AD-A048 788 STANFORD UNIV CALIF DEPT OF COMPUTER SCIENCE F/G 12/1
C TO THE M-TH DERIVATIVE CONVERGENCE OF TRIGONOMETRIC INTERPOLA--ETC(U)
OCT 77 K P BUBE N00014-75-C-1132
UNCLASSIFIED STAN-CS-77-636 NL

1 OF 1
AD
A048788

END
DATE
FILMED
2-78
DDC

C^m CONVERGENCE OF TRIGONOMETRIC INTERPOLANTS

by

Kenneth P. Bube

STAN-CS-77-636
OCTOBER 1977

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: STAN-CS-77-636 ✓ See Field 30
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): (6) C^m CONVERGENCE OF TRIGONOMETRIC INTERPOLANTS.

5. TYPE OF REPORT & PERIOD COVERED: Technical, October 1977

6. PERFORMING ORG. REPORT NUMBER: (14) STAN-CS-77-636

7. AUTHOR(s): (10) Kenneth P. Bube

8. CONTRACT OR GRANT NUMBER(s): (15) N00014-75-C-1132

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Stanford University
Computer Science Department
Stanford, Ca. 94305

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (12) 27p.

11. CONTROLLING OFFICE NAME AND ADDRESS:
Office of Naval Research
Department of the Navy
Arlington, Va. 22217

12. REPORT DATE: (11) Oct[redacted]977

13. NUMBER OF PAGES: 25

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
ONR Representative: Philip Surra
Durand Aeronautics Bldg., Rm. 165
Stanford University
Stanford, Ca. 94305

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

Releasable without limitations on dissemination.

(9) Technical rept.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For $m \geq 0$, we obtain sharp estimates of the uniform accuracy of the m-th derivative of the n-point trigonometric interpolant of a function for two classes of periodic functions on $\mathbb{R}$. As a corollary, the n-point interpolant of a function in C^k uniformly approximates the function to order $o(n^{1/2-k})$, improving the recent estimate of $O(n^{1-k})$. These results remain valid if we replace the trigonometric interpolant by its K-th partial sum, replacing n by K in the estimates.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified 094 120

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

C^m CONVERGENCE OF TRIGONOMETRIC INTERPOLANTS

Kenneth P. Bube*

ABSTRACT

For $m \geq 0$, we obtain sharp estimates of the uniform accuracy of the m-th derivative of the n-point trigonometric interpolant of a function for two classes of periodic functions on $\mathbb{R}$. As a corollary, the n-point interpolant of a function in C^k uniformly approximates the function to order $o(n^{1/2-k})$, improving the recent estimate of $\mathcal{O}(n^{1-k})$. These results remain valid if we replace the trigonometric interpolant by its K-th partial sum, replacing n by K in the estimates.

*Stanford University, Stanford, California 94305.
Supported in part by the Office of Naval Research under Contract N00014-75-C-1132.

1. Introduction and Notation

Using the concept of aliasing, Snider [6] obtains an $\mathcal{O}(n^{1-k})$ estimate of the uniform accuracy of the n-point trigonometric interpolants of periodic C^k functions for $k \geq 2$, improving the $\mathcal{O}(n^{-1/2})$ estimate for C^2 functions presented in Isaacson and Keller [2]. Kreiss and Oliger [4] use aliasing to show that if the Fourier coefficients $\hat{v}(\xi)$ of a periodic function $v(x)$ satisfy $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-\beta})$ with $\beta > 1$, then the trigonometric interpolants of v uniformly approximate v to order $\mathcal{O}(n^{1-\beta})$. This also gives an $\mathcal{O}(n^{1-k})$ estimate for C^k functions since the largest β we can use in general is $\beta = k$. We use aliasing and a different property of the Fourier coefficients of C^k functions--the fact that C^k is contained in the Sobolev space H^k-- to obtain an $o(n^{1/2-k})$ estimate for $k \geq 1$.

In [5], Kreiss and Oliger estimate the L^2 accuracy of trigonometric interpolants and their derivatives for functions in Sobolev spaces. This paper applies their approach and an extension of a theorem appearing in Zygmund [7] to obtain an $o(n^{1/2+m-s})$ estimate of the uniform accuracy of the m-th derivatives of trigonometric interpolants of functions in the Sobolev spaces H^s for $s > \frac{1}{2} + m$. By similar methods we obtain an $o(n^{m-k})$ estimate for functions in C^k whose k-th derivatives have absolutely converging Fourier series if $k \geq m$, and we show that these two estimates are sharp. We also obtain an $\mathcal{O}(n^{1/2+m-k-\alpha})$ estimate for functions in the Hölder space $C^{k,\alpha}$ if $0 < \alpha \leq 1$ and $k + \alpha > \frac{1}{2} + m$. These results remain valid if we replace the trigonometric interpolant by its K-th partial sum, replacing n by

K in the estimates.

All functions considered will be assumed to be defined on $\mathbb{R}$ and one-periodic. We use the following notation.

$\|v\|_\infty$ denotes $\sup|v(x)|$.

L^2 is the set of complex-valued measurable functions $v(x)$ for which

$$\|v\|_2^2 = \int_0^1 |v(x)|^2 dx < \infty .$$

The Fourier series of a function $v(x) \in L^2$ is

$$\sum_{\xi=-\infty}^{\infty} \hat{v}(\xi) e^{2\pi i \xi x}$$

where $$\hat{v}(\xi) = \int_0^1 v(x) e^{-2\pi i \xi x} dx .$$

$D^k v$ denotes $d^k v/dx^k$. If we say that $D^k v \in B$ for some space of functions B, we mean that $D^{k-1} v$ is an indefinite integral of the function $D^k v$ in B. C^k is the set of functions with k continuous derivatives.

$$\|v\|_{C^k} = \sum_{j=0}^{k} \|D^j v\|_\infty$$

For a real number $s > 0$, H^s is the set of functions $v(x) \in L^2$ such that

$$\|v\|_{H^s}^2 = |\hat{v}(0)|^2 + \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 < \infty .$$

A is the set of functions $v(x) \in L^2$ with absolutely converging Fourier series, i.e.,

$$\sum_{\xi=-\infty}^{\infty} |\hat{v}(\xi)| < \infty$$

For $0 < \alpha \leq 1$, let

$$[v]_\alpha = \sup_{x,y \in \mathbb{R}} \frac{|v(x)-v(y)|}{|x-y|^\alpha}$$

For an integer $k \geq 0$, $C^{k,\alpha}$ is the set of functions $v(x) \in C^k$ such that $[D^k v]_\alpha < \infty$.

If $v \in A$, then v is equal a.e. to a continuous function. Since we are interested in interpolation, we will tacitly assume that $A \subset C^0$ and similarly that $H^s \subset C^0$ for $s > \frac{1}{2}$. For an integer $k \geq 1$, H^k is the set of functions $v(x)$ such that $D^k v \in L^2$ and thus $C^k \subset H^k$. See Agmon [1] for a discussion of L^2 derivatives.

2. Trigonometric Interpolation

We state some well known results on trigonometric interpolation. These appear in this form for odd n in Kreiss and Oliger [4]. See also Isaacson and Keller [2] and Zygmund [7].

A. n is odd. Let $N > 0$ be an integer and $h = \frac{1}{2N+1}$ and let $x_\nu = \nu h$ for $\nu = 0,1,2,\ldots,2N$. There is a unique trigonometric polynomial $I_N v(x)$ of order at most N which interpolates $v(x)$ at the points x_ν for $0 \leq \nu \leq 2N$ given by

$$(1) \qquad I_N v(x) = \sum_{\xi=-N}^{N} a(\xi) e^{2\pi i \xi x}$$

where

$$(2) \qquad a(\xi) = h \sum_{\nu=0}^{2N} v(x_\nu) e^{-2\pi i \xi x_\nu} .$$

The effect called aliasing is the fact that

$$(3) \qquad a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}(\xi + j(2N+1)) \qquad |\xi| \leq N$$

provided that the Fourier series for $v(x)$ converges at the points x_ν for $0 \leq \nu \leq 2N$.

Following the notation of Zygmund, define for $1 \leq K \leq N$

$$(4) \qquad I_{N,K} v(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i \xi x}$$

where $a(\xi)$ is given by (2). $I_{N,K} v$ is the K-th partial sum of $I_N v$, and $I_{N,N} v = I_N v$. If $v(x)$ is real-valued, so is $I_{N,K} v$.

B. N is even. Let $N > 0$ be an integer and $h = \frac{1}{2N}$ and let $x_\nu = \nu h$ for $0 \leq \nu \leq 2N-1$. There is a unique trigonometric polynomial $E_N v(x)$ of order at most N which interpolates $v(x)$ at the points x_ν for $0 \leq \nu \leq 2N-1$ given by

$$(5) \qquad E_N v(x) = \sum_{\xi=-N}^{N}{}' a(\xi) e^{2\pi i \xi x}$$

which also satisfies

$$a(-N) = a(N) .$$

The Σ' notation indicates that the first and last terms are multiplied by 1/2. The coefficients are given by

$$(6) \qquad a(\xi) = h \sum_{\nu=0}^{2N-1} v(x_\nu) e^{-2\pi i \xi x_\nu} .$$

Provided that the Fourier series for $v(x)$ converges at the points x_ν for $0 \leq \nu \leq 2N-1$, we have

$$(7) \qquad a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}(\xi + j(2N)) \qquad |\xi| \leq N$$

Define for $1 \leq K < N$

$$(8) \qquad E_{N,K} v(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i \xi x}$$

where $a(\xi)$ is given by (6), and let $E_{N,N} v = E_N v$. If $v(x)$ is real-valued, so is $E_{N,K} v$ for $K \leq N$. If $w(x)$ is a trigonometric polynomial of order at most N and $\hat{w}(N) = \hat{w}(-N)$, then $E_N w = w$.

3. Accuracy Estimation

Define

$$\delta(v,m,N,K) = \|D^m v - D^m(I_{N,K}v)\|_\infty$$

$$\epsilon(v,m,N,K) = \|D^m v - D^m(E_{N,K}v)\|_\infty$$

The $m = 0$ case of the following lemma appears in Theorem 5.16 of Chapter 10 in Zygmund [7].

Lemma 1. Let $m \geq 0$ be an integer, and suppose that $u = D^m v \in A$. Then

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi|>K} |\hat{u}(\xi)|$$

Proof. Let

(9) $$v_L(x) = \sum_{\xi=-K}^{K} \hat{v}(\xi)e^{2\pi i\xi x} \qquad v_R(x) = \sum_{|\xi|>K} \hat{v}(\xi)e^{2\pi i\xi x}$$

(10) $$w_L = I_{N,K}v_L \qquad w_R = I_{N,K}v_R$$

Then $v = v_L + v_R$ and $I_{N,K}v = w_L + w_R$. Since $w_L = v_L$,

(11) $$v - I_{N,K}v = v_R - w_R$$

so

(12) $$\delta(v,m,N,K) \leq \|D^m v_R\|_\infty + \|D^m w_R\|_\infty \ .$$

By (3),

$$w_R(x) = \sum_{\xi=-K}^{K} \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N+1))e^{2\pi i\xi x}$$

$$\|D^m w_R\|_\infty \le \sum_{\xi=-K}^{K} |2\pi\xi|^m \sum_{j=-\infty}^{\infty} |\hat{v}_R(\xi + j(2N+1))|$$

$$\le \sum_{\xi=-K}^{K} \sum_{j=-\infty}^{\infty} |2\pi(\xi + j(2N+1))|^m |\hat{v}_R(\xi + j(2N+1))|$$

$$\le \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^m |\hat{v}_R(\xi)|$$

So

(13) $$\|D^m w_R\|_\infty \le \sum_{|\xi| > K} |\hat{u}(\xi)|$$

Also

(14) $$\|D^m v_R\|_\infty \le \sum_{|\xi| > K} |\hat{u}(\xi)|$$

Combining (12), (13), and (14) gives the lemma.

<u>Lemma 2</u>. Let $m \geq 0$ be an integer, and suppose that $u = D^m v \in A$. Then

$$\epsilon(v,m,N,K) \leq 2 \sum_{|\xi| > K} |\hat{u}(\xi)| \qquad \text{for} \qquad K < N$$

$$\epsilon(v,m,N,N) \leq 2 \sum_{|\xi| \geq N} |\hat{u}(\xi)|$$

<u>Proof</u>. For $K < N$, the proof is the same as in Lemma 1.

Using (9) with $K = N - 1$ and replacing (10) by

$$(15) \qquad w_L = E_N v_L \qquad\qquad w_R = E_N v_R$$

we obtain

$$(16) \qquad \epsilon(v,m,N,N) \leq \|D^m v_R\|_\infty + \|D^m w_R\|_\infty$$

By (7),

$$w_R(x) = \sum_{\xi=-N}^{N}{}' \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N))e^{2\pi i \xi x}$$

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-N}^{N}{}' \sum_{j=-\infty}^{\infty} |2\pi(\xi + j(2N))|^m |\hat{v}_R(\xi + j(2N))|$$

$$= \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^m |\hat{v}_R(\xi)|$$

and the lemma follows as in the proof of Lemma 1.

<u>Theorem 1</u>. Let $m \geq 0$ be an integer and $v \in H^s$ with $s > \frac{1}{2} + m$. Then for each K,

$$(17) \qquad \sup_{N \geq K} \delta(v,m,N,K) \leq C R_K(v) K^{1/2 + m - s}$$

where

$$C = \frac{2\,(2\pi)^{m-s}}{\sqrt{s - \frac{1}{2} - m}}$$

and

$$R_K(v) = \left(\sum_{|\xi| > K} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 \right)^{1/2} .$$

Also

(18) $$\sup_{N > K} \epsilon(v,m,N,K) \leq C R_K(v) K^{1/2+m-s}$$

and

(19) $$\epsilon(v,m,K,K) \leq C R_{K-1}(v)(K-1)^{1/2+m-s}$$

Note that since $v \in H^s$, $R_K(v) \to 0$ as $K \to \infty$.

Proof. By Lemma 1, for $N \geq K$ we have

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi|>K} |2\pi\xi|^m |\hat{v}(\xi)|$$

$$\leq 2\left(\sum_{|\xi|>K} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 \right)^{1/2} \left(\sum_{|\xi|>K} |2\pi\xi|^{2(m-s)} \right)^{1/2}$$

$$\leq 2 R_K(v)(2\pi)^{m-s} \left(2 \frac{K^{1+2(m-s)}}{2(s-m) - 1} \right)^{1/2}$$

and (17) follows. (18) and (19) follow similarly from Lemma 2.

Theorem 2. *Let* $k \geq m \geq 0$ *be integers, and suppose* $D^k v \in A$. *Then for each* K,

(20) $$\sup_{N \geq K} \delta(v,m,N,K) \leq Cr_K(v)K^{m-k}$$

where

$$C = 2(2\pi)^{m-k}$$

and

$$r_K(v) = \sum_{|\xi| > K} |2\pi\xi|^k |\hat{v}(\xi)| .$$

Also

(21) $$\sup_{N > K} \epsilon(v,m,N,K) \leq Cr_K(v)K^{m-k}$$

and

(22) $$\epsilon(v,m,K,K) \leq Cr_{K-1}(v)K^{m-k}$$

Note that since $D^k v \in A$, $r_K(v) \to 0$ as $K \to \infty$.

<u>Proof</u>. By Lemma 1, for $N \geq K$ we have

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi| > K} |2\pi\xi|^m |\hat{v}(\xi)|$$

$$\leq 2(2\pi K)^{m-k} \sum_{|\xi| > K} |2\pi\xi|^k |\hat{v}(\xi)|$$

and (20) follows. (21) and (22) follow similarly from Lemma 2.

<u>Theorem 3</u>. Let $m \geq 0$ be an integer and $v \in C^{k,\alpha}$ with $k + \alpha > \frac{1}{2} + m$. Then for each K,

(23) $$\sup_{N \geq K} \delta(v,m,N,K) \leq C[D^k v]_\alpha K^{1/2+m-k-\alpha}$$

where

$$C = \frac{2^{\alpha+1/2}\pi^{m-k}}{1-2^{1/2+m-k-\alpha}}$$

Also

(24) $$\sup_{N \geq K} \epsilon(v,m,N,K) \leq C[D^k v]_\alpha K^{1/2+m-k-\alpha}$$

Proof. The method of proof is similar to that of Bernstein's theorem that $C^{0,\alpha} \subset A$ for $\alpha > \frac{1}{2}$. See Katznelson [3]. Let $u = D^m v$ and $f = D^k v$. If $t = \frac{1}{3}2^{-\nu}$ and $2^\nu \leq |\xi| \leq 2^{\nu+1}$, then $|e^{2\pi i\xi t} - 1| \geq \sqrt{3}$, *so since*

$$f(x+t) - f(x) = \sum_{\xi=-\infty}^{\infty} (e^{2\pi i\xi t} - 1)\hat{f}(\xi)e^{2\pi i\xi x}$$

Parseval's relation implies that

$$\sum_{2^\nu \leq |\xi| \leq 2^{\nu+1}} |\hat{f}(\xi)|^2 \leq \frac{1}{3} \sum_{2^\nu \leq |\xi| \leq 2^{\nu+1}} |e^{2\pi i\xi t} - 1|^2 |\hat{f}(\xi)|^2$$

$$\leq \frac{1}{3} \|f(x+t) - f(x)\|_2^2$$

$$\leq \frac{1}{3} \|f(x+t) - f(x)\|_\infty^2$$

$$\leq \frac{1}{3} t^{2\alpha}[f]_\alpha^2$$

$$\leq \frac{1}{3} 2^{-2\nu\alpha}[f]_\alpha^2$$

By the Schwarz inequality,

$$\sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)| \le (2^{\nu+1} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)|^2)^{1/2}$$

$$= (2^{\nu+1} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} \frac{|\hat{f}(\xi)|^2}{|2\pi\xi|^{2(k-m)}})^{1/2}$$

$$\le (2\pi)^{m-k} \, 2^{\nu(1/2+m-k)} (2 \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{f}(\xi)|^2)^{1/2}$$

$$\le (2\pi)^{m-k} \, 2^{\nu(1/2+m-k-\alpha)} [f]_\alpha$$

Given K, let j satisfy $2^j \le K < 2^{j+1}$. Then by Lemma 1, for $N \ge K$ we have

$$\delta(v,m,N,K) \le 2 \sum_{|\xi| \ge K} |\hat{u}(\xi)|$$

$$\le 2 \sum_{\nu=j}^{\infty} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)|$$

$$\le 2(2\pi)^{m-k} [f]_\alpha \sum_{\nu=j}^{\infty} 2^{\nu(1/2+m-k-\alpha)}$$

$$\le 2(2\pi)^{m-k} [f]_\alpha \frac{(2^j)^{1/2+m-k-\alpha}}{1 - 2^{1/2+m-k-\alpha}}$$

and (23) follows since $\frac{K}{2} \ge 2^j$ and $\frac{1}{2} + m - k - \alpha < 0$. (24) follows similarly from Lemma 2.

4. Sharpness of Estimates

Theorem 1 shows that if $v \in H^s$ and $s > \frac{1}{2} + m$, then $\delta(v,m,N,K)$ and $\epsilon(v,m,N,K)$ are $o(K^{1/2+m-s})$, independent of $N \geq K$. Theorem 2 shows that if $D^k v \in A$ and $k \geq m$, then $\delta(v,m,N,K)$ and $\epsilon(v,m,N,K)$ are $o(K^{m-k})$, independent of $N \geq K$. We prove in this section that these estimates are sharp: they cannot be improved for these two classes of functions.

Theorem 4. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $m \geq 0$ be an integer, and $s > \frac{1}{2} + m$. Then there exists a $v \in H^s$ such that

$$\limsup_{K \to \infty} \frac{\inf_{N \geq K} \delta(v,m,N,K)}{\gamma_K K^{1/2+m-s}} = \infty \tag{25}$$

Proof. Let $p_0 = 1$ and define a strictly increasing sequence $\{p_j\}$ of positive integers inductively such that for $j \geq 1$, if j is odd $p_j = 2p_{j-1}$, and if j is even p_j is a power of 2 such that

$$\gamma_\nu \leq 2^{-j} \qquad \text{for} \qquad \nu \geq p_j/4 \ . \tag{26}$$

Define the sequence $\{b_\nu\}$ for $\nu \geq 1$ by

$$b_\nu = \left(\frac{2^{-j}}{p_{j+1} - p_j}\right)^{1/2} \qquad \text{for} \qquad p_j \leq \nu < p_{j+1} \tag{27}$$

Then $\sum_{\nu=1}^{\infty} b_\nu^2 = \sum_{j=0}^{\infty} \sum_{p_j \leq \nu < p_{j+1}} b_\nu^2 = \sum_{j=0}^{\infty} 2^{-j} < \infty.$

Note that $b_\nu \geq b_{\nu+1}$ for $\nu \geq 1$ since $p_j \geq 2p_{j-1}$ for $j \geq 0$. Let

(28) $$v(x) = \sum_{\nu=1}^{\infty} (-1)^\nu \frac{1}{(2\pi\nu)^s} b_\nu e^{2\pi i\nu x}$$

Since $\sum_{\xi=-\infty}^{\infty} |2\pi\xi|^{2s}|\hat{v}(\xi)|^2 = \sum_{\nu=1}^{\infty} b_\nu^2 < \infty$, $v \in H^s$. Define v_L, v_R, w_L, and w_R as in (9) and (10). By (11),

(29) $$\delta(v,m,N,K) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty \ .$$

Now

$$|D^m v_R(\tfrac{1}{2})| = |\sum_{|\xi|>K} (2\pi i\xi)^m \hat{v}(\xi) e^{\pi i\xi}| = \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu$$

so

(30) $$\|D^m v_R\|_\infty \geq \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu \ .$$

By (3),

$$w_R(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i\xi x}$$

where for $|\xi| \leq K$,

$$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N+1)) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N+1))$$

Since $2N + 1$ is odd, this last series is an alternating series of terms decreasing in absolute value, so

$$|a(\xi)| \leq |\hat{v}(\xi + 2N + 1)| .$$

Hence

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-K}^{K} |2\pi\xi|^m |a(\xi)|$$

$$\leq \sum_{\xi=-K}^{K} |2\pi(\xi + 2N + 1)|^m |\hat{v}(\xi + 2N + 1)|$$

$$= \sum_{\nu=2N+1-K}^{2N+1+K} (2\pi\nu)^{m-s} b_\nu$$

$$\leq \sum_{\nu=K+1}^{3K+1} (2\pi\nu)^{m-s} b_\nu$$

since the b_ν's form a decreasing sequence. Combining this with (29) and (30) yields

$$\delta(v,m,N,K) \geq \sum_{\nu=3K+2}^{\infty} (2\pi\nu)^{m-s} b_\nu \quad .$$

For even $j \geq 4$, let $K_j = p_j/4$. Then since $p_{j+1} = 2p_j$,

$$\delta(v,m,N,K_j) \geq \sum_{\nu=p_j}^{\infty} (2\pi\nu)^{m-s} b_\nu$$

$$\geq \sum_{p_j \leq \nu < p_{j+1}} (2\pi\nu)^{m-s} (p_j 2^j)^{-1/2}$$

$$\geq (p_j 2^j)^{-1/2} (2\pi)^{m-s} \int_{p_j}^{2p_j} \frac{dx}{x^{s-m}}$$

Now $\int_{p_j}^{2p_j} \frac{dx}{x^\beta} = c_\beta p_j^{1-\beta}$ where

$$c_\beta = \begin{cases} \dfrac{2^{1-\beta}-1}{1-\beta} & \text{for} \quad \beta \neq 1 \\ \log 2 & \text{for} \quad \beta = 1 \end{cases}$$

so if $d_\beta = 2^{1-3\beta}\pi^{-\beta}c_\beta$,

$$\delta(v,m,N,K_j) \geq c_{s-m}\, 2^{-j/2}(2\pi)^{m-s}p_j^{\,1/2+m-s}$$

$$= d_{s-m}2^{-j/2}K_j^{1/2+m-s}$$

Thus (26) implies that

$$\frac{\delta(v,m,N,K_j)}{\gamma_{K_j}K_j^{1/2+m-s}} \geq d_{s-m}2^{j/2}$$

and the theorem follows.

<u>Theorem 5</u>. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $k \geq m \geq 0$ be integers. Then there exists a v with $D^k v \in A$ such that

$$\lim\sup_{K\to\infty} \frac{\inf_{n\geq K} \delta(v,m,N,K)}{\gamma_K K^{m-k}} = \infty \quad . \tag{31}$$

<u>Proof</u>. Same as the proof of Theorem 4 with the following alterations. Replace s by k throughout the proof. Replace (26) by

$$\gamma_\nu \leq 2^{-2j} \qquad \text{for} \qquad \nu \geq p_j/4 \quad . \tag{26'}$$

Define $\quad b_\nu = \dfrac{2^{-j}}{p_{j+1} - p_j} \quad$ for $\quad p_j \le \nu < p_{j+1}$.

Then $\sum_{\nu=1}^{\infty} b_\nu < \infty$ and $\sum_{\xi=-\infty}^{\infty} |2\pi\xi|^k |\hat{v}(\xi)| < \infty$ so $D^k v \in A$. We have for even $j \ge 4$

$$\begin{aligned}
\delta(v,m,N,K_j) &\ge \sum_{\nu=p_j}^{\infty} (2\pi\nu)^{m-k} b_\nu \\
&\ge \sum_{p_j \le \nu < p_{j+1}} (2\pi\nu)^{m-k} (p_j 2^j)^{-1} \\
&\ge (p_j 2^j)^{-1} (2\pi)^{m-k} \int_{p_j}^{2p_j} \frac{dx}{x^{k-m}} \\
&= c_{k-m} 2^{-j} (2\pi)^{m-k} p_j^{m-k} \\
&= \frac{1}{2} d_{k-m} 2^{-j} K_j^{m-k}
\end{aligned}$$

Thus (26') implies that

$$\frac{\delta(v,m,N,K_j)}{\gamma_{K_j} K_j^{m-k}} \ge \frac{1}{2} d_{k-m} 2^j$$

and the theorem follows.

The following lemma is geometrically obvious.

<u>Lemma 3</u>. Let $\{\beta_\nu\}$ be a decreasing sequence of positive numbers converging to 0. Then $\sum_{\nu=1}^{\infty} \beta_\nu e^{2\pi i\nu/3}$ converges and

$$\left| \sum_{\nu=1}^{\infty} \beta_\nu e^{2\pi i\nu/3} \right| \le \beta_1 .$$

<u>Theorem 6</u>. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $m \geq 0$ be an integer, and $s > \frac{1}{2} + m$. Then there exists a $v \in H^s$ such that

$$(32) \qquad \limsup_{K\to\infty} \frac{\inf_{N>K,\, 3\nmid N} \epsilon(v,m,N,K)}{\gamma_K K^{1/2+m-s}} = \infty$$

and

$$(33) \qquad \limsup_{N\to\infty} \frac{\epsilon(v,m,N,N)}{\gamma_N N^{1/2+m-s}} = \infty .$$

If k is an integer with $k \geq m$, then there exists a v with $D^k v \in A$ such that

$$(34) \qquad \limsup_{K\to\infty} \frac{\inf_{N>K,\, 3\nmid N} \epsilon(v,m,N,K)}{\gamma_K K^{m-k}} = \infty$$

and

$$(35) \qquad \limsup_{N\to\infty} \frac{\epsilon(v,m,N,N)}{\gamma_N N^{m-k}} = \infty .$$

<u>Proof</u>. The proof of (32) is the same as the proof of Theorem 4 with the following alterations. Replace (28) by

$$v(x) = \sum_{\nu=1}^{\infty} e^{2\pi i\nu/3} \frac{1}{(2\pi\nu)^s} b_\nu e^{2\pi i\nu x} .$$

For $N > K$, we have

$$\epsilon(v,m,N,K) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty$$

where v_R is given by (9) and $w_R = E_{N,K}v_R$. Now

$$|D^m v_R(\tfrac{2}{3})| = |\sum_{|\xi|>K} (2\pi i\xi)^m \hat{v}(\xi) e^{4\pi i\xi/3}| = \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu$$

so

$$\|D^m v_R\|_\infty \geq \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu \ .$$

By (7),

$$w_R(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i\xi x}$$

where for $|\xi| \leq K$,

$$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N)) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N)) \ .$$

Suppose $3 \not| N$. Then $j(2N)$ cycles through the equivalence classes mod 3, so by Lemma 3,

$$|a(\xi)| \leq |\hat{v}(\xi + 2N)| \ .$$

Hence, as before,

$$\|D^m w_R\|_\infty \leq \sum_{\nu=K+1}^{3K+1} (2\pi\nu)^{m-s} b_\nu$$

and the rest of the proof goes through, establishing (32).

To prove (33) for this v, imitate the proof of Theorem 4 as above with the following changes. Define v_L and v_R by (9) with $K = N - 1$, and define w_L and w_R by (15). Then

$$\epsilon(v,m,N,N) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty \ .$$

As above,

$$\|D^m v_R\|_\infty \geq \sum_{\nu \geq N} (2\pi\nu)^{m-s} b_\nu \quad .$$

By (7),

$$w_R(x) = \sum_{\xi=-N}^{N}{}' \, a(\xi) e^{2\pi i \xi x}$$

where

$$a(\xi) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N)) \qquad \text{for} \quad |\xi| < N$$

$$a(-N) = a(N) = \sum_{j=0}^{\infty} \hat{v}(N + j(2N))$$

For $N = K_j$ for even $j \geq 4$, $3 \nmid N$, so by Lemma 3,

$$|a(\xi)| \leq |\hat{v}(\xi + 2N)| \qquad \text{for} \quad |\xi| < N$$

$$|a(-N)| = |a(N)| \leq |\hat{v}(N)| \quad .$$

Hence

$$\begin{aligned} \|D^m w_R\|_\infty &\leq \sum_{\xi=-N}^{N}{}' \, |2\pi\xi|^m |a(\xi)| \\ &\leq \sum_{\xi=-N+1}^{N-1} |2\pi(\xi + 2N)|^m |\hat{v}(\xi + 2N)| + |2\pi N|^m |\hat{v}(N)| \\ &= \sum_{\nu=N}^{3N-1} (2\pi\nu)^{m-s} b_\nu \end{aligned}$$

So

$$\epsilon(v,m,N,N) \geq \sum_{\nu=3N}^{\infty} (2\pi\nu)^{m-s} b_\nu$$

and (33) follows.

(34) and (35) follow by similar alterations to the proof of Theorem 5.

<u>Remarks</u>. Theorem 4 shows that the $o(K^{1/2+m-s})$ estimate of $\delta(v,m,N,K)$ given by Theorem 1 is sharp by showing that there is no function $g(K)$ going to 0 faster than $K^{1/2+m-s}$ for which $\delta(v,m,N,K) = \mathcal{O}(g(K))$ for all $v \in H^s$. Note that we can obtain a real-valued function in H^s satisfying (25): since the trigonometric interpolants of real-valued functions are real-valued, at least one of the real or imaginary parts of the v constructed must also satisfy (25). Similar statements hold for Theorem 5 and 6. Also, many of the details of the constructions are for convenience, e.g. making the p_j's powers of 2, and placing the singularities at $x = \frac{1}{2}$ in the odd case and at $x = \frac{2}{3}$ in the even case.

5. Corollaries and Summary

Let w_n denote the n-point trigonometric interpolant of v. i.e., if $n = 2N + 1$, $w_n = I_N v$ and if $n = 2N$, $w_n = E_N v$.

Corollary 1. Let $m \geq 0$ be an integer. If $v \in H^s$ with $s > \frac{1}{2} + m$, then

$$\|v - w_n\|_{C^m} = o(n^{1/2+m-s})$$

If $D^k v \in A$ and $k \geq m$, then

$$\|v - w_n\|_{C^m} = o(n^{m-k})$$

If $v \in C^{k,\alpha}$ and $k + \alpha > \frac{1}{2} + m$, then

$$\|v - w_n\|_{C^m} = \mathcal{O}(n^{1/2+m-k-\alpha}) \ .$$

The $m = 0$ case gives the improved estimate for C^k functions:

Corollary 2. If $v \in C^k$ and $k \geq 1$, then

$$\|v - w_n\|_{\infty} = o(n^{1/2-k}) \ .$$

These corollaries also hold for the K-th partial sums of w_n if we replace n by K in the estimates.

Although we gain an extra half power of n in the estimate for general C^k functions over the recent $\mathcal{O}(n^{1-k})$ estimate, there are other classes of functions for which Kreiss and Oliger's $\mathcal{O}(n^{1-\beta})$ estimate for functions satisfying $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-\beta})$ yields better results. For example, if $D^k v$ is not necessarily continuous but is of bounded variation, then $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-k-1})$, so $\|v - w_n\|_\infty = \mathcal{O}(n^{-k})$. Or, if $D^{k-1}v$ is absolutely continuous (or equivalently if $D^k v \in L^1$), then $\hat{v}(\xi) = o(|\xi|^{-k})$, and Kreiss and Oliger's proof shows that $\|v - w_n\|_\infty = o(n^{1-k})$ if $k > 1$. See Katznelson [3] and Zygmund [7] for discussions of the growth of Fourier coefficients. We conclude with a table of estimates.

If $D^k v \in$	then $\|v - w_n\|_\infty =$	for
L^1	$o(n^{1-k})$	$k \geq 2$
L^2	$o(n^{1/2-k})$	$k \geq 1$
$C^{0,\alpha}$	$\mathcal{O}(n^{1/2-k-\alpha})$	$k + \alpha > \frac{1}{2}$
H^s	$o(n^{1/2-k-s})$	$k + s > \frac{1}{2}$
B.V.	$\mathcal{O}(n^{-k})$	$k \geq 1$
A	$o(n^{-k})$	$k \geq 0$.

Acknowledgement. The author would like to thank Dr. Joseph Oliger for the helpful suggestions he made during the preparation of this paper.

REFERENCES

[1] S. Agmon. Lectures on Elliptic Boundary Value Problems. Princeton: D. Van Nostrand Company, Inc., 1965.

[2] E. Isaacson and H. B. Keller. Analysis of Numerical Methods. New York: John Wiley and Sons, Inc., 1966.

[3] Y. Katznelson. An Introduction to Harmonic Analysis. New York: John Wiley and Sons, Inc., 1968.

[4] H.-O. Kreiss and J. Oliger. Methods for the Approximate Solution of Time Dependent Problems. GARP Publications Series, No. 10. Geneva: World Meteorological Organization, 1973.

[5] H.-O. Kreiss and J. Oliger. Stability of the Fourier Method. Report STAN-CS-77-616, Computer Science Department, Stanford University, 1977.

[6] A. D. Snider, "An Improved Estimate of the Accuracy of Trigonometric Interpolation," SIAM J. Numer. Anal., v. 9, 1972. pp. 505-508.

[7] A. Zygmund. Trigonometric Series. Cambridge University Press, 1959.