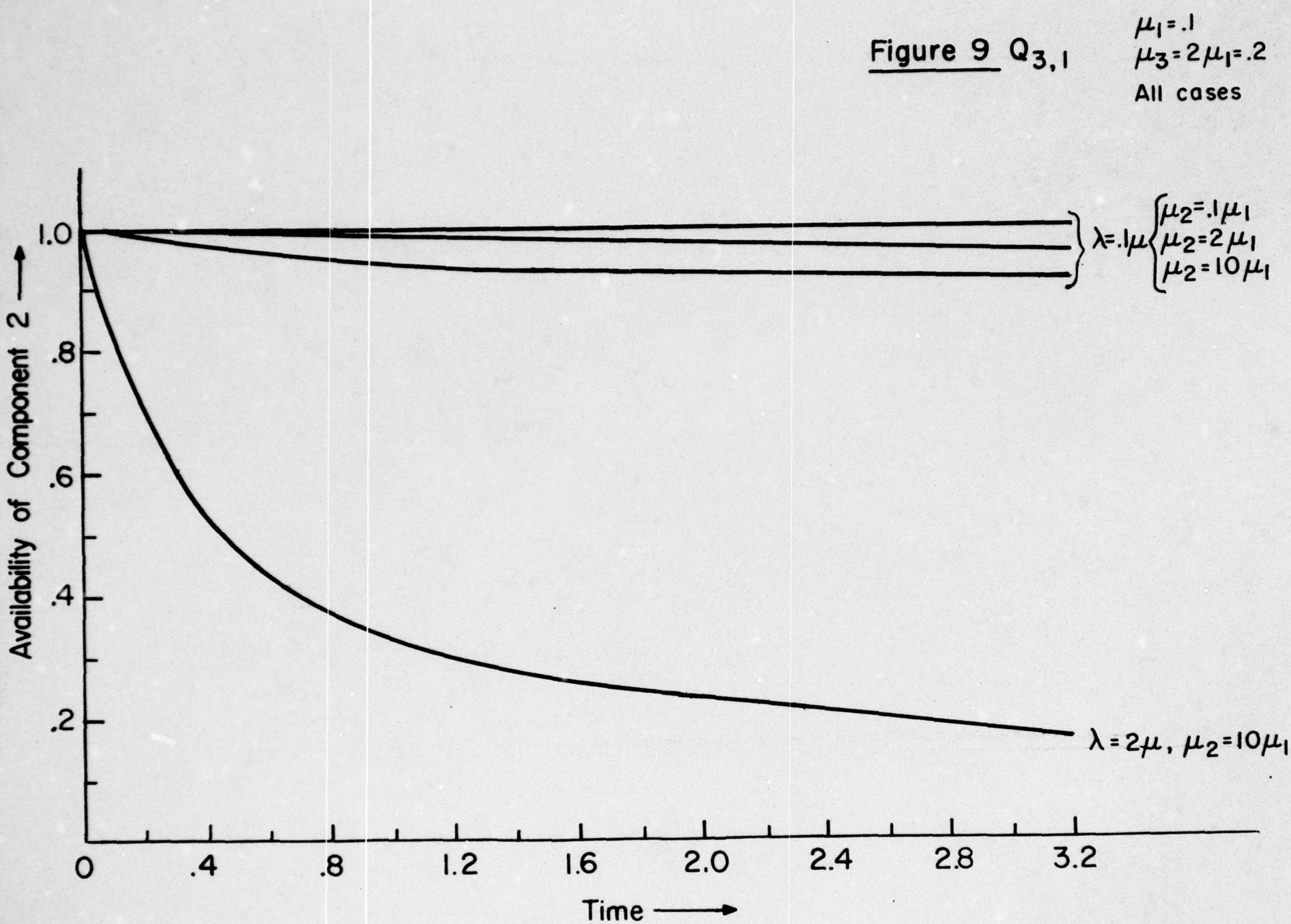


Figure 8 $Q_{2,1}$
STEADY STATE
$\mu_2 = \mu_1 + .1$
$\lambda_1 = \mu_1 - .05$
$\lambda_2 = \mu_1$
Note for this case
$\mu_1 = 0$ is not defined.
Component Availability
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
μ_1
$\tilde{y}_1$
$\tilde{y}_2$

Figure 9 $Q_{3,1}$



μ1 = .1
μ3 = 2μ1 = .2
All cases
λ = .1μ
μ2 = .1μ1
μ2 = 2μ1
μ2 = 10μ1
λ = 2μ, μ2 = 10μ1
Availability of Component 2
1.0
.8
.6
.4
.2
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
Time

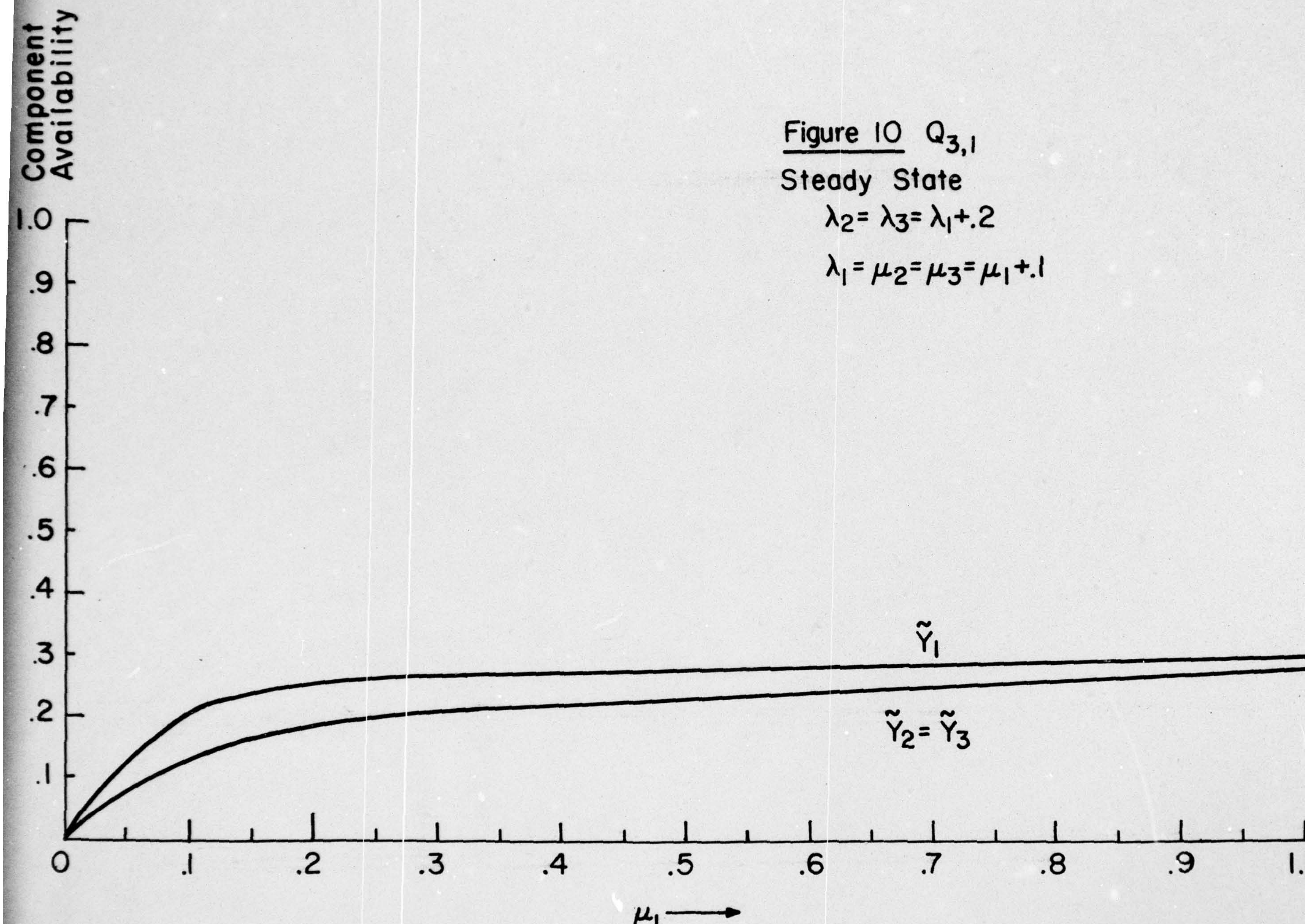


Figure 10 $Q_{3,1}$
Steady State
$\lambda_2 = \lambda_3 = \lambda_1 + .2$
$\lambda_1 = \mu_2 = \mu_3 = \mu_1 + .1$
Component Availability
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\tilde{Y}_1$
$\tilde{Y}_2 = \tilde{Y}_3$
μ_1 ⟶

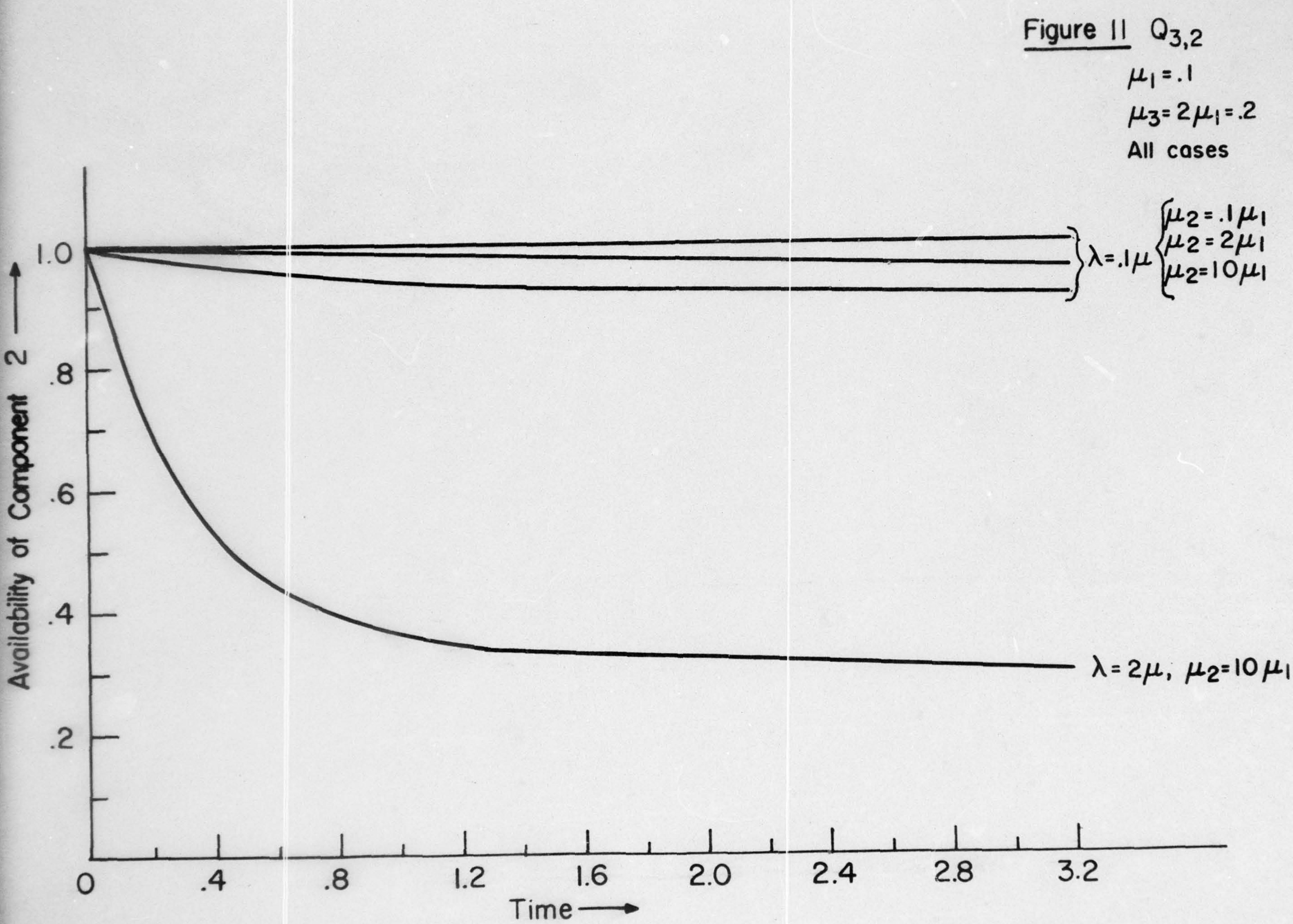


Figure 11 Q3,2
μ1 = .1
μ3 = 2μ1 = .2
All cases
λ = .1μ
μ2 = .1μ1
μ2 = 2μ1
μ2 = 10μ1
λ = 2μ, μ2 = 10μ1
Availability of Component 2
1.0
.8
.6
.4
.2
0
.4
.8
1.2
1.6
2.0
2.4
2.8
3.2
Time

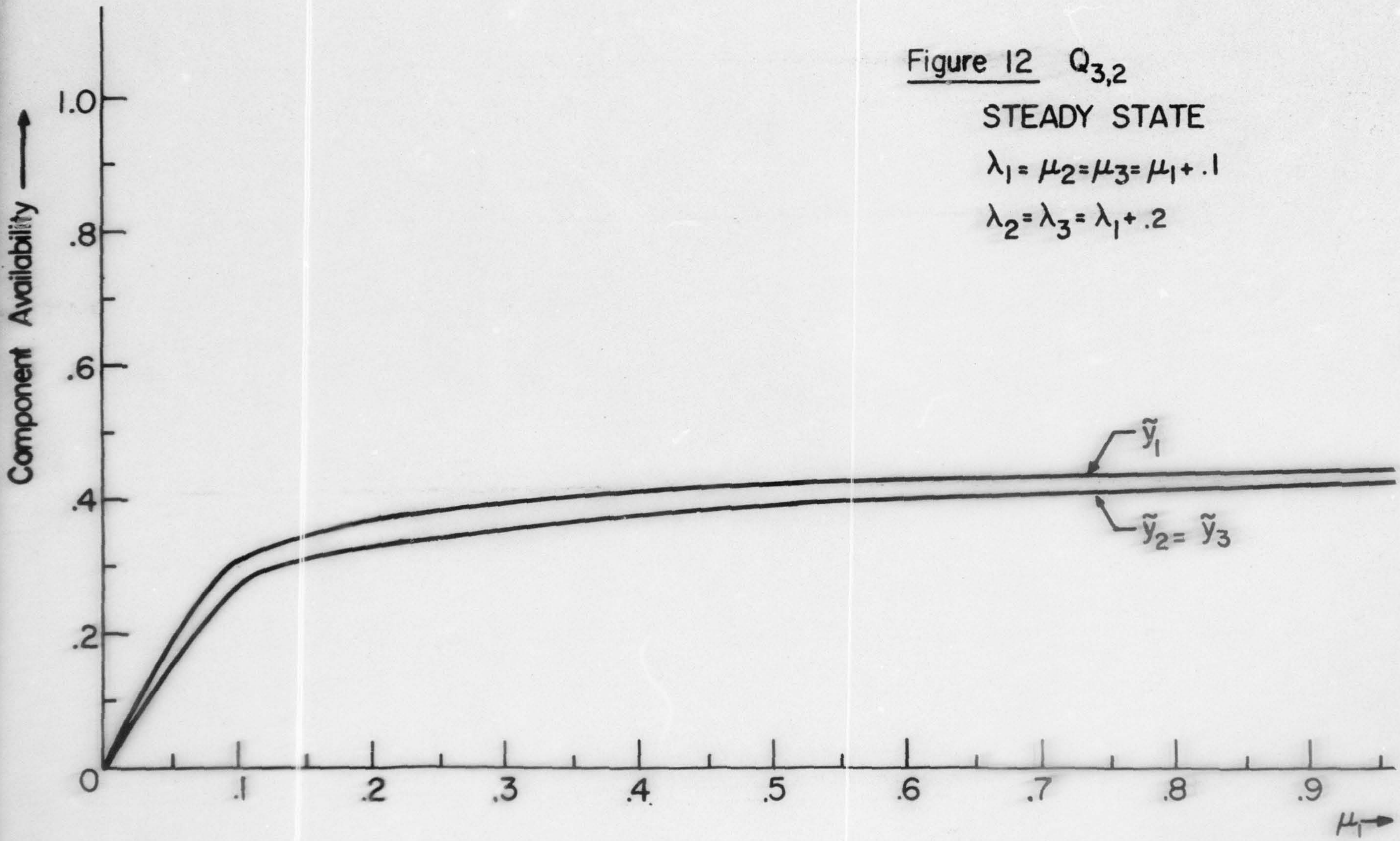


Figure 12 $Q_{3,2}$

STEADY STATE

$\lambda_1 = \mu_2 = \mu_3 = \mu_1 + .1$

$\lambda_2 = \lambda_3 = \lambda_1 + .2$

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER Poly EE/EP 76-004	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Contribution to the Theory of Component Availability		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER EER 121
7. AUTHOR(s) Clifford Marshall		8. CONTRACT OR GRANT NUMBER(s) N00014-67-A-0438-0013
9. PERFORMING ORGANIZATION NAME AND ADDRESS Polytechnic Institute of New York 333 Jay Street Brooklyn, New York 11201		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-301
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (Code 436) 800 N. Quincy Street Arlington, VA 22217		12. REPORT DATE February 1976
		13. NUMBER OF PAGES 63
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Availability
Reliability
Differential Equations
Queue

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper develops the concept of component availability as intrinsically dependent upon the system in which the component functions. The connection between the repair aspects of the system and component availability is developed in two ways. An integro-differential equation is derived that relates the two concepts. Alternatively the repair process is treated as a combinatorial queue in which each element of the system is tracked through its states of availability. Examples are given in some detail for small number of elements. Numerical solutions are given for illustrative purposes. Both

next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (Continued)

time varying and steady state solutions are obtained. Some approaches to a trade off analysis for availability specification or design are given.