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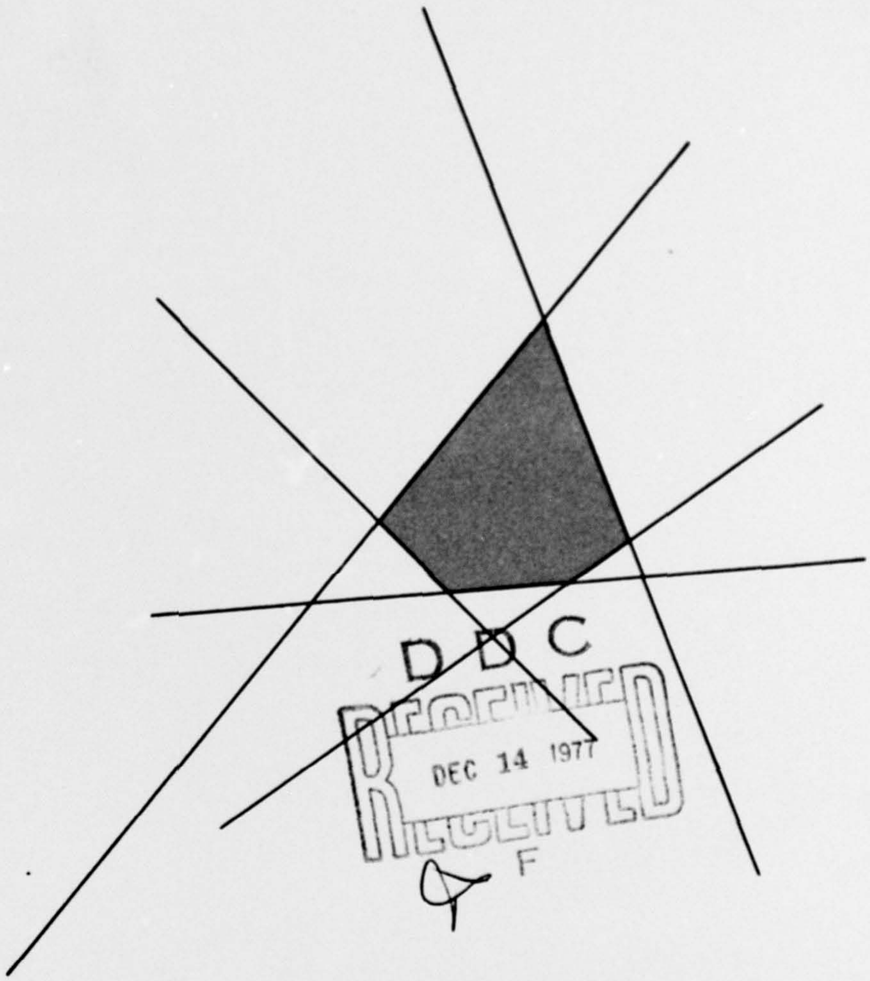
by

DAVID GALE



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FAIR DIVISION OF A RANDOM HARVEST

by

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ABSTRACT

A resource such as an orchard is owned jointly by m agents, the i^{th} agent's *share* of the resource being θ_i. The yield of the resource, (the *harvest*) and the utilities of each agent are functions of the state of nature. A *fair distribution scheme* is one which is (1) (Pareto) optimal and which (2) gives each agent an expected consumption proportional to his share of the resource. ~~We show that~~ with the usual concavity assumptions on utilities there always exists one and only one fair distribution scheme. The proof is achieved by constructing a suitable social welfare function which is maximized at the desired distribution scheme.

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1. INTRODUCTION

A number of agents who may be thought of as individuals, companies or countries are joint owners of some productive resource such as an orchard or an ocean. The yield of this resource is a random variable depending on the state of nature. Thus, the size of an apple harvest may depend upon the amount of rainfall, temperature fluctuations etc. or the catch of tuna may vary with prevailing winds and currents. Each agent, A_i, is assumed to have his own utility function which may also depend on the state of nature. Thus, one's appetite for apples may be different in a dry hot summer from what it is in a cold wet one. The agents wish to arrive at some sort of a *distribution scheme*, that is an arrangement for dividing up the harvest in each possible state of nature. Now there is one property that any such scheme ought to have, namely it should be (Pareto) optimal, meaning that no other scheme should provide a higher *expected utility* to all of the agents. While this seems a fairly obvious requirement it should be pointed out that it sharply limits the set of acceptable schemes. For example, the simple minded arrangement in which each of m agents receives one m^{th} of the harvest regardless of the state of nature will in general be nonoptimal. In the very simplest case of two agents and two states of nature if the agents are not identical it will almost always be the case that both will prefer to have one agent get more than half the harvest in one state if the other gets more than half in the other as opposed to a fifty-fifty split in both. There are, nevertheless, an infinity of possible optimal distribution schemes but some of these are clearly "unfair," e.g., always giving the entire harvest to the first agent. To arrive at a notion of fairness we will suppose there is given a set of positive numbers $\theta_1, \ldots, \theta_m$, which sum to one, where θ_i is A_i's *share* of the resource. Thus, if the agents own equal shares then $\theta_i = \frac{1}{m}$ for all i. If the agents are thought of as countries then θ_i might be proportional to the population of the i^{th} country. If the agents are firms the θ_i might reflect the amount each firm has invested in developing the resource. Given the shares θ_i there is now a rather natural way to define fairness. Let $h(s)$ be the size of the harvest in state s and let $\bar{h}$ be the expected value of $h(s)$. Let $c_i(s)$ be the amount consumed by A_i in state s under some distribution scheme. We call this distribution scheme *fair* if $\bar{c}_i = \theta_i \bar{h}$ where $\bar{c}_i$ is the expected consumption of A_i under this scheme. Such a criterion would seem reasonable to an outsider who knew nothing of the utility function of the agents, or, to put it the other way, any scheme which violated this condition would appear to give some agents more or less than their share. For further discussion of this notion of fairness see the final section.

In the present paper we consider only the case in which there are only a finite number of states of nature. A subsequent paper will take up the case where s belongs to a general probability space. For the finite case our result can be stated concisely as follows: *If all utility function are strictly concave and increasing then there exist exactly one distribution scheme which is both optimal and fair.* The proof is achieved by exhibiting a particular social welfare function ρ defined over the set of all fair distribution schemes. This function being strictly concave has a unique maximum. Our proof then consists in showing that a fair scheme is optimal if and only if it maximizes ρ. The argument consists of repeated application (4 times) of the Kuhn-Tucker theorem.

Since the social welfare function ρ is the center piece of our exposition we will define it here for the reader to contemplate. Let $u_{ij}(c)$ be the utility of c units of consumption to A_i in state s_j and define

(1) $$\rho_{ij}(c) = \int_1^c \log u'_{ij}(x)dx .$$

Let $C = (c_{ij})$ be the $m \times n$ matrix corresponding to a fair distribution scheme where c_{ij} denotes the amount consumed by A_i in s_j. Then

(2) $$\rho(C) = \sum_{i,j} \rho_{ij}(c_{ij}) .$$

We would welcome any suggestions as to the economic interpretation of this rather curious function.

2. PRELIMINARY SIMPLIFICATIONS

Let p_j denote the probability that state s_j occurs and let h_j be the size of the harvest in this case. A fair distribution is then an $m \times n$ matrix $C = (c_{ij})$ such that

$$\sum_i c_{ij} = h_j \quad \text{for all } j , \tag{3}$$

$$\sum_j p_i c_{ij} = q_i \quad \text{for all } i \tag{4}$$

where $q_i = \theta_i \sum_j p_j h_j$.

It is convenient to reduce our problem to the special case where all the p_j are equal. For this purpose introduce new variables $x_{ij} = p_j c_{ij}$ and let $k_j = p_j h_j$. Then (3) is equivalent to

$$\sum_i x_{ij} = k_j \tag{3'}$$

and (4) is equivalent to

$$\sum_j x_{ij} = q_i . \tag{4'}$$

Note that from our definition $\sum_j k_j = \sum_i q_i$.

Finally define functions v_{ij} by

$$v_{ij}(x) = p_j u_{ij}(x/p_j) . \tag{5}$$

Then the expected utility of A_i under the distribution C is $\sum_j p_j u_{ij}(c_{ij}) = \sum_j p_j u_{ij}(x_{ij}/p_j) = \sum_j v_{ij}(x_{ij})$. We denote this last sum by $v_i(X)$ where X is the $m \times n$ matrix of x_{ij}'s .

Note that the functions v_{ij} inherit the relevant properties of the u_{ij} , i.e., concavity differentiality, etc.

We will call a distribution matrix X feasible if it satisfies (3') and *fair* if it satisfies (4'). A feasible X is called *optimal* if there is no other feasible X' such that $v_i(X') \geqq v_i(X)$ for all i with strict inequality for at least one i . Our result now becomes the following

Theorem 1:

If the functions v_{ij} are differentiable, concave and increasing then there exists a unique distribution matrix which is both fair and optimal.

3. THE KUHN-TUCKER THEOREM

We present a version of the Kuhn-Tucker theorem needed for the present application.

Let f be a strictly concave differentiable function from R_+^n into $\bar{R}$, the reals including $\pm\infty$. Let A be an $m \times n$ matrix and b an m-vector and define K by

$$K = \left\{ x \mid Ax = b,\ x \in R_+^n \right\} .$$

Denote by f_j the partial derivative of f with respect to x_j and denote by a_j the j^{th} column of A.

Theorem: (K - T)

If K contains a strictly positive vector then $\bar{x}$ maximizes f in K if and only if there exists an m-vector u such that

$$\begin{aligned} f_j(\bar{x}) - ua_j &\leqq 0 \quad \text{for all } j \\ &= 0 \quad \text{if } \bar{x}_j > 0 . \end{aligned}$$

For a proof see for example [1].

4. PROOF OF THE MAIN THEOREM

As in the introduction define functions ρ_{ij} by the rule

$$\rho_{ij}(x) = \int_1^x \log v'_{ij}(t)dt \quad \text{for} \quad x \geqq 0 .$$

Since we allow the possibility that $\lim_{x\to 0} v_{ij}(x) = \infty$ it is possible also that the $\rho_{ij}(x)$ decrease to $-\infty$ as x approaches 0 in which case we define $\rho_{ij}(0) = -\infty$.

Lemma:

The functions ρ_{ij} are differentiable and concave.

Proof:

Since v'_{ij} is positive and continuous ρ_{ij} exists and its derivative is $\log v'_{ij}$ which is decreasing because v'_{ij} is decreasing and log is increasing, so ρ_{ij} is strictly concave.■

Our main result follows from

Theorem 2:

The distribution matrix X is fair and optimal if and only if it maximizes $\rho(X) = \sum_{i,j} \rho_{ij}(x_{ij})$ among all nonnegative matrices satisfying (3') and (4').

This theorem implies our result for since ρ is strictly concave and the set of solutions of (3') and (4') is compact ρ attains a maximum at only one point.

Proof:

Let $\bar{X} = (\bar{x}_{ij})$ maximize ρ . Note that the positve matrix $X = \left(q_i k_j / \sum_i q_i\right)$ gives a positive solution of (3'), (4') so the hypotheses of our Kuhn-Tucker theorem are satisfied. Therefore there exist numbers λ_i and μ_j such that

$$(6) \qquad \log v'_{ij}(\bar{x}_{ij}) - \lambda_i \leqq \mu_j \quad \text{for all } i, j$$
$$= \mu_j \quad \text{if } \bar{x}_{ij} > 0 .$$

Letting $\alpha_i = e^{-\lambda_i}$, $\beta_j = e^{\mu_j}$ this gives

$$(7) \qquad \alpha_i v_{ij}(\bar{x}_{ij}) \leqq \beta_j \quad \text{for all } i, j$$
$$= \beta_j \quad \text{if } \bar{x}_{ij} > 0 .$$

Applying (7) for a fixed j and using K - T in the other direction we see that $(\bar{x}_j)$ maximizes $\sum_i \alpha_i v_{ij}(x_{ij})$ subject to (3') where $\bar{x}_j = (x_{1j}, \ldots, x_{mj})$. Thus if X is any distribution matrix satisfying (3') then

$$(8) \qquad \sum_i \alpha_i v_{ij}(\bar{x}_{ij}) \geqq \sum_i \alpha_i v_{ij}(x_{ij}) .$$

Recall that by definition $v_i(X) = \sum_j v_{ij}(x_{ij})$. Then summing (8) on j gives

$$(9) \qquad \sum_i \alpha_i v_i(\bar{X}) \geqq \sum_i \alpha_i v_i(X)$$

and since $\alpha_i > 0$ for all i we cannot have $v_i(X) \geqq v_i(\bar{X})$ for all i, and $v_i(X) > v_i(\bar{X})$ for some i, so $\bar{X}$ is optimal.

Conversely suppose $\bar{X}$ is fair and optimal. Let U be the set of all points $u \varepsilon R^m$ such that there exists a feasible X with $v_i(X) \geqq u_i$ for all i. From the concavity of the v_i the set U is convex and $(v_1(\bar{X}), \ldots, v_m(\bar{X}))$ is an efficient point of U since $\bar{X}$ is optimal so there exist nonnegative numbers α_i such that $\bar{X}$ maximizes $\sum_i \alpha_i v_i(X) = \sum_i \alpha_i \sum_j v_{ij}(x_{ij}) = \sum_j \sum_i \alpha_i v_{ij}(x_{ij})$ so for each j

$$(10) \qquad \bar{x}_j \text{ maximizes } \sum_i \alpha_i v_{ij}(x_{ij}) \text{ subject to } \sum x_{ij} = k_j .$$

The above means that $\alpha_i > 0$ for all i, for suppose, say, $\alpha_1 = 0$. Then since $q_1 > 0$ we must have $x_{1j} > 0$ for some j say $j = 1$, but since v_{11} is increasing it is clear that to maximize $\sum \alpha_i v_{i1}(x_{i1})$ one would set $x_{11} = 0$, a contradiction. Applying K - T to (10) for each j gives the existence of numbers β_j such that

$$\alpha_i v'_{ij}(\bar{x}_{ij}) \leqq \beta_j$$
$$= \beta_j \text{ if } \bar{x}_{ij} > 0$$

which is exactly (7) and since $\alpha_i > 0$ and $v'_{ij} > 0$ we have $\beta_j > 0$. Setting $\lambda_i = -\log \alpha_i$ and $\mu_j = \log \beta_j$ we get again (6), and applying K - T backwards once more we see that $\bar{X}$ maximizes ρ. ∎

REFERENCE

[1] Luenberger, D. G., INTRODUCTION TO LINEAR AND NONLINEAR PROGRAMMING, Addison-Wesley, (1965).