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NON-LINEAR STURM-LIOUVILLE PROBLEMS WITH NO SECONDARY BIFURCATION--ETC(U)

SEP 77 J B MCLEOD, C A STUART

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MRC Technical Summary Report # 1792

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WITH NO SECONDARY BIFURCATION

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ABSTRACT

The paper is concerned with giving sufficient conditions that in the non-linear boundary-value problem

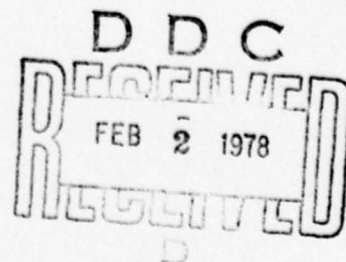
$$\left. \begin{aligned} u''(x) + \{q(x) + G(x, u(x), \lambda)\}u(x) &= 0 \\ u(0) = 0, \quad u(1) = 0 \text{ or } u'(1) &= 0 \end{aligned} \right\}$$

there should be no secondary bifurcation, i. e. that, given a branch of solutions (u, λ) bifurcating from the trivial solution, there should be no further bifurcation on that branch. Sufficient conditions on G are given which include, for example, Kolodner's problem of the motion of a heavy rotating string.

AMS(MOS) Subject Classification: 34B15; 47H15

Key Words: Bifurcation, Non-linear ordinary differential equations, Non-linear Sturm-Liouville problems, Secondary bifurcation.

Work Unit No. 1 - Applied Analysis



SIGNIFICANCE AND EXPLANATION

We give a typical physical example where the mathematical analysis in this paper is relevant. The simplest model for the buckling of a rod under an axial load is the classical Euler theory, in which the buckling load is given by the eigenvalue of a linear differential equation.

A more realistic model leads to a nonlinear boundary value problem in which the buckling load is given by the eigenvalue of a linear problem derived from the nonlinear equation - a bifurcation point. In contrast to the linear theory, the nonlinear theory also enables us to follow the behavior of the rod after buckling occurs. Under certain circumstances it is possible for a structure to buckle; then, when the deformation due to buckling has progressed so far, the structure may buckle a second time. When the behavior of the system is governed by a second order nonlinear differential equation, the present paper gives conditions under which this secondary buckling cannot occur.

NON-LINEAR STURM-LIOUVILLE PROBLEMS WITH NO SECONDARY BIFURCATION

J. B. McLeod & C. A. Stuart

1. Introduction

Let $G : (0,1) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function and let G_i denote the i -th partial derivative of G . We consider the non-linear eigenvalue problem

$$\left. \begin{aligned} u''(x) + q(x)u(x) + u(x)G(x, u(x), \lambda) &= 0 \\ u(0) = u(1) &= 0 \end{aligned} \right\} \quad (1)$$

and simultaneously the problem in which the boundary condition $u(1) = 0$ is replaced by $u'(1) = 0$. We suppose that $q : (0,1) \rightarrow \mathbb{R}$ is a continuously differentiable function.

Since both the function u and the constant λ are unknowns in (1), we regard a solution of (1) as a pair $(u, \lambda) \in C^2(0,1) \cap C^1[0,1] \times \mathbb{R}$. We wish to give conditions on the function G which will imply that, for any solution (u_0, λ_0) of (1) with $u_0 \not\equiv 0$, there is a neighbourhood of (u_0, λ_0) in $C^1[0,1] \times \mathbb{R}$ such that all the solutions of (1) lying inside this neighbourhood form a single smooth curve which can be parameterized by λ .

A result of this kind can be obtained directly from the implicit function theorem as follows. Let X be the Banach space consisting of the set $\{u \in C^1[0,1] : u(0) = 0\}$ together with the usual norm for $C^1[0,1]$. Define an operator $A : X \times \mathbb{R} \rightarrow X$ by

$$A(u, \lambda)(x) = \int_0^1 g(x, y)[q(y)u(y) + u(y)G(y, u(y), \lambda)]dy$$

where g is the Green's function for the problem

$$-u''(x) = h(x), \quad u(0) = u(1) = 0 \quad (\text{respectively } u'(1) = 0).$$

Let us assume that for each $\lambda \in \mathbb{R}$, $A(\cdot, \lambda) : X \rightarrow X$ is a completely continuous (i.e. continuous and compact) operator. This is certainly the case if q is continuous on $[0,1]$ and G is continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$, but it is also true even if q and G have mild singularities at $x = 0$. With this notation, the problem (1) is equivalent to

$$u + A(u, \lambda) = 0 \quad \text{for } (u, \lambda) \in X \times \mathbb{R}. \quad (2)$$

Supposing that $(u_0, \lambda_0) \in X \times \mathbb{R}$ is a solution of (2), we shall be able to invoke the implicit function theorem to yield the desired conclusion provided that we can show that the linear mapping $I + A_u(u_0, \lambda_0) : X \rightarrow X$ has a bounded inverse. We use $A_u(u_0, \lambda_0)$ to denote the Fréchet derivative of A at (u_0, λ_0) . Since $A_u(u_0, \lambda_0)$ is a compact linear operator (by the compactness of A), $I + A_u(u_0, \lambda_0)$ has a bounded inverse provided that $I + A_u(u_0, \lambda_0) : X \rightarrow X$ is one-to-one. Proving that $I + A_u(u_0, \lambda_0) : X \rightarrow X$ is one-to-one is equivalent to proving that the boundary-value problem

$$\left. \begin{aligned} v''(x) + q(x)v(x) + v(x)\{G(x, u_0(x), \lambda_0) + u_0(x)G_2(x, u_0(x), \lambda_0)\} &= 0 \\ v(0) = 0, \quad v(1) = 0 \quad (\text{respectively } v'(1) = 0) \end{aligned} \right\} \quad (3)$$

has only the solution $v \equiv 0$. The aim of the paper is to give conditions on G which ensure that this is the case for every solution (u_0, λ_0) of (1) such that $u_0 \not\equiv 0$.

Of course, we may also consider the same problem for solutions of (2) of the form $(0, \lambda)$. This is the problem of bifurcation of non-trivial solutions of (1) from the line $\{(0, \lambda) \in X \times \mathbb{R} : \lambda \in \mathbb{R}\}$ of trivial solutions. In keeping with what has been said above, bifurcation cannot occur at a value $\lambda = \lambda_0$ which is not an eigenvalue of (3) with $u_0 \equiv 0$. Furthermore it is well-known [1] that at the eigenvalues of

$$\left. \begin{aligned} v''(x) + q(x)v(x) + v(x)G(x, 0, \lambda) &= 0 \\ v(0) = 0, \quad v(1) = 0 \quad (\text{respectively } v'(1) = 0) \end{aligned} \right\},$$

bifurcation does indeed occur provided that the transversality condition (d) of Theorem 1.7 of [1] is satisfied. In fact, in a neighbourhood of the bifurcation point in $X \times \mathbb{R}$, the set of all solutions of (1) consists of two simple curves which intersect only at the bifurcation point. One of these curves is a section of the line of trivial solutions and the only trivial solution on the other curve is the bifurcation point. It is a corollary of our result that, under our hypotheses, this curve of non-trivial solutions can be continued to infinity in $X \times \mathbb{R}$. Furthermore there is no branching from this curve (i.e. no secondary bifurcation) and the curve can be parameterized by λ . The conditions under which we establish these results are considerably more general than those formulated previously which seem restricted either to autonomous

equations [2] or to problems with small non-linearities [3].

Our method is to compare (3) with two other problems, one being (1) and the other a specially constructed linear problem the number of nodes of solutions of which can be related to the number for u_0 . The Sturm comparison theorems then give (from each comparison) information about the number of nodes of a solution of (3), and if this information is self-contradictory, we can conclude that (3) has only the trivial solution.

Before giving our result we remark that our method will apply to problems not of the form (1), for example to problems in which $u'(x)$ appears explicitly in the differential equation. For this reason we have arranged the proof so as to show what must be done in order to construct the third comparison problem, and in so doing we show also that our result is more or less the best possible one obtainable by our methods for problems which have the form (1).

2. The main theorem

Theorem. Suppose that $(u_0, \lambda_0) \in X \times \mathbb{R}$ is a solution of (1) with $u_0 \neq 0$. Then the boundary-value problem (3) has only the solution $v \equiv 0$ in X provided that G has the following form:

$$G(x, p, \lambda) = f(x) F(p e^{\int_1^x h(y) dy}, \lambda) \text{ for } x \in (0, 1) \text{ and } p, \lambda \in \mathbb{R} \quad (4)$$

where $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is any C^1 function such that $p F_1(p, \lambda_0) < 0$ for all $p \neq 0$, $f: (0, 1] \rightarrow \mathbb{R}$ is any positive C^2 function, with $f(x)$ bounded from zero for $x \in (0, 1]$ and with $\int_0^1 x f(x) dx < \infty$, $h: (0, 1] \rightarrow \mathbb{R}$ is any C^2 solution of the Riccati equation $h' - h^2 = q$ in $(0, 1)$, with the restriction that $h(1) \leq 0$ if we have the boundary condition $u'(1) = 0$, $q: (0, 1] \rightarrow \mathbb{R}$ is any C^1 function with $\int_0^1 x |q(x)| dx < \infty$, and where we have the additional inequality,

$$\frac{3}{4} f^{-5/2} f'^2 - f^{-3/2} (f'h + \frac{1}{2} f'') + 2f^{-1/2} h' \geq 0 \text{ throughout } (0, 1). \quad (5)$$

Remark. 1. The conditions allow the functions f , h and q to have singularities at 0. This is essential because there are relevant examples in which this arises, in particular Example 2 which we discuss later. The singularities can be handled analytically because the boundary condition at 0 has the form $u(0) = 0$; and if we take problem (1) with the boundary condition $u(1) = 0$ rather than with the alternative $u'(1) = 0$, we can allow singularities in f , h , and q at 1 as well. We have not bothered to make this extension.

2. While the theorem is a theorem about the non-existence of non-trivial solutions of a certain boundary-value problem, the introduction has pointed out the relevance of this for secondary bifurcation, and it should therefore be remarked that the conditions of the theorem are sufficient to guarantee that the operator A of the introduction is completely continuous. This is done in an appendix to the paper, and it will be seen there that the conditions are not only sufficient but also close to being necessary.

3. The proof and truth of the theorem remain unchanged if we generalize F so as to allow zeros of $F_1(p, \lambda_0)$ for isolated values of p . If $F_1 \equiv 0$, the problem of course ceases to be non-linear.

4. It is possible to construct a theorem in which the inequality $pF_1(p, \lambda_0) < 0$ is reversed to $pF_1(p, \lambda_0) > 0$ and the inequality (5) is also reversed. The comparisons in the proof that follows are then reversed and a contradiction can be obtained much as before. We leave the necessary modifications of statement and proof to the reader.

We make a final remark on generalizations at the end of the proof of the theorem.

5. As a consequence of Remark 1, it becomes part of the proof of the theorem to discuss the singularities and show that they can be handled. We do this in two simple lemmas which we state now and prove at the conclusion of the proof of the theorem.

Lemma 1. Let h be as in the statement of the theorem. Then, as $x \downarrow 0$, either $h(x) = o(x^{-1})$

or

$$h(x) \sim -x^{-1} \text{ and } \int_1^x h(y)dy \text{ converges to a finite limit.}$$

Lemma 2. Let u_0 be as in the statement of the theorem. Then u_0 has only a finite number of zeros in $(0, 1)$.

Proof of Theorem. Setting

$$\tilde{G}(x, p, \lambda) = q(x) + G(x, p, \lambda) \quad \text{for } x \in (0, 1) \text{ and } p, \lambda \in \mathbb{R},$$

we have that u_0 satisfies the equation

$$w''(x) + \tilde{G}(x, u_0(x), \lambda_0)w(x) = 0 \quad \text{for } 0 < x < 1.$$

We suppose that $v \not\equiv 0$ is a solution of (3) in X and show that this leads to a contradiction.

Since v satisfies the equation

$$w''(x) + \{\tilde{G}(x, u_0(x), \lambda_0) + u_0(x) G_2(x, u_0(x), \lambda_0)\}w(x) = 0 \quad \text{for } 0 < x < 1$$

and since

$$u_0(x) G_2(x, u_0(x), \lambda_0) = f(x)u_0(x)e^{\int_1^x h} F_1(u_0(x)e^{\int_1^x h}, \lambda_0) \leq 0$$

by our hypotheses (with equality only when $u_0(x) \equiv 0$), it follows from the Sturm comparison theorem that u_0 vanishes at least once strictly between any two consecutive zeros of v .

If we have the boundary conditions $u_0(0) = u_0(1) = 0$ and u_0 has no zeros in $(0, 1)$, this already leads to a contradiction since v will have zeros at 0 and 1.

Let us now suppose that u_0 has $n-1$ zeros in $(0, 1)$ where $n \geq 2$ and that u_0 satisfies

$u_0(0) = u_0(1) = 0$. We denote the zeros of u_0 by a_i where $0 = a_0 < a_1 < \dots < a_{n-1} < 1$. By comparing (3) with another linear problem we shall show that this again leads to a contradiction.

To obtain the desired equation for comparison, let $\alpha, \beta \in C^2(0,1]$, and consider, in $(0,1)$,

$$\phi = \alpha u_0 + \beta u_0'.$$

Then

$$\begin{aligned} \phi'' &= \alpha'' u_0 + (2\alpha' + \beta'') u_0' + (\alpha + 2\beta') u_0'' + \beta u_0''' \\ &= \{\alpha'' - (\alpha + 2\beta') \tilde{G} - \beta \tilde{G}_1\} u_0 + \{2\alpha' + \beta'' - \beta \tilde{G} - \beta u_0 \tilde{G}_2\} u_0'. \end{aligned}$$

Hence, provided that

$$\beta\{\alpha'' - (\alpha + 2\beta') \tilde{G} - \beta \tilde{G}_1\} = \alpha\{2\alpha' + \beta'' - \beta \tilde{G} - \beta u_0 \tilde{G}_2\}, \quad (6)$$

we see that ϕ satisfies the equation

$$\phi'' + \left\{ \tilde{G} + u_0 \tilde{G}_2 - \frac{2\alpha' + \beta''}{\beta} \right\} \phi = 0, \quad (7)$$

provided that β does not vanish in $(0,1)$.

Now the condition (6) can be regarded as a partial differential equation for $\tilde{G}$, if we regard the arguments in G , i.e. x and $u_0 = p$, as the independent variables and α and β as given functions of x . Indeed, it is a first-order linear equation with the general solution

$$\tilde{G}(x, p, \lambda_0) = \left(\frac{\alpha}{\beta}\right)' - \left(\frac{\alpha}{\beta}\right)^2 + \frac{1}{\beta^2} F\left(p e^{\int_1^x \frac{\alpha}{\beta}}, \lambda_0\right), \quad (8)$$

where $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function, and we observe that the particular $\tilde{G} = q + G$ which arises from G as given in the statement of the present theorem is indeed of the form (8) if we take

$$\beta = f^{-1/2}, \quad \alpha = f^{-1/2} h. \quad (9)$$

(We note that α and β so defined satisfy the conditions so far used in the analysis, that $\alpha, \beta \in C^2(0,1]$ and $\beta > 0$ in $(0,1]$.)

The proof of (8) is just undergraduate mathematics, but for the purposes of the proof of the present theorem we are in any case interested in the converse, that if $\tilde{G}$ is of the form given

by (8), and in particular if $\tilde{G}$ is as given in the statement of the present theorem, then (6) holds (with α and β given by (9)) and so ϕ satisfies (7), and this converse is proved just by differentiating (8). The Sturm comparison theorem can now be applied to compare zeros of v and ϕ , provided that $2\alpha' + \beta''$ is of constant sign, and for α and β of the form (9), this is achieved by equation (5), which just states that

$$2\alpha' + \beta'' \geq 0.$$

Hence between any two consecutive zeros of ϕ in $(0,1)$ there must lie at least one zero of v . (This statement has to be suitably interpreted if there are intervals in which $2\alpha' + \beta'' \equiv 0$, but we leave this to the reader.)

We are now ready to achieve the required contradiction. We know that u_0 has precisely $n-1$ zeros in $(0,1)$, and that consequently, from the comparison between u_0 and v , v has at most $n-2$ zeros in $(0,1)$. (For if v has $n-1$ interior zeros at $b_1, \dots, b_{n-1}$, then u_0 has at least one in each of the n intervals $(0, b_1), (b_1, b_2), \dots, (b_{n-1}, 1)$.) But u'_0 has opposite signs at consecutive zeros of u_0 (it is part of the proof of Lemma 2 that $u'_0(0) \neq 0$), and so ϕ changes sign at least once in each of the intervals $(a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, 1)$ of u_0 . (Since it is possible that $\beta(0) = 0$, we have to consider the interval $(0, a_1)$ as a special case.) Thus ϕ has at least $n-1$ zeros in $(0,1)$.

In fact, ϕ has a further zero, either at 0 or in $(0, a_1)$. There are two cases to consider, depending upon whether $h(x) = o(x^{-1})$ as $x \downarrow 0$ or $h(x) \sim -x^{-1}$.

In the first case, we can suppose without loss of generality that $u'_0(0) > 0$, $u_0 > 0$ in $(0, a_1)$, $u'_0(a_1) < 0$. Then $\phi(a_1) < 0$, while, as $x \downarrow 0$,

$$\phi(x) = f^{-1/2}(x) \{ h(x) u_0(x) + u'_0(x) \},$$

which is certainly positive for sufficiently small x if $h(x) = o(x^{-1})$ as $x \downarrow 0$. Hence ϕ has a zero in $(0, a_1)$.

If $h(x) \sim -x^{-1}$ as $x \downarrow 0$, then

$$\begin{aligned} \phi(x) &= f^{-1/2}(x) \{ h(x) u_0(x) + u'_0(x) \} \\ &= f^{-1/2}(x) \{ -x^{-1} [x u'_0(0) + o(x)] + u'_0(0) + o(1) \} \\ &= o(1) \end{aligned}$$

since $f^{1/2}$ is bounded. Hence $\phi(0) = 0$.

The comparison between v and ϕ now assures us that v has at least $n-1$ zeros in $(0,1)$, and since we established earlier that v had at most $n-2$ zeros in $(0,1)$, the required contradiction is obtained.

We deal finally with the case where u_0 has $n-1$ zeros in $(0,1)$, with $n \geq 1$, and the boundary conditions are $u_0(0) = 0$, $u'_0(1) = 0$. (In the case $n = 1$ the argument that follows has to be suitably modified and this is left to the end.) We note as before that v has at most $n-2$ zeros in $(0,1)$. The function ϕ is constructed as before and has at least one zero in each of the intervals $[0, a_1)$, (a_1, a_2) , $\dots$, (a_{n-2}, a_{n-1}) . If we can show that ϕ also has a zero in $(a_{n-1}, 1]$, we can make the comparison with v and obtain a contradiction as before. To show the existence of this last zero, we suppose without loss of generality that $u'_0(a_{n-1}) > 0$, $u_0 > 0$ in $(a_{n-1}, 1]$, with $u'_0(1) = 0$. Then $\phi(a_{n-1}) > 0$ while $\phi(1) \leq 0$ since $h(1) \leq 0$, and this gives the required result.

If $n = 1$, there is no need to introduce the function ϕ . Since u_0 has no internal zeros, neither has v , and we may assume $u_0(x) > 0$, $v(x) > 0$ for $x \in (0,1)$. Then

$$u''_0(x) v(x) - v''(x) u_0(x) = u_0^2(x) v(x) G_2(x, u_0(x), \lambda_0),$$

and on integration

$$[u'_0(x) v(x) - v'(x) u_0(x)]_0^1 < 0,$$

which contradicts the boundary values for u_0, v .

Remark. If we do not assume G of the form (4), it is still possible to consider $\phi = \alpha u_0 + \beta u'_0$, where, α, β are suitable functions of x , and to construct a differential equation for ϕ of the form

$$\phi'' + A\phi' + B\phi = 0,$$

for functions A, B which depend on $x, u_0, \alpha, \beta, G, q$. By the standard transformation

$$\phi = e^{-1/2 \int A dx} \psi,$$

we obtain an equation of the form

$$\psi'' + C\psi = 0$$

for ψ , and this can be used for comparison with the equation for v provided that C satisfies a certain inequality. This inequality is to be regarded as a condition on G , but in this degree of generality seems to be too complicated to be made much of.

3. Proof of Lemma 1

If $z(x)$ satisfies the equation

$$z'' + qz = 0, \quad (10)$$

then it is a standard result (and easily proved) that $h = -z'/z$ satisfies the Riccati equation

$$h' - h^2 = q; \quad (11)$$

and, conversely, every solution of the Riccati equation can be obtained in this way. Further, one solution of (10), z_1 say, can be obtained by solving the integral equation

$$z_1 = x - \int_0^x (x-t) q(t) z_1(t) dt$$

by iteration, the iterative process converging, at least for x sufficiently small, because of the assumptions on q ; and this solution has the asymptotic behaviour as $x \downarrow 0$ that

$$z_1(x) \sim x, \quad z_1'(x) \sim 1,$$

and consequently the corresponding function h satisfies

$$h(x) \sim -x^{-1}. \quad (12)$$

The full solution of (10) can now be given, at least for x sufficiently small, by the formula

$$z = Az_1 + Bz_1 \int_{\delta}^x \{z_1(t)\}^{-2} dt,$$

where A and B are arbitrary constants and δ is a fixed positive number sufficiently small that z_1 does not vanish in $(0, \delta]$. If $B = 0$, then z is a multiple of z_1 , and we have already seen that then $h(x) \sim -x^{-1}$. If $B \neq 0$, it is an easy calculation, using the asymptotic behaviour of z_1 , to check that $h(x) = o(x^{-1})$, as required by the statement of the lemma.

If $h(x) \sim -x^{-1}$, then for x sufficiently small we can divide (11) by h and integrate to obtain

$$\log h - \int_1^x h(t) dt = \int_0^x \frac{q(t)}{h(t)} dt + \text{constant},$$

the integral on the right existing because of (12) and the conditions on q . The proof of the lemma is then completed by taking exponentials of both sides and letting $x \downarrow 0$.

4. Proof of Lemma 2

Since the differential equation satisfied by u_0 is singular only at 0, we see that 0 is the only possible limit point for an infinity of zeros of u_0 , if such exists. Further, since $u_0 \in C^1[0,1]$, if 0 is such a limit point, then $u'_0(0) = 0$. It will therefore be sufficient to show that $u'_0(0) \neq 0$.

Suppose then for contradiction that $u'_0(0) = 0$. Integrating the equation for u_0 , we have

$$u'_0(x) = - \int_0^x \{q(t) + f(t) F(u_0(t) e^{\int_1^t h}, \lambda_0)\} u_0(t) dt,$$

the integral existing because $u_0(t) = O(t)$ as $t \downarrow 0$ and

$$\int_0^1 t f(t) dt < \infty, \quad \int_0^1 t |q(t)| dt < \infty.$$

(Note that, from Lemma 1, and for $t \in [0,1]$,

$$|u_0(t) e^{\int_1^t h}| \leq K_1 |u_0(t)/t| \leq K_2,$$

for suitable constants K_1, K_2 .) If now, for some $\delta > 0$,

$$M = \sup_{t \in [0, \delta]} |u'_0(t)|,$$

then we have

$$M \leq M \int_0^\delta \{|q(t)| + K f(t)\} t dt, \quad (13)$$

where

$$K = \sup_{t \in [0,1]} |F(u_0(t) e^{\int_1^t h}, \lambda_0)|,$$

and clearly (13) implies $M = 0$ if δ is chosen sufficiently small. Hence $u_0 \equiv 0$ in $[0, \delta]$

and so throughout $[0,1]$.

5. Two examples

Example 1. Suppose that $q \equiv 0$. Then we may take $h \equiv 0$ and G then has the form

$$G(x, p, \lambda) = f(x) F(p, \lambda),$$

where F is an arbitrary smooth function such that $p F_1(p, \lambda_0) < 0$ for all $p \neq 0$, and where f is any C^2 function on $(0, 1]$, bounded from zero and with

$$\int_0^1 x f(x) dx \leq \infty$$

and

$$\frac{3}{2} f'^2 - f f'' \geq 0 \quad \text{in } (0, 1).$$

A particular case of this result is due to Professor D. Henry. Let $f \equiv 1$ and F have the form

$$F(p, \lambda) = g(p, \lambda)/p,$$

so that the equation under discussion is

$$u'' + g(u, \lambda) = 0.$$

If $g(p, \lambda)$, regarded as a function of p , is $C^2[0, 1]$, with

$$g(0, \lambda) = g(1, \lambda) = 0 \quad \text{and} \quad g''(p, \lambda) < 0 \quad \text{for } 0 < p < 1,$$

and if we are interested in solutions u for which $0 \leq u \leq 1$, then, for $0 < p \leq 1$,

$$\begin{aligned} p F_1(p, \lambda) &= g'(p, \lambda) - \frac{g(p, \lambda)}{p} \\ &= \frac{1}{p} \{ p g'(p, \lambda) - g(p, \lambda) \} \\ &= \xi g''(\xi, \lambda)/p \quad (0 < \xi < p) \\ &< 0, \end{aligned}$$

which is the required condition on F .

Example 2. Suppose again that $q \equiv 0$. We may take $h(x) = -x^{-1}$ to satisfy the required Riccati equation, and it is easily checked that (5) is satisfied if we take $f(x) = x^{-1}$. A possible choice for F satisfying the requisite conditions is

$$F(t, \lambda) = \lambda(1 + t^2)^{-1/2}, \quad \lambda > 0.$$

This leads to

$$G(x, p, \lambda) = \frac{1}{x} \frac{\lambda}{(1+p^2/x^2)^{1/2}} = \frac{\lambda}{(x^2+p^2)^{1/2}},$$

and the corresponding equation is

$$u'' + \frac{\lambda u}{(x^2 + u^2)^{1/2}} = 0,$$

which has been discussed by Kolodner [4] as a model for the motion of a rotating string.

Appendix

Here it is our object to prove that under the conditions of our theorem the operator

$A: X \times \mathbb{R} \rightarrow X$, defined in the introduction, is completely continuous.

Let

$$Lu(x) = \int_0^1 g(x, y) u(y) dy,$$

where g is the Green's function defined in the introduction. By the Ascoli-Arzelà theorem,

it is easy to check that $L: L'(0, 1) \rightarrow X$ is a compact linear operator. For $u \in X$, let

$$Q(u)(x) = q(x) u(x) \text{ for } x > 0$$

and

$$R(u, \lambda)(x) = G(x, u(x), \lambda) u(x) \text{ for } x > 0.$$

Then it will be sufficient to prove that $Q: X \rightarrow L'(0, 1)$ and $R: X \times \mathbb{R} \rightarrow L'(0, 1)$ are bounded and continuous.

Now, for $u \in X$,

$$\sup_{0 < x \leq 1} \frac{|u(x)|}{x} \leq \sup_{0 \leq x \leq 1} |u'(x)|,$$

and so

$$\int_0^1 |q(x) u(x)| dx \leq \left\{ \sup_{0 \leq x \leq 1} |u'(x)| \right\} \int_0^1 x |q(x)| dx,$$

proving that $Q: X \rightarrow L'(0,1)$ is a bounded linear operator.

Also, from Lemma 1, $x e^{\int_1^x h}$ is a continuous function of x on $(0,1]$, and $\lim_{x \downarrow 0} x e^{\int_1^x h}$ exists. Thus

$$N(u)(x) = u(x) e^{\int_1^x h}$$

defines a bounded linear operator from X into $C[0,1]$, and consequently $F(u(x) e^{\int_1^x h}, \lambda)$

defines a bounded and continuous mapping of $X \times \mathbb{R}$ into $C[0,1]$. Setting $M(u)(x) = f(x) u(x)$ for $x > 0$ and repeating the argument given for Q , we see that $M: X \rightarrow L'(0,1)$ is a bounded linear operator. Since

$$G(x, u(x), \lambda) u(x) = M(u)(x) F(u(x) e^{\int_1^x h}, \lambda),$$

we deduce $R: X \times \mathbb{R} \rightarrow L'(0,1)$ is bounded and continuous, and the required result is proved.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The paper is concerned with giving sufficient conditions that in the non-linear boundary-value problem $u''(x) + \{q(x) + G(x, u(x), \lambda)\}u(x) = 0$ $u(0) = 0, \quad u(1) = 0 \quad \text{or} \quad u'(1) = 0$ with there should be no secondary bifurcation, i.e. that, given a branch of solutions (u, λ) bifurcating from the trivial solution, there should be no further bifurcation on		

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that branch. Sufficient conditions on G are given which include, for example, Kolodner's problem of the motion of a heavy rotating string.

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