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RATIONAL BOUNDS FOR THE T-TAIL AREA WITH AN APPLICATION TO BONF--ETC(U)
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RATIONAL BOUNDS FOR THE t-TAIL AREA WITH AN APPLICATION TO BONFERRONI t-PERCENTILES

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With an Application to Bonferroni t-Percentiles

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Abstract

The bounds of Boyd (1959) for the normal distribution are extended to the t-distribution. It is shown how these bounds can be used to calculate the Bonferroni descriptive level as well as Bonferroni t-percentiles. The adequacy of the approximation is discussed and numerical examples provided.

Key words: Bonferroni percentiles, bounds, descriptive Bonferroni level, t-tail area, t-distribution.

AMS(MOS) Subject Classification: 60E05

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SIGNIFICANCE AND EXPLANATION

In experimental work it is sometimes desired to make n different inferences from a given data set. Associated with the data is a hypothesis called the overall null hypothesis which is to be tested, and under the assumption that the overall null hypothesis is true it is desired to have a preassigned high probability $1-\alpha$ of having all the individual inferences be correct. If the proper assumptions are made concerning the statistical distributions involved, this requires finding the abscissa $t_{\alpha/2n}$ for a specified t-distribution having area $\alpha/(2n)$ to the right of it. Such abscissas are called Bonferroni t-percentiles. Since statistical tables only give abscissas corresponding to α, it is sometimes troublesome to find $t_{\alpha/2n}$. Whereas previous work of the author is applicable if $\alpha/(2n)$ is small, the present paper gives a method that works well for any $\alpha/(2n)$ between .0 and .5 and is readily implemented on a computer.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

Rational Bounds for the t-Tail Area

With an Application to Bonferroni t-Percentiles

Andrew P. Soms

The purpose of this paper is to complete the extension to the t-distribution of results known for the normal (see, e.g., Johnson and Kotz 1970, Chapter 33). In Soms (1976) an asymptotic expansion for the tail area of the t-distribution was given good for small areas and in Soms (1977) some improved bounds were given, none of which, however, had simultaneously the right limiting behavior as $x \to 0$ or $x \to \infty$. In this paper lower and upper bounds are given with this property, the accuracy of the approximation is studied, and the results applied to the calculation of Bonferroni t-percentile and descriptive levels.

2. Extensions of Boyd's Results

Boyd (1959) showed that if $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$, $\bar{F}_k(x) = \int_x^\infty \phi(t)dt$, and $R_x = \bar{F}_k(x)/\phi(x)$, $x > 0$, then

$$p(x,\gamma_{min}) < R_x < p(x,\gamma_{max}),$$

where $p(x,\gamma) = (\gamma+1)/[(x^2 + \frac{2}{\pi}(\gamma+1)^2)^{1/2} + \gamma x]$, $\gamma_{max} = 2/(\pi-2)$, $\gamma_{min} = \pi-1$, and the bounds are the best possible in the class $\{p(x,\gamma),\ \gamma > -1\}$. This is also discussed in Johnson and Kotz (1970, Chapter 33). Here we will obtain analogous results for the t-distribution.

Let

$$f_k(t) = c_k(1+t^2/k)^{-(k+1)/2}, \quad c_k = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)(\pi k)^{\frac{1}{2}}},$$

k an integer ≥ 1, and

$$\bar{F}_k(x) = 1 - F_k(x) = \int_x^\infty f_k(t)dt,$$

for $x > 0$ here and throughout the paper. Also let

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$$R_x = \overline{F}_k(x)/((1+x^2/k)f_k(x)) .$$

It was shown in Soms (1976) that

$$1/x - k/((k+2)x^3) < R_x < 1/x$$

and

$$R_x = 1/x - k/((k+2)x^3) + O(x^{-5}) . \qquad (2.1)$$

In Soms (1977) it was shown that

$$\left[\frac{(k-1)x}{2k} + \left(1 + \left(\frac{(k+1)x}{2k}\right)^2\right)^{\frac{1}{2}}\right]^{-1} < R_x < \left[\frac{(3k-1)x}{4k} + \left(\frac{k-1}{2k} + \left(\frac{(k+1)x}{4k}\right)^2\right)^{\frac{1}{2}}\right]^{-1} , \qquad (2.2)$$

where the lower bound is valid for $k \geq 1$ and the upper for $k \geq 2$. It can be seen that these bounds do not have the right limiting behavior as $x \to 0$. Motivated by (2.2), consider the problem of finding best upper and lower bounds of the form

$$\frac{\alpha}{(x^2+\beta)^{\frac{1}{2}} + \gamma x} .$$

Since $\lim_{x\to 0} R_x = 1/2c_k$ and $R_x \sim 1/x$ (as $x \to \infty$), one must have

$$\alpha = 1 + \gamma \quad \text{and} \quad \alpha/\beta^{\frac{1}{2}} = 1/2c_k .$$

Thus $\beta = 4c_k^{\,2}(1+\gamma)^2$ and the approximating functions $p(x,\gamma)$ must be of the form

$$p(x,\gamma) = \frac{1+\gamma}{(x^2+4c_k^{\,2}(1+\gamma)^2)^{\frac{1}{2}}+\gamma x} .$$

For small x,

$$R_x = 1/2c_k - x + O(x^2)$$

and

$$p(x,\gamma) = 1/2c_k - \gamma x/(4c_k^2(1+\gamma)) + O(x^2) .$$

For large x, from (2.1),

$$R_x = 1/x - k/((k+2)x^3) + O(x^{-5})$$

and

$$p(x,\gamma) = 1/x - 2c_k^2(1+\gamma)/x^3 + O(x^{-5}) .$$

So if $p(x,\gamma)$ is to maximize R_x it is necessary that

$$\frac{\gamma}{4c_k^2(\gamma+1)} \le 1 \quad \text{and} \quad \frac{k}{k+2} \ge 2c_k^2(1+\gamma) ,$$

or

$$-1 < \gamma \le \frac{4c_k^2}{1-4c_k^2} \quad \text{and} \quad -1 < \gamma \le \frac{k}{2(k+2)c_k^2} - 1 ,$$

or

$$-1 < \gamma \le \min\left(\frac{4c_k^2}{1-4c_k^2}, \ \frac{k}{2(k+2)c_k^2} - 1\right) . \tag{2.3}$$

Note that from Lemmas A2 and A3 and the direct evaluation of $\frac{k}{2(k+2)c_k^2} - 1$ for $k = 1$ the interval in (2.3) is non-empty. Similarly, if $p(x,\gamma)$ is to minimize R_x, it is necessary that

$$\frac{\gamma}{4c_k^2(\gamma+1)} \ge 1 \quad \text{and} \quad \frac{k}{k+2} \le 2c_k^2(\gamma+1) ,$$

or

$$\gamma \ge \max\left(\frac{4c_k^2}{1-4c_k^2}, \ \frac{k}{2(k+2)c_k^2} - 1\right) .$$

Now

$$\frac{4c_k^2}{1-4c_k^2} \gtreqless \frac{k}{2(k+2)c_k^2} - 1$$

is equivalent to

$$c_k^2 \gtreqless k/(6k+4) .$$

Hence from Lemma A4, if $k \geq 3$, then for maximizing it is necessary that

$$\gamma \leq 4c_k^2/(1-4c_k^2) ,$$

and for minimizing

$$\gamma \geq \frac{k}{2(k+2)c_k^2} - 1 .$$

If $k = 2$, then $4c_k^2/(1-4c_k^2) = k/(2(k+2)c_k^2) - 1$, and it is verified by differentiation that (here and throughout "'" will denote differentiation)

$$-\left[(1 + \frac{x^2}{k})f_k(x)p(x,\gamma)\right]' = f_k(x) ,$$

and hence in this case $R_x = p(x,\gamma)$, where $\gamma = 4c_k^2/(1-4c_k^2) = k/(2(k+2)c_k^2) - 1$. Therefore for $k = 2$ the approximating problem has been solved in the best possible way and this case will not be considered further. If $k = 1$, then for maximizing it is necessary that

$$-1 < \gamma \leq \frac{k}{2(k+2)c_k^2} - 1 ,$$

and for minimizing

$$\gamma \geq 4c_k^2/(1-4c_k^2) ,$$

again using Lemma A4.

Using the above as motivation, the first result is

<u>Theorem 2.1</u>: Let $\gamma = 4c_k^2/(1-4c_k^2)$. Then for $k \geq 3$, $R_x < p(x,\gamma)$ and for $k = 1$, $R_x > p(x,\gamma)$.

<u>Proof</u>: Note that $R'_x = (\frac{(k-1)x}{k} R - 1)/(1+x^2/k)$ and $R''_x = ((\frac{k-1}{k})(\frac{(k-2)x^2}{k} + 1)R_x - \frac{(k-3)x}{k})/(1+x^2/k)$. Let $f(x) = p((x,\gamma) - R_x$. Then $f(0) = f'(0) = 0$ and

$$f''(0) = \frac{-1+2\gamma^2}{8c_k^3(1+\gamma)^2} - \frac{k-1}{2kc_k}$$

$$= \frac{16kc_k^4 + (4k+4)c_k^2 - k}{8c_k^3(1+\gamma)^2 k} .$$

It is verified directly that $f''(0) < 0$ for $k = 1$. For $k \geq 3$, $f''(0) > 0$ is equivalent to

$$16kc_k^4 + (4k+4)c_k^2 > k , \tag{2.4}$$

which will certainly be true if

$$16c_k^4 + 4c_k^2 > 1 . \tag{2.5}$$

In Lemma A1 it is shown that the even and odd c_k's form an increasing sequence. It follows by direct verification that (2.4) is true for $3 \leq k \leq 16$ and that (2.5) is true for $k = 17$ and $k = 18$ and hence (2.4) is true for $k \geq 3$. So for $k \geq 3$, $f''(0) > 0$. Consider

$$h(x) = \frac{k-1}{k} xf/(1+x^2/k) - f' .$$

By some algebra, it follows that

$$(1+x^2/k)\left[(x^2+4c_k^2(1+\gamma)^2)^{\frac{1}{2}} + \gamma x\right]^2 (x^2+4c_k^2(1+\gamma)^2)^{\frac{1}{2}}(1-4c_k^2)^3 h(x)$$

$$= x\left[4c_k^2(8c_k^2-1) \left(\frac{x}{(x^2+4c_k^2/(1-4c_k^2)^2)^{\frac{1}{2}} + x}\right) - \left(-1 + 4\,\frac{k+1}{k}\,c_k^2 + 16c_k^4\right)\right]. \tag{2.6}$$

By Lemma A2 and A3, $1-4c_k^2 > 0$ for $k \geq 1$, and $8c_k^2 - 1 > 0$ for $k \geq 3$ and $8c_k^2 - 1 < 0$ for $k = 1$. Also, as $x \to \infty$,

$$4c_k^2(8c_k^2-1)\frac{x}{(x^2+4c_k^2/(1-4c_k^2)^2)^{\frac{1}{2}}+x)} \to 2c_k^2(8c_k^2-1) , \tag{2.7}$$

and the convergence is monotone. For $k \geq 3$,

$$2c_k^2(8c_k^2-1) > -1 + 4\frac{k+1}{k}c_k^2 + 16c_k^4 \tag{2.8}$$

and for $k = 1$,

$$2c_k^2(8c_k^2-1) < -1 + 4\frac{k+1}{k}c_k^2 + 16c_k^4 , \tag{2.9}$$

since (2.8) and (2.9) are equivalent to $c_k^2 < \frac{k}{6k+4}$ and $c_k^2 > \frac{k}{6k+4}$, respectively, and this is true by Lemma A5. Since the constant term in the parentheses of the right-hand side of (2.6) has the same sign as $f''(0)$, it follows from (2.7) and (2.8) that for $k \geq 3$

$$h(x) < 0 , \quad x < x_0 , \tag{2.10}$$

$$h(x) = 0 , \quad x = x_0 ,$$

and

$$h(x) > 0 , \quad x > x_0 . \tag{2.11}$$

Suppose now that $f(x) \leq 0$ for some $x \leq x_0$. Then since $f(x)$ is increasing at the origin, it must have a maximum in $(0,x_0)$ for which $f > 0$ and $f' = 0$, contradicting (2.10). If $f(x) \leq 0$ for some $x > x_0$, then f must have a minimum ($\lim_{x\to\infty} f(x) = 0$) for which $f \leq 0$ and $f' = 0$, contradicting (2.11). So $R_x < p(x,\gamma)$ for $k \geq 3$.

Consider now $k = 1$. By (2.7), (2.9), and the fact that $f''(0) < 0$ for $k = 1$,

$$h(x) > 0 , \quad x < x_0 , \tag{2.12}$$

$$h(x) = 0 , \quad x = x_0 ,$$

and

$$h(x) < 0 \,, \qquad x > x_0 \,. \tag{2.13}$$

Suppose that $f(x) \geq 0$ for some $x \leq x_0$. Then, since f is decreasing at the origin there must be an $x < x_0$ for which $f < 0$ and $f' = 0$, contradicting (2.12). If $f(x) \geq 0$ for $x > x_0$, there must be an $x > x_0$ for which $f \geq 0$ and $f' = 0$, contradicting (2.13). Hence for $k = 1$, $R_x > p(x,\gamma)$, completing the proof.

Consider now the problem of minimizing R_x for $k \geq 3$ and maximizing for $k = 1$. Using similar reasoning to that used to arrive at the statement of Theorem 2.1, we have

<u>Theorem 2.2</u>: Let $\gamma = \frac{k}{2(k+2)c_k^2} - 1$. Then for $k \geq 3$, $R_x > p(x,\gamma)$ and for $k = 1$, $R_x < p(x,\gamma)$.

<u>Proof</u>: Let $g(x) = R_x - p(x,\gamma)$. Then $g(0) = 0$ and

$$g'(0) = -1 + \left(\frac{k}{2(k+2)c_k^2} - 1\right) \Big/ \left(4c_k^2 \frac{k}{2(k+2)c_k^2}\right)$$

$$= \frac{(-6k-4)c_k^2 + k}{4c_k^2} \,,$$

and hence by Lemma A5, $g'(0) > 0$ for $k \geq 3$, and $g'(0) < 0$ for $k = 1$. Consider

$$h(x) = \frac{k-1}{k} \, xg/(1+x^2/k) - g' \,.$$

After some algebra,

$$h = \Big[-(x^2 + 4c_k^2(1+\gamma)^2)^{\frac{1}{2}}((\gamma-1)x^2 + (1+\gamma)(-4c_k^2(1+\gamma) + \gamma))$$

$$+ x\left\{(\gamma-1)x^2 + (1+\gamma)(4c_k^2(1+\gamma)((1+1/k)\gamma - 1 + 1/k) - 1)\right\}\Big]$$

$$\Big/ \Big[((x^2 + 4c_k^2(1+\gamma)^2)^{\frac{1}{2}} + \gamma x)^2 (x^2 + 4c_k^2(1+\gamma)^2)^{\frac{1}{2}}(1+x^2/k)\Big] \,,$$

or, upon some further algebraic simplification,

$$((x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}}+\gamma x)^2(x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}}(1+x^2/k)((x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}}+x)h(x)/(1+\gamma)^2$$

$$= -4c_k{}^2\left[x^2(\gamma-1)+(1+\gamma)(-4c_k{}^2(1+\gamma)+\gamma)\right]$$

$$+ x^2\left[4c_k{}^2((1+1/k)\gamma+1/k)-1\right] + (x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}}x$$

$$\cdot \left[4c_k{}^2((1+1/k)\gamma+1/k)-1\right] . \tag{2.14}$$

Note that

$$4c_k{}^2((1+1/k)\gamma+1/k)-1 = 4c_k{}^2(\gamma-1)-4c_k{}^2((1+1/k)\gamma+1/k)+1 . \tag{2.15}$$

Hence, using (2.15), the right-hand side of (2.14), apart from the constant term, can be written as

$$-cx^2 + cx(x^2+4c_k{}^2(1+\gamma)^2)^{\frac{1}{2}} = \frac{c^2\ 4c_k{}^2(1+\gamma)^2}{c[1+(1+4c_k{}^2(1+\gamma)^2/x^2)^{\frac{1}{2}}]} , \tag{2.16}$$

where $c = 4c_k{}^2((1+1/k)\gamma+1/k)-1$. *Also,* $-4c_k{}^2(1+\gamma)+\gamma \gtrless 0$ *is equivalent to* $c_k{}^2 \lessgtr k/(6k+4)$, and hence by Lemma A5, $-4c_k{}^2(1+\gamma)+\gamma > 0$ for $k \geq 3$ and $-4c_k{}^2(1+\gamma)+\gamma < 0$ for $k = 1$. Further, $4c_k{}^2((1+1/k)\gamma+1/k)-1 > 0$ is equivalent to

$$c_k{}^2 < k/(4k+8) ,$$

and since

$$k/(4k+8) > k/(6k+4) , \quad k \geq 3 ,$$

by Lemma A5

$$4c_k{}^2((1+1/k)\gamma+1/k)-1 > 0 , \quad k \geq 3.$$

It is verified by direct evaluation that $4c_k{}^2((1+1/k)\gamma+1/k)-1 < 0$ for $k = 1$.

Consider the limit $2cc_k^2(1+\gamma)^2$ of (2.16) as $x \to \infty$. For $k \geq 3$,

$$2cc_k^2(1+\gamma)^2 > 4c_k^2(1+\gamma)(-4c_k^2(1+\gamma)+\gamma) \tag{2.17}$$

is equivalent to

$$c_k^2 > \frac{(k)(k+4)}{8(k+1)(k+2)} \ . \tag{2.18}$$

(2.18) will certainly be true if

$$c_k^2 > \frac{k+4}{8(k+2)} \ , \tag{2.19}$$

and it is verified directly that (2.19) holds for $k = 9$ and $k = 10$ and hence (2.17) is true for $k \geq 9$ by Lemma A1. It is also verified directly that (2.17) holds for $3 \leq k \leq 8$ and that the inequality is reversed for $k = 1$. Hence it follows that for $k \geq 3$

$$h(x) < 0 \ , \qquad x < x_0 \ , \tag{2.20}$$

and

$$h(x) = 0 \ , \qquad x = x_0 \ ,$$

$$h(x) > 0 \ , \qquad x > x_0 \ , \tag{2.21}$$

and for $k = 1$

$$h(x) > 0 \ , \qquad x < x_0 \ , \tag{2.22}$$

and

$$h(x) = 0 \ , \qquad x = x_0 \ ,$$

$$h(x) < 0 \ , \qquad x > x_0 \ . \tag{2.23}$$

Consider $k \geq 3$ first and suppose that $g(x) \leq 0$ for some $x \leq x_0$. Then, since g is increasing at the origin, there must be an x in $(0,x_0)$ such that $g > 0$ and $g' = 0$, which contradicts (2.20). If $g(x) \leq 0$ for some $x > x_0$ then there must be an $x > x_0$ for which $g \leq 0$ and $g' = 0$ and this contradicts (2.21). Hence $R_x > p(x,\gamma)$ for $k \geq 3$. If $k = 1$ and $g(x) \geq 0$ for some $x \leq x_0$, then, since g is decreasing at the origin, there must be an x

in $(0,x_0)$ such that $g(x) < 0$ and $g'(x) = 0$, contradicting (2.22). If $g(x) \geq 0$ for some $x > x_0$, then there must be an $x > x_0$ for which $g \geq 0$ and $g' = 0$, contradicting (2.23) and completing the proof of Theorem 2.2.

We summarize the results of Theorem 2.1 and 2.2 below in an algebraically equivalent form. For $k \geq 3$, let $\gamma_{max} = 4c_k^2/(1-4c_k^2)$ and $\gamma_{min} = \dfrac{k}{2(k+2)c_k^2} - 1$. Then

$$\left[((x/(1+\gamma_{min}))^2+4c_k^2)^{\frac{1}{2}} + \gamma_{min}x/(1+\gamma_{min})\right]^{-1} < R_x < \left[((x/(1+\gamma_{max}))^2+4c_k^2)^{\frac{1}{2}} + \gamma_{max}\, x/(1+\gamma_{max})\right]^{-1} . \tag{2.24}$$

If $k = 2$, then $\gamma_{max} = \gamma_{min} = \gamma$ and

$$R_x = \left[((x/(1+\gamma))^2 + 4c_k^2)^{\frac{1}{2}} + \gamma x/(1+\gamma)\right]^{-1} .$$

For $k = 1$, (2.24) holds with the definitions of γ_{max} and γ_{min} interchanged.

Since $\partial p(x,\gamma)/\partial\gamma < 0$ for $x > 0$, it follows that for $k \geq 3$, γ_{max} is the best value of γ for which $R_x < p(x,\gamma)$ for all x and γ_{min} is the best value for which $R_x > p(x,\gamma)$ for all x. A similar statement holds for $k = 1$ with the definitions of γ_{max} and γ_{min} interchanged. Equivalently, the bounds given in Theorems 2.1 and 2.2 are uniformly best in the class considered.

The accuracy of the lower and upper bounds are related to the maximum value of the ratio $p(x,\gamma_{max})/p(x,\gamma_{min})$. Call the ratio $r(x)$ and consider the maximum value of $r(x)$, which must be attained for some positive x at which $r'(x) = 0$. By differentiation, $r'(x) = 0$ is equivalent to

$$f(x) = (a_1^2-a_2^2)x + (1-a_1)(a_1^2x^2+4c_k^2)^{\frac{1}{2}} - (1-a_2)(a_2^2x^2+4c_k^2)^{\frac{1}{2}} = 0 , \tag{2.25}$$

where $a_1 = 2(k+2)c_k^2/k$ and $a_2 = 1-4c_k^2$ if $k \geq 3$, and the definitions of a_1 and a_2 are interchanged if $k = 1$. It will now be shown that f as defined by (2.25) is a monotonically decreasing function, $f(0) > 0$, $f(\infty) < 0$, and hence the maximum is the unique solution of (2.25). By differentiation,

$$f'(x) = a_1^2 - a_2^2 + \frac{(1-a_1)a_1^2 x}{(a_1^2 x^2 + 4c_k^2)^{\frac{1}{2}}} - \frac{(1-a_2)a_2^2 x}{(a_2^2 x^2 + 4c_k^2)^{\frac{1}{2}}} . \qquad (2.26)$$

Consider $k \geq 3$, a similar argument holding for $k = 1$. $a_1 < a_2$ is equivalent to $c_k^2 < k/(6k+4)$, true by Lemma A5. Hence $a_1^2 - a_2^2 < 0$ and

$$f'(x) < a_1^2 - a_2^2 + \frac{a_2 x}{(a_2^2 x^2 + 4c_k^2)^{\frac{1}{2}}}(a_1(1-a_1) - a_2(1-a_2)) < 0 .$$

By direct evaluation, $f(0) > 0$ and $f(\infty) < 0$, giving the conclusion. The maximizing value x_M can be conveniently obtained by a very short FORTRAN program that does interval halving, and Table 1 was so obtained.

Table 1

Maxima of r(x)

k	x_M	Maximum
3	.711	1.003
5	.781	1.008
10	.847	1.012
20	.885	1.015
50	.911	1.017
100	.919	1.017
1000	.928	1.018

Suppose k is fixed and M is the maximum value of $r(x)$, obtained as above. Then

$$(1+x^2/k)f_k(x)p(x,\gamma_{min}) < \overline{F}_k(x) < (1+x^2/k)f_k(x)p(x,\gamma_{max}) \qquad (2.27)$$

and the absolute error for either approximation is bounded by $(M-1)(1+x^2/k)f_k(x)p(x,\gamma_{min})$ and for any given x is less than $(1+x^2/k)f_k(x)(p(x,\gamma_{max})-p(x,\gamma_{min}))$, and the absolute value of the relative error is bounded by M-1 and for any given x is less than $p(x,\gamma_{max})/p(x,\gamma_{min}) - 1$. Some numerical examples are given in Table 2. Here U_{max} is the right-hand and U_{min} the left-hand expression of (2.27).

Table 2

Upper and Lower Bounds to t-Tail Probabilities

	x											
	.1		.5		1.0		2.0		4.0		6.0	
k	U_{min}	U_{max}	U_{min}	U_{max}	U_{min}	U_{max}	U_{min}	U_{max}	U_{min}	U_{max}	U_{min}	U_{max}
3	.463	.463	.325	.326	.195	.196	.696-01	.698-01	.140-01	.140-01	.464-02	.464-02
5	.461	.462	.318	.320	.181	.182	.509-01	.512-01	.516-02	.517-02	.923-03	.924-03
10	.460	.461	.312	.315	.169	.171	.366-01	.369-01	.126-02	.126-02	.660-04	.662-04
20	.459	.461	.308	.312	.163	.166	.295-01	.298-01	.352-03	.353-03	.362-05	.365-05
50	.458	.461	.306	.311	.160	.162	.254-01	.257-01	.105-03	.105-03	.109-06	.110-06
100	.458	.460	.306	.310	.158	.161	.240-01	.243-01	.607-04	.611-04	.159-07	.159-07
1000	.458	.460	.305	.310	.157	.160	.228-01	.231-01	.340-04	.342-04	.138-08	.138-08

3. An Application to the Calculation of Bonferroni t-Percentiles

Let $T_1,\ldots,T_n$ be statistics associated with an experiment such that under the overall null hypothesis H_0 each T_i has a t-distribution with k degrees of freedom, i = 1,...,n. Then if it is desired to control the probability error rate (the probability of 1 or more false statements under H_0), one possibility is to reject the null-hypothesis associated with T_i if

$$|T_i| > t_{\alpha/2n} \; ,$$

where $\overline{F}_k(t_{\alpha/2n}) = \alpha/2n$, and to accept H_0 if

$$|T_i| \leq t_{\alpha/2n} \; , \quad 1 \leq i \leq n \; .$$

Here $t_{\alpha/2n}$ is called the 100 α/2n Bonferroni t-percentile. Similarly we may define the descriptive Bonferroni level by

$$nP_{H_0}[|T_1| > |t_m|] \; , \tag{3.1}$$

where $|t_m|$ is the maximum of the observed values of $|T_1|,\ldots,|T_n|$, and the actual descriptive level, $P_{H_0}[\max_{1\leq i\leq n} |T_i| > |t_m|]$, will be less than or equal to the descriptive Bonferroni level. Hence it is of interest to find $t_{\alpha/2n}$ and to evaluate probabilities of the type in (3.1). Recently Bailey (1977) has given a table of $t_{\alpha/2n}$ for small n and n associated with pairwise comparisons. The method to be described below has the advantages of simple computer implementation (short program, no tables to be stored), and of working for any α and n.

Denote by t_{max} the unique solution in t of

$$(1+t^2/k)f_k(t)p(t,\gamma_{max}) = \alpha/2n \; , \tag{3.2}$$

and by t_{min} the unique solution in t of

$$(1+t^2/k)f_k(t)p(t,\gamma_{min}) = \alpha/2n \; . \tag{3.3}$$

Then it immediately follows from (2.27) that

$$t_{min} < t_{\alpha/2n} < t_{max} \; ,$$

and hence either may be used as an approximation, or the digits common to both. The solutions to (3.2) and (3.3) can be obtained very simply by a short FORTRAN program that does interval halving. Table 3 was generated this way. If the entries in Bailey's Table are rounded to three decimals, they all fall within the intervals in Table 3. The maximum difference between the upper and lower bound in Table 3 is .006 and this occurs for α = .05, n = 1. In general, the smaller α and the larger n, the better the approximation. The n and α for Table 3 were chosen deliberately to correspond to Bailey's Table.

Table 3

Approximations to Bonferroni t-Percentiles

	α															
	.01								.05							
	n								n							
	1		20		91		190		1		20		91		190	
k	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}	t_{min}	t_{max}
3	5.841	5.842	16.326	16.327	27.131	27.131	34.698	34.698	3.182	3.184	9.465	9.465	15.816	15.817	20.253	20.253
5	4.032	4.034	7.976	7.976	10.962	10.962	12.758	12.758	2.570	2.573	5.604	5.605	7.817	7.817	9.136	9.137
10	3.169	3.172	5.049	5.050	6.139	6.140	6.715	6.716	2.227	2.231	4.004	4.006	4.985	4.986	5.496	5.498
20	2.845	2.848	4.146	4.148	4.796	4.798	5.116	5.118	2.084	2.090	3.455	3.457	4.105	4.107	4.421	4.422
50	2.677	2.680	3.723	3.725	4.199	4.201	4.423	4.424	2.007	2.012	3.184	3.186	3.692	3.694	3.928	3.929
100	2.625	2.629	3.598	3.600	4.027	4.029	4.226	4.227	1.982	1.988	3.101	3.104	3.570	3.572	3.784	3.785
1000	2.580	2.583	3.492	3.494	3.883	3.884	4.061	4.062	1.960	1.966	3.030	3.033	3.466	3.468	3.662	3.664

The probabilities in (3.1) may be bounded by (2.27) and numerical examples have been given in Table 2.

The listings of the three very short computer programs referred to in this paper (maximum, bounds, percentiles) are available on request from the author.

Appendix: Results for the c_k

For ease of reference, all the necessary results involving c_k's are given here in the form of lemmas.

<u>Lemma A1</u>: The even and the odd c_k's form increasing sequences with limit $1/(2\pi)^{\frac{1}{2}}$.

<u>Proof</u>: That the limit is $1/(2\pi)^{\frac{1}{2}}$ is well-known. By the recursion formula for the Γ function,

$$\frac{c_{2i+3}}{c_{2i+1}} = \left(\frac{i+1}{i+1/2}\right)\left(\frac{2i+1}{2i+3}\right)^{\frac{1}{2}} > 1 , \quad i \geq 0 .$$

Similarly

$$\frac{c_{2i+2}}{c_{2i}} = \left(\frac{i+1/2}{i}\right)\left(\frac{i}{i+1}\right)^{\frac{1}{2}} > 1 , \quad i \geq 1 ,$$

completing the proof.

<u>Lemma A2</u>: For $k \geq 1$, $1-4c_k^{\ 2} > 0$.

<u>Proof</u>: This is proved differently in Soms (1977), but for the sake of completeness a shorter proof is given here. By Lemma A1, it suffices to show that

$$1/(2\pi) < 1/4 ,$$

which is true.

<u>Lemma A3</u>: For $k \geq 3$, $8c_k^{\ 2} - 1 > 0$, and for $k = 1$, $8c_k^{\ 2} - 1 < 0$.

<u>Proof</u>: By direct computation, $8c_1^2 - 1 < 0$. From Lemma A1, it suffices to show that $8c_3^2 = 16/(\pi\sqrt{3}) > 1$ and $8c_4^2 = 9/8 > 1$, which is true.

<u>Lemma A4</u>: For $k \geq 3$, $4c_k^{\ 2}/(1-4c_k^{\ 2}) < k/(2(k+2)c_k^{\ 2}) - 1$, and for $k = 1$ the inequality is reversed.

Proof: For $k \geq 3$, the inequality is equivalent to

$$c_k^2 < k/(6k+4) \,, \qquad \text{(A1)}$$

and for $k = 1$, to

$$c_1^2 > 1/10 \,. \qquad \text{(A2)}$$

(A1) will follow from Lemma A5 and (A2) is verified directly.

Lemma A5: For $k \geq 3$, $c_k^2 < k/(6k+4)$ and for $k = 1$, $c_k^2 > k/(6k+4)$.

Proof: The case $k = 1$ was shown to hold in Lemma A4. Since $1/2\pi < 1/(6+4/16)$, $c_k^2 < k/(6k+4)$ for $k \geq 16$. It is verified directly that it holds for $3 \leq k \leq 15$.

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The bounds of Boyd (1959) for the normal distribution are extended to the t-distribution. It is shown how these bounds can be used to calculate the Bonferroni descriptive level as well as Bonferroni t-percentiles. The adequacy of the approximation is discussed and numerical examples provided.

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