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THE MOTION OF A DRIVEN OSCILLATOR. PART 1 NONDECREASING INPUT V--ETC(U)

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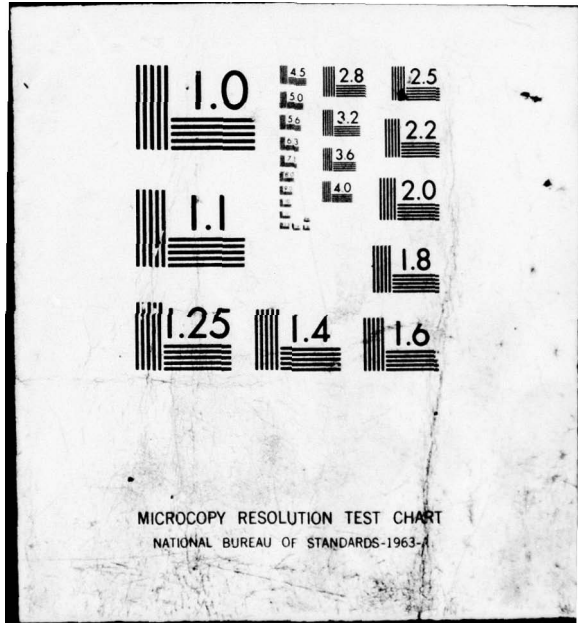
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REPORT

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THE MOTION OF A DRIVEN OSCILLATOR
PART 1: NONDECREASING INPUT VELOCITIES

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J.G. Ternan

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DEPARTMENT OF DEFENCE
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PART 1: NONDECREASING INPUT VELOCITIES,

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ABSTRACT



An inertia-actuated impact fuze is represented by a cantilever mounted on an accelerating frame. The lever is a driven, normally harmonic, oscillator and operates a secondary mechanism if it is moving fast enough at a particular deflection.

The sets of possible motion are determined when the input frame velocity is nondecreasing. These motion sets lead to sufficient conditions, in terms of the input velocity at a particular time, for no operation up to that time, for certain operation and, when operation is certain, to the maximum operating time.

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16. ABSTRACT (if this is security classified, the announcement of this report will be similarly classified):

An inertia-actuated impact fuze is represented by a cantilever mounted on an accelerating frame. The lever is a driven, normally harmonic, oscillator and operates a secondary mechanism if it is moving fast enough at a particular deflection.

The sets of possible motion are determined when the input frame velocity is nondecreasing. These motion sets lead to sufficient conditions, in terms of the input velocity at a particular time, for no operation up to that time, for certain operation and, when operation is certain, to the maximum operating time.

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THE MOTION OF A DRIVEN OSCILLATOR

PART 1: NONDECREASING INPUT VELOCITIES

Fig. 1 shows an undamped cantilever with resonant radian frequency ω mounted on a frame of acceleration a . Small-amplitude oscillations with $x \leq x_0$ satisfy

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x + a \end{aligned} \tag{1}$$

(1) is a model of an inertia-actuated impact fuze in which the cantilever contains a striking pin and the wall at $x = x_0$ is the detonator. The physical background of the model is discussed in Ref. 1.

The lever, then, with initial conditions $x(0) = 0$, $v(0) = 0$, operates the secondary mechanism if $v > v_0 > 0$ when $x = x_0$. If $v = v_1 \leq v_0$ when $x = x_0$ the lever will be returned from the side wall without operating the secondary mechanism and its motion then satisfies (1) with initial conditions $(x_0, -\alpha v_1)$ where $0 \leq \alpha \leq 1$. It may then operate on a second or later contact. The lever may also have a back wall at $x = 0$ and then (1) applies only for $0 \leq x \leq x_0$. In this paper only motion up to the first contact with either wall will be considered and the derived conditions for operation will be sufficient but not necessary.

The lever has a wide range of input accelerations a with broad common properties which allow the definition of input sets such as, for example, a positive or a increasing. The solution-point and trajectory sets, defined by (1), corresponding to the input sets can then be found. When $x = x_0$ the minimum and maximum v can be determined for all allowable a . Sufficient conditions on the input members can then be given for the operation, or non-operation up to a particular time, of the mechanism. The effect of some input members will remain unknown.

Sach (Ref. 1) concentrated on positive triangular input accelerations as a reasonable approximation to actual accelerations and found minimum velocities at particular displacements of the lever over a range of triangles. He also discussed how the input velocity could be measured at particular times for practical fuzes. This velocity information is needed to apply the present work.

In (1) take $\omega = 1$ which is equivalent to considering the variables ωt , a/ω , v , ωx . Then, with the definition $z = v + ix \equiv re^{i\theta}$, (1), with zero initial conditions, has the solution

$$z(t) = \int_0^t e^{i(t-\tau)} a(\tau) d\tau \quad (2)$$

The work would rely heavily on delta functions with the solution in this form. To avoid this difficulty, and since only the (weighted) integral properties of a are relevant, (2) is replaced by

$$z(t) = \int_0^t e^{i(t-\tau)} dw(\tau) \quad (3)$$

which will be taken as the starting point; w is the input velocity. (3) is a Riemann-Stieltjes integral (Refs. 2 & 3) and, since the exponential function is continuous, exists if w has bounded variation in $[0, t]$. If w is the Riemann integral of some function a , then (3) is equivalent to (2).

In Part 1 w is assumed nondecreasing so that a , if it exists, is not negative. In later Parts w will be taken in turn as convex (a nondecreasing), concave (a nonincreasing), or unimodal (a nondecreasing up to a particular time and nonincreasing thereafter). Results are also given in the convex, concave and unimodal cases when w is, in addition, nondecreasing. This expands the results of Part 1 where w is simply nondecreasing.

The normalised solution-point and trajectory sets of (3), with no x -constraints, are derived in the appendix by elementary methods. However, some of the proofs are intricate and need careful reading. These results are then used in the main text to give operating conditions and times, expressed in unnormalised variables, for the oscillator.

1. Operating Bounds, w Nondecreasing

The lever has parameters ω , x_0 , v_0 defined in the introduction and the derived parameters θ_0 , r_0 , defined by $\omega x_0 = r_0 \sin \theta_0$, $v_0 = r_0 \cos \theta_0$, will be used frequently.

Suppose, for the moment, that the walls are transparent so there are no reflections. Then, from (R4) of the appendix, sufficient conditions for the lever not to operate before time T for w nondecreasing over $[0, T]$, $w(0) = 0$, are :

$$(i) \quad w(T) \leq r_0$$

$$\text{or} \quad (ii) \quad w(T) > r_0, \quad w(T) \sin \omega T < r_0 \sin \theta_0$$

The two cases are shown in Fig. 2(a). The trajectory set Σ_1 in this figure is explained in the appendix. In (i) the maximum radius of $\Sigma_1(\omega T)$ is too small, so that the lever, if it hits the wall $x = x_0$, has insufficient velocity for first-time operation. In (ii) the maximum x -component is too small and the lever has not hit either wall.

It is easy to show from (3) that $r(t) \leq w(t)$, if w is nondecreasing, for any number of passive reflections $\alpha, |\alpha| \leq 1$; $w(T) \leq r_0$ therefore implies no operation before T independently of reflections. Thus condition (i) or (ii) is sufficient for all passive walls and the two conditions have been used in Fig. 2(b) to give a region of no operation for $(T, w(T))$.

The trajectory corresponding to $w(t) = w(T)h(t)$, where h is a unit step, for a point $(T, w(T))$ outside the shaded region of Fig. 2(b) will give first-time operation inside $[0, T]$. The sufficient conditions (i) and (ii) cannot be improved without more assumptions about w and, in this sense, are therefore necessary.

Fig. 2(b) can be made more general, for it is implicit in the proof of (R4) that the trajectory set $\Sigma_1(t)$ for $t < T$, $\omega = 1$, $w(T) = 1$, is $\Sigma_1(t) = \{z: 0 \leq \theta \leq t, |z| \leq 1\}$. It follows that if w is nondecreasing over $[0, T]$ the conditions for no operation before t are $w(T) \leq r_0 \sin \theta_0 \operatorname{cosec} \omega t$ for $t \leq \theta_0/\omega$ and $w(T) \leq r_0$ for $t > \theta_0/\omega$.

(R6) of the appendix will now be used to give conditions for certain operation. By (R6) the lever will be operated by a trajectory that has not left the first quadrant for w nondecreasing, $w(0) = 0$, if there is a T satisfying $\omega T \leq \theta_0$ at which $w(T) \cos(\omega T/2) > r_0$. The lever operates on the first contact at $x = x_0$ and reflections play no part. The conditions are shown in Fig. 3(a) and give the region of certain operation in Fig. 3(b). Note that T is not the operating time.

For every point $(T, w(T))$ outside the shaded region of Fig. 3(b) a trajectory may be constructed which either never reaches the wall or has $v \leq v_0$ on the first contact. The sufficient conditions of Fig. 3 are therefore the weakest possible for certain operation on the first contact. Conditions for operation on subsequent contacts are not known.

The maximum exit time \hat{t}_2 from $X_0 = \{z: x < x_0\}$, for w nondecreasing, becomes the maximum operating time over the region of certain operation, where $w(T) \cos(\omega T/2) > r_0 > \omega x_0$. Thus, from (R7) of the appendix, $\hat{t}_2 = T + \omega^{-1} \arcsin(r_0 \sin \theta_0 / w(T))$. This is shown in Fig. 4. Since $w(T) > r_0$, $T \leq \theta_0/\omega$, over the certain operating region, then $\hat{t}_2 < 2\theta_0/\omega$.

CONCLUSION

For some applications the assumption that w is nondecreasing is unnecessarily wide. By (R1) the minimum magnitude of the solution-point set is realised by $w(T) = (h(t) + h(t-T))/2$, $a(t) = (\delta(t) + \delta(t-T))/2$, and this double-peaked acceleration may be unreasonable. In later Parts w will be taken in turn as convex, concave and unimodal. If w is also non-decreasing these additional restrictions reduce the solution-point and trajectory sets and increase the operating region of Fig. 3(b).

The maximum operating time is realised by $w(t) = h(t-T)$ which is convex for $t \leq T$. Imposing the extra condition, w concave over $[0, T]$, must decrease the maximum operating time.

If $w(\infty) > r_0$ a point $(T, w(T))$ not in the region of certain operation may, or may not, operate the lever. It is impossible to say what will happen without more information about w , and the results given here cannot be improved without this information.

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APPENDIX

SOLUTION-POINT AND TRAJECTORY SETS

Suppose w belongs to some set W . The corresponding solution of (3) is z_w and the solution-point set is defined by

$$S(t) = \{z_w(t) : w \in W\} \quad (a1)$$

The set $\sigma_w(t) = \{z_w(\alpha) : \alpha \leq t\}$ is the trajectory up to time t corresponding to w . The set of all trajectories is

$$\Sigma(t) = \{\sigma_w(t) : w \in W\} \quad (a2)$$

which may also be regarded as a union of sets of points:

$$\Sigma(t) = \bigcup_{\alpha \leq t} S(\alpha) \quad (a3)$$

In the applications, motion of the oscillator which is confined to the fixed set $v \leq 0$ is of particular interest. Define $p_w = \{z_w(\tau) : v_w(\alpha) \geq 0 \text{ for } \alpha \leq \tau\}$, so that p_w is that part of the trajectory $\sigma_w(\infty)$ along which $v_w(t)$, starting at $t = 0$, remains non-negative. Suppose W is such that $p_w \neq \emptyset$ for any $w \in W$. For example, W could not contain 0 or $-t$. Then define

$$P = \{p_w : w \in W\} \quad (a4)$$

Thus P is only defined when all trajectories, corresponding to W , start off into $\{z : v > 0\}$.

The operating time of the oscillator is important. To this end suppose, for all $w \in W$, that all trajectories eventually leave $X_0 = \{z : x < x_0\}$. Since, from (3), the position x is continuous in t then t_w may be defined as the time to leave X_0 ; t_w is called the exit time and \hat{t} is the supremum over $w \in W$. The subscript w specifying the input velocity will often be omitted.

The w to be considered will be in turn nondecreasing, convex, concave and unimodal. The sets of these input velocity functions have the outstanding common property of convexity and the whole treatment is based on this fact. A set W containing w_1, w_2 , is convex if it also contains the

line segment $\{\alpha w_1 + (1-\alpha)w_2 : 0 \leq \alpha \leq 1\}$. It follows from (3) and (a1) that the point set $S(\bar{t})$ is convex. The trajectory sets $\Sigma(t)$, P need not be convex.

(3) is unchanged if a constant is added to w . Put $w(0) = 0$ and also $w(T) = 1$ which normalises z and is equivalent to using the variable $z/w(T)$. T is an arbitrary fixed time. When finding solutions beyond T it is sometimes convenient to fix $w(\infty)$. We put $w(\infty) = 1$ in this case.

A unit step function h will be used frequently. $h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t > 0$ and $h(0)$ can have any convenient value in $[0,1]$.

A1. w Nondecreasing

Let

$$W_1 = \{w : w \text{ nondecreasing over } [0,T], w(0)=0, w(T)=1\} \quad (\text{a5})$$

W_1 is convex and this will now be used to find the solution-point set $S_1(T)$, which has been given the same subscript as the input velocity set. $S_1(T)$ is illustrated in Fig. A1 where the axes are left-handed, and the real axis is vertical to conform to the phase-plane convention for position and velocity.

When $w \in W_1$ let z_1 be the point of minimum modulus

r_1 of the corresponding solution-point set $S_1(T)$.

z_1 is unique. If $T \leq \pi$ then $z_1 = (1 + e^{iT})/2$,

$r_1 = \cos(T/2)$. If $T \geq \pi$ then $z_1 = 0$. (R1)

Proof: Since $r(T) = |z(T) e^{i\alpha}|$ for all α , then $r(T) \geq \int_0^T \cos(\tau-\alpha) dw(\tau)$ for all α .

If $T \leq \pi$, and in particular $\alpha = T/2$, then $r(T) \geq \cos(T/2)$. This lower bound is realised by $w(t) = (h(t) + h(t-T))/2$ in (3). Thus

$r_1 = \cos(T/2)$ and $z_1 = (1 + e^{iT})/2$ which point is unique.

For suppose there is another point z_2 of the same modulus r_1 belonging to $S_1(T)$. Then, because W_1 convex implies that $S_1(T)$ is convex, $S_1(T)$ would contain the line segment $\{az_1 + (1-a)z_2 : 0 < a < 1\}$ which lies inside the circle $r = \cos(T/2)$, contradicting the definition of z_1 .

If $T \geq \pi$ put $w(t) = (h(t) + h(t-\pi))/2$ to give the unique point $z_1 = 0$./

(Here and elsewhere a solidus indicates the end of the proof of a result).

If $w \in W_1$, $T < \pi$, the solution-point set at $t = T$

is $S_1(T) = \{z = e^{iT/2}(a + ib) : a \geq \cos(T/2), |z| \leq 1\}$.

$S_1(T)$ is contained in the segment $0 \leq \theta \leq T$. (R2)

Proof: Since $S_1(T)$ is convex it contains no point inside the tangent to the circle $r = r_1$, at $z = z_1$; r_1 and z_1 are given in (R1). Such a point would imply the existence of a third point of modulus less than r_1 which contradicts the definition of z_1 .

$w(t) = (1-\alpha)h(t) + \alpha h(t-T)$, $0 \leq \alpha \leq 1$, gives $z(T) = \alpha + (1-\alpha)e^{iT}$ which is a line segment belonging to $S_1(T)$ and containing z_1 at $\alpha = 1/2$. This line must be tangent to the circle $r = r_1$, at $z = z_1$, and is part of the boundary of $S_1(T)$. From (3) $|z(T)| \leq 1$ and so

$S_1(T) \subset \{z = e^{iT/2}(a+ib) : a \geq r_1, |z| \leq 1\}$. The boundary of the set on the right is contained in $0 \leq \theta \leq T$ and belongs to $S_1(T)$ which is a convex and so has no holes. Hence the sets are equal./

(R2) can now be used to find $S_1(T)$ for $T \geq \pi$. The construction, using $S_1(T/2)$ and $S_1(T/2)e^{iT/2}$, is shown in Fig. A1(b). $S_1(T)$ retains the same form for $T \leq 2\pi$ but the angle bound, $0 \leq \theta \leq T$, of (R2) is no longer true if $\pi < T < 2\pi$. The proof used for (R2) obviously breaks down since $r_1 = 0$ when $T \geq \pi$ and there is no unique tangent.

If $w \in W_1$, $T \leq 2\pi$ the solution-point set at $t = T$

is $S_1(T) = \{z = e^{iT/2}(a+ib) : a \geq \cos(T/2), |z| \leq 1\}$

If $T \geq 2\pi$ then $S_1(T) = \{z : |z| \leq 1\}$. (R3)

Proof: The case $T < \pi$ is covered by (R2). When $\pi \leq T < 2\pi$ write

$$z(T) = z(T/2)e^{iT/2} + \int_0^{T/2} e^{i(T/2-\tau)} dw(\tau + T/2)$$

$z(T/2)$ is the response at $T/2 < \pi$ to a w which is nondecreasing over $[0, T/2]$ with $0 \leq w(T/2) \leq 1$. The integral is the response to a w_1 , where where $w_1(t) = w(t+T/2)$, which is nondecreasing over $[0, T/2]$ with $0 \leq w_1(T/2) - w_1(0) \leq 1$. Hence, $z(T)$ has the form $z(T) = \alpha z_\gamma + (1-\alpha)z_\beta$, $0 \leq \alpha \leq 1$, where, by (R2), $z_\gamma \in S_1(T/2)e^{iT/2}$, $z_\beta \in S_1(T/2)$. $z(T)$ belongs to the union of line segments which join points in $S_1(T/2)$ to points in $S_1(T/2)e^{iT/2}$. These two sets are convex and the line-segment union is their convex hull with $|z| = |\alpha z_\gamma + (1-\alpha)z_\beta| \leq 1$ and, from (R2),

$$z(T) = \alpha e^{iT/4}(a_1 + ib_1) + (1-\alpha)e^{i3T/4}(a_2 + ib_2)$$

where $a_1 \geq \cos(T/4)$, $a_2 \geq \cos(T/4)$, $a_1^2 + b_1^2 \leq 1$, $a_2^2 + b_2^2 \leq 1$. If this is to have the form $e^{iT/2}(a + ib)$ then

$$a = \alpha(a_1 \cos(T/4) + b_1 \sin(T/4)) + (1-\alpha)(a_2 \cos(T/4) - b_2 \sin(T/4))$$

Since $a_1 \geq \cos(T/4)$, $a_2 \geq \cos(T/4)$, $|b_1| \leq \sin(T/4)$, $|b_2| \leq \sin(T/4)$, it then follows that $a \geq \cos(T/2)$, which proves the first part of (R3).

At $t=T=2\pi$, $S_1(2\pi) = \{z: |z| \leq 1\}$ can be constructed from $S_1(\pi)$.

Since, from (3) and (a5), $S_1(T)$ is nondecreasing with T and $S_1(T) \subset S_1(2\pi)$ then $S_1(T) = S_1(2\pi)$ for $T \geq 2\pi$.

(R2) and (R3) can be used to find solution-point sets for w nondecreasing over semi-closed or open intervals. For example, if w is nondecreasing over $(0, T)$ and $w(0) = 0$, $w(T) = 1$, then w can only fail to be nondecreasing over $[0, T]$ by having negative steps at the end points. Hence w can always be written in the form $\alpha h(t) + \beta h(t-T) + (1-\alpha-\beta)w_1(t)$ where $\alpha \leq 0$, $\beta \leq 0$, $w_1 \in W_1$. Conversely, since $\alpha \leq 0$, $\beta \leq 0$, a function of this form is nondecreasing over $(0, T)$. Hence

$$S(T) = \{\alpha e^{iT} + \beta + (1-\alpha-\beta)z : \alpha \leq 0, \beta \leq 0, z \in S_1(T)\}$$

which is easily constructed.

$$S(T) = \{e^{iT/2}(a+ib) : a \geq \cos(T/2)\}, T \leq 2\pi, \text{ which is a half-plane.}$$

(R2) and (R3) will now be used to find the trajectory sets of (a3) and (a4).

If $w \in W_1$, $T < \pi$, the trajectory set at $t = T$ is

$$\Sigma_1(T) = \{z: 0 \leq \theta < T, |z| \leq 1\} \cup e^{iT}. \quad \text{If } T \geq \pi \text{ then}$$

$$\Sigma_1(T) = S_1(T) \text{ of (R3).}$$

(R4)

Proof: For a particular $w(t)$, $t < T$, (R3) gives the solution-point set at t as

$$S(t; w(t)) = \{e^{it/2}(a+ib)w(t) : a \geq \cos(t/2), |z| \leq w(t)\}$$

For $t < T < \pi$ the union of the $S(t; w(t))$ over all allowable $w(t)$, $0 \leq w(t) \leq 1$, is $S_1(t) = \{z : 0 \leq \theta \leq t, |z| \leq 1\}$ and the union of the $S_1(t)$, $t < T$, with $S_1(T)$ gives

$$\Sigma_1(T) = \{z : 0 \leq \theta < T, |z| \leq 1\} \cup e^{iT}.$$

$S_1(T)$ contributes the extra point e^{iT} to the union.

If $T \geq \pi$ then $S_1(t) \subset S_1(T)$ for $t \leq T$. This proves $\Sigma_1(T) = S_1(T)$.

To find P of (a4) a bound will be needed for $r(t)$ when $v \geq 0$.

If w is nondecreasing and $v \geq 0$ over $[t_1, t_2]$ then

r is nondecreasing.

(R5)

Proof: If w is also continuous over $[t_1, t_2]$ then, using (3),

$$\int_{t_1}^{t_2} z^*(t) dz(t) = \int_{t_1}^{t_2} z^*(t) (iz(t)dt + dw(t))$$

Taking the real part gives

$$(r(t_2))^2 - (r(t_1))^2 = 2 \int_{t_1}^{t_2} v(t) dw(t) \quad (a6)$$

This is an energy balance and shows that r is nondecreasing for $v \geq 0$.

If w is discontinuous at points of $[t_1, t_2]$, v is also discontinuous at the same points. The integral (a6) does not exist and work done on the

oscillator cannot be expressed as $\int_{t_1}^{t_2} v(t)dw(t)$.

In this case define $v_1(t) = v(t_1)$ for $t < t_1$, $v_1(t) = v(t)$ for $t_1 \leq t \leq t_2$, $v_1(t) = v(t_2)$ for $t > t_2$. Then

$$(r(t_2))^2 - (r(t_1))^2 = \int_{t_1}^{t_2} (v_1(t+\epsilon) + v_1(t-\epsilon))dw(t) + o(\epsilon) \quad (a7)$$

providing $\epsilon > 0$ can be chosen so that the integral exists (Ref. 3, Theorem 14).

(a7) is established by developing the expression

$$|z(t_2) - z(t_1)e^{i(t_2-t_1)}|^2 = \int_{t_1}^{t_2} dw(\tau) \int_{t_1}^{t_2} \cos(\tau-\eta)dw(\eta)$$

The integration range ($t_1 \leq \tau \leq t_2$, $t_1 \leq \eta \leq t_2$) is split in two about the line $\eta = \tau + \epsilon$. The integral for $\eta \geq \tau + \epsilon$ is

$$I_1 = \int_{t_1}^{t_2} v_1(\eta-\epsilon)dw(\eta) - Rz(t_1)e^{i(t_2-t_1)}(z(t_2)-z(t_1)e^{i(t_2-t_1)})^* + o(\epsilon)$$

where R denotes "real part". The integral for $\eta \leq \tau + \epsilon$ is

$$I_2 = \int_{t_1}^{t_2} v_1(\tau+\epsilon)dw(\tau) - Rz(t_1)e^{i(t_2-t_1)}(z(t_2)-z(t_1)e^{i(t_2-t_1)})^* + o(\epsilon)$$

$|z(t_2)-z(t_1)e^{i(t_2-t_1)}|^2 = I_1 + I_2$ then gives (a7), and it only remains to prove existence.

The set of discontinuity points of w is countable (Ref. 4) and so may be denoted by $\{t_j\}$. The set $\{|t_i - t_j|\}$ of all differences magnitudes is also countable. Hence, for each $\delta > 0$, there is an ϵ , $0 < \epsilon < \delta$, which does not belong to $\{|t_i - t_j|\}$, since otherwise the set would contain the interval $(0, \delta)$ and not be countable. Thus for any null sequence $\{\delta_n\}$ there is a null sequence $\{\epsilon_n\}$ for which $|t_i - t_j| \neq \epsilon_n$, all, i, j, n . (For example, if w is discontinuous at the rationals let $\epsilon_n = \alpha/n$ where α is irrational). $v_1(t + \epsilon_n) + v_1(t - \epsilon_n)$ and $w(t)$ are never discontinuous for the same t which justifies (a7) and proves

$$(r(t_2))^2 - (r(t_1))^2 = \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} (v_1(t + \epsilon_n) + v_1(t - \epsilon_n)) dw(t)$$

This limit is not negative./

A subset of W_1 , of (a5), is defined by

$$W_2 = \{w : w \text{ nondecreasing, } w(0) = 0, w(T) = 1, w(\infty) = I\} \quad (\text{a8})$$

If $w \in W_2$, $T \leq \pi/2$, the trajectory set over which v is

not negative is $P_2 = \{z : 0 < \theta < T, r < \cos(T/2)\} \cup$

$$\{z : \theta = 0, r \leq 1\} \cup \{z : 0 < \theta \leq \pi/2, \cos(T/2) \leq r \leq I\} \quad (\text{R6})$$

If $T > \pi/2$ then $P_2 = \{z : \theta = 0, r \leq 1\} \cup$

$$\{z : 0 < \theta \leq \pi/2, r \leq I\}.$$

P_2 , and its defining subsets, are shown in Fig. A2 for $T < \pi/2$.

Proof: It follows immediately from (3) that if $w(t_1) = 0$, $w(t) > 0$, and $t - t_1 > 0$ is sufficiently small then $v(\tau) \geq 0$ for $\tau < t$ and $v(t) > 0$. Thus all the trajectories corresponding to W_2 start off into $\{z : v > 0\}$ which is the necessary condition for the definition of P_2 by (a4).

Since $v \geq 0$ then $x(t) = \int_0^t v(\tau) d\tau \geq 0$ and so $P_2 \subset \{z : 0 \leq \theta \leq \pi/2\}$.

Moreover since x is nondecreasing with t then $x(t) = 0$ implies $x(\tau) = 0$ for $0 \leq \tau \leq t$ and since w is nondecreasing then $w(\tau) = 0$ for $0 \leq \tau < t$. w must have the form $w(\tau) = w(t) h(\tau-t)$ for $0 \leq \tau \leq t$ and so $v(t) = w(t)$. Since $w(T) = 1$ then $x(t) = 0$ implies $t \leq T$ and so $0 \leq v(t) \leq 1$. This proves $P_2 \subset \{z : 0 < \theta \leq \pi/2\} \cup \{z : \theta = 0, r \leq 1\}$.

Since $r(t) \leq 1$ and, by (R1), $r(T) \geq \cos(T/2)$ it follows from (R5) that if $t \geq T$ then $\cos(T/2) \leq r(t) \leq 1$ for trajectories with $v \geq 0$. Thus $P_2 \subset \Sigma_1(T) \cup \{z : \cos(T/2) \leq r \leq 1\}$.

Combining these two conditions on P_2 and using (R4) gives, for $T \leq \pi/2$,

$$P_2 \subset \{z : 0 < \theta < T, r < \cos(T/2)\} \cup \{z : \theta = 0, r \leq 1\} \cup \{z : 0 < \theta \leq \pi/2, \cos(T/2) \leq r \leq 1\} \quad (a9)$$

To complete (R6), for the case $T \leq \pi/2$, it remains to show that all points of the superset of P_2 in (a9) are on trajectories, with $v \geq 0$, for $w \in W_2$.

When $w(t) = h(t-T/2)$ then $z(T/2) = h(0)$ which generates $\{z : \theta = 0, r \leq 1\}$ by choice of $h(0)$. $w(t) = \alpha h(t-T)$, $1 \leq \alpha \leq 1$, generates $\{z : 0 < \theta \leq \pi/2, 1 \leq r \leq 1\}$. $w(t) = \alpha h(t) + (1-\alpha)h(t-T)$, $0 \leq \alpha \leq 1$, generates $\{z : 0 < \theta < T, r < \cos(T/2)\} \cup \{z : 0 < \theta \leq \pi/2, \cos(T/2) \leq r \leq 1\}$, the remaining part of the superset.

When $T > \pi/2$ the two conditions on P_2 give, corresponding to (a9),

$$P_2 \subset \{z : \theta = 0, r \leq 1\} \cup \{z : 0 < \theta \leq \pi/2, r \leq 1\}$$

Trajectories, with $v \geq 0$, corresponding to $w(t) = \alpha h(t-T)$, $1 \leq \alpha \leq 1$, or $w(t) = \alpha h(t-T+\pi/2) + (1-\alpha)h(t-T)$, $0 \leq \alpha \leq 1$, can be made to pass through any point of this superset of P_2 . The sets are therefore equal./

If $w \in W_2$, $T < \pi$, $x_0 \leq \cos(T/2)$, the exit time t_2 from

$$\underline{X_0 = \{z : x < x_0\} \text{ has the maximum } \hat{t}_2 = T + \arcsin x_0.} \quad (R7)$$

\hat{t}_2 is realised by a trajectory belonging to P_2 of (R6).

The conclusion is not true if $x_0 > \cos(T/2)$. For example, take $w(t) = \alpha h(t) + (1-\alpha)h(t-T)$. The trajectory, for the case $\alpha = 1/2$, does not leave X_0 . If $0 < \alpha < 1/2$ and $x_0 = r(T) > \cos(T/2)$ then the trajectory leaves X_0 , but $t_2 > T + \arcsin x_0$.

Proof: From (R2), $z = 1 \in S_1(T)$ and so $\hat{t}_2 > T$. For $t > T$

$$z(t) = z(T)e^{i(t-T)} + \int_T^t e^{i(t-\tau)} dw(\tau) \quad (a10)$$

To find \hat{t}_2 consider only those w for which $x(T) < x_0$ since the time to leave X_0 for the remaining w is not greater than T . By (R2), $z(T) = e^{iT/2}(a+ib)$, $a \geq \cos(T/2)$, which implies $v(T) > 0$, $|z(T)| \geq x_0$, if $x(T) < x_0 \leq \cos(T/2)$. Thus, in this case, there is a t , say t_1 , with $0 \leq t_1 - T \leq \theta(T) + t_1 - T \leq \pi/2$ for which the first term of (a10) satisfies $x_0 = |z(T)| \sin(\theta(T) + t_1 - T)$. Since $|z(T)| \sin(\theta(T) + t_1 - T)$ is increasing with t_1 , and the x -component of the integral term of (a10) is not negative over $[T, T + \pi/2]$, the addition of the integral term can only make the exit time $t_2 \leq t_1$.

Therefore, towards maximising t_2 , force the integral term to zero by putting $w(t) = 1$ for $T \leq t \leq T + \pi/2$. t_1 then becomes the exit time t_2 . (This is only valid when t_1 exists which is the reason for the assumption $x_0 \leq \cos(T/2)$).

(a10) simplifies to $z(t) = z(T)e^{i(t-T)}$ and $x_0 = |z(T)| \sin(\theta(T) + t_2 - T)$, $0 \leq \theta(T) + t_2 - T \leq \pi/2$, shows, for fixed $\theta(T)$, that t_2 increases as $|z(T)|$ decreases. Using this and (R2) gives $z(T) = e^{iT/2}(\cos(T/2) + ib)$, $|b| \leq \sin(T/2)$, which leads to

$$x_0 = \cos(T/2) \sin(t_2 - T/2) + b \cos(t_2 - T/2), \quad 0 \leq t_2 - T/2 \leq \pi \quad (a11)$$

$t - T/2 = \pi/2$ gives $x(t) = \cos(T/2) \geq x_0$ and so the exit time must satisfy $t_2 - T/2 \leq \pi/2$; (a11) then implies $x_0 \geq \sin(t_2 - T)$. The maximum exit time is then given by $x_0 = \sin(\hat{t}_2 - T)$ which corresponds to $z(T) = 1$. By combination with the earlier condition on w , $w(T) = 1$ for $T \leq t \leq T + \pi/2$, \hat{t}_2 is realised by $w(t) = h(t-T)$ for $t \leq T + \pi/2$. The corresponding trajectory belongs to P_2 of (R6)./

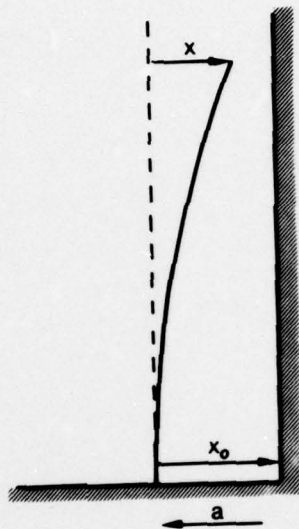


FIG. 1 - Driven Cantilever.

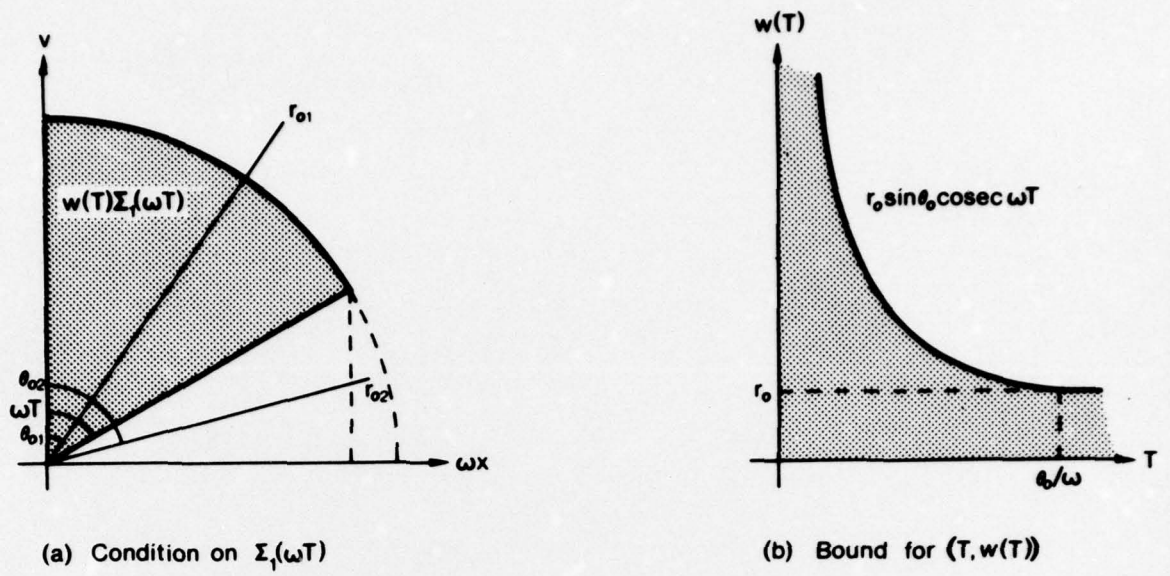


FIG. 2 - No operation before T , w nondecreasing over $[0, T]$.

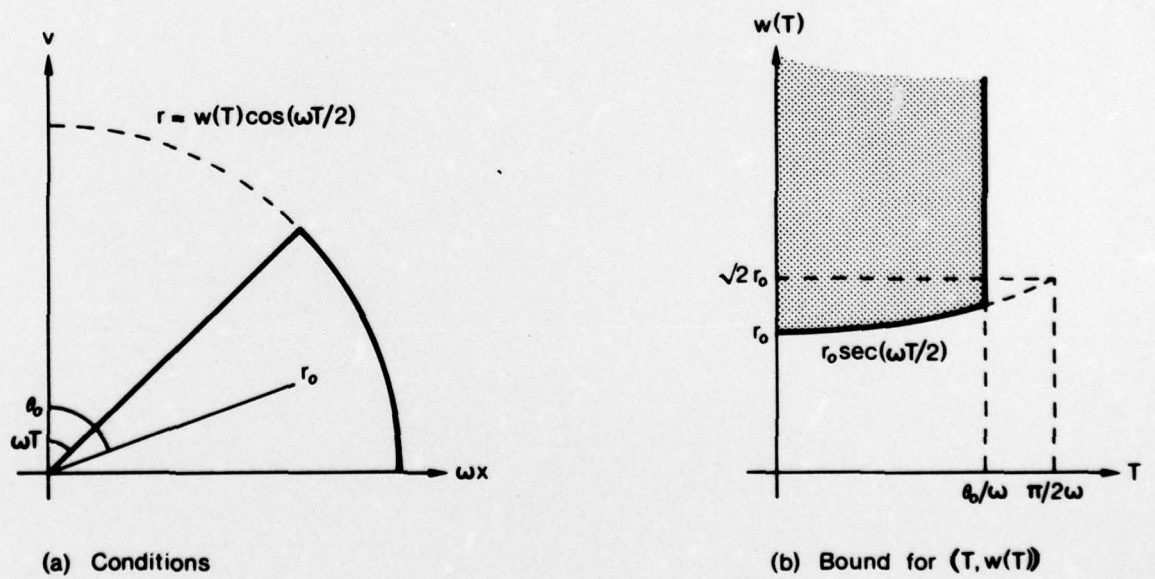


FIG. 3 - Certain operation, w nondecreasing.

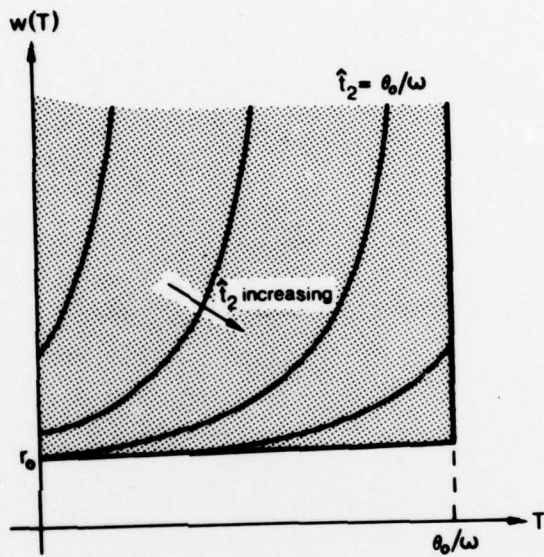
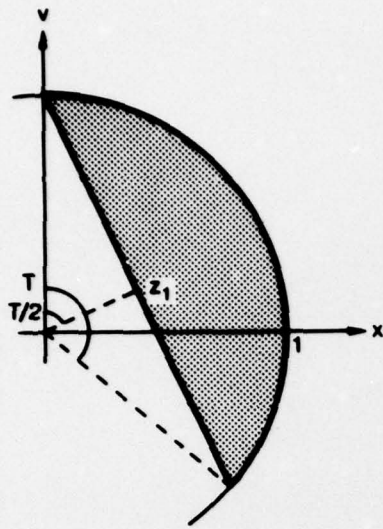
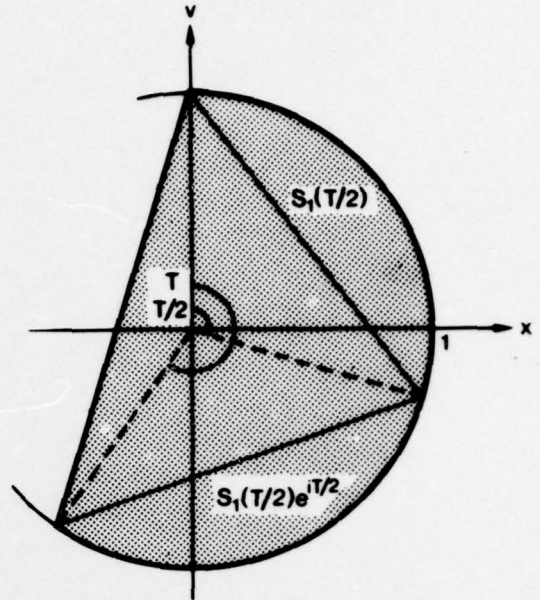


FIG. 4 - Maximum operating time \hat{t}_2 , w nondecreasing.



(a) $T < \pi$



(b) $\pi \leq T \leq 2\pi$. Construction using $S_1(T/2)$ and $S_1(T/2)e^{iT/2}$

FIG. A1 - Solution-point set $S_1(T)$.

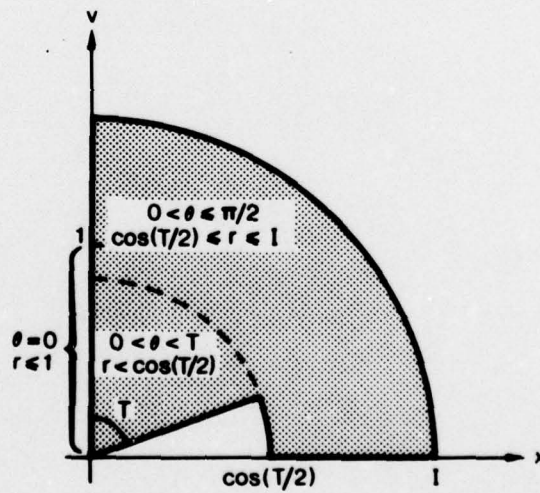


FIG. A2 - Trajectory set P_2 , and its defining subsets, $T < \pi/2$.

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