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REGENERATIVE SIMULATION FOR EXTREME VALUES*

by

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1. Introduction

Let $\underset{\sim}{X} = \{X_t : t \geq 0\}$ be a regenerative process which we wish to simulate. Under mild regularity conditions the distribution of X_t converges to the distribution of some limiting random variable (or vector) X. This type of convergence is known as weak convergence and written $X_t \Rightarrow X$, as $t \uparrow \infty$. Simulators speak of X as the "steady-state" configuration of the system and are often interested in estimating the constant $r = E\{f(X)\}$, where f is a given real-valued function defined on the state-space of the process $\underset{\sim}{X}$. The regenerative method of simulation provides a means of constructing point and interval estimates for r; see IGLEHART (1977) for an expository summary of this method.

The problem we consider in this paper does not involve estimation of r, but rather the estimation of extreme values of the regenerative process $\underset{\sim}{X}$. Suppose, for the sake of discussion, we are simulating a stable GI/G/1 queue in order to estimate the maximum waiting time among the first $n+1$ customers; call this random variable W_n^*. As n grows, so will W_n^*. However, W_n^* does not converge to a finite limit, but rather

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diverges to $+\infty$. We will be interested in estimating the distribution function of W_n^* for finite, but large n. By the same token we might wish to estimate the maximum queue length during the interval $[0,t]$. While this problem of estimating extreme values would seem to be of great practical importance to simulators, we know of no papers in the simulation literature which offer any guidance on the subject. This paper will attempt to partially fill the gap.

We begin in Section 2 by summarizing a series of probabilistic results in extreme value theory which will provide the theoretical basis for the methods we propose. Section 3 discusses two methods for estimating extreme values for the general regenerative simulation. In Sections 4 and 5 we treat the special cases of the GI/G/1 queue and birth-death processes respectively. Theoretical results are available for these two classes of regenerative processes that are useful in assessing the accuracy of the simulation methods proposed. Section 6 contains the numerical results for simulations of the M/M/1 queue carried out to illustrate the estimation methods proposed.

2. Probabilistic Background

Let $\{F_n : n \geq 1\}$ be a sequence of distribution functions (d.f.'s) on the real line, $\mathbb{R} = (-\infty, +\infty)$. This sequence converges weakly to a d.f. G if $\lim_{n \to \infty} F_n(x) = G(x)$ for all $x \in \mathbb{R}$ which are continuity points of G. We write $F_n \Rightarrow G$ to denote this type of convergence. If X_n (resp. X) is a random variable (r.v.) with d.f. F_n (resp. G), we also write $X_n \Rightarrow X$ to denote this weak convergence. Sometimes it is convenient to write $X_n \Rightarrow G$ to connote the same thing. The material presented in this section can be found for the most part in deHAAN (1970), currently the best comprehensive treatment of the subject.

Now let $\{X_n : n \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) r.v.'s and denote the maximum of the first n r.v.'s by $M_n = \max\{X_k : 1 \leq k \leq n\}$. If each of the X_k's has d.f. F, then M_n will have d.f. F^n. We shall say that F belongs to the domain of attraction of the non-degenerate d.f. G, and write $F \in \mathcal{D}(G)$, if we can choose two sequences of constants $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ with $a_n > 0$ such that

$$F^n(a_n x + b_n) \to G(x) \tag{2.1}$$

as $n \to \infty$ for all $x \in \mathbb{R}$ for which G is continuous. Equivalently, $F \in \mathcal{D}(G)$ if $(M_n - b_n)/a_n \Rightarrow G$ as $n \to \infty$. Thus for large n we would approximate $P\{M_n \leq x\}$ by $G((x-b_n)/a_n)$. If a r.v. X has d.f. $F \in \mathcal{D}(G)$, we also write $X \in \mathcal{D}(G)$.

A famous result in extreme value theory states that the only d.f.'s G which can arise in (2.1) are of one of the following three types:

$$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-x^{-\alpha}), & x > 0 \end{cases} \tag{2.2}$$

$$(2.3) \qquad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$(2.4) \qquad \Lambda(x) = \exp(-e^{-x}), \qquad x \in \mathbb{R},$$

where in (2.2) and (2.3) α is a positive constant. Recall that two d.f.'s G_1 and G_2 are said to be of the same type if there exists two constant a and b, $a > 0$, such that $G_1(x) = G_2(ax + b)$ for all $x \in \mathbb{R}$. Thus aside from translations and scaling by a positive constant the three d.f.'s given in (2.2) - (2.4) are the only ones that can appear in (2.1). This result on the three types of limit d.f.'s is usually attributed to GNEDENKO (1943), however it was first formulated in this way by FISHER and TIPPETT (1928).

The next logical result to seek is necessary and sufficient conditions for $F \in \mathcal{D}(G)$, where G of necessity is one of the three d.f.'s given in (2.2) - (2.4). Furthermore, if $F \in \mathcal{D}(G)$ we need a method for selecting the two sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$. To this end we first define the right endpoint, $x_0 \leq +\infty$, of the d.f. F as

$$x_0 = \sup\{x : F(x) < 1\} .$$

A d.f. $F \in \mathcal{D}(\Phi_\alpha)$ if and only if for all $x > 0$

$$(2.5) \qquad \lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha} .$$

If $F \in \mathcal{D}(\Phi_\alpha)$, then we can take

$$(2.6) \qquad a_n = \inf\{x : 1 - F(x) \leq 1/n\}$$

and $b_n = 0$. A d.f. $F \in \mathcal{D}(\Psi_\alpha)$ if and only if $x_0 < \infty$ and for all $x > 0$

$$(2.7) \qquad \lim_{t \to \infty} \frac{1 - F[x_0 - (tx)^{-1}]}{1 - F(t)} = x^{-\alpha} .$$

If $F \in \mathcal{D}(\Psi_\alpha)$, then we can take $b_n = x_0$ and

$$a_n = x_0 - \inf\{x : 1 - F(x) \leq 1/n\} .$$

The final case, $F \in \mathcal{D}(\Lambda)$, is the most important one for our simulation applications. A d.f. $F \in \mathcal{D}(\Lambda)$ if and only if

$$(2.8) \qquad \lim_{t \uparrow x_0} \frac{1 - F(t + xf(t))}{1 - F(t)} = e^{-x} , \qquad \text{for all } x \in \mathbb{R} ,$$

where for $t < x_0$

$$f(t) = \frac{\int_t^{x_0} (1 - F(s))ds}{1 - F(t)} .$$

If $F \in \mathcal{D}(\Lambda)$, then we can take

$$b_n = \inf\{x : 1 - F(x) \le 1/n\}$$

and

$$a_n = \frac{\int_{b_n}^{x_0} [1 - F(t)]dt}{1 - F(b_n)} \quad .$$

Alternative expressions are available for a_n and b_n. Let $Q_n(p)$ denote the p-quantile of the d.f. F^n: for $0 < p < 1$,

$$Q_n(p) = \inf\{x : F^n(x) \ge p\} \quad .$$

Then if $F \in \mathcal{D}(\Lambda)$, we can alternatively select

(2.9) $$b_n = Q_n(e^{-1})$$

and

(2.10) $$a_n = Q_n(e^{-e^{-1}}) - Q_n(e^{-1}) \quad .$$

Furthermore, if $F \in \mathcal{D}(\Lambda)$ and $x_0 = +\infty$, $M_n/a_n \Rightarrow 1$ as $n \to \infty$. Many of the classical d.f.'s such as the exponential, gamma, normal, lognormal, logistic, and Cauchy belong to $\mathcal{D}(\Lambda)$.

Suppose F has $x_0 = +\infty$ and possesses an exponential tail:

(2.11) $$1 - F(x) \sim b \exp(-ax) \ , \qquad \text{as } x \to \infty \ ,$$

where a and b are two positive constants. Then it is easy to check that (2.8) holds and $F \in \mathcal{D}(\Lambda)$. Using the expressions (2.9) and (2.10) it can be shown that b_n and a_n can be selected as follows:

$$b_n = a^{-1} \ln(nb) \ ,$$

and

$$a_n = a^{-1} \ .$$

An interesting (and practical) situation arises if F is a discrete d.f. as, for example, the geometric d.f. $F(x) = 1 - \exp(-[x])$, $x \geq 0$, where $[x]$ is the integer part of x. In this case neither (2.5) nor (2.8) hold, and since $x_0 = +\infty$, F does not belong to the domain of attraction of any of the three types (2.2) - (2.4). However, a result has been salvaged by ANDERSON (1970). Let $\mathcal{A}$ be the class of all d.f.'s whose support consists of all sufficiently large positive integers. Then if $F \in \mathcal{A}$,

(2.12) $$\limsup_{n \to \infty} F^n(\alpha^{-1}x + b_n) \leq \exp(-e^{-x})$$

and

(2.13) $$\liminf_{n \to \infty} F^n(\alpha^{-1}x + b_n) \geq \exp(-e^{-(x-\alpha)})$$

for some $\alpha > 0$, all x, and some sequence $\{b_n : n \geq 1\}$ if and only if

(2.14) $$\lim_{n \to \infty} \frac{1 - F(n)}{1 - F(n+1)} = e^{\alpha} \ .$$

When this condition holds, the constants b_n can be selected as follows. For $F \in \mathcal{Q}$ and each positive integer n let $h(n) = -\log(1-F(n))$ and define h_c to be the extension of h obtained by linear interpolation for $x \geq 1$. Then define for $x \geq 1$

$$F_c(x) = 1 - \exp(-h_c(x)) \quad .$$

Clearly F_c is a continuous d.f. and for sufficiently large x is strictly increasing since $F \in \mathcal{Q}$. For $x \leq 1$ the F_c can be defined arbitrarily just so long as it is a d.f. In terms of F_c we can define b_n for large n as the unique root of

$$1 - F_c(b_n) = 1/n \quad .$$

If $F \in \mathcal{Q}$ and $1 - F(n) \sim b \exp(-an)$ as $n \to \infty$ $(a,b > 0)$, then for $b_n = a^{-1} \ln(nb)$ it can easily be shown using the method followed by HEYDE (1970) that for integer k

$$\lim_{n \to \infty} \left[P\{M_n - [b_n] \leq k\} - \exp(-e^{-a(k-d_n)}) \right] = 0 \, , \tag{2.15}$$

where $d_n = b_n - [b_n]$. Thus for n large we would use the approximation

$$P\{M_n \le k + [b_n]\} \cong \exp(-e^{-a(k-d_n)}) \quad ,$$

or

$$(2.16) \qquad P\{M_n \le k\} \cong \exp(-e^{-a(k-b_n)}) \quad .$$

Suppose now that we also have defined on the probability triple $(\Omega,\mathfrak{F},P)$ that supports the i.i.d. sequence $\{X_n : n \ge 1\}$ a renewal process $\{\ell(t) : t \ge 0\}$ with mean time between renewals m $(0 < m < \infty)$. Then the weak law of large numbers for renewal processes states that $\ell(t)/t \Rightarrow m^{-1}$ as $n \to \infty$. Next set

$$M_t' = \max\{X_k : 1 \le k \le \ell(t)\} \quad .$$

The following useful result for this situation was obtained by BERMAN (1962). If $(M_n - b_n)/a_n \Rightarrow G$, one of the three extreme value d.f.'s (2.2) - (2.4), then as $t \to \infty$

$$(2.17) \qquad (M_t' - b_{[t]})/a_{[t]} \Rightarrow G^{1/m} \quad .$$

This result provides a useful tool for extreme values of regenerative processes. To be explicit suppose $\underset{\sim}{X} = \{X_t : t \ge 0\}$ is a regenerative process defined on $(\Omega,\mathfrak{F},P)$ and T_k, $k \ge 1$, is the time of the kth regeneration point of $\underset{\sim}{X}$ with $T_0 = 0$. Then the renewal process $\{\ell(t) : t \ge 0\}$ which counts the number of regeneration points in $(0,t]$ is defined by

$$\ell(t) = \max\{k : T_k \leq t\}$$

with $\ell(0) = 0$. For $k \geq 1$, let

$$M_k^+ = \sup\{X_t : T_{k-1} \leq t < T_k\} .$$

Since $\underset{\sim}{X}$ is regenerative, the sequence of maxima, $\{M_k^+ : k \geq 1\}$, will be i.i.d. Then if $L_t = \sup\{X_t : 0 \leq s \leq t\}$, clearly

$$\max\{M_k^+ : 1 \leq k \leq \ell(t)\} \leq L_t \leq \max\{M_k^+ : 1 \leq k \leq \ell(t) + 1\} . \tag{2.18}$$

Combining the inequalities of (2.15) with the limit theorem of (2.14) enables us to show that

$$(L_t - b_{[t]})/a_{[t]} \Rightarrow G^{1/m} , \tag{2.19}$$

where $m = E\{T_1\}$, provided $M_1^+ \in \mathcal{D}(G)$. Of course if $M_1^+ \in \mathcal{Q}$, then the weaker results of Anderson or Heyde are all that can be expected.

We conclude this section by summarizing the problems confronting us for a regenerative processes with continuous state space. If $M_+^1 \in \mathcal{D}(G)$, then we can use (2.19) to obtain the asymptotic (for large t) approximation

$$P\{L_t \leq x\} \cong G^{1/m}\left(\frac{x - b_{[t]}}{a_{[t]}}\right) . \tag{2.20}$$

If the simulation is run for n cycles, then (2.20) should be replaced by

(2.21) $$P\left\{\max_{1 \leq k \leq n} M_k^+ \leq x\right\} \cong G\left(\frac{x-b_n}{a_n}\right) \quad .$$

For (2.20) or (2.21) to be useful, we must estimate m, a_n, and b_n. The expected cycle length, m, can of course be estimated by the sample mean of the cycle lengths observed. Two methods for estimating a_n and b_n will be discussed in Section 3. Finally, we must assess whether $M_1^+ \in \mathcal{D}(G)$ for one of the three d.f.'s (G's) given in (2.2) - (2.4). For most simulations in which extreme values are being estimated, the limit d.f.'s G will be either Λ or Φ_α, since the maxima arising are unbounded $(x_0 = +\infty)$. Our experience with specific examples indicates that if the regenerative process is stable (converges to a non-degenerate limit), then $G = \Lambda$. While if the process is "null-recurrent" $(m = E\{T_1\} = +\infty)$, then $G = \Phi_\alpha$. However, in this case $G^{1/m}(x) = 0$ for all $x > 0$ which indicates that a different normalization must be used to obtain a non-degenerate limit. In any case, we note that if $X \in \mathcal{D}(\Phi_\alpha)$ with constants $a_n > 0$ and $b_n = 0$, then $\ln X \in \mathcal{D}(\Lambda)$ with constants $a_n' = \alpha^{-1}$ and $b_n' = \alpha^{-1} \ln a_n$. For the balance of this paper we shall assume that $G = \Lambda$ for continuous state space processes. We note in passing that the extreme value behavior of some function of a regenerative process can be handled in the same way. If the state space of the regenerative process is discrete, then we shall only consider the situation in which the d.f. of $M_1^+ \in \mathcal{C}$ and $P\{M_1^+ > n\} \sim b \exp(-an)$

as $n \to \infty$ for some $a, b > 0$. In this case we can approximate the d.f. of L_t or $\max\{M_k^+ : 1 \leq k \leq n\}$ by using (2.12) and (2.13) or (2.15) and (2.16).

3. Statistical Estimation Problem

We have seen in (2.20) and (2.21) that the key to estimating the d.f. of extreme values occurring in regenerative processes is the constants a_n and b_n. These constants are in turn determined by the tail behavior of the d.f. of M_1^+, the maximum value of the process within a cycle. Following the remarks at the end of Section 2, we shall first assume the state space of the regenerative process is continuous and that $M_1^+ \in \mathcal{D}(\Lambda)$. Two methods are proposed here for estimating a_n and b_n. The first is based on the asymptotic relation (2.11).

Assume that n cycles are simulated and the n maxima, $\{M_k^+ : 1 \leq k \leq n\}$, are arranged in decreasing order. Call them $Y_{j,n}$; i.e., $Y_{1,n} \geq Y_{2,n} \geq \cdots \geq Y_{n,n}$. Now form the tail of the empirical d.f., namely,

$$E_n(x) = n^{-1} \# \{i : 1 \leq i \leq n,\ Y_{i,n} > x\} .$$

Plot $\log E_n(x)$ versus x. If this graph is roughly linear in x (at least for large x), then we can assume (2.11) holds and fit a linear regression line. To do this some judgment will have to be made as to how large x need be before the relationship is linear. Using the standard point and interval estimates for the slope and y-intercept of a regression line the parameters a and b of (2.11) can be estimated by $\tilde{a}$ and $\tilde{b}$, say. In particular, if the regression $y = cx+d$ is fitted to the plot of $\log E_n(x)$ vs. x, then $\tilde{a} = -c$ and $\tilde{b} = \exp(d)$. This in turn provides estimates for $b_n = a^{-1} \ln(nb)$ and $a_n = a^{-1}$, namely $\tilde{b}_n = \tilde{a}^{-1} \ln(n\tilde{b})$ and $\tilde{a}_n = \tilde{a}^{-1}$. Should the plot of $\log E(x)$ vs. x not be linear, our only suggestion is to first take logarithms of the M_k^+'s and try again. Maybe the underlying situation was $M_1^+ \in \mathcal{D}(\Phi_\alpha)$ for some $\alpha > 0$.

A second method for estimating a_n and b_n is that given by FEIGIN and WEISSMAN (1977) which is based on the k (a fixed positive integer) largest $Y_{j,n}$'s. They assume that $M_1^+ \in \mathcal{D}(\Lambda)$. Assuming that n is large enough so that $P\{Y_{1n} \leq x\} \cong \Lambda((x-b_n)/a_n)$ WEISSMAN (1976) has shown that the asymptotic maximum likelihood estimators (AMLE) of a_n and b_n are given by

$$\hat{a}_n = \bar{Y}_{k,n} - Y_{k,n} \;, \quad \hat{b}_n = \hat{a}_n \ln k + Y_{k,n}$$

and the asymptotic uniformly minimum variance unbiased estimators (AUMVUE) by

$$a_n^* = \bar{Y}_{k-1,n} - Y_{k,n} \;, \quad b_n^* = a_n^*(\beta_k-\gamma) + Y_{k,n} \;,$$

where $\bar{Y}_{j,n} = (1/j) \sum_{i=1}^{j} Y_{i,n}$, $\beta_k = \sum_{j=1}^{k-1} j^{-1}$, and $\gamma = .577216\ldots$ is Euler's constant. Furthermore, FEIGIN and WEISSMAN (1977) have shown that

$100(1-\alpha)\%$ (asymtotic) confidence intervals for a_n and b_n are given by

$$\left[2(k-1)a_n^*/\chi^2_{2k-2}(1-\tfrac{\alpha}{2}),\ 2(k-1)a_n^*/\chi^2_{2k-2}(\alpha/2)\right]$$

and

$$\left[Y_{k,n} - a_n^* U_{k,1}(1-\tfrac{\alpha}{2}),\ Y_{k,n} - a_n^* U_{k,1}(\alpha/2)\right],$$

where $\chi^2_r(p)$ is the 100p% point of the χ^2 distribution with r degrees of freedom. Also $U_{k,1}$ is the distribution of the ratio (m_k/Y) in which m_k and Y are independent, $\exp(-m_k)$ has a gamma distribution with parameters k and 1, and $(k-1)Y$ is gamma with parameters $(k-1)$ and 1. The quantiles, $U_{k,1}(p)$, have been computed by FEIGIN and WEISSMAN (1977) and are reproduced in Table 1.

Assume now that the state space of the regenerative proces is discrete, $M_1^+ \in \mathcal{Q}$, and $P\{M_1^+ > n\} \sim b\exp(-an)$ as $n \to \infty$ for some $a,b > 0$. Then we can use the approximation given in (2.16), $P\{Y_{1,n} \le k\} \cong \Lambda(a(k-b_n))$, where $b_n = a^{-1}\ln(bn)$. Thus again we can apply either the regression or Feigin-Weissman method to estimate a_n and b_n.

Once the constants a_n and b_n have been estimated, the d.f. of extreme values is estimated using either (2.20) or (2.21) depending on whether the simulation was run for a fixed length of time, t, or for a fixed number of cycles n.

These methods, which seems to be the simplest, for estimating a_n and b_n will be illustrated in Section 6 for the M/M/1 queue. Before leaving this section we point out some other relevant references. The

TABLE 1

Quantiles of $U_{k,1}$

k	p = 0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
2	-58.3	-23.0	-11.2	-5.32	-0.543	1.06	2.71	6.00	15.9
3	-14.7	- 8.90	- 5.99	-3.92	-1.04	-0.093	0.225	0.649	1.48
4	- 9.69	- 6.77	- 5.07	-3.70	-1.37	-0.470	-0.275	-0.0825	0.216
5	- 8.00	- 5.99	- 4.72	-3.66	-1.61	-0.724	-0.552	-0.404	- 0.221
6	- 7.15	- 5.60	- 4.59	-3.67	-1.81	-0.925	-0.755	-0.620	- 0.467
8	- 6.38	- 5.25	- 4.47	-3.75	-2.10	-1.24	-1.06	-0.931	- 0.791
10	- 6.02	- 5.10	- 4.46	-3.83	-2.33	-1.48	-1.31	-1.170	- 1.03
12	- 5.83	- 5.04	- 4.47	-3.91	-2.51	-1.67	-1.50	-1.36	- 1.22
14	- 5.71	- 5.02	- 4.50	-3.98	-2.67	-1.85	-1.66	-1.53	- 1.39
16	- 5.64	- 5.01	- 4.53	-4.05	-2.79	-1.99	-1.82	-1.68	- 1.53
18	- 5.59	- 5.02	- 4.57	-4.11	-2.91	-2.13	-1.94	-1.80	- 1.66
20	- 5.57	- 5.02	- 4.60	-4.18	-3.02	-2.24	-2.05	-1.92	- 1.77
25	- 5.54	- 5.06	- 4.70	-4.32	-3.24	-2.47	-2.31	-2.17	- 2.02
30	- 5.54	- 5.10	- 4.79	-4.44	-3.42	-2.69	-2.52	-2.38	- 2.23

reliability theory literature contains many references on the problem of testing whether observations come from an exponential or extreme value d.f. and of estimating the associated parameters. Two useful places to find such papers are EPSTEIN (1960) and MANN, SCHAFER, and SINGPURWALLA (1974), Chapter 5. PICKANDS (1975) has developed a method for determining which G d.f. is appropriate for a given set of observations. His method uses a random and increasing number of the $Y_{j,n}$'s as n increases. The method is expensive computationally and exphasizes an aspect of the extreme value problem which is not of great concern for simulation.

4. The GI/G/1 Queue

The GI/G/1 queue and the birth-death processes treated in Section 5 are among the very few regenerative processes for which we know the values of a_n and b_n. For this reason these processes are excellent candidates for testing the effectiveness of the estimation procedures proposed in Section 3.

In the GI/G/1 queue we assume customer 0 arrives at $t_0 = 0$, finds a free server, and experiences a service time v_0. Customer n arrives at time t_n and experiences a service time v_n. Customers are served in their order of arrival and the server is never idle if customers are waiting. Let the interarrival times $t_n - t_{n-1} = u_n$, $n \geq 1$. We assume the two sequences $\{v_n : n \geq 0\}$ and $\{u_n : n \geq 1\}$ each consist of i.i.d. r.v.'s and are themselves independent. Let $E\{u_n\} = \lambda^{-1}$ and $E\{v_n\} = \mu^{-1}$, where $0 < \lambda, \mu < \infty$. The traffic intensity $\rho = \lambda/\mu$ is assumed to be less than one. We exclude the deterministic system in which both the v_n's and u_n's are degenerate. Let the waiting time of the nth customer be W_n, the workload (or virtual waiting time) at time t be V_t, and the number of customers in the system at time t be Q_t. Also set $W_n^* = \max\{W_k : 0 \leq k \leq n\}$, $V_t^* = \sup\{V_s : 0 \leq s \leq t\}$, and $Q_t^* = \sup\{Q_s : 0 \leq s \leq t\}$. Let $X_n = v_{n-1} - u_n$, $n \geq 1$, and set $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $S_0 = 0$. If n_k denotes the number of customers served in the kth busy period, then n_1 is related to the partial sum process $\{S_n : n \geq 0\}$ since

$$n_1 = \inf\{n > 0 : S_n \leq 0\} .$$

When $\rho < 1$, we have $m = E\{n_1\} < \infty$. Also $-S_{n_1}$ is the length of the first idle period. We assume that X_1 has an aperiodic d.f. (support is <u>not</u> concentrated on a set of points of the form $0, \pm h, \pm 2h, \pm 3h, \ldots$), that there exists a positive number κ such that $E\{\exp(\kappa X_1)\} = 1$, and $0 < \mu_\kappa = E\{X_1 \exp(\kappa X_1)\} < \infty$. These assumptions will normally be satisfied if the d.f. of v_0 has an exponentially decaying tail; e.g., when v_0 has a gamma distribution. Under these conditions we know (see IGLEHART (1972)) that

(4.1) $$(W_n^* - \kappa^{-1} \log b_1 n)/\kappa^{-1} \Rightarrow \Lambda^{1/m}(x) .$$

and

(4.2) $$(V_t^* - \kappa^{-1} \log b_2 t)/\kappa^{-1} \Rightarrow \Lambda^{\lambda/m}(x) ,$$

where

$$b_1 = \frac{[1 - E\{e^{\kappa S_{n_1}}\}]^2}{\kappa \mu_\kappa m}$$

and

$$b_2 = E\{e^{\kappa V_0}\} b_1 \quad .$$

Thus to use (4.1) and (4.2) for estimating the d.f.'s of W_n^* and V_t^* we need only estimate m and $E\{e^{\kappa S_{n_1}}\}$, assuming that κ, μ_κ, and $E\{e^{\kappa V_0}\}$ can be calculated numerically. In the special case of M/G/1 queues no estimation is required, since $m = (1-\rho)^{-1}$ and $E\{e^{\kappa S_{n_1}}\} = \lambda/(\lambda+\kappa)$. If the simulation is carried out for a fixed number of cycles, then counterparts of (4.1) and (4.2) hold with the exponents of Λ removed.

The queue-length process $\{Q_t : t \geq 0\}$ is discrete-valued and the associated d.f. of $M_1^+ \in \mathcal{C}$. Hence a limit theorem comparable to (4.1) or (4.2) does not exist. Instead we must seek results like (2.12) and (2.13) or (2.15) and (2.16). Unfortunately, these results are only known for the M/G/1 and GI/M/1 queues; see COHEN (1968), Theorems 7.2 and 7.5. Let $M_k^+ = \sup\{Q_s : T_{k-1} \leq t < T_k\}$. Then for an M/G/1 queue the counterpart of (2.16) is

$$(4.3) \qquad P\{\max_{1 \leq j \leq n} M_j^+ \leq k\} \cong \exp(-e^{-a(k-b_n)}) \quad ,$$

where

$$a = \log((\lambda+\kappa)/\lambda)$$

and

$$b_n = a^{-1} \log(b_2 n) \quad .$$

On the other hand, for GI/M/1 queues (4.3) holds with $a = \log((\mu-\kappa)/\mu)$ and the same value for b_n. Tables 2 and 3 contain the values of m, κ, μ_κ, b_1, and b_2 for the M/M/1 and M/E_2/1 queues as a function of the traffic intensity ρ.

5. Birth-Death Processes

A second class of regenerative processes for which theoretical results are available is birth-death processes in discrete or continuous time. Let $\{X_n : n \geq 0\}$ be a discrete time Markov chain with state-space $E = \{0, 1, 2, \ldots\}$ and transition probabilities given by

$$p_{ij} = \begin{cases} q_i, & j = i-1 \\ p_i, & j = i+1 \\ 0\ , & \text{other } j, \end{cases} \tag{5.1}$$

where $q_0 = 0$, $p_0 = 1$ and the other q_i's and p_i's are positive. This chain will automatically be both irreducible and periodic. Furthermore, recall that it will be recurrent if and only if

$$\sum_{j=1}^{\infty} (\pi_i\ p_j)^{-1} = \infty$$

where $\pi_0 = 1$ and $\pi_j = (p_0 \cdots p_{j-1})/(q_1 \cdots q_j)$. We assume the chain is recurrent. It will be positive recurrent if and only if

TABLE 2

Parameter Values for M/M/1 Queue with $\mu = 10$

ρ	m	κ	μ_κ	b_1	b_2
.1	1.11	9	.900	.09	.9
.2	1.25	8	.400	.16	.8
.3	1.43	7	.233	.21	.7
.4	1.67	6	.150	.24	.6
.5	2.00	5	.100	.25	.5
.6	2.50	4	.067	.24	.4
.7	3.33	3	.043	.21	.3
.8	5.00	2	.025	.16	.2
.9	10.00	1	.011	.09	.1
.95	20.00	0.5	.005	.0475	.05
.99	100.00	0.1	.001	.0099	.01

TABLE 3

Parameter Values for $M/E_2/1$ Queue with $\mu = 10$

ρ	m	κ	μ_κ	b_1	b_2
.1	1.11	15.00	.3375	.1562	2.5
.2	1.25	12.60	.2016	.2346	1.7119
.3	1.43	10.61	.1395	.2874	1.3038
.4	1.67	8.83	.1012	.3179	1.0201
.5	2.00	7.19	.0741	.3263	0.7957
.6	2.50	5.64	.0534	.3118	0.6050
.7	3.33	4.16	.0367	.2733	0.4357
.8	5.00	2.73	.0227	.2094	0.2809
.9	10.00	1.35	.0106	.1188	0.1366
.95	20.00	0.67	.0051	.0630	0.0674
.99	100.00	0.13	.0010	.0132	0.0134

$$\sum_{j=0}^{\infty} \pi_j < \infty \ .$$

Next define

$$\tau_1(k) = \inf\{n > 0 : X_n = k\}, \qquad k \in E \ ,$$

the first entrance time to state k. Let $P_i\{\cdot\} = P\{\cdot | X_0 = i\}$, the conditional probability of an event, given $X_0 = i$. Then our concern here will be in the probability, given $X_0 = i$, of the Markov chain entering state n before it enters state 0. Let this probability be denoted by

$$r_i(n) = P_i\{\tau_1(n) < \tau_1(0)\} \ , \qquad i \in \{1, 2, \ldots, n-1\} \ .$$

Fortunately, this probability has been calculated and in particular

$$r_0(n) = r_1(n) = (1 + \sum_{i=1}^{n-1} (\pi_i p_i)^{-1})^{-1} \ ;$$

see CHUNG (1960), p. 68. Note that $\lim_{n \to \infty} r_0(n) = 0$ when the chain is recurrent, in keeping with our intuition. Define

$$M_1^+ = \sup\{X_n : 0 \leq n \leq \tau_0 - 1\} \ .$$

Then

$$\text{(5.2)} \qquad P_0\{M_1^+ > n\} = r_0(n+1) = \left(\sum_{i=0}^{n} (\pi_i p_i)^{-1} \right)^{-1} \ , \qquad \text{as } n \to \infty \ .$$

Suppose now that we have a birth-death process $\{X_t : t \geq 0\}$: a continuous time Markov chain with state space $E = \{0, 1, \ldots\}$ and embedded jump chain whose probabilities are given by (5.1). As above, define the first entrance time to state k and the maximum in the first cycle by

$$\tau_1(k) = \inf\{s > 0 : X_{s-} \neq j,\ X_s = j\}$$

and

$$M_1^+ = \sup\{X_t : 0 \leq t < \tau_1(0)\} .$$

Because of the path structure of the birth-death process, it is easy to see from (5.2) that

$$(5.3) \qquad P_0\{M_1^+ > n\} = \left[1 + \lambda_0 \sum_{i=1}^{n} (\pi_1 \lambda_i)^{-1}\right]^{-1} ,$$

where λ_i [resp. μ_i] are the birth [resp. death] parameters and $\pi_0 = 1$, $\pi_i = (\lambda_0 \lambda_1 \cdots \lambda_{i-1} / \mu_1 \mu_2 \cdots \mu_i)$. The same argument can of course be used to show that (5.3) also holds for semi-Markov processes with embedded jump chain whose probabilities are given by (5.1).

(5.4) EXAMPLE. M/M/s <u>queue</u>. The queue-length process, $\{Q_t : t \geq 0\}$, is a birth-death process with parameters $\lambda_j = \lambda$ and $\mu_j = \mu\,(j \wedge s)$, $j \geq 0$. Assume the queue has traffic intensity $\rho = \lambda/\mu s \leq 1$, a necessary and sufficient condition for recurrence. Then from (5.3)

$$P_0\{M_1^+ > n\} = \left[\sum_{i=0}^{n} \pi_i^{-1}\right]^{-1}$$

$$= \left[\sum_{i=0}^{s} \left(\frac{\mu}{\lambda}\right)^i i! + \frac{s!}{s^s \rho^{s+1}} \sum_{i=0}^{n-(s+1)} \rho^{-i}\right]^{-1} .$$

Asymptotically, as $n \to \infty$

$$P_0\{M_1^+ > n\} \sim \begin{cases} (s^s/s!)n^{-1} , & \rho = 1 \\ (s^s(1-\rho)/s!)\rho^n , & \rho < 1 . \end{cases}$$

Thus for $\rho < 1$ we can use (2.16) to obtain

$$P_0\{\max_{1 \leq j \leq n} M_j^+ \leq k\} \cong \exp(e^{-a(k-b_n)}) ,$$

where $a = \log \rho^{-1}$ and $b_n = a^{-1} \log(n\, s^s(1-\rho)/s!)$. Note that this is consistent with (4.3). ◀

6. Numerical Results

A simulation of the M/M/1 queue was performed to help assess the effectiveness of the estimation methods proposed in Section 3. Our goal is to estimate the d.f. of $\max_{1 \leq k \leq n} M_k^+$, where M_k^+ is either the maximum waiting time, virtual waiting time, or queue-length in the kth cycle and n is the number of cycles simulated. These d.f.'s will all be estimated by $\Lambda((x-b_n)/a_n)$. Thus our principal task is to estimate the appropriate a_n and b_n for the three processes mentioned above; theoretical values of a_n and b_n are available from Section 4.

The random number generator used was subroutine GGU3 available in the IMSL package. This generator is the congruential generator developed by LEARMONTH and LEWIS (1973). Regression lines were fitted to the plots of $\log E_n(x)$ versus x beginning at $x = 0$. For the $\{W_n : n \geq 0\}$ and $\{V_t : t \geq 0\}$ processes observations were grouped in 150 cells of length 0.02 [0.05] for $\rho = 0.5$ [0.9]. Tables 4-9 contain the results of the simulation for estimating a_n and b_n. Two values of ρ (0.5 and 0.9) were used along with several combinations of run lengths (number of cycles simulated) and number of replications. The entries contained in the tables are the sample means of the various estimates over the number of replications and the half-length of a symmetric 90% confidence interval about the sample mean. For example, in Table 4 take $\rho = 0.5$, 500 cycles, and 100 replications. Then for $k = 20$ in the Feigin-Weissman procedure 0.1899 is the sample mean of 100 estimates, a^*_{500}, of a_{500} and [.1899 - .0077, .1899 + .0077] is the corresponding 90% confidence interval based on the 100 estimates. In this case the true value of a_n is 0.2.

Here are some general observations based on Tables 4-9. The AUMVUE estimators a^*_n and b^*_n do in general have smaller bias than the AMLE estimators $\hat{a}_n$ and $\hat{b}_n$, however, the mean square errors (MSE), that is bias squared plus variance, are comparable. In general, the bias decreases with increasing k. The regression estimator in general yields a smaller bias than the other estimators for the $\{W_n : n \geq 0\}$ and $\{V_t : t \geq 0\}$ processes. In those cases where the bias is larger the MSE is usually quite a bit smaller. For the $\{Q_t : t \geq 0\}$ process the smallest

TABLE 4

Estimates of a_n for $\{W_n : n \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{a}_n$	k = 10 $\hat{a}_n$	k = 10 a_n^*	k = 20 $\hat{a}_n$	k = 20 a_n^*	k = 30 $\hat{a}_n$	k = 30 a_n^*
250/200	0.5	0.2	.2046 .0059	.1678 .0070	.1864 .0078	.1807 .0052	.1903 .0055	.1799 .0041	.1861 .0043
500/100	0.5	0.2	.2008 .0065	.1704 .0090	.1893 .0100	.1804 .0073	.1899 .0077	.1818 .0063	.1881 .0065
1000/50	0.5	0.2	.1972 .0068	.1721 .0109	.1912 .0121	.1832 .0088	.1928 .0093	.1793 .0081	.1855 .0083
1000/80	0.9	1.0	.8746 .0398	.8172 .0508	.9080 .0564	.8313 .0391	.8751 .0411	.8207 .0319	.8490 .0330
2000/40	0.9	1.0	.8739 .0486	.8515 .0821	.9461 .0912	.8667 .0607	.9123 .0639	.8636 .0500	.8934 .0517
4000/20	0.9	1.0	.8865 .0591	.8823 .1298	.9803 .1442	.9312 .0895	.9802 .0942	.8984 .0773	.9293 .0799

TABLE 5

Estimates of b_n for $\{W_n : n \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{b}_n$	k = 10 $\hat{b}_n$	k = 10 b_n^*	k = 20 $\hat{b}_n$	k = 20 b_n^*	k = 30 $\hat{b}_n$	k = 30 b_n^*
250/200	0.5	0.8270	0.8496	0.7727	0.8061	0.7883	0.8120	0.7848	0.8028
			0.0188	0.0178	0.0190	0.0162	0.0168	0.0146	0.0150
500/100	0.5	0.9657	0.9764	0.9068	0.9408	0.9189	0.9425	0.9201	0.9382
			0.0239	0.0247	0.0263	0.0236	0.0245	0.0221	0.0227
1000/50	0.5	1.1043	1.1016	1.0424	1.0767	1.0554	1.0795	1.0456	1.0635
			0.0280	0.0285	0.0304	0.0277	0.0287	0.0284	0.0292
1000/80	0.9	4.4998	4.4067	4.1663	4.3292	4.1605	4.2695	4.1311	4.2131
			0.1498	0.1446	0.1541	0.1312	0.1361	0.1204	0.1235
2000/40	0.9	5.1930	5.0102	4.8474	5.0172	4.8464	4.9600	4.8320	4.9183
			0.2170	0.2460	0.2614	0.2182	0.2259	0.1967	0.2015
4000/20	0.9	5.8861	5.6715	5.5718	5.7477	5.6337	5.7558	5.5565	5.6463
			0.2812	0.3362	0.3598	0.3089	0.3202	0.2943	0.3019

TABLE 6

Estimates of a_n for $\{V_t : t \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{a}_n$	k = 10 $\hat{a}_n$	k = 10 a^*_n	k = 20 $\hat{a}_n$	k = 20 a^*_n	k = 30 $\hat{a}_n$	k = 30 a^*_n
250/200	0.5	0.2	.1951 .0051	.1690 .0070	.1877 .0078	.1793 .0052	.1887 .0054	.1815 .0043	.1878 .0044
500 100	0.5	0.2	.1944 .0058	.1704 .0090	.1893 .0100	.1804 .0073	.1899 .0077	.1818 .0063	.1881 .0065
1000/50	0.5	0.2	.1929 .0067	.1677 .0091	.1863 .0102	.1787 .0068	.1881 .0072	.1821 .0062	.1884 .0064
1000/80	0.9	1.0	.8456 .0377	.8182 .0527	.9091 .0585	.8296 .0393	.8732 .0414	.8165 .0321	.8447 .0332
2000/40	0.9	1.0	.8485 .0461	.8523 .0808	.9470 .0898	.8640 .0627	.9095 .0660	.8684 .0490	.8983 .0507
4000/20	0.9	1.0	.8865 .0591	.8823 .1298	.9803 .1442	.9312 .0895	.9802 .0942	.8984 .0773	.9293 .0799

TABLE 7

Estimates for b_n for $\{V_t : t \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{b}_n$	k = 10 $\hat{b}_n$	k = 10 b_n^*	k = 20 $\hat{b}_n$	k = 20 b_n^*	k = 30 $\hat{b}_n$	k = 30 b_n^*
250/200	0.5	0.9657	0.9797 0.0180	0.9140 0.0178	0.9477 0.0191	0.9257 0.0162	0.9492 0.0168	0.9293 0.0150	0.9475 0.0154
500/100	0.5	1.1043	1.1103 0.0240	1.0428 0.0243	1.0762 0.0259	1.0550 0.0226	1.0785 0.0234	1.0606 0.0219	1.0788 0.0224
1000/50	0.5	1.2429	1.2384 0.0304	1.1827 0.0335	1.2171 0.0359	1.1889 0.0305	1.2125 0.0317	1.1973 0.0283	1.2157 0.0291
1000/80	0.9	4.6052	4.4714 0.1474	4.2752 0.1455	4.4383 0.1533	4.2645 0.1319	4.3733 0.1368	4.2278 0.1206	4.3094 0.1237
2000/40	0.9	5.2983	5.0695 0.2117	4.9571 0.2450	5.1271 0.2602	4.9495 0.2210	5.0628 0.2290	4.9512 0.1959	5.0380 0.2006
4000/20	0.9	5.9915	5.6715 0.2812	5.6769 0.3420	5.8530 0.3654	5.7239 0.3126	5.8447 0.3242	5.6716 0.2977	5.7616 0.3055

TABLE 8

Estimates of a_n for $\{Q_t : t \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{a}_n$	k = 10 $\hat{a}_n$	k = 10 a_n^*	k = 20 $\hat{a}_n$	k = 20 a_n^*	k = 30 $\hat{a}_n$	k = 30 a_n^*
250/200	0.5	1.4427	1.3404 0.0317	1.2544 0.0566	1.3938 0.0629	1.3549 0.0477	1.4262 0.0502	1.1527 0.0404	1.1925 0.0418
500/100	0.5	1.4427	1.3338 0.0367	1.1919 0.0748	1.3244 0.0831	1.3600 0.0668	1.4315 0.0703	1.2116 0.0613	1.2534 0.0635
1000/50	0.5	1.4427	1.3297 0.0395	1.2020 0.1120	1.3355 0.1244	1.3640 0.0903	1.4358 0.0950	1.3126 0.0923	1.3579 0.0949
1000/80	0.9	9.4912	7.9714 0.4069	7.6224 0.5067	8.4694 0.5630	7.7724 0.3883	8.1815 0.4088	7.8412 0.3279	8.1116 0.3392
2000/40	0.9	9.4912	7.9621 0.5117	8.2575 0.8411	9.1750 0.9345	8.0462 0.6418	8.4697 0.6756	8.0466 0.5252	8.3241 0.5433
4000/20	0.9	9.4912	8.2406 0.6612	8.7600 1.2651	9.7333 1.4057	8.4350 0.8397	8.8789 0.8839	8.6533 0.7232	8.9517 0.7481

TABLE 9

Estimates of b_n for $\{Q_t : t \geq 0\}$ in the M/M/1 queue with $\mu = 10$

# cycles/ # repl.	ρ	true value	regression $\tilde{b}_n$	k = 10 $\hat{b}_n$	k = 10 b^*_n	k = 20 $\hat{b}_n$	k = 20 b^*_n	k = 30 $\hat{b}_n$	k = 30 b^*_n
250/200	0.5	6.9658	6.9196 0.1222	7.0985 0.1313	7.3486 0.1417	7.2691 0.1278	7.4468 0.1338	6.8159 0.1202	6.9311 0.1241
500/100	0.5	7.9658	7.8141 0.1603	7.9746 0.1769	8.2122 0.1900	8.2041 0.1890	8.3825 0.1974	7.8811 0.1727	8.0021 0.1786
1000/50	0.5	8.9658	8.7129 0.1911	8.9077 0.2436	9.1473 0.2637	9.1611 0.2558	9.3450 0.2671	9.0646 0.2457	9.1958 0.2545
1000/80	0.9	43.7087	42.3106 1.5365	40.7508 1.4367	42.2704 1.5325	40.6961 1.2861	41.7154 1.3353	40.7690 1.1925	41.5525 1.2245
2000/40	0.9	50.2875	47.7921 2.2903	47.5382 2.5132	49.1844 2.6746	47.0291 2.2174	48.0841 2.2985	46.9179 2.0136	47.7221 2.0646
4000/20	0.9	56.8663	54.5220 3.0902	54.7205 3.4052	56.4669 3.6425	54.0188 2.9162	55.1249 3.0220	54.3815 2.7475	55.2463 2.8183

bias when estimating a_n was obtained from a_n^* with $k = 10$ or 20. Again, however the MSE of the regression estimator was generally smallest. When estimating b_n for $\{Q_t : t \geq 0\}$ the smallest bias was obtained by four estimators in different cases. Percentage-wise the differences in bias were small and again the MSE of the regression estimator was often the lowest.

To summarize these results, we recommend using the a_n^* and b_n^* estimators rather than $\hat{a}_n$ and $\hat{b}_n$. The value of k should be as large as possible, realizing of course that the amount of computer time required will increase with k. The regression estimators $\tilde{a}_n$ and $\tilde{b}_n$ performed very well. They achieved the smallest bias in about one-third of the cases and had the smallest MSE in virtually all cases. In practice, we suggest that a plot of $\log E_n(x)$ versus x be made before the regression line is fitted. This will enable the simulator to eyeball a straightline fit and the x-value at which to begin fitting.

Our ultimate objective is to estimate the extreme value distribution $P\{\max_{1 \leq k \leq n} M_k^+ \leq x\}$ for the three queueing processes. Each of these distributions will be approximated by $\Lambda((x-b_n)/a_n)$ and then the constants a_n and b_n estimated by one of the methods discussed above. The approximation of $P\{\max_{1 \leq k \leq n} M_k^+ \leq x\}$ by $\Lambda((x-b_n)/a_n)$ is a large sample one valid for large n from the limit theorem (2.1). As the simulation results for the waiting time and virtual waiting time processes were quite similar, we only present the figures for the waiting time and queue length processes. For the Feigin-Weissman (F-W) method only the $k = 30$ case is presented,

TABLE 10

Estimates of $P\{\max_{1 \le k \le n} M_k^+ \le x\}$ for $\{W_n : n \ge 0\}$

in the M/M/1 queue, $\rho = 0.5$, $\mu = 10$

method	# cycles/ # repl.	$\Lambda((x-b_n)/a_n)/x$				
		.25/.7617	.50/.9003	.75/1.0762	.90/1.2771	.99/1.7471
regression	250/200	.2876 .0261	.4894 .0289	.6994 .0246	.8486 .0168	.9721 .0054
F-W, k = 30	250/200	.3358 .0254	.5578 .0256	.7666 .0192	.8983 .0112	.9867 .0023
		$\Lambda((x-b_n)/a_n)/x$				
		.25/.9003	.50/1.0390	.75/1.2148	.90/1.4157	.99/1.8857
regression	500/100	.2870 .0352	.4962 .0373	.7148 .0354	.8647 .0199	.9788 .0054
F-W, k = 30	500/100	.3448 .0387	.5586 .0380	.7637 .0280	.8946 .0161	.9858 .0032
		$\Lambda((x-b_n)/a_n)/x$				
		.25/1.0390	.50/1.1776	.75/1.3535	.90/1.5544	.99/2.0243
regression	1000/50	.2921 .0422	.5147 .0459	.7382 .0370	.8832 .0226	.9845 .0047
F-W, k = 30	1000/50	.3615 .0512	.5820 .0501	.7842 .0361	.9071 .0202	.9882 .0038

TABLE 11

Estimates of $P\{\max_{1 \leq k \leq n} M_k^+ \leq x\}$ for $\{W_n : n \geq 0\}$

in the M/M/1 queue, $\rho = 0.9$, $\mu = 10$

method	# cycles/ # repl.	$\Lambda((x-b_n)/a_n)/x$ .25/4.1732	.50/4.8663	.75/5.7457	.90/6.7502	.99/9.1000
regression	1000/80	.3495 .0475	.5654 .0484	.7674 .0380	.8939 .0248	.9837 .0068
F-W, k = 30	1000/80	.4003 .0437	.6241 .0423	.8141 .0308	.9225 .0175	.9906 .0032
		$\Lambda((x-b_n)/a_n)/x$ .25/4.8663	.50/5.5595	.75/6.489	.90/7.4433	.99/9.7931
regression	2000/40	.3732 .0627	.5948 .0628	.7901 .0520	.9046 .0379	.9842 .0120
F-W, k = 30	2000/40	.4074 .0701	.6037 .0716	.7719 .0613	.8788 .0461	.9702 .0194
		$\Lambda((x-b_n)/a_n)/x$ .25/5.5595	.50/6.2526	.75/7.1320	.90/8.1365	.99/10.4862
regression	4000/20	.3691 .0851	.5936 .0839	.7923 .0691	.9094 .0449	.9877 .0099
F-W, k = 30	4000/20	.3881 .0942	.5928 .0924	.7798 .0720	.8979 .0450	.9842 .0105

TABLE 12

Estimates of $P\{\max_{1 \le k \le n} M_k^+ \le x\}$ for $\{Q_t : t \ge 0\}$

in the M/M/1 queue, $\rho = 0.5$, $\mu = 10$

method	# cycles/ # repl.	$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$ .3744/7	.6128/8	.7831/9	.9408/11	.9924/14
regression	250/200	.4289 .0276	.6317 .0261	.7785 .0208	.9272 .0105	.9868 .0031
F-W, k = 30	250/200	.4431 .0314	.6529 .0281	.7975 .0218	.9358 .0106	.9884 .0030
		$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$ .3755/8	.6132/9	.7832/10	.9408/12	.9924/15
regression	500/100	.4451 .0369	.6494 .0343	.7944 .0265	.9364 .0121	.9897 .0029
F-W, k = 30	500/100	.4251 .0447	.6305 .0404	.7781 .0313	.9264 .0156	.9860 .0048
		$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$ .3761/9	.6135/10	.7833/11	.9408/13	.9924/16
regression	1000/50	.4621 .0462	.6707 .0419	.8136 .0311	.9467 .0127	.9924 .0025
F-W, k = 30	1000/50	.3823 .0582	.5891 .0573	.7427 .0480	.9063 .0261	.9801 .0080

TABLE 13

Estimates of $P\{\max_{1 \le k \le n} M_k^+ \le x\}$ for $\{Q_t : t \ge 0\}$ in the M/M/1 queue, $\rho = 0.9$, $\mu = 10$

method	# cycles/ # repl.	$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$				
		.2599/41	.5272/48	.7599/56	.9088/66	.9906/88
regression	1000/80	.3907 .0506	.6156 .0491	.7927 .0388	.9080 .0262	.9836 .0088
F-W, k = 30	1000/80	.3994 .0436	.6336 .0433	.8101 .0331	.9215 .0189	.9893 .0038
		$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$				
		.2800/48	.5074/54	.7693/63	.9034/72	.9900/94
regression	2000/40	.4396 .0700	.6321 .0657	.8207 .0531	.9126 .0410	.9830 .0144
F-W, k =30	2000/40	.4379 .0697	.6233 .0661	.8114 .0505	.9093 .0344	.9849 .0096
		$P\{\max_{1 \le k \le n} M_k^+ \le x\}/x$				
		.2575/54	.5231/61	.7568/69	.9074/79	.9905/101
regression	4000/20	.4033 .0905	.6318 .0902	.8027 .0790	.9116 .0537	.9853 .0133
F-W, k = 30	4000/20	.3712 .0968	.5860 .0906	.7696 .0691	.8978 .0421	.9842 .0104

the results in general being better than for smaller k. As an example of the entries in Tables 10-13, consider in Table 10 the regression estimate for 500 cycles and 100 replications when $\Lambda((x-b_n)/a_n) = 0.75$ with $x = 1.2148$. The mean of the 100 estimates in this case is 0.7148 and the 90% confidence interval has half-length 0.0354. For the queue length process the exact value of $P\{\max_{1 \le k \le n} M_k^+ \le x\}$ can be calculated from the results in Example (5.4), and these values are indicated in Tables 12 and 13. A comparison of the bias and MSE for the regression and F-W estimates in Tables 10-13 shows them to be quiet comparable. For both estimators the bias is in general smaller for the larger percentiles. For the smallest percentiles both estimators over estimate the quantity in question.

Recommendations

The first step in estimating extreme values for regenerative simulations is to plot the function $\log E_n(x)$ versus x. This enables the simulator to ascertain whether the tail of the distribution of the maximum value within a cycle is exponential. Based on the examples we can calculate, we would expect this tail to be exponential for regenerative processes with finite mean cycle length. If the plot of $\log E_n(x)$ is roughly linear, we would then recommend estimating a_n and b_n by both the regression and Feigin-Weissman AUMVUE methods. For the F-W method k should be at least equal to 30, and perhaps even larger if increased computer time is not a big problem. We would expect the two methods to give farily comparable results. Using both methods gives the simulator a rough check against inadvertant programming errors. Based on our simulations of the M/M/1 queue we would select the regression method when the program is debugged, if forced to choose just one method. Once estimates for a_n and b_n are available, the distribution of extreme values for n cycles ($P\{\max_{1 \leq k \leq n} M_k^+ \leq x\}$) can be approximated by $\Lambda((x-b_n)/an)$ where the appropriate estimates of a_n and b_n are used.

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Let $\{X_t : t \geq 0\}$ denote the regenerative process being simulated and assume that X_t converges weakly (in distribution) to a limit random variable X. Our conern ~~in this paper~~ here is in estimating the extreme values of the process $\{X_t : t \geq 0\}$. Suppose we are interested in the largest value attained in the interval $(0,t)$: $X_t^* = \sup(X_s : 0 \leq s \leq t)$. Examples of this are the maximum queue lengths or waiting times in a queueing system. As t increases so will X_t^*, without bound if the state space of $\{X_t : t \geq 0\}$ is unbounded. This report developes two methods for estimating the distribution of X_t^*. When the regenerative process is either the GI/G/1 queue or a birth-death process theoretical results are available for the distribution of X_t^*. The waiting time, queue length, and virtual waiting time for an M/M/1 queue were simulated. The two methods for estimating the distribution of X_t^* were employed and the simulation results compared with the theoretical results.

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