JOINT DISTRIBUTIONS FOR TOTAL PROGENY IN A CRITICAL BRANCHING PROCESS

BY

HOWARD J. WEINER

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Joint Distributions for Total Progeny in a Critical Branching Process

by Howard J. Weiner *

I. Introduction. Let

(1.1) $Z(t)$ denote the number of cells alive at time t in a critical age-dependent branching process ([1], Ch. 4) as follows. At time $t=0$, a new cell starts the process and has random lifetime with continuous distribution function

$$0 \leq G(t) < 1, \quad G(0+) = 0. \tag{1.2}$$

Assume

$$t^2(1-G(t)) \to 0 \quad \text{as } t \to \infty \tag{1.3}$$

and denote

$$0 < \mu \equiv \int_0^\infty t dG(t). \tag{1.4}$$

At the end of its life the cell is replaced by k new cells with probability p_k. Define

$$h(s) = \Sigma p_k s^k. \tag{1.5}$$

Assume, for some $\epsilon > 0$,

$$h(1+\epsilon) < \infty. \tag{1.6}$$

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This allows for differentiation of $h(s)$, $0 \leq s \leq 1$, under the summation sign, and also implies that

$$(1.7) \qquad \sum_{k=1}^{\infty} k^n p_k < \infty \quad \text{for all } n \geq 1.$$

The basic assumption of criticality is that

$$(1.8) \qquad m \equiv \sum_{k=1}^{\infty} k\, p_k = 1 .$$

Each new cell proceeds as the parent cell, independent of the past and of other cells.

Let

(1.9) $N(t)$ denote the number of total progeny born by t in a critical age-dependent branching process satisfying (1.1) - (1.8).

It is the purpose of this note to obtain a limit theorem for the joint distribution of $N(\alpha t)/t^2$ and $N(t)/t^2$ given $Z(t) > 0$, where $0 < \alpha < 1$, and to indicate an extension. The method involves comparison with a corresponding Galton-Watson process and fractional linear generating function for number of offspring so that iterates may be explicitly computed.

II. Iterations and Approximations.

Definitions

$$(2.1) \qquad F(s_1,s_2,t_0,t_1) \equiv E\left[s_1^{N(t_0)}\; s_2^{N(t_0+t_1)};\; Z(t_0+t_1)=0\right]$$

$$(2.2) \qquad H(s_1,s_2,t_0,t_1) \equiv E\left[s_1^{N(t_0)}\; s_2^{N(t_0+t_1)}\right].$$

By the law of total probability,

$$(2.3) \qquad F(s_1,s_2,t_0,t_1) = s_1 s_2 \left[\int_0^{t_0} h(F(s_1,s_2,t_0-u,t_1))dG(u) + \int_{t_0}^{t_0+t_1} h(F(1,s_2,0,t_0+t_1-u))dG(u) \right],$$

$$F(s_1,s_2,0,0) = 0$$

and

$$(2.4) \qquad H(s_1,s_2,t_0,t_1) = s_1 s_2 \left[\int_0^{t_0} h(H(s_1,s_2,t_0-u,t_1))dG(u) + \int_{t_0}^{t_0+t_1} h(H(1,s_2,0,t_0+t_1-u))dG(u) + 1 - G(t_0+t_1) \right].$$

<u>Definitions</u>

$$(2.5) \qquad F(s,t) \equiv E(s^{N(t)};\ Z(t)=0).$$

$$(2.6) \qquad H(s,t) = E(s^{N(t)}).$$

Then

$$(2.7) \qquad F(s,t) = s\int_0^t h(F(s,t-u))dG(u)$$

$$F(s,0) = 0$$

and

$$(2.8) \qquad H(s,t) = s\left[1 - G(t) + \int_0^t h(H(s,t-u))dG(u) \right].$$

Define the iterative schemes

$$(2.9) \qquad F_{n+1}(s_1,s_2,t_0,t_1) = s_1 s_2 \int_0^{t_0} h(F_n(s_1,s_2,t_0-u,t_1))dG(u) + s_1 s_2 \int_{t_0}^{t_0+t_1} h(F(1,s_2,0,t_0+t_1-u))dG(u)$$

with

$$(2.10) \qquad F_0(s_1,s_2,t_0,t_1) \equiv F(s_1 s_2,t_1) = F(1,s_1 s_2,0,t_1),$$

and

$$(2.11) \qquad H_{n+1}(s_1,s_2,t_0,t_1) = s_1 s_2 \int_0^{t_0} h(H_n(s_1,s_2,t_0-u,t_1))dG(u) + s_1 s_2 \int_{t_0}^{t_0+t_1} h(H(1,s_2,0,t_0+t_1-u))dG(u) + s_1 s_2 (1 - G(t_0+t_1)),$$

with

$$(2.12) \qquad H_0(s_1,s_2,t_0,t_1) = s_1 H(s_2,t_1) \equiv H(s_1,s_2,0,t_1)$$

$$(2.13) \qquad D_{n+1}(s,t) = s\int_0^t h(D_n(s,t-u))dG(u)$$

with

$$(2.14) \qquad D_0(s,t) = 0$$

$$(2.15) \qquad C_{n+1}(s,t) = s\left[\ 1 - G(t) + \int_0^t h(C_n(s,t-u))dG(u)\right]$$

with

$$(2.16) \qquad C_0(s,t) = s$$

$$K_{n+1}(s_1,s_2) = s_1 s_2 h(K_n(s_1,s_2))G(t_0) + 1 - G(t_0) \tag{2.17}$$

with

$$K_0(s_1,s_2) = s_1 F(s_2,t_1) + 1 - G(t_0) \tag{2.18}$$

$$J_{n+1}(s_1,s_2) = s_1 s_2 h(J_n(s_1,s_2)) \tag{2.19}$$

with

$$J_0(s_1,s_2) = s_1 H(s_2,t_1) = F(s_1,s_2,0,t_1) \tag{2.20}$$

$$L_{n+1}(s) = sh(L_n(s)) \tag{2.21}$$

with

$$L_0(s) = 0 \tag{2.22}$$

$$R_{n+1}(s) = sh(R_n(s)) \tag{2.23}$$

with

$$R_0(s) = s. \tag{2.24}$$

Denote

$$G^{(n)}(t) \tag{2.25}$$

to be the n-th convolution of G evaluated at t.

Lemma 1.

$$0 \le F(s,t) - D_n(s,t) \le G^{(n)}(t) \tag{2.26}$$

$$0 \le L_n(s) - D_n(s,t) \le 1 - G^{(n)}(t) \tag{2.27}$$

$$0 \le C_n(s,t) - H(s,t) \le G^{(n)}(t) \tag{2.28}$$

$$(2.29)\qquad 0 \le C_n(s,t) - R_n(s) \le 1 - G^{(n)}(t)$$

$$(2.30)\qquad 0 \le H_n(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1) \le G^{(n)}(t_0)$$

$$(2.31)\qquad 0 \le H_n(s_1,s_2,t_0,t_1) - J_n(s_1,s_2) \le 1 - G^{(n)}(t_0)$$

$$(2.32)\qquad 0 \le F(s_1,s_2,t_0,t_1) - F_n(s_1,s_2,t_0,t_1) \le G^{(n)}(t_0)$$

$$(2.33)\qquad 0 \le K_n(s_1,s_2) - F_n(s_1,s_2,t_0,t_1) \le 1 - G^{(n+1)}(t_0)$$

Proof. Only (2.32) and (2.33) will be explicitly proved. The other results are similar or simpler.

For (2.32), let $n=0$. Then, assuming $t_1 > t_0$

$$(2.34)\qquad \begin{aligned} F(s_1,s_2,t_0,t_1) &\equiv E(s_1^{N(t_0)}\, s_2^{N(t_0+t_1)};\ Z(t_0+t_1)=0) \\ &\ge E(s_1^{N(t_0)}\, s_2^{N(t_1)}\, s_2^{\sum_{i=1}^{Z(t_1)} N_i(t)};\ Z(t_1)=0) \\ &\ge E((s_1 s_2)^{N(t_1)};\ Z(t_1)=0) \equiv F_0(s_1,s_2,t_0,t_1), \end{aligned}$$

since path considerations yield that

$$(2.35)\qquad N(t_0+t_1) \ge N(t_1) + \sum_{i=1}^{Z(t_1)} N_i(t_0),$$

where $\{N_i(t_0)\}$ are I.I.D. as $N(t_0)$ and independent of the $(Z(t_1),N(t_1))$ process.

Similarly, if $t_0 > t_1$,

$$(2.35)\qquad F(s_1,s_2,t_0,t_1) \geq E\left(s_1^{N(t_0)}\, s_2^{N(t_0)}\, s_3^{\sum_{i=1}^{Z(t_0)} N_i(t_0)} ;\ Z(t_0)=0\right)$$

$$= E\left((s_1 s_2)^{N(t_0)} ;\ Z(t_0)=0\right) \geq E\left[(s_1 s_2)^{N(t_0)} ;\ Z(t_1)=0\right]$$

$$= E\left[(s_1 s_2)^{N(t_1)} ;\ Z(t_1)=0\right].$$

By induction, as

$$(2.36)\qquad 0 \leq F - F_0 \leq 1 = G^{(0)}(t_0)$$

and

$$(2.37)\qquad 0 \leq F - F_1 = s_1 s_2 \int_0^{t_0} (h(F) - h(F_0))dG \leq \int_0^{t_0} (F - F_0)dG \leq G(t_0),$$

if it is assumed that

$$(2.38)\qquad 0 \leq F - F_n \leq G^{(n)}(t_0),$$

then

$$(2.39)\qquad 0 \leq F - F_{n+1} = s_1 s_2 \int_0^{t_0} (h(F) - h(F_n))dG \leq \int_0^{t_0} (F - F_n)dG \leq G^{(n+1)}(t_0)$$

proving (2.32).

To show (2.33), for $n=0$,

$$(2.40)\qquad K_0 - F_0 = s_1 E\left[s_2^{N(t_1)} ;\ Z(t_1)=0\right] - E\left[(s_1 s_2)^{N(t_1)} ;\ Z(t_1)=0\right] + 1 - G(t_0)$$

and hence

$$(2.41) \qquad 0 \leq K_0 - F_0 \leq 1 - G(t_0).$$

Also, for $n = 1$,

$$(2.42) \qquad 0 \leq K_1 - F_1 = s_1 s_2 \int_0^{t_0} h(K_0) - h(F_0) dG(u) + 1 - G(t_0)$$

$$- s_1 s_2 \int_{t_0}^{t_0+t_1} h(F) dG(u)$$

and

$$(2.43) \qquad K_1 - F_1 \leq \int_0^{t_0} (1 - G(t_0-u) dG(u) + 1 - G(t_0) = 1 - G^{(2)}(t_0).$$

By induction, assume (2.33) for n. Then

$$(2.44) \qquad 0 \leq K_{n+1} - F_{n+1} \leq \int_0^{t_0} (h(K_n) - h(F_n)) dG + 1 - G(t_0)$$

$$K_{n+1} - F_{n+1} \leq \int_0^{t_0} (K_n - F_n) dG + 1 - G(t_0)$$

$$\leq \int_0^{t_0} (1 - G^{(n)}(t_0-u)) dG(u) + 1 - G(t_0) = 1 - G^{(n+1)}(t_0),$$

completing (2.33).

<u>Lemma 2</u>. Let $h_1(s)$, $h_2(s)$ satisfy (1.5) - (1.8) and assume

$$(2.45) \qquad \sigma_1^2 \equiv h_1''(1) < h_2''(1) \equiv \sigma_2^2 .$$

Then there exists an $0 < s_0 < 1$, and an integer $M > 0$ such that for $s_1 > s_0$, $s_2 > s_0$ and all $n > m > M$,

$$E_1(s_1^{N_m} s_2^{N_n}) \leq E_2(s_1^{N_m} s_2^{N_n}) \tag{2.46}$$

and

$$E_1(s_1^{N_m} s_2^{N_n};\ Z_n = 0) \leq E_2(s_1^{N_m} s_2^{N_n};\ Z_n = 0) \tag{2.47}$$

where N_m, N_n, Z_n are from G-W processes governed by $h_1(s)$ and $h_2(s)$, respectively.

Proof. As $n > m \to \infty$, for $h_i(s)$, $i = 1,2$

$$E_i\left[s_1^{N_m} s_2^{N_n}\right] \to E_i\left[(s_1 s_2)^N\right] \tag{2.48}$$

and

$$E_i(s_1^{N_m} s_2^{N_n};\ Z_n = 0) \to E_i\left[(s_1 s_2)^N;\ Z = 0\right] = E_i(s_1 s_2)^N \tag{2.49}$$

where N, Z are bona-fide r.v.s. and

$$P[Z = 0] = 1 \tag{2.50}$$

for the critical case.

To prove the lemma, it therefore suffices to show that there exists an $1 > s_0 > 0$ such that for $s > s_0$,

$$E_1(s^{N_n}) < E_2(s^{N_n}). \tag{2.51}$$

This proof, due to N. Knueppel, will now be given.

A Taylor expansion of p. 22 of [1] shows that for $s > s_1 > 0$,

$$h_1(s) < h_2(s). \tag{2.52}$$

Since

$$E_i(s^{N_n}) \downarrow E_i(s^N), \quad i = 1,2, \tag{2.53}$$

for $s > s_0$,

$$E_i(s^N) > s_1. \tag{2.54}$$

For $n = 1$ and $s > s_0 > s_1$,

$$E_1 s^{N_1} = sh_1(s) < sh_2(s). \tag{2.55}$$

Assume that for $s > s_0$,

$$s_1 < E_1(s^{N_n}) < E_2(s^{N_n}). \tag{2.56}$$

Then for $s > s_0$,

$$\begin{aligned} E_1(s^{N_{n+1}}) &= sh_1(E_1(s^{N_n})) < sh_2(E_1(s^{N_n})) \\ &< sh_2(E_2(s^{N_n})) = E_2(s^{N_{n+1}}), \end{aligned} \tag{2.57}$$

completing the proof of lemma 2.

Define the iterations

$$(2.58) \qquad T(s_1,s_2,m,n) \equiv E(s_1^{N_m} s_2^{N_n};\ Z_n = 0)$$

with

$$(2.59) \qquad T(s_1,s_2,0,n\text{-}m) = s_1 E(s_2^{N_{n-m}};\ Z_{n-m} = 0) = s_1 L_{n-m}(s_2).$$

$$(2.60) \qquad U(s_1,s_2,m,n) = E\left[s_1^{N_m} s_2^{N_n} \right]$$

with

$$(2.61) \qquad U(s_1,s_2,0,n\text{-}m) = s_1 E\left[s_2^{N_{n-m}} \right] = s_1 R_{n-m}(s_2),$$

where Z_n, N_m, N_n are from a critical G-W process with $h''(1) = \sigma^2$.

<u>Lemma 3</u>.

$$(2.62) \qquad F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\leq T(s_1,s_2,n,r+n) - U(s_1,s_2,n,r+n) + 2G^{(r)}(t_1)$$

$$+ 2G^{(n)}(t_0) + 1 - (G(t_0))^{n+1},$$

and

$$(2.63) \qquad F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\geq T(s_1,s_2,n,r+n) - U(s_1,s_2,n,r+n) - 2(1 - G^{(r)}(t_1))$$

$$- 2(1 - G^{(n)}(t_0)).$$

Proof. From (2.30) - (2.33),

$$K_n - J_n - 2(1 - G^{(n)}(t_0)) \le F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1) \le 2G^{(n)}(t_0) + K_n - J_n. \tag{2.64}$$

From (2.26) to (2.29), for any $r \ge 1$,

$$s_1(L_r(s_2) - (1 - G^{(r)}(t))) + 1 - G(t_0) \le K_0(s_1,s_2) \le s_1(L_r(s_2) + G^{(r)}(t)) + 1 - G(t_0), \tag{2.65}$$

and

$$s_1(R_r(s_2) - G^{(r)}(t)) \le J_0(s_1,s_2) \le s_1(R_r(s_2) + 1 - G^{(r)}(t)). \tag{2.66}$$

For a critical generating function h, note that for $a > 0$, $b > 0$, $a + b \le 1$, the mean value theorem yields that

$$\begin{aligned} h(a+b) &\le h(a) + b \\ h(a) - b &\le h(a-b). \end{aligned} \tag{2.67}$$

Note that

$$T(s_1,s_2,m+1,n+1) = s_1 s_2 h(T(s_1,s_2,m,n)) \tag{2.68}$$

$$U(s_1,s_2,m+1,n+1) = s_1 s_2 h(U(s_1,s_2,m,n)). \tag{2.69}$$

Then (2.58) - (2.61), (2.64) - (2,69) together with (2.17) - (2.20) upon successive application of (2.67) yield, for $r \geq 1$,

$$(2.70) \qquad K_1 = s_1 s_2 h(K_0) G(t_0) + 1 - G(t_0)$$

$$\leq s_1 s_2 G(t_0) h(s_1 L_r(s_2)) + 1 - (G(t_0))^2 + G^{(r)}(t),$$

or

$$K_1 \leq T(s_1, s_2, 1, r+1) + G^{(r)}(t) + 1 - (G(t_0))^2.$$

$$(2.71) \qquad K_1 \geq s_1 s_2 G(t_0) h(s_1 L_r(s_2) - (1 - G^{(r)}(t))) + 1 - G(t_0),$$

from which it follows that

$$(2.72) \qquad K_1 \geq T(s_1, s_2, 1, r+1) - (1 - G^{(r)}(t)) s_1 s_2$$

$$(2.73) \qquad K_2 = s_1 s_2 h(K_1) G(t_0) + 1 - G(t_0)$$

$$\leq s_1 s_2 G(t_0) h(T(s_1, s_2, 1, r+1)) + G^{(r)}(t) + 1 - (G(t_0))^3.$$

or

$$(2.74) \qquad K_2 \leq T(s_1, s_2, 2, r+2) + G^{(r)}(t) + 1 - (G(t_0))^3.$$

From (2.72),

$$(2.75) \qquad K_2 \geq s_1 s_2 h(K_1) \geq s_1 s_2 h(T(s_1, s_2, 1, r+1)) - (s_1 s_2)^2 (1 - G^{(r)}(t))$$

or

$$(2.76) \qquad K_2 \geq T(s_1,s_2,2,r+2) - (s_1s_2)^2(1 - G^{(r)}(t)).$$

By induction, assume that

$$(2.77) \qquad K_n \leq T(s_1,s_2,n,r+n) + G^{(r)}(t) + 1 - (G(t_0))^{n+1}$$

and

$$(2.78) \qquad K_n \geq T(s_1,s_2,n,r+n) - (s_1s_2)^n(1 - G^{(r)}(t)).$$

Then

$$(2.79) \qquad K_{n+1} = s_1s_2G(t_0)h(K_n) + 1 - G(t_0)$$

$$\leq s_1s_2h(T(s_1,s_2,n,r+n)) + G^{(r)}(t) + 1 - (G(t_0))^{n+2}$$

or

$$(2.80) \qquad K_{n+1} \leq T(s_1,s_2,n+1,r+n+1) + G^{(r)}(t) + 1 - (G(t_0))^{n+2},$$

completing the induction started by (2.77).

In the other direction, using (2.78),

$$(2.81) \qquad K_{n+1} \geq s_1s_2h(K_n) \geq s_1s_2h(T(s_1,s_2,n,r+n)) - (s_1s_2)^{n+1}(1 - G^{(r)}(t))$$

or

(2.82) $K_{n+1} \geq T(s_1, s_2, n+1, r+n) - (s_1 s_2)^{n+1}(1 - G^{(r)}(t))$,

completing the induction started by (2.78).

A similar argument to that of (2.70) - (2.82) yields

(2.83) $U(s_1, s_2, n, r+n) - G^{(r)}(t) \leq J_n \leq U(s_1, s_2, n, r+n) + 1 - G^{(r)}(t)$.

Hence (2.64), (2.77), (2.78), (2.83) yield

(2.84)
$$F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \geq T(s_1, s_2, n, r+n) - U(s_1, s_2, n, r+n)$$
$$- 2(1 - G^{(r)}(t)) - 2(1 - G^{(n)}(t_0))$$

and, omitting the same arguments as in (2.84),

(2.85) $F - H \leq T - U + 2G^{(r)}(t) + 1 - (G(t_0))^{n+1} + 2G^{(n)}(t_0)$.

Now set $t = t_1$. This completes lemma 3.

Let

(2.86) $$h_0(s, \sigma^2) \equiv \frac{\sigma^2 + (2 - \sigma^2)s}{\sigma^2(1-s) + 2} .$$

Let

(2.87) $U_0(s_1, s_2, m, n)$

and

$T_0(s_1, s_2, m, n)$

denote the respective quantities U, T obtained for h_0 of (2.86).

Lemma 4. For the critical generating function (2.86), it follows that

(2.88)
$$\lim_{n\to\infty} \left(\frac{n\sigma^2}{2}\right)\left(U_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha)) - T_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha))\right)$$

$$= \lim_{n\to\infty} E\left[\exp\{-\frac{1}{n^2}(\theta_1 N_{0,[\alpha n]} + \theta_2 N_{0,n})\} \,|\, Z_{on} > 0\right]$$

$$= \frac{4\sqrt{2\sigma^2\theta_2}\,(\theta_1+\theta_2)}{(\sqrt{\theta_2}+\sqrt{\theta_1+\theta_2})^2\sinh\{\alpha\sqrt{2\sigma^2(\theta_1+\theta_2)} + (1-\alpha)\sqrt{2\sigma^2\theta_2}\}}$$

$$+ (\sqrt{\theta_1+\theta_2} - \sqrt{\theta_2})^2\sinh\{(1-\alpha)\sqrt{2\sigma^2\theta_2} - \alpha\sqrt{2\sigma^2(\theta_1+\theta_2)}\}$$

$$+ 2\theta_1\sinh\{(1-\alpha)\sqrt{2\sigma^2\theta_2}\}$$

where N_{0m} is the total progeny and number alive, respectively, in a critical G-W process at generation m with offspring generating function $h_0(s,\sigma^2)$.

Proof. The proof follows the method of Lindvall ([2] pp. 318-319).

For $0 < m < n$, with N_{0m}, N_{0n}, Z_{0n} from a critical G-W process with offspring generating function $h_0(s,\sigma^2)$, one may write

(2.89)
$$E(s_1^{N_{0m}} s_2^{N_{0n}} s_3^{Z_{0n}}) = E\left[(s_1 s_2)^{N_{0m}} E\left(s_2^{\sum_{i=1}^{Z_{0m}} N_{0,n-m,i}} s_3^{\sum_{i=1}^{Z_{0m}} Z_{0,n-m,i}} \,\Big|\, Z_{0m}\right)\right]$$

where $\{N_{0,n-m,i}\}$ are I.I.D. as $N_{0,n-m}$, the $\{Z_{0,n-m,i}\}$ are I.I.D. as $Z_{0,n-m}$, and both sets of r.v.s. are independent of the (Z_{0m},N_{0m}) part of the process, and $N_{0,n-m,i}$ and $Z_{0,n-m,j}$ are independent for $i \neq j$, with $N_{0,n-m,i}$ and $Z_{0,n-m,i}$ from the same critical G-W process. Hence

(2.90) $$E(s_1^{N_{0m}} s_2^{N_{0n}} s_3^{Z_{0n}}) = h_m(s_1 s_2, h_{n-m}(s_2, s_3))$$

where

(2.91) $$h_r(s_1, s_2) \equiv E(s_1^{N_{0r}} s_2^{Z_{0r}}).$$

To express $h_r(s_1, s_2)$ in terms of $h_0(s) \equiv h_0(s, \sigma^2)$ and its iterates, note that

(2.92) $$h_1(s_1, s_2) = s_1 h_0(s_1 s_2)$$

and

(2.93) $$h_{r+1}(s_1, s_2) \equiv E(s_1^{N_{0,r+1}} s_2^{Z_{0,r+1}}) = E\left[E(s_1^{\sum_{i=1}^{Z_{0r}} N_{0r,i}} s_2^{\sum_{i=1}^{Z_{or}} Z_{0r,i}} | Z_{01})\right]$$

where $N_{0r,i}$ and $Z_{0r,i}$ are from the same process, and N_{0rj} and Z_{0ri} are independent for $i \neq j$, and the $\{N_{0ri}\}$ are I.I.D. as N_{0r}, and $\{Z_{0ri}\}$ are I.I.D. as Z_{0r}.

Hence

(2.94) $$h_{r+1}(s_1, s_2) = s_1 h_0(h_r(s_1, s_2)).$$

A tedious but straight-forward induction using (2.90) yields that

(2.95) $$h_n(s_1, s_2) = \frac{P_{1,n}(s_1) + s_2 P_{2,n+1}(s_1)}{P_{3,n-1}(s_1) + s_2 P_{4,n}(s_1)}$$

where P_{in}, denote the n-th degree polynomials to be determined. Relation (2.95) yields

(2.96) (a) $P_{1,n+1}(s) = spP_{3,n-1}(s) - (2p-1)sP_{1,n}(s)$

(b) $P_{2,n+2}(s) = spP_{3,n}(s) - (2p-1)sP_{2,n+1}(s)$

(c) $P_{3,n}(s) = P_{3,n-1}(s) - pP_{1,n}(s)$

(d) $P_{4,n+1}(s) = P_{4,n}(s) - pP_{2,n+1}(s)$.

From the theory of difference equations one may solve pairs (2.96) (a) and (c) and pair (2.96) (b) and (d) and from initial conditions obtained from explicit formulas for $h_1(s_1,s_2)$ and $h_2(s_1,s_2)$ one substitutes a solution

(2.97) $$P_{in} = A_i r^n, \quad 1 \leq i \leq 4$$

to obtain

(2.98) $$P_{in} = A_{0i} r_1^{\,n} + A_{1i} r_2^{\,n}, \quad 1 \leq i \leq 4$$

where $\{A_{0i}\}$, $\{A_{1i}\}$, r_1, r_2 are explicitly determined.

Writing $n\alpha$ instead of $[n\alpha]$, which will not affect a limit, it follows that

(2.99) $$\lim_{n\to\infty} E\left[\exp\{-\frac{1}{n^2}(\theta_1 N_{0,n\alpha} + \theta_2 N_{0n})\} \,|\, Z_{0n} > 0\right]$$

$$= \lim_{n\to\infty} \frac{h_{n\alpha}(h_{n(1-\alpha)}(1,e^{-\theta_2/n^2}),e^{-(\theta_1+\theta_2)/n^2}) - h_{n\alpha}(h_{n(1-\alpha)}(0,e^{-\theta_2/n^2}),e^{-(\theta_1+\theta_2)/n^2})}{1 - h_n(1,0)}$$

A tedious but straightforward computation using (2.95), (2.98) in (2.99) yields the result of lemma 4.

Let

$$\frac{2p}{q} \equiv \sigma^2 \tag{2.100}$$

where $0 < p < 1$ and $q = 1 - p$.

For $0 < \epsilon \ll p$, denote

$$\begin{aligned} p_1 &= p + \epsilon \\ q_1 &= 1 - p_1 \end{aligned} \tag{2.101}$$

and

$$\begin{aligned} p_2 &= p - \delta(\epsilon) \\ q_2 &= 1 - p_2 \end{aligned} \tag{2.102}$$

where

$$p_1 q_1 = p_2 q_2 \ . \tag{2.103}$$

Denote

$$\sigma_i^{\,2} = 2p_i/q_i \ , \quad i = 1,2. \tag{2.104}$$

<u>Corollary</u>. For $1 \le i,\ j \le 2$, $i \ne j$, and $0 < \epsilon \le \epsilon_0 \ll p$, and $0 < \alpha < 1$, and if (2.100) - (2.103) hold, then

$$\begin{aligned} \overline{\lim_{n,\epsilon \le \epsilon_0}}\ n \Big| U_0\big(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha), \sigma_i^{\,2}\big) \\ - T_0\big(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha), \sigma_j^{\,2}\big) \Big| \le C \end{aligned} \tag{2.105}$$

where $C < \infty$ is a positive constant.

Proof. This is a straightforward if tedious computation of U_0, T_0 using the method of difference equations of the previous lemma, noting that from (2.103), the constant term in the expansion of $U_0 - T_0$ cancels out, leaving terms of order $\frac{1}{n}$ and lower in n.

Theorem. Under the assumptions (1.1) to (1.8)

$$(2.106)\qquad \lim_{t\to\infty} E\left[\exp\left\{-\frac{1}{t^2}(\theta_1 N(\alpha t)+\theta_2 N(t))\,\middle|\, Z(t)>0\right\}\right]$$
$$= \left(\frac{4\sqrt{2\sigma^2\theta_2(\theta_1+\theta_2)}}{\mu}\right)\Bigg[(\sqrt{\theta_2}+\sqrt{\theta_1+\theta_2})^2\sinh\left\{\frac{\alpha\sqrt{2\sigma^2(\theta_1+\theta_2)}+(1-\alpha)\sqrt{2\sigma^2\theta_2}}{\mu}\right\}$$
$$+(\sqrt{\theta_1+\theta_2}-\sqrt{\theta_2})^2\sinh\left\{\frac{(1-\alpha)\sqrt{2\sigma^2\theta_2}-\alpha\sqrt{2\sigma^2(\theta_1+\theta_2)}}{\mu}\right\}$$
$$+2\theta\sinh\left\{\frac{(1-\alpha)\sqrt{2\sigma^2\theta_2}}{\mu}\right\}\Bigg]^{-1}.$$

Proof. From lemma 3, let, for $0<\varepsilon<\varepsilon_0\ll p$, where $\sigma^2=\frac{2p}{q}$,

$$(2.107)\qquad \text{(a)}\quad r_1=\left[\frac{t_1(1+\varepsilon)}{\mu}\right]$$

$$\text{(b)}\quad r_2=\left[\frac{t_1(1-\varepsilon)}{\mu}\right]$$

$$\text{(c)}\quad n_1=\left[\frac{t_0(1+\varepsilon)}{\mu}\right]$$

$$\text{(d)}\quad n_2=\left[\frac{t_0(1-\varepsilon)}{\mu}\right].$$

Then by lemma 3 and the lemma 3 of Ch. 4 of [1], pp. 158--160, (2.108) $$F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\leq T(s_1,s_2,n_1,r_1+n_1) - U(s_1,s_2,n_1,r_1+n_1) + o(t_0^{-1}) + o(t_1^{-1})$$

and

(2.109) $$F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\geq T(s_1,s_2,n_2,r_2+n_2) - U(s_1,s_2,n_2,n_2+r_2) + o(t_0^{-1}) + o(t_1^{-1}).$$

Now, using lemma 2 in (2.108), (2.109) yields, with assumptions (2.100) - (2.104), for r_i, n_i sufficiently large, $i = 1,2$, and $\epsilon < \epsilon_0 \ll p$,

(2.110) $$F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\leq T_0(s_1,s_2,n_1,r_1+n_1,\sigma_1^2) - U_0(s_1,s_2,n_1,r_1+n_1,\sigma_2^2) + o(t_0^{-1} + t_1^{-1})$$

and

(2.111) $$F(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1)$$

$$\geq T_0(s_1,s_2,n_2,r_2+n_2,\sigma_2^2) - U_0(s_1,s_2,n_2,r_2+n_2,\sigma_1^2) + o(t_0^{-1} + t_1^{-1})$$

where

(2.112) $$\sigma_1^2 > \sigma^2 > \sigma_2^2$$

and

(2.113) $$\sigma_i^2 = 2p_i/q_i, \quad i = 1,2$$

with

(2.114) $p_i = p \pm \epsilon_i$, as in (2.101) - (2.103).

Now, set, for $0 < \alpha < 1$,

$$(2.115) \quad \text{(a) } t = n\mu$$

$$\text{(b) } t_0 = n\alpha\mu$$

$$\text{(c) } t_1 = n(1-\alpha)\mu$$

$$\text{(d) } s_1 = e^{-\theta_1/n^2}, \quad s_2 = e^{-\theta_2/n^2}.$$

Multiply (2.110) and (2.111) by n.

Then let $\epsilon \to 0$, then $n \to \infty$, noting that by the corollary, these limits may be interchanged.

Since, for fixed $\sigma^2 > 0$,

$$(2.116) \quad E\left[\exp\left\{ -\frac{1}{t^2}(\theta_1 N(\alpha t) + \theta_2 N(t)) \middle| Z(t) > 0 \right\} \right] = \frac{H(e^{-\theta_1/t^2}, e^{-\theta_2/t^2}, \alpha t, (1-\bar{\alpha})t) - F(e^{-\theta_1/t^2}, e^{-\theta_2/t^2}, \alpha t, (1-\alpha)t)}{P[Z(t) > 0]}$$

and by [1], Ch. 4,

$$(2.117) \quad \lim_{t \to \infty} tP[Z(t) > 0] = \frac{2\mu}{\sigma^2},$$

then lemma 4 and the corollary together with (2.116), (2.117) and the substitution of θ_i/μ^2 for θ_i, $i = 1, 2$, then yields the result of the theorem.

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[1] ATHREYA, K. and NEY, P.E. (1970). Branching Processes. Springer-Verlag, New York.

[2] LINDVALL, T. (1974). Limit theorems for some functionals of certain Galton-Watson branching processes. Adv. Appl. Prob. 6, 309-321.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $N(t)$ denote the total progeny born by time t in a critical age-dependent branching process. A limit law for the joint distribution of $N(\alpha t)/t^2$ and $N(t)/t^2$ conditioned on the event that the process is not extinct at t is obtained, where $) < \alpha < 1$.