AD-A048 056 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 12/1

A DYNAMIC FORMULATION OF THE LAW OF DIMINISHING RETURNS.(U)

OCT 77 R FAERE N00014-76-C-0134

UNCLASSIFIED ORC-77-29 NL

| OF | ADA048056

END

DATE FILMED

1 -78

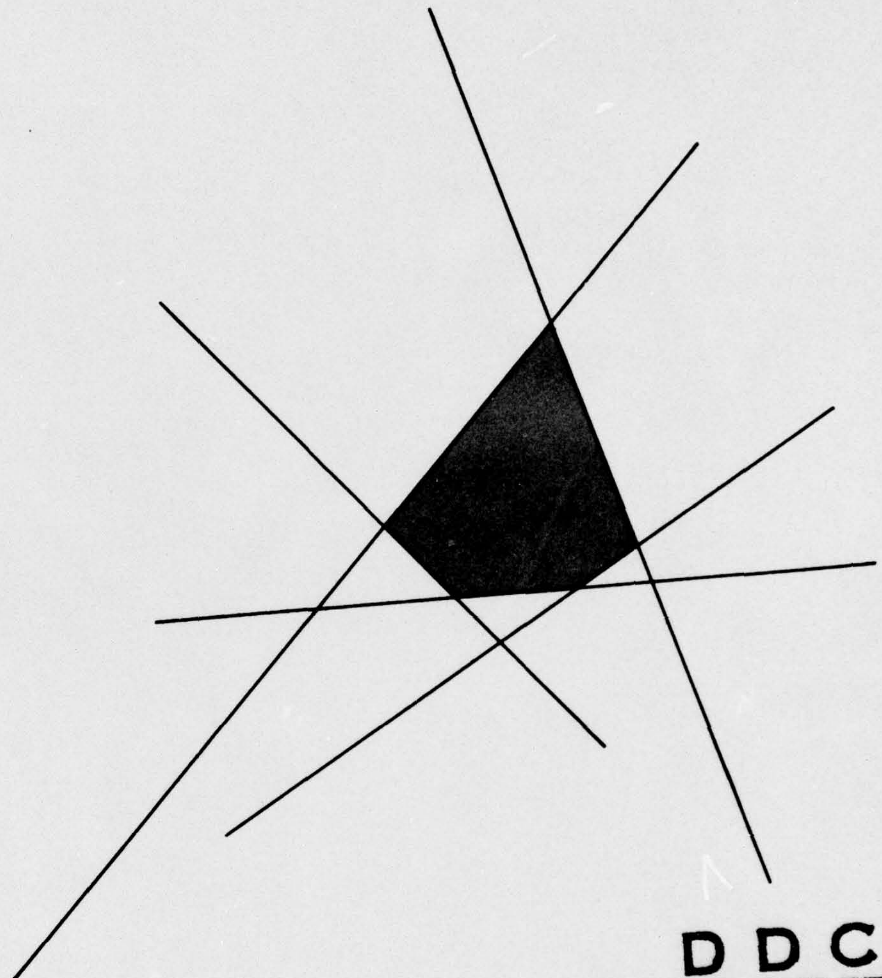
DDC

(12)

ORC 77-29
OCTOBER 1977

A DYNAMIC FORMULATION OF THE LAW OF DIMINISHING RETURNS

by
ROLF FÄRE



OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

A DYNAMIC FORMULATION OF THE LAW OF DIMINISHING RETURNS[†]

by

Rolf Färe[††]
Operations Research Center
University of California, Berkeley

OCTOBER 1977 ORC 77-29

[†]This research has been partially supported by the Office of Naval Research under Contract N00014-76-C-0134 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[††]The author is grateful to the Swedish Counsil for Social Science Research for partially sponsoring this research.

(14) REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC-77-29	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) A DYNAMIC FORMULATION OF THE LAW OF DIMINISHING RETURNS.		5. TYPE OF REPORT & PERIOD COVERED (9) Research Rep[redacted]t.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Rolf [redacted] Faere		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-76-C-0134
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047 033
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, Virginia 22217		12. REPORT DATE (11) Oct[redacted]77
		13. NUMBER OF PAGES 25 (12) 26p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Dynamic Production Structure
Dynamic Neoclassical Production Function
Law of Diminishing Returns
Essential
Output Rate Limitational
Output Time Limitational

270750

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

ACKNOWLEDGEMENT

I am very grateful to Professor R. W. Shephard for his comments and suggestions on this paper.

ABSTRACT

Based on an axiomatic framework for a dynamic production structure a law of diminishing returns is derived. This law shows that for output to be bounded in rate (i.e., norm) and in time availability (i.e., support) it is necessary and sufficient that an essential subvector of inputs is likewise bounded in rate and time availability.

A DYNAMIC FORMULATION OF THE LAW OF DIMINISHING RETURNS

by

Rolf Färe

1. INTRODUCTION

The law of diminishing returns, originally formulated by Turgot [10] for agriculture, has recently been investigated within a steady state framework for production (see [2], [3], [4] and [6]). In his work on the law Shephard (see [4]) showed, for a single net output production technology, that for a bounded factor combination to limit output it is necessary and sufficient that it is essential. By essentiality it is understood that only zero output is obtainable when the essential factors are null. He also showed that in general, not every positive bound on the essential factors leads to bounded output. On this issue Färe (see [2])(1) gave a sufficient condition. The work by Shephard on the law of diminishing returns was generalized in [6] to hold for steady state multi-output production technologies.

A dynamic theory for production correspondences is being developed by Shephard and Färe, see Section 2 for details. Inputs and net outputs are treated as functions of time. For such production structures, two questions are important in relation to the law of diminishing returns. First, does there exist a positive bound on the rate (i.e., the norm) of an essential subvector of inputs such that the rate of net output is

---

(1) Note that the condition given is not necessary as claimed.

bounded? Second, does there exist a bound on the time availability (i.e., the support) of an essential subvector of inputs such that net output is not available after a finite time horizon?

This paper is addressed to each of the two questions and it is shown that there are bounds both on the rate and the time availability of essential inputs such that net output rate is bounded and such that net output is not available after a finite horizon. Consequently the smallest of these bounds serve to bound net output rate and net output (time) availability. The law of diminishing returns is understood to mean the existence of these bounds.

Necessary and sufficient conditions on the production structure, beyond the axioms, such that each bound on an essential factor combination bounds output rate and output (time) availability, are also given. In particular an input homothetic production structure satisfies these conditions.

2. THE GENERAL TECHNOLOGY

A general dynamic production structure is modelled here as in [7] by an output correspondence $x \to \mathbb{P}(x)$ of input (vector) histories $x \in BM_+^n$(2) to subsets of output histories $u \in BM_+^1$ (i.e., BM_+ for short) or inversely by an input correspondence $u \to \mathbb{L}(u)$ of output histories u to subsets of input (vector) histories x. $\mathbb{P}(x)$ denotes the set of all output histories obtainable from a vector of input histories $x \in BM_+^n$ and $\mathbb{L}(u)$ all input histories yielding at least the output history $u \in BM_+$. The two correspondences are inversely related by

$$\mathbb{L}(u) := \left\{x \in BM_+^n \mid u \in \mathbb{P}(x)\right\} \quad \text{and} \quad \mathbb{P}(x) := \{u \in BM_+ \mid x \in \mathbb{L}(u)\} .$$

$u \in BM_+$ denotes a net output history with $u(t)$ equal the number of units per unit time at time $t \in [0,+\infty)$. Similar for each $i \in \{1,2, \ldots, n\}$, $(x_1,x_2, \ldots, x_i, \ldots, x_n) \in BM_+^n$, $x_i(t)$ denotes the number of units per unit time at $t \in [0,+\infty)$.

The basic axioms taken for the dynamic production structure follow essentially those of [7] and they are:

- $\mathbb{P}.1$ $\quad \mathbb{P}(0) = 0$;
- $\mathbb{P}.2$ $\quad \mathbb{P}(x)$ is bounded for $||x||$ finite;
- $\mathbb{P}.3$ $\quad \mathbb{P}(\lambda \cdot x) \supset \mathbb{P}(x)$ for $\lambda \geqq 1$, $x \in BM_+^n$;

---

(2) $BM_+^\alpha := \{f \in BM^\alpha \mid f(t) \geqq 0 , t \in [0,+\infty)\}$, $\alpha = n$ or 1, where $BM^\alpha := \{f := (f_1,f_2, \ldots, f_\alpha) \mid f_i : [0,+\infty) \to \mathbb{R}$, f_i is bounded and measurable with $||f_i|| := \sup\{|f_i(t)| \mid t \in [0,+\infty)\}$ and the Euclidean product norm$\}$. BM^α is a Banach space, i.e., complete normed linear (see [7]).

$\mathbb{P}.4$ If $u \in \mathbb{P}(\lambda \cdot x)$ for some $\lambda > 0$ and $x \in BM_+^n$, then for each scalar $\theta \in (0,+\infty)$, there is a λ_θ such that $(\theta \cdot u) \in \mathbb{P}(\lambda_\theta \cdot x)$;

$\mathbb{P}.5$ The correspondence $x \to \mathbb{P}(x)$ is closed (i.e., $[\{x_n\} \to x_o , \{u_n\} \to u_o$ with $u_n \in \mathbb{P}(x_n)$ for all $n] \Rightarrow u_o \in \mathbb{P}(x_o))$;

$\mathbb{P}.6$ $u \in \mathbb{P}(x) \Rightarrow \{v \mid 0 \leqq v \leqq u\} \subset \mathbb{P}(x)$, $x \in BM_+^n$;

$\mathbb{P}.A.A.$ The Asymmetric Axiom: The efficient subset of input histories, $\mathbb{E}\mathrm{ff}\ \mathbb{L}(u) := \left\{x \in BM_+^n \mid x \in \mathbb{L}(u) , y \leq x \Rightarrow y \notin \mathbb{L}(u)\right\}$ [3], is totally bounded[4] for $u \in BM_+^n$, $u \neq 0$, and $\mathbb{E}\mathrm{ff}\ \mathbb{L}(0) := 0$.

The first property of the output correspondence $x \to \mathbb{P}(x)$ states that for null inputs, there can be only null output a self-evident axiom. For bounded input histories only bounded output is obtainable i.e., property $\mathbb{P}.2$ excludes e.g., the possibility of infinite accumulation. The third axiom is a statement concerning disposability of inputs. It says that individual input (vector) histories are disposable. Axiom four models obtainability and is motivated by the possibility of "doubling size of operation." The closeness axiom ($\mathbb{P}.5$) guarantees that these are efficient input and output histories. Note that $\mathbb{P}.5$ implies and is implied by that the input correspondence $u \to \mathbb{L}(u)$ is also closed.

---

(3) $y \leq x$ means $y_i \leqq x_i$ $i = 1,2, \ldots, n$ and $y_i < x_i$ for some i . $y_i < x_i$ means $y_i(t) \leqq x_i(t)$ for all $t \in [0,+\infty)$ and $y_i(t) < x_i(t)$ for some $t \in [0,+\infty)$.

(4) A set in BM_+^n is totally bounded if and only if every infinite sequence in the set contains a Cauchy subsequence.

Disposability of output histories is modelled by axiom $\mathbb{P}.6$. This disposability axiom may be weakened to read $u \in \mathbb{P}(x)$ implies $\{v \mid v = \theta \cdot u , \theta \in [0,1]\} \subset \mathbb{P}(x)$, but for pedagogical reasons the stronger form $\mathbb{P}.6$ is applied in this paper. Finally, the asymmetric axiom is used to put a limit to input histories to be termed efficient.

Throughout this paper the above axioms are used as the basic model for production. Although frequently in the sequel the equivalent axioms on the input correspondence are used, they are all easy to derive and thus they are not listed here.

If only constant input and output histories are considered, the subspaces so obtained from $\mathbb{BM}^n$ and $\mathbb{BM}$ are isometrically isomorphic (i.e., equivalent) to $\mathbb{R}^n$ and $\mathbb{R}$, respectively. Consequently the steady state models discussed in [4] and [5] are special cases of the dynamic production structure.

3. PROPERTIES OF THE GENERAL TECHNOLOGY

A major interest in economic theory is to find efficient allocations. It is therefore important to show that the above production technology guarantees the existence of efficient input and output histories.

Proposition 1:

$\mathbb{L}(u)$ nonempty implies $\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$ nonempty, $u \in BM_+^n$.

Proof:

Assume $\mathbb{L}(u)$ nonempty for some $u \neq 0$ ($\mathbb{E}\mathrm{ff}\ \mathbb{L}(0) := 0$) and let $x^o \in \mathbb{L}(u)$. Define $F_0 := \left(\mathbb{L}(u) \cap \left\{x \in BM_+^n \mid x \leqq x^o\right\}\right)$. F_0 is a closed set as the intersection of two closed set. Furthermore, define $f(x^o) := \sup \left\{\|x^o - x\| \mid x \in F_0\right\}$. $f(x^o) < +\infty$ is the diameter of F_0 measured from x^o . If $f(x^o) = 0$, then $x^o \in \mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$ and the proof is done. Thus assume $f(x^o) > 0$ and let x^* be an input vector such that $f(x^o) = \|x^o - x^*\|$. Define $\tilde{x} := (x^o - x^*)$ and consider $F_1 := \left(\mathbb{L}(u) \cap \left\{x \in BM_+^n \mid x \leqq x^* + \tilde{x}/2\right\}\right)$. Clearly $F_1 \subset F_0$ and F_1 is closed. By repeating this procedure one obtains;

$F_n := \left(\mathbb{L}(u) \cap \left\{x \in BM_+^n \mid x \leqq x^* + \tilde{x}/2^n\right\}\right)$, $F_n \subset F_{n+1}$, $n = 1,2, \ldots$, F_n is closed and the diameter of F_n (i.e., $d(F_n) := \sup \{\|x - y\| \mid x,y \in F_n\}$) tends to zero as $n \to \infty$.

It now follows from Cantor's Intersection Theorem, (see [8], p. 73), that $\left(\bigcap_{n=1}^{\infty} F_n\right) = \{\bar{x}\}$ is a singleton. Consequently, $\bar{x} \in \mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$ and the proposition holds. Q.E.D.

It is useful for the sequel to show that the input set $\mathbb{L}(u)$ is contained in a decomposition of input histories into those that are efficient and those that belong to BM_+^n .

Proposition 2:[5]

$$\mathbb{L}(u) \subset \left(\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} + BM_+^n\right),^{(6)} \quad u \in BM_+^n .$$

Proof:

Define $\mathbb{L}^S(u) := \mathbb{L}(u) + BM_+^n := \left\{x \mid x = y + z \ , \ y \in \mathbb{L}(u) \ , \ z \in BM_+^n\right\}$ and note that $\mathbb{L}(u) \subset \mathbb{L}^S(u)$. It is first shown that $\mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u) = \mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$. This is clearly true for $u = 0$ and for u such that $\mathbb{L}(u)$ empty. Thus assume $u \neq 0$ with $\mathbb{L}(u)$ nonempty, and let $x \in \mathbb{E}\mathrm{ff}\ \mathbb{L}(u) \subset \mathbb{L}(u) \subset \mathbb{L}^S(u)$. For $y \leq x$, $y \notin \mathbb{L}(u)$ implying $y \notin \mathbb{L}^S(u)$. Hence $\mathbb{E}\mathrm{ff}\ \mathbb{L}(u) \subset \mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u)$. Conversely, let $x \in \mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u)$. Then $x = y + z$, $y \in \mathbb{L}(u)$ and $z \in BM_+^n$. For $x \in \mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u)$ it is necessary that $z = 0$, consequently $x \in \mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$. Hence $\mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u) = \mathbb{E}\mathrm{ff}\ \mathbb{L}(u)$.

The following lemma serves the proof:

Lemma 1:

$\mathbb{L}^S(u)$ is a connected subset of BM_+^n .

Proof:

Let $x , y \in \mathbb{L}(u)$, $x \neq y$. Since $\mathbb{L}^S(u) := \mathbb{L}(u) + BM_+^n$, the intersection $\left(x + BM_+^n\right) \cap \left(y + BM_+^n\right)$ is nonempty. Also

(5) This and Proposition 3 are also proved in [7].

(6) For a set S , $\bar{S}$ denotes its closure.

$\mathbb{L}^S(u) = \bigcup_{x \in \mathbb{L}(u)} \left(x + BM^n_+\right)$ is the union of connected subsets with non-empty intersection. Therefore $\mathbb{L}^S(u)$ is connected. Q.E.D.

To continue the proof of Proposition 2, consider $x^o \in \mathbb{L}^S(u)$ and define $F_0 : = \left\{x \in BM^n_+ \mid x \leqq x^o\right\}$. F_0 is closed and $\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}^S(u)} \cap F_0 = \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} \cap F_0$ is a compact subset of BM^n_+ (see asymmetric axiom). $\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} \cap F_0$ is nonempty because if not, $\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}$ and F_0 are disjoint contradicting the connectedness of $\mathbb{L}^S(u)$. Let x^* yield $\min \{||x|| \mid x \in (\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} \cap F_0)\}$. The point x^* exists since the norm is continuous and $(\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} \cap F_0)$ is compact. Since $x^* \in \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}$, and $y = x^* + (y - x^*)$ with $(y - x^*) \in BM^n_+$, $y \in \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} + BM^n_+$, and $\mathbb{L}^S(u) \subset \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} + BM^n_+$. Finally since $\mathbb{L}(u) \subset \mathbb{L}^S(u)$, $\mathbb{L}(u) \subset \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} + BM^n_+$. Q.E.D.

Frequently in economics, like in the theory *of exhaustible resources* (see [9]), dynamic neoclassical production functions are applied. It is therefore of interest to determine their existence, hence introduce:

Definition 1:

The functional $\phi\!\!\phi$ defined pointwise by $\phi\!\!\phi(x(t)) : = \max \{u(t) \in \mathbb{R}_+ \mid u \in \mathbb{P}(x)\}$, $t \in [0,+\infty)$, is called a dynamic neoclassical production function.

Proposition 3:

There exists a dynamic neoclassical production function $\phi\!\!\phi(x) \in \mathbb{P}(x)$, $x \in BM^n_+$, if and only if the efficient subset of output histories $(\mathbb{E}\mathrm{ff}\ \mathbb{P}(x))$ is a single output history.

Before proving this proposition define the efficient subset of output histories by:

$$\mathbb{E}\text{ff }\mathbb{P}(x) : = \begin{cases} \{u \mid u \in \mathbb{P}(x) , v > u \Rightarrow v \notin \mathbb{P}(x)\} , & \mathbb{P}(x) \neq 0 \\ 0 \text{ for } \mathbb{P}(x) = 0 . \end{cases}$$

The output correspondence $x \to \mathbb{P}(x)$ is bounded and closed (see properties $\mathbb{P}.2$ and $\mathbb{P}.5$) thus by argument like those of Proposition 1, it follows that $\mathbb{E}\text{ff }\mathbb{P}(x)$ nonempty for $x \in \mathbb{BM}^n_+$.

Proof of Proposition 3:

Assume there is a dynamic neoclassical production function $\phi(x) \in \mathbb{P}(x)$, $x \in \mathbb{BM}^n_+$. Then $\phi(x(t)) \geqq u(t)$ for all $t \in [0,+\infty)$ and $u \in \mathbb{P}(x)$, implying that $\mathbb{E}\text{ff }\mathbb{P}(x) = \{\phi(x)\}$. Conversely assume $\mathbb{E}\text{ff }\mathbb{P}(x) = \{u\}$ thus for all $v \in \mathbb{P}(x)$, $u(t) \geqq v(t)$, $t \in [0,+\infty)$ and hence $\phi(x) : = u$ is a neoclassical dynamic production function. Q.E.D.

4. ESSENTIALITY OF PRODUCTION FACTORS AND LIMITATIONALITY OF OUTPUT RATES

The first step in characterizing a dynamic law of diminishing returns is dealt with in this section. The aim here is to find conditions under which there are bounds on the rates of a subvector of input histories such that output rates are bounded even when the other inputs may freely vary. To pursue this issue introduce:

Definition 2:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, is essential if $\mathbb{P}(x) = 0$ for all $x \in \left\{x \in \mathbb{BM}_+^n \mid x_{\nu_i} = 0 , i = 1,2, \ldots, k\right\} =: D(\nu_1, \nu_2, \ldots, \nu_k)$.

Definition 3:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, is output rate weak limitational if there exists a positive scalar B such that $\mathbb{P}(x)$ is bounded for all $x \in \left\{x \in \mathbb{BM}_+^n \mid ||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}$.

Definition 4:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, is output rate strong limitational if for each positive scalar B , $\mathbb{P}(x)$ is bounded for all $x \in \left\{x \in \mathbb{BM}_+^n \mid ||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}$.

Note that if a factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is essential, then the intersection $(\mathbb{L}(u) \cap D(\nu_1, \nu_2, \ldots, \nu_k))$ is empty for all $u \neq 0$. Also note that an output rate strong limitational factor combination is weak limitational.

The relationship between essentiality and weak limitationality is clear from:

Proposition 4:

A factor combination $\{\nu_1,\nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, *is essential if and only if it is output rate weak limitational.*

Proof:

Assume first that the factor combination $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is not essential, then there is an input history $x^o \in D(\nu_1,\nu_2, \ldots, \nu_k)$ such that there is a nonzero output history $u \in \mathbb{P}(x^o)$, hence by property $\mathbb{P}.4$, that factor combination is not output rate strong nor weak limitational.

To prove the converse, assume that $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is an essential factor combination. Then for any nonzero $u \in BM_+$, $\mathbb{L}(u) \cap D(\nu_1,\nu_2, \ldots, \nu_k)$ is empty. Also since $\mathrm{Eff}\ \mathbb{L}(u) \subset \mathbb{L}(u)$ and $\mathbb{L}(u)$ is closed (property $\mathbb{P}.5$) the intersection $\overline{\mathrm{Eff}\ \mathbb{L}(u)} \cap D(\nu_1,\nu_2, \ldots, \nu_k)$, where $\overline{\mathrm{Eff}\ \mathbb{L}(u)}$ denotes the closure of $\mathrm{Eff}\ \mathbb{L}(u)$, is empty. The set $D(\nu_1,\nu_2, \ldots, \nu_k)$ is nonempty and closed thus for $x \in \overline{\mathrm{Eff}\ \mathbb{L}(u^o)}$, with $u^o \in BM_+$, $u^o \neq 0$ and $\mathbb{L}(u^o)$ nonempty, the distance

$$d(x,D(\nu_1,\nu_2, \ldots, \nu_k)) := \inf \{||x - y|| \mid y \in D(\nu_1,\nu_2, \ldots, \nu_k)\}$$

is strictly positive. The function d is continuous in x (see [1], p. 84) and since $\overline{\mathrm{Eff}\ \mathbb{L}(u^o)}$ is a nonempty compact set (see asymmetric axiom) there is an input vector $x^o \in \overline{\mathrm{Eff}\ \mathbb{L}(u^o)}$ such that x^o minimize

$$0 < \delta := \min \left\{ d(x,D(\nu_1,\nu_2, \ldots, \nu_k)) \mid x \in \overline{\mathrm{Eff}\ \mathbb{L}(u^o)} \right\} .$$

Choose as the positive bound $B = \delta/2$. By Proposition 2, $\mathbb{L}(u^o) \subset \left(\overline{\mathbb{E}\text{ff } \mathbb{L}(u^o) + BM_+^n}\right)$ thus it follows from property $\mathbb{P}.6$ that the intersections $\left(\mathbb{L}(u) \cap \left\{x \in BM_+^n \mid ||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}\right)$ are empty for all $u \geqq u^o$. Consequently, the essential factor combination is output rate weak limitational. Q.E.D.

In order to show that essentiality not necessarily implies output rate strong limitationality, consider the following production function:

$$(1)\quad \phi(x_1,x_2) := \begin{cases} x_1(t_1) + x_2(t_2) & \text{for } t_i > 0 ,\ i = 1,2 ,\ x_1(t_1) \geqq B > 0 , \\ & \qquad t_1 \geqq T > 0 ,\ x_2(t_2) \geqq 0 \\ 0 & \text{for } t_i = 0 ,\ i = 1,2 ,\ B > x_1(t_1) \geqq 0 ,\ T > t_1 \geqq 0 , \\ & \qquad x_2(t_2) \geqq 0 , \end{cases}$$

where $x_i(t_i)$, $i = 1,2$, are step functions for $t \geqq T$, $T \geqq t$. Clearly the first factor is essential and for positive bound on $||x_1||$ less than B , it is limitational. On the other hand, for $x_1^o(t_1) \geqq B$ and $t_i \geqq T$, $i = 1,2$, $||\phi\left(x_1^o, x_2\right)|| \to +\infty$ as $||x_2|| \to +\infty$. Hence, x_1 is essential but not output rate strong limitational.

For the special case of a homothetic input correspondence it will be shown that an essential factor combination is output rate strong limitational. Therefore introduce:

<u>Definition 5:</u>

The dynamic input correspondence $u \to \mathbb{L}(u)$ is homothetic if $\mathbb{L}(u) := F(u) \cdot \mathbb{L}(1)$, where the functional $F : BM_+ \to \mathbb{R}_+$ satisfies: F.1 $F(u) > 0$ for $u \neq 0$. F.2 $F(u)$ is finite for $||u|| < +\infty$ and $\mathbb{L}(u)$ not empty, $+\infty$ for $\mathbb{L}(u)$ empty. F.3 $F(u) \geqq F(u')$ for

$u \geqq u'$. F.4 F is lower semi-continuous. F.5 $F(u) \to +\infty$ as $||u|| \to +\infty$ with $\mathbb{L}(1)$ being a fixed input set, closed and for $x \in \mathbb{L}(1)$, $\lambda \cdot x \in \mathbb{L}(1)$, $\lambda \geqq 1$.

Proposition 5:

If $u \to \mathbb{L}(u)$ is homothetic, a factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, is essential if and only if it is output rate strong limitational.

Proof:

From the first part of the proof of Proposition 4 it is clear that (in general without homotheticity) output rate strong limitationality of a factor combination implies that it is essential. To prove the converse let B^o be an arbitrarily chosen bound on the essential factors of production. Then clearly from property F.5 there is a u^o such that the intersection $\left(F(u) \cdot \mathbb{L}(1) \cap \left\{x \in BM_+^n \mid ||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}\right)$ is empty for any $u \geqq u^o$. Q.E.D.

It is also of interest to give a complete characterization of when essentiality of a factor combination is equivalent to output rate strong limitationality. The following proposition does this.

Proposition 6:

A necessary and sufficient for an essential factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, to be output rate strong limitational is that for each positive bound B , there is a $u(B) \in BM_+$ such that the intersection $\left(\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)} \cap \left\{x \in BM_+^n \mid ||x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}|| \leqq B\right\}\right)$ is empty for $u \geqq u(B)$.

<u>Proof</u>:

From Proposition 2 and property $\mathbb{P}.6$ of the technology the sufficiency clearly follows. To prove the necessity, assume that an essential factor combination is output rate strong limitational, then for each positive bound B , there is a $u(B) \in \mathbb{R}_+$ such that $\left(\mathbb{L}(u) \cap \left\{x \in \mathbb{R}^n_+ \mid \|x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\| \leqq B\right\}\right)$ is empty for any $u \geqq u(B)$. Consequently since $\mathbb{E}\mathrm{ff}\ \mathbb{L}(u) \subset \mathbb{L}(u)$ and $\mathbb{L}(u)$ is closed the proposition holds. Q.E.D.

The economic interpretation of the condition stated in Proposition 6 is that for efficient increases in production, more of the essential factors must be used.

5. A REFINEMENT OF THE GENERAL TECHNOLOGY

The next step in explaining a dynamic law of diminishing returns is to show how time bounds on the essential factors i.e., $x_{\nu_i}(t) = 0$ for $t \geq T_i$ $i = 1,2, \ldots, k$ relates to time bounds on net output. The purpose of this is to model the situation where some (essential) factor is exhausted and to analyze its consequences on the availability of net outputs.

In order to pursue this issue some additional axioms will be introduced. For this reason consider the following notations.

For $f \in BM_+^\alpha$ and $F \subset BM_+^\alpha$ ($\alpha = 1$ or n in this paper), define:

$$\text{supp } f := \overline{\{t \in \mathbb{R}_+^\alpha \mid f_i(t) > 0 , i = 1, \ldots, \alpha\}} ,$$

$$\text{supp } F := \overline{\{t \in \mathbb{R}_+^\alpha \mid f_i(t) > 0 , i = 1, \ldots, \alpha , f \in F\}} ,$$

$$\text{sup supp } F := \Big\{ t \in \prod_{i=1}^{n} (\bar{\mathbb{R}}_+ \cup \{-1\})_i \mid t_i := \sup \{\tau_i \mid \tau_i \in \text{supp } f_i\} , \; i = 1, \ldots, \alpha , f \in F \Big\} .^{(7)}$$

Note that sup supp for $f_i = 0$ is taken equal to -1 and for supp f_i not bounded sup supp $f_i := +\infty$.

The following three axioms on the time structure of production, not found in [7], are applied:

(7) $\bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$, the positively extended nonnegative real numbers.

T.1 $\mathbb{P}(x) = 0$ for $x \in BM_+^n$ with $x_i(t) = 0$, $t > 0$, $i = 1,2, \ldots, n$;

T.2 If $\operatorname{supp} \mathbb{P}(x) \subset [t^0,t^1]$ nonempty for $x \in BM_+^n$, then for each $\tau > 0$, there is an input vector $\tilde{x} \in BM_+^n$, $\tilde{x}_i(t) := x_i(t + \tau)$, $t \in \mathbb{R}_+$, $i = 1,2, \ldots, n$, such that $\operatorname{supp} \mathbb{P}(\tilde{x}) \subset [t^0 + \tau, t^1 + \tau]$;

T.3 For supp u bounded, $\operatorname{supp} \mathbb{E}\mathrm{ff}\, \mathbb{L}(u)$ is bounded.

The first axiom states that if positive inputs is only applied at $t = 0$, there can be no net output. This axiom is motivated by the fact that production takes time. Note that T.1 dominates $\mathbb{P}$.1 above i.e., if T.1 applies so does $\mathbb{P}$.1. Axiom T.2 is motivated by nonforgetfulness and says that if net output can be produced within a time period, by delaying the use of inputs, output can be produced at any translation of that time interval. The last axiom states that for bounded production periods, it can not be efficient to apply inputs indefinitely.

With these additional axioms on the production technology, time bounds on inputs and net output are next studied.

6. ESSENTIALITY OF PRODUCTION FACTORS AND LIMITATIONALITY OF OUTPUT TIME

Two forms of output time limitationality are distinguished between, namely:

Definition 6:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leq k < n$, is output time weak limitational if there is a positive time bound $T \in (0, +\infty)$, such that sup supp $\mathbb{P}(x)$ is bounded for all $x \in \left\{ x \in BM_+^n \mid x_{\nu_i}(t) = 0 , t \in [T, +\infty) , i = 1, 2, \ldots, k \right\}$, and

Definition 7:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leq k < n$, is output time strong limitational if for each positive time bound $T \in (0, +\infty)$, sup supp $\mathbb{P}(x)$ is bounded for all $x \in \left\{ x \in BM_+^n \mid x_{\nu_i}(t) = 0 , t \in [T, +\infty) , i = 1, 2, \ldots, k \right\}$.

Clearly if a factor combination is output time strong limitational it is weakly so. Next the relationship between essentiality and output time weak limitationality is shown.

Proposition 7:

A factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leq k < n$, is essential if and only if it is output time weak limitational.

Proof:

Assume first that the factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is not essential. Then there is an input history $x^o \in D(\nu_1, \nu_2, \ldots, \nu_k)$ such that $\mathbb{P}(x^o) \neq 0$. If sup supp $\mathbb{P}(x^o)$ not bounded then there is nothing to prove thus assume sup supp $\mathbb{P}(x^o)$ bounded. Then it follows from property T.2 that $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is not output time weak nor strong limitational, proving the second part.

In proving the converse the following lemma is useful.

Lemma 2:

If the intersection $\left(\overline{\text{sup supp } \overline{\mathbb{E}\text{ff } \mathbb{L}(u)}} \cap \text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0 ,\right.\right.$ $\left.\left. t \geqq T , i = 1,2, \ldots, k\right\}\right)$ is empty so is $\left(\text{sup supp } \mathbb{L}(u) \cap \text{supp}\left\{x \in BM_+^n \mid\right.\right.$ $\left.\left. x_{\nu_i}(t) = 0 , t \geqq T , i = 1,2, \ldots, k\right\}\right)$, $u \in BM_+$.

Proof:

It is first shown that sup supp $\mathbb{L}(u) \subset \left(\overline{\text{sup supp } \overline{\mathbb{E}\text{ff } \mathbb{L}(u)}} + \mathbb{R}_+^n\right)$. Thus let $t \in$ sup supp $\mathbb{L}(u)$, then $t =$ sup supp x for some $x \in \mathbb{L}(u)$ and by Proposition 2, $x = y + z$ where $y \in \mathbb{E}\text{ff } \mathbb{L}(u)$ and $z \in BM_+^n$. Let t_y : sup supp y, then since $\overline{\mathbb{E}\text{ff } \mathbb{L}(u)} \subset \mathbb{L}(u)$, $t - t_y \geqq 0$ and consequently, $t = t_y + (t - t_y)$ where $t_y \in \overline{\text{sup supp } \overline{\mathbb{E}\text{ff } \mathbb{L}(u)}}$ and $(t - t_y) \in \bar{\mathbb{R}}_+^n$. It is clear that supp $\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0 ,\right.$ $\left. t \geqq T , i = 1,2, \ldots, k\right\} \subset \left\{t \in \prod_{i=1}^{n} (\bar{\mathbb{R}}_+ \cup \{-1\})_i \mid t_{\nu_i} \leqq T , i = 1,2, \ldots, k ,\right.$ $\left. t_{\nu_i} \in \bar{\mathbb{R}}_+ \cup \{-1\} , i = k+1,k+2, \ldots, n\right\}$ and hence the lemma holds. Q.E.D.

To continue the proof of Proposition 7 it is next shown that intersection $\left(\overline{\text{sup supp } \overline{\mathbb{E}\text{ff } \mathbb{L}(u)}} \cap \text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0 , t > 0 ,\right.\right.$ $\left.\left. i = 1,2, \ldots, k\right\}\right)$ is empty for $u \neq 0$. For this reason assume that t^o belongs to the intersection. Then there is a sequence $\{t^n\} \subset \overline{\text{sup supp } \overline{\mathbb{E}\text{ff } \mathbb{L}(u)}}$ with $t^n \to t^o$ as $n \to +\infty$.

Consequently there is a sequence of input histories $\{x^n(t^n)\} \subset \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}$. It then follows from the compactness of $\overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}$ that $x^{n_\ell} \to x^o \in \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}$ for some subsequence x^{n_ℓ} of $x^n(t^n)$. Hence by essentiality of the factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$ and by property T.1 of the production technology x^o must have $x_{\nu_i}(t) > 0$ for some $t > 0$, and $i = 1,2, \ldots, k$ a contradiction since t^o was picked from the intersection $\left(\overline{\text{sup supp } \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u)}} \cap \text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0,\ t > 0,\ i = 1,2, \ldots, k\right\}\right)$. Now let $u^o \in BM_+^n$, $u^o \neq 0$, $\text{supp}\, u^o$ bounded and $\mathbb{L}(u^o)$ not empty. Then, $\text{sup supp } \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u^o)}$ is a nonempty compact subset of $\prod_{i=1}^{n} (\bar{\mathbb{R}}_+ \cup \{-1\})_i$ (see property T.3), the set $\text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0,\ t > 0,\ i = 1,2, \ldots, n\right\}$ is nonempty and by definition closed, consequently by arguments like those used to prove Proposition 4, there is a positive time bound T such that $\overline{\text{sup supp } \overline{\mathbb{E}\mathrm{ff}\ \mathbb{L}(u^o)}}$ has an empty intersection with $\text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0,\ t \geqq T,\ i = 1,2, \ldots, k\right\}$. Consequently by Lemma 2 and property $\mathbb{P}$.6, Proposition 7 holds. Q.E.D.

The production function (1) above also satisfies properties T.1 - T.3. The first factor is essential and for time bounds on x_1 less that T, it is output time limitational. However, it is not output time strong limitational, see above conditions.

For a homothetic input correspondence the following proposition is valid:

Proposition 8:

If the input correspondence $u \to \mathbb{L}(u)$ is homothetic an essential factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, is output time strong limitational.

If for efficiently increased production time of net output (i.e., sup supp u), the use of the essential factors have to be extended in time, then a factor combination is output time strong limitational if and only if it is essential. Formally:

Proposition 9:

A necessary and sufficient condition for an essential factor combination $\{\nu_1, \nu_2, \ldots, \nu_k\}$, $1 \leqq k < n$, to be output time strong limitational is that for each positive time bound T there is an output history $u(T)$ such that the intersection $\left(\overline{\text{sup supp } \mathbb{E}\text{ff } \mathbb{L}(u)} \cap \text{supp}\left\{x \in BM_+^n \mid x_{\nu_i}(t) = 0 , t \geqq T , i = 1,2, \ldots, k\right\}\right)$ is empty for $u \geqq u(T)$.

The proof is immediate from Lemma 2 and property $\mathbb{P}.6$ of the production structure.

REFERENCES

[1] Berge, C., TOPOLOGICAL SPACES, McMillan Company, (1963).

[2] Färe, R., "Strong Limitationality of Essential Proper Subsets of Factors of Production," Zeitschrift für Nationalökonomie, Vol. 34, pp. 417-424, (1972).

[3] Färe, R. and L. Jansson, "Joint Inputs and the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 36, pp. 407-416, (1976).

[4] Shephard, R. W., "Proof of the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 30, pp. 7-34, (1970).

[5] Shephard, R. W., THEORY OF COST AND PRODUCTION FUNCTIONS, Princeton University Press, (1970).

[6] Shephard, R. W. and R. Färe, "The Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 34, pp. 69-90, (1974).

[7] Shephard, R. W. and R. Färe, "A Dynamic Theory of Production Correspondences," ORC 75-13, Operations Research Center, University of California, Berkeley, California, (1975).

[8] Simmons, G. F., INTRODUCTION TO TOPOLOGY AND MODERN ANALYSIS, McGraw-Hill Book Company, Inc., (1963).

[9] Symposium on Exhaustible Resources, Review of Economic Studies, (December 1974).

[10] Turgot, A. R. J., "Observations sur le Memoire de M. Saint-Peravy 1967," in OEUVES DE TURGOT, Daire, Editor, Vol. 1, pp. 418-483, Paris (1844).